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Introduction

Experiments of lepton-hadron scattering are an efficient and direct tool to investigate the structure of the hadrons. The experiments are based on the principle of letting a beam of leptons scatter on the hadronic target and then measuring the angle and energy of the scattered lepton. The analysis of these empirical quantities can contribute to draw a picture of the target. Taking as an example the electron nucleon elastic case, where the final hadronic state is also a nucleon, we will arrive at the definition of Lorentz invariant structures called Form Factors, which are related to the electric and magnetic properties of the nucleon. In particular, a combination of these quantities brings us to the definition of the so called Sachs Form Factors, which can be extracted from measurements of cross sections and thus represent a direct connection to phenomenology.

The first aim of the present work is the study of these structures within the framework of Chiral Perturbation Theory (ChPT), an effective field theory with validity in the low energy domain. We will present a non-relativistic approach called Heavy Baryon Chiral Perturbation Theory (HBChPT), which is based on the idea that baryons are heavy static sources to be treated non-relativistically. By performing the calculations up to the order $O(p^3)$, we will derive the $q^2$ dependence of the form factors and we compare the results that come out of this approximation with a parametrization derived from the best fit to empirical data. This will enable us to prove the validity of the theory.

A second aim will arise when we introduce a new class of functions known under the name of Generalized Parton Distributions (GPDs). These distributions are very useful tools in the field of nuclear physics, since they are characterized by a wide range of applicability and connected to several physical and measurable quantities. Focusing on the Form Factors, we will find that Pauli and Dirac Form Factors can be defined as zeroth moments in $x$ of GPDs. At this point a generalization comes straightforward: what happens when we go further to higher moments of GPDs? Can we relate the first moments of GPDs to Generalized Form Factors as we did with the zeroth moments and the Dirac and Pauli Form Factors? With the aim of answering to these questions and by working on the properties of GPDs, we will
introduce three new $q^2$ dependent structures which are defined by analogy with the usual Form Factors and that probably hide interesting information on the structure of the nucleon.

The second step is the analysis of the Generalized Form Factors through the path we followed when dealing with Dirac and Pauli Form Factors. By making use of HBChPT, we will perform calculation up to order $\mathcal{O}(p^4)$, mainly concentrating the analysis on the forward limit. When possible, the results will be compared with phenomenology and the lattice data, even if a good correspondence between these two theories is hard to achieve.

The present work is organized as follows: the first chapter will deal with Chiral Perturbation Theory. The mathematical formalism needed to perform calculations is provided with special regards to the heavy baryon approximation. The second chapter will be dedicated to the Pauli and Dirac Form Factors, their origin, their evaluation in HBChPT and their definition starting from GPDs, together with a brief excursus on these functions. In the third chapter we will introduce the Generalized Form Factors and by analogy we will follow the same pattern that lies under the second chapter and we will try to come to interesting conclusions. Given our interest in the momentum dependence of the Generalized Form Factors, a further development of this work will be necessary, including new calculations at higher chiral order. Concluding, the present work paves the way towards more detailed studies within the frame of GPDs. Guidelines on relevant information will be given in the final outlook.
Chapter 1

HBChPT

Quantum Chromo Dynamics (QCD) is the fundamental quantum field theory of strong interactions. Thanks to asymptotic freedom high energy regions are probed by means of perturbative techniques. At low momenta and energies ($Q < 1$ GeV) the running coupling $\alpha_s(Q)$ is of order one so that an expansion in power of $\alpha_s$ is no longer practical. Moreover, QCD is written in terms of the wrong degrees of freedom (quarks and gluons), while low energy experiments are performed with hadronic bound states.

Chiral Perturbation Theory (ChPT) fills the gap left by QCD describing low-energy properties and processes in the framework of an effective field theory (EFT). This allows for a perturbative treatment in terms of momentum instead of a coupling constant expansion.

Throughout this work I will make use of the so-called Heavy-Baryon Formulation of Chiral Perturbation Theory (HBChPT), which, together with the developed theory for mesons, provides us with the required stuff.

1.1 EFT and Chiral Symmetry

The idea of effective field theories derives from a Weinberg theorem stating that a perturbative description in terms of the most general effective Lagrangian containing all possible terms compatible with assumed symmetry principles yields the most general $S$ matrix consistent with the fundamental principles of quantum field theory and the assumed symmetry principles.[16]
In the present case, this means to construct a Lagrangian which replaces the original QCD Lagrangian in the low-energy sector but which respects all the symmetries and symmetry breaking models of QCD. The most important invariance for low-energy QCD is chiral symmetry.

The strong interaction as described by the Standard Model involves six different quark flavors: three 'light' \((u,d,s)\) and three 'heavy'. The light and heavy adjectives are defined with respect to a typical hadronic scale of 1 GeV. Typical values for the light quark masses, measured at a scale of 1 GeV are \cite{17}

\[
m_d = 5 \pm 2 \text{ MeV} \quad m_s = 175 \pm 55 \text{ MeV},
\]

while \(m_c > 1 \text{ GeV}\) and \(m_b,t \gg 1 \text{ GeV}\). Working in the three-flavor sector but restricting to the lightest \(u\) and \(d\) quarks, the QCD Lagrangian takes the form:

\[
L_{QCD} = \overline{q}(i\slashed{D} - M)q - \frac{1}{2} Tr(G_{\mu\nu}G^{\mu\nu})
\]

(1.1)

where \(q^T(x) = (u(x),d(x))\) is the quark wave function, \(G_{\mu\nu}\) the gluon field strength tensor, \(M = \text{diag}(m_u,m_d)\) the quark mass matrix and \(D_\mu = (\partial_\mu - ig_sG_\mu)\) the gauge covariant derivative with \(g_s\) strong coupling constant and \(G_\mu\) the gluon field.

In this work I will always make use of the standard notation

\[
\slashed{D} = \gamma^\mu D_\mu
\]

where \(\gamma_\mu\) are Dirac matrices (see Appendix B).

Introducing the projection operators

\[
P_{L,R} = \frac{1}{2}(1 \pm \gamma_5) = \frac{1}{2} \begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix}
\]

(1.2)

and applying them to quark fields, \(q_{L,R} = (1/2)(1 \mp \gamma_5)q\), we can rewrite the quark component of the QCD Lagrangian in terms of chiral components \(q_{R,L}\):

\[
\overline{q}(i\slashed{D} - M)q = \overline{q}_L i\slashed{D}q_L + \overline{q}_R i\slashed{D}q_R - \overline{q}_L mq_R - \overline{q}_R mq_L.
\]

(1.3)
Then we can observe that in the limit as $m \to 0$

$$\mathcal{L}_{QCD} \to \bar{q}(i\slashed{D} - M)q = \bar{q}_L i\slashed{D} q_L + \bar{q}_R i\slashed{D} q_R$$

would be invariant under independent left and right handed rotations

$$q_L \to \exp(i\frac{\sigma^a}{2}\alpha^a_L)q_L, \quad q_R \to \exp(i\frac{\sigma^a}{2}\alpha^a_R)q_R, \quad (1.4)$$

where $\sigma^a$ are the Pauli matrices defined in Appendix B, generators of $SU(2)$. The following conserved chiral currents correspond to this global $SU(2)_L \times SU(2)_R$ symmetry of $\mathcal{L}^0_{QCD} \equiv \mathcal{L}_{QCD}(m = 0)$:

$$L^a_\mu = \bar{q}\gamma_\mu \frac{\tau^a}{2} q, \quad \partial_\mu L^{\mu,a} = 0, \quad (1.5)$$

$$R^a_\mu = \bar{q}\gamma_\mu \frac{\tau^a}{2} q, \quad \partial_\mu R^{\mu,a} = 0. \quad (1.6)$$

with $\tau^a = \overline{\sigma}^a$.

Instead of right- and left-handed currents we introduce the vector and axial currents

$$V^a_\mu = R^a_\mu + L^a_\mu = \bar{q}\gamma_\mu \frac{\tau^a}{2} q, \quad (1.7)$$

$$A^a_\mu = R^a_\mu - L^a_\mu = \bar{q}\gamma_\mu \gamma^5 \frac{\tau^a}{2} q \quad (1.8)$$

with parity properties

$$P: V^a_\mu(\vec{x},t) \mapsto V^a_\mu(-\vec{x},t),$$

$$P: A^a_\mu(\vec{x},t) \mapsto -A^a_\mu(-\vec{x},t)$$

and corresponding charges

$$Q^a_V = \int d^3x q^\dagger(x) \frac{\tau^a}{2} q(x), \quad Q^a_A = \int d^3x q^\dagger(x) \gamma_5 \frac{\tau^a}{2} q(x). \quad (1.9)$$

If the ground state of QCD were chirally symmetric, both vector and axial charge operators would annihilate the vacuum

$$Q^a_V|0\rangle = Q^a_A|0\rangle = 0. \quad (1.10)$$
Since $Q_V$ and $Q_A$ have different parity, one consequence of Eq. (1.10) would be the existence of parity doublets in the hadron spectrum. There would be a correspondence between states of the same mass but opposite parity. In the real world such a degeneracy is not observed. Taking as an example the $J^P = \frac{1}{2}^+$ nucleon with a mass of about 1 GeV, we experimentally know that the nearest $\frac{1}{2}^-$ resonance is 600 MeV heavier. This phenomenological evidence together with the approximate validity of $SU(2)$ flavor symmetry suggests that chiral symmetry is spontaneously broken down to its vectorial subgroup:

$$SU(2)_L \times SU(2)_R \rightarrow SU(2)_V.$$ 

According to Goldstone’s theorem this gives rise to $N_f^2 - 1 = 3$ massless pseudoscalar bosons. Because of the small quark mass terms which explicitly break the exact chiral invariance, such massless $0^-$ bosons don’t exist. The best candidates to be identified with these pseudo-Goldstone bosons are the three pions $\pi^0, \pi^\pm$, whose main properties are collected in Table 1.1.

<table>
<thead>
<tr>
<th>Particle</th>
<th>Antip.</th>
<th>$q\bar{q}$</th>
<th>mass (MeV/$c^2$)</th>
<th>S</th>
<th>C</th>
<th>B</th>
<th>$\tau(s)$</th>
<th>Main decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+$</td>
<td>$\pi^-$</td>
<td>ud</td>
<td>139.6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.60 $10^{-8}$</td>
<td>$\mu^+ \nu_\mu$</td>
</tr>
<tr>
<td>$\pi^0$</td>
<td>self</td>
<td>$\frac{u\bar{d} + d\bar{u}}{\sqrt{2}}$</td>
<td>135.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.83 $10^{-16}$</td>
<td>$2\gamma$</td>
</tr>
</tbody>
</table>

Table 1.1: Pion’s main properties

Another consequence coming from Goldstone’s theorem is the non-vanishing transition matrix element of the axial current sandwiched between the pion field and the vacuum:

$$\langle 0 | A_\mu^a(x) | \pi_b(p) \rangle = ip_\mu F \delta_{ab} e^{-ip \cdot x},$$

(1.11)

where $F$ is the pion decay constant in the chiral limit, related to the physical decay constant

$$F_\pi = (92.4 \pm 0.3)\text{MeV}$$

by

$$F_\pi = F(1 + O(m_q)).$$

Since the difference between $F_\pi$ and $F$ is linear in the small quark mass, $F_\pi$ can be identified with the chiral limit value.
The idea from which ChPT takes its origin is the development of an effective field theory with pions (i.e. the pseudo-Goldstone bosons of the theory) as the explicit degrees of freedom. Since pion interaction at low energy is very weak (and vanishes for massless pions), a perturbative approach is now possible.

### 1.2 Chiral Perturbation Theory

Chiral Perturbation Theory has been first developed for mesons ($B = 0$) and later extended to the baryon sector ($B = 1$) in order to include also pion-nucleon interactions. The Lagrangian considered here therefore reads

\[ \mathcal{L}_{\text{ChPT}} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N}. \] (1.12)

The most general effective Lagrangian has to respect all the symmetry properties of the system. In particular, in the chiral limit $\mathcal{L}_{\text{ChPT}}$ has to be invariant under $SU(2)_V \times SU(2)_A$ while the ground state has to be invariant only under the subgroup $SU(2)_V$.

#### 1.2.1 Mesonic Sector

We denote the dynamical variables representing the pion field by the matrix field $U(x) \in SU(2)_{\text{flavour}}$

\[ U = \exp \left[ \frac{i \vec{r} \cdot \vec{\pi}}{F_\pi} \right], \] (1.13)

with properties

\[ UU^\dagger = 1, \quad \det U = 1 \]

and which transforms linearly under chiral symmetry $SU(2)_L \times SU(2)_R$

\[ U \mapsto RUL^\dagger \]

with $L, R$ element of $SU(2)_{L,R}$.

Going from a theory (QCD) with quark and gluon variables to one wrt-
ten in terms of Goldstone bosons variables, we build up a chiral effective
Lagrangian $\mathcal{L}_{\text{eff}}$ organized in powers of derivatives $\partial^\mu U$:

$$\mathcal{L}_{\text{QCD}} \longrightarrow \mathcal{L}_{\text{eff}}(U, \partial^\mu U, \ldots).$$

The effective Lagrangian $\mathcal{L}_{\text{eff}}$ describes only the Goldstone bosons but in-
corporates the full chiral symmetry of QCD. Its general expression is

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{eff}}^{(4)} + \ldots, \quad (1.14)$$

where the subscripts $(n = 2, 4, \ldots)$ mark the low energy dimension, corre-
sponding to the number of derivatives. Since Lorentz invariance is required,
only terms with an even number of derivatives are allowed. Chiral orders in
the mesonic sector are therefore always even $[\mathcal{O}(p^{2n})]$.

Starting from the leading term ($n = 2$, the only term used in the present
work), the most general, chirally and Lorentz invariant Lagrangian, consis-
tent with parity, $G$ parity and charge conjugation takes the form

$$\mathcal{L}_{\pi \pi}^{(2)} = \frac{1}{4} F_\pi^2 \left\{ Tr \left[ \nabla_\mu U^\dagger \nabla^\mu U + \chi^\dagger U + \chi U^\dagger \right] \right\}. \quad (1.15)$$

The covariant derivative $\nabla_\mu U$ defined as

$$\nabla_\mu U = \partial_\mu U - i (v_\mu + a_\mu) U + i U (v_\mu - a_\mu) \quad (1.16)$$

serves the purpose of promoting the global symmetry of the effective La-
grangian to a local one and contains the couplings to the external vector
$(v_\mu)$ and axial $(a_\mu)$ fields. Like the matrix $U$ on which it acts, the covariant
derivative transforms linearly under chiral symmetry.

The field $\chi$ appearing in the Lagrangian is a linear combination of the
scalar $(s)$ and pseudoscalar $(p)$ fields

$$\chi = 2B (s + ip). \quad (1.17)$$

The constant $B$ is related to the spontaneous symmetry breaking of the
theory, measuring the strength of the chiral condensate:

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = -F_\pi^2 B (1 + \mathcal{O}(M)).$$

(1.18)

However, it is also connected to the explicit symmetry breaking via the explicit symmetry breaking part of the Lagrangian $\mathcal{L}^{(2)}_{\pi\pi}$. Setting $p = 0$, $s = M$ ($\longrightarrow \chi = 2BM$) and expanding in powers of the pion fields we find indeed:

$$\mathcal{L}^{(2)}_{SB} = \frac{1}{2} F_\pi^2 B Tr[M(u+u^\dagger)] = (m_u + m_d) B \left[ F_\pi^2 - \frac{\pi^2}{2} + \frac{\pi^4}{24 F_\pi^2} + \mathcal{O}(\pi^6) \right],$$

(1.19)

where the first term represents the vacuum energy while the second and third ones are respectively related to the meson mass and the interaction.

From equation (1.19) we can also derive the form of the pseudoscalar mass terms. In particular, assuming isospin symmetry $(m_u = m_d = \tilde{m})$ we find:

$$m_\pi^2 = 2\tilde{m}B \{1 + \mathcal{O}(M)\}.$$  

(1.20)

### 1.2.2 The Heavy-Baryon Formulation

Chiral Perturbation Theory can be extended to include baryons. The goal is the construction of a theory describing the dynamics of baryons, their interaction with the pions and external fields at low energies. In particular, we are interested in the pion-nucleon ($\pi N$) system. In analogy with the mesonic sector we search for a general structure of the effective Lagrangian with respect to all symmetry properties of the system.

We still make use of the matrix field $U(x)$ representing the pions and we introduce the isospinor $\Psi$

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

(1.21)

to collect proton ($p$) and neutron ($n$) fields. The covariant derivative of the nucleon field takes the form

$$D_\mu \Psi = \partial_\mu \Psi + \Gamma_\mu \Psi,$$

(1.22)
where
\[ \Gamma_{\mu} = \frac{1}{2} [u^\dagger, \partial_{\mu} u] - \frac{i}{2} (v_{\mu} + a_{\mu}) u - \frac{i}{2} u (v_{\mu} - a_{\mu}) u^\dagger \]  
(1.23)
is the so-called chiral connection expressed in terms of the external field \( v_{\mu}, a_{\mu} \) and of the square root of \( U \) denoted by \( u(x), u^2(x) = U(x) \).

Together with the covariant derivative, another building block acting at \( O(p) \) (like \( \Gamma_{\mu} \) it contains only one derivative) is the so-called vielbein,
\[ u_{\mu} \equiv i (u^\dagger \nabla_{\mu} u - u \nabla_{\mu} u^\dagger) = i \{ u^\dagger, \nabla_{\mu} u \} = i u^\dagger \nabla_{\mu} U u^\dagger, \]
(1.24)
which under parity transforms as an axial vector.

Since the present work deals exclusively with processes involving a single nucleon in the initial and final state and no nucleon loops, the effective Lagrangian we need reduces to
\[ \mathcal{L}_{\text{eff}}[\pi, \Psi, \bar{\Psi}] = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi\Psi} + \mathcal{L}_{\pi\Psi\bar{\Psi}} + \ldots \]
(1.25)
\[ = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N}. \]

The most general lowest order effective \( \mathcal{L}_{\pi N} \) reads
\[ \mathcal{L}_{\pi N}^{(1)} = \bar{\Psi} \left( i \gamma^\mu D_\mu - m_N + \frac{g_A}{2} \gamma^\mu \gamma_5 u_{\mu} \right) \Psi \]
(1.26)
where \( \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \), \( m_N \) is the nucleon mass and \( g_A \) the axial-vector coupling constant, both taken in the chiral limit.

We observe that unlike the mesonic case the chiral orders increase in units of one.

Working at low energy a very useful and simplified formulation of baryon ChPT is possible. In this new framework, called Heavy Baryon ChPT, baryons are treated non-relativistically. Seen as very heavy sources surrounded by a cloud of light particles (the pions), their momentum is split up into a large piece of the order of their mass and a small residual component.

Starting from the field \( \Psi \) representing the low-energy nucleon we then
decompose the four-momentum $p_\mu$ into

$$p_\mu = mv_\mu + r_\mu \quad (1.27)$$

where the four-velocity $v_\mu$ satisfies $v^2 = 1$ and $v \cdot r \ll m$. For the special choice $v_\mu = (1, 0, 0, 0)$ we have

$$v \cdot r = -\frac{r^2}{2m} = r_0 = E - m \ll m.$$  

It is useful to introduce the velocity projection operators $P_{v_\pm}$

$$P_{v_\pm} \equiv \frac{1 \pm \psi}{2} \quad (1.28)$$

which obey the relations

$$P_{v_\pm}^2 = P_{v_\pm}, \quad P_{v_\pm} P_{v_\mp} = 0, \quad P_{v_+} + P_{v_-} = 1.$$  

Applying them to the nucleon field we find the velocity-dependent fields $N_v$ and $h_v$,

$$N_v \equiv e^{-imv \cdot x}P_{v_+} \Psi, \quad h_v \equiv e^{-imv \cdot x}P_{v_-} \Psi \quad (1.29)$$

and we can write $\Psi$ as

$$\Psi = e^{-imv \cdot x}(N_v + h_v). \quad (1.30)$$

The fields $N_v$ and $h_v$ are eigenstates of the projector operators

$$\psi N = N, \quad \psi h = -h \quad (1.31)$$

and working in the framework $v_\mu = (1, 0, 0, 0)$ they reduce to the upper and lower component of the spinor in the non-relativistic reduction.

Inserting the expression (1.30) into $L_{\pi N}^{(1)}$ and dealing with the properties of the fields $N_v$, $h_v$ we find that the ”heavy” component $h_v$ is suppressed by powers of $1/m$ relative to the ”light” component $N_v$. The lowest order
Lagrangian finally reads

$$\mathcal{L}_{\pi N}^{(1)} = \overline{N_v}(i\gamma \cdot D + g_A S \cdot u)N_v,$$

(1.32)

where the spin operator $S^\mu_v = \frac{i}{2} \gamma_5 \sigma^\mu v\nu u\nu$ (for definitions and useful properties see Appendix B) in $d$ dimensions has the properties

$$S_v \cdot v = 0, \quad S_v^2 = \frac{1 - d}{4}$$

(1.33)

Comparing $\mathcal{L}_{\pi N}^{(1)}$ in HBChPT with the corresponding Lagrangian of the relativistic framework (1.25) we observe that the nucleon mass has disappeared and the spinor $\Psi$ has been replaced by the light component $N_v$. Since both the covariant derivative $D_\mu$ and the chiral vielbein $u_\mu$ count as $\mathcal{O}(p)$, the Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ is of chiral order $\mathcal{O}(p)$ too.

Applying the same ideas to the next to leading order we can derive the most general expression for $\mathcal{L}_{\pi N}^{(2)}$, which is given in Appendix C. The Lagrangian is written in terms of seven low-energy constants ($c_i$), whose value is not determined by chiral symmetry but by comparison with experimental information. As regards the present work, the main contribution of $\mathcal{L}_{\pi N}^{(2)}$ will arise in the next chapters when dealing with mass, self energy and the derivation of $Z$-factor.

HBChPT provides a Power Counting Scheme to derive the chiral dimension $D$ of a given Feynman diagram. Defining:

- $N_L \equiv$ number of independent loop momenta,
- $I_M \equiv$ number of internal pion lines,
- $N_{2n}^M$ number of pion vertices originating from $\mathcal{L}_{2n}$,
- $N_M = \sum_{n=1}^{\infty} 2n N_{2n}^M \equiv$ total number of pion vertices,
- $I_B \equiv$ number of internal nucleon lines,
- $N_n^B \equiv$ number of baryonic vertices originating from $\mathcal{L}_{\pi N}^{(n)}$,
the chiral dimension $D$ is given by the expression [30]

$$D = 4N_L - 2I_M - I_B + \sum_{n=1}^{\infty} 2n N_n^M + \sum_{n=1}^{\infty} n N_n^B.$$ (1.34)

We will often make use of this relation with the aim of finding out which diagrams contribute to a specific order. What we need to keep in mind is that $D \geq 1$ and that loops start contributing at $D = 3$. As we will see in the next paragraph, this becomes very important when dealing with the divergences of the theory.

### 1.3 Renormalization

Calculations involving one loop graphs yield divergent integrals in momentum space. As discussed in the previous paragraph, loops start to appear at order $O(p^3)$ ($D = 3$). This implies that the low-energy constants appearing in lower order Lagrangians are not affected by loop effects and are therefore finite.

Regularization of the theory is achieved by the introduction in the Lagrangian of contact interactions acting as counterterms. The low-energy constants which delete the infinities are scale dependent and are decomposed into a finite and an infinite part [14]:

$$B_i = B_i^r(\lambda) + \beta_i 16\pi^2 L,$$ (1.35)

where

$$L = \frac{\lambda^{d-4}}{16\pi^2} \left[ \frac{1}{d-4} + \frac{1}{2}(\gamma_E - 1 - \ln 4\pi) \right]$$ (1.36)

with $\lambda$ renormalization scale and $\gamma_E$ Euler-Mascheroni constant. The value of the renormalized low-energy constants $B_i^r(\lambda)$ can be fixed from a fit to observables. The function $\beta_i$ depends on coupling constants of the theory like $g_A$.

The $B_i^r(\lambda)$ at two different scales $\lambda_1, \lambda_2$ are connected by the following
relation
\[ B'_i(\lambda_2) = B'_i(\lambda_1) - \beta_i \log \frac{\lambda_2}{\lambda_1}. \]  

(1.37)

All the calculations made within the present work assume \( \lambda = 1 \text{GeV} \).

A regularized effective field theory based on chiral symmetry properties of QCD in the zero quark mass limit has been introduced. Working in a non-relativistic framework we have written the main Lagrangians describing the system in the so called Heavy Baryon approximation. We are now ready to analyse the Form Factors of the nucleon from the point of view of Heavy Baryon Chiral Perturbation Theory (HBChPT).
Chapter 2

The Form Factors of the nucleon to $\mathcal{O}(p^3)$

Elastic electron-nucleon and (anti)neutrino-nucleon scattering show that the nucleon exhibits an electromagnetic and an axial structure, which is reflected by $q^2$ dependent functions (with $q^2$ the squared momentum transfer) appearing in the interaction vertex. These functions are called Form Factors and can be determined experimentally. Since several data over a large range of momentum transfer are nowadays available, the analysis of these functions could provide important information to our understanding of the structure of the nucleon.

We study the structure of the nucleon when probed by virtual photons concentrating ourselves on the electromagnetic Form Factors in the low-energy domain, studying them in the heavy baryon formulation discussed in chapter 1.2.2 up to order $\mathcal{O}(p^3)$.

Finally, we introduce a new class of functions known under the name of Generalized Parton Distributions (GPDs) and we analyse their connection to the Form Factors. A generalization will be then necessary, linking us to the third chapter.
2.1 Definition

The quark vector current \( V_\mu^a \) = \( \bar{q} \gamma_\mu \frac{q}{2} q \) (see paragraph 1.2) sandwiched between nucleon states can be written in terms of a nucleonic current operator \( \Gamma^\mu \):

\[
\langle N(p_2)|V_\mu^a|N(p_1)\rangle = \bar{u}(p_2) \Gamma_\mu u(p_1) \times \eta^i \frac{\tau^a}{2} \eta, \tag{2.1}
\]

where \( u(p) \) is a Dirac spinor with isospin component \( \eta \). Requiring Lorentz invariance of the right hand side of the equation, \( \Gamma^\mu \) can only be a combination of gamma matrices and momenta of the system. Defining \( q = p_2 - p_1 \), \( \tilde{p} = p_1 + p_2 \), the current operator \( \Gamma_\mu \) takes the form

\[
\Gamma_\mu = \gamma_\mu \cdot A + \tilde{p}_\mu \cdot B + q_\mu \cdot C \tag{2.2}
\]

The coefficients \( A, B, C \) could be built up of terms like \( \gamma_1, \gamma_2 \) but because of Dirac equation (which states \( \gamma_1 u(p_1) = m \bar{u}(p_1), \gamma_2 (p_2) \gamma_2 = \vec{\gamma}(p_2)m \) they can be identified with numbers without loss of generality. Since the only available scalar is the squared momentum transfer \( q^2 \), the coefficients can only be function of it and we have \( A, B, C \rightarrow A(q^2), B(q^2), C(q^2) \). Applying the Ward Identity \( q_\mu \Gamma^\mu = 0 \) to eq.(2.2) we find that, when inserted in the matrix element, both first and second terms vanish, so that \( C \) must be identically null. Making then use of the Gordon Identity

\[
\bar{u}(p_2)\gamma^\mu u(p_1) = \bar{u}(p_2) \left[ \frac{\tilde{p}_\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \right] u(p) \tag{2.3}
\]

we can finally rewrite \( \Gamma_\mu \) as

\[
\Gamma_\mu = F_1(q^2)\gamma_\mu + \frac{i}{2m} F_2(q^2)\sigma_{\mu\nu}q^\nu \tag{2.4}
\]

and the matrix element of the quark vector current reads

\[
\langle N(p_2)|V_\mu^a|N(p_1)\rangle = \bar{u}(p_2) \left[ F_1^{(v,s)}(q^2)\gamma_\mu + \frac{i}{2m_N} F_2^{(v,s)}(q^2)\sigma_{\mu\nu}q^\nu \right] u(p_1) \times \eta^i \frac{\tau^a}{2} \eta, \tag{2.5}
\]

where \( m_N \) is the nucleon mass and \( F_1(q^2), F_2(q^2) \) are respectively the Dirac and Pauli Form Factors. The superscript \( ^{(v,s)} \) denotes isovector, respectively
Figure 2.1: Probabilistic interpretation of the Form Factors; $b_T$ denotes the impact parameter space (for details see the discussion in chapter 2.3.2) and $ho(b_T)$ is the distribution (for the electric case the charge distribution) coming from the three-dimensional Fourier integral of the Form Factors. [17]

isoscalar and refers to the response of the nucleon to the isovector ($\tau^{1,2,3} = \tau'$) and isoscalar ($\tau^4 = I$) components of an external vector current,

$$F^v_i = F_p - F_n, \quad F^s_i = F_p + F_n, \quad (i = 1, 2),$$

where $p, n$ stand for proton and neutron. For completeness we also introduce the so-called Sachs Form Factors $G_{E,M}^{p,n}(q^2)$ related to $F_{1,2}(q^2)$ via

$$G_E(q^2) = F_1(q^2) + \tau F_2^2(q^2)$$

$$G_M(q^2) = F_1(q^2) + F_2(q^2),$$

with $\tau = \frac{q^2}{4m_N^2}$. The Sachs Form Factors play an important role in the connection to experimental values. Measurements of elastic electron-nucleon differential cross section at low momentum transfer allow indeed a simple determination of $G_{E,M}(q^2)$. The analysis of the data on their $q^2$ dependence shows an approximate dipole behavior at low momentum transfer. As for
the proton, the experimental data can be described by the following dipole ansatz for the magnetic Form Factor [2][24].

\[ G_M^p(q^2) = \left( 1 - \frac{q^2}{0.71\text{GeV}^2} \right)^{-2} G_M^p(0) \]

and a similar behavior for the electric one.

Taking into account the given dipole fit and by Fourier transforming it to position space, one can predict charge and magnetization distributions of the nucleon. As depicted in Fig.2.1, the Form Factors have a probabilistic meaning. This important feature of the Sachs factors is revealed when working in the so called Breit frame. Denoting with \( p(p') \) the electron momentum before (after) the scattering and with \( P(P') \) the same quantity for the nucleon we have:

\[
\begin{align*}
\vec{p} &= +\vec{q}/2, \quad \vec{P} = -\vec{q}/2 \\
\vec{p}' &= -\vec{q}/2, \quad \vec{P}' = +\vec{q}/2
\end{align*}
\]

This implies that the four momentum transfer has components \( q^\mu = (0, \vec{q}) \) and \( Q^2 = \vec{q}^2 \). In such a reference frame no energy is transfered during the process, the momentum transfer is purely spacelike. The Sachs factors \( G_E \) and \( G_M \) studied in this domain are nothing but the Fourier-transforms of the charge and the magnetization distribution, respectively.

Their normalization at \( Q^2 = -q^2 = 0 \) is given by the nucleon charges and magnetic moments [28]:

\[
\begin{align*}
G_E^p &= 1, & G_E^n &= 0 \\
G_M^p &= \mu_p = 2.793, & G_M^n &= \mu_n = -1.913
\end{align*}
\]

where \( \mu_p, n \) are given in units of the nuclear magneton \( \mu_N = e/2m_p \). Keeping in mind the relation (2.7), (2.8), the Form Factors \( F_1, F_2 \) normalizations therefore read

\[
\begin{align*}
F_1^p(0) &= 1, & F_1^n(0) &= 0 \\
F_2^p(0) &= \kappa_p = \mu_p - 1, & F_2^n(0) &= \kappa_n = \mu_n
\end{align*}
\]
and in particular
\[ F_1^v(0) = 1 \quad F_2^v(0) = \kappa_v \]  
with \( \kappa_v = 3.71 \) the isovector nucleon anomalous magnetic moment.

The slope of the Form Factors at \( q^2 = 0 \) is usually expressed in terms of a nucleon radius \( \langle r^2 \rangle^{1/2} \),
\[ F_i(t) = F(0) \left( 1 + \frac{1}{6} \langle r_i^2 \rangle q^2 + \mathcal{O}(q^4) \right), \]
\[ \langle r_i^2 \rangle = \frac{6}{F_i(0)} \left. \frac{dF_i(q^2)}{dq^2} \right|_{q^2=0} \]

For an infinitely heavy, spherical charge distribution in the Breit frame the nucleon radius coincides with the usual definition
\[ \langle r^2 \rangle = 4\pi \int_0^\infty dr \, r^2 \rho(r). \]

Empirical values for the proton charge and magnetic radii and for the magnetic one of the neutron are [19] [27] [24]:
\[ \langle r_E^2 \rangle_{p}^{1/2} = (0.86 \pm 0.01) \text{ fm}, \quad \langle r_M^2 \rangle_{p}^{1/2} = (0.86 \pm 0.06) \text{ fm}, \]
\[ \langle r_M^2 \rangle_{n}^{1/2} = (0.89 \pm 0.07) \text{ fm} \]
which are all equal within experimental errors. Because of the normalization (2.9), the electric mean square radius for the neutron is defined without \( 1/G_E^n(0) \). The measurements of this quantity provide interesting results. We have
\[ 6 \left. \frac{dG_E^n(q^2)}{dq^2} \right|_{q^2=0} = (-0.113 \pm 0.003 \pm 0.004) \text{ fm}^2 \]
where together with experimental error systematic uncertainties are also given. Unlike \( G_E^p \), this slope shows a positive sign.

The analysis of the moments of the Form Factors with respect to \( q^2 \) therefore provides a new connection to phenomenology and could help with the understanding of the low energy structure of the nucleon.
### 2.2 Isovector vector Form Factors

We focus our research on the isovector vector Form Factors of the nucleon. As derived in the previous paragraph, the isovector component of the quark vector current sandwiched between nucleon states gives the expression

\[
\langle N(p_2)|V^a_{\mu}|N(p_1)\rangle = \bar{u}(p_2) \left[ F^a_1(q^2)\gamma_{\mu} + \frac{i}{2m_N} F^a_2(q^2)\sigma_{\mu\nu}q^\nu \right] u(p_1) \times \eta^+ \frac{\tau^a}{2} \eta ,
\]

where the Dirac and Pauli Form Factors obey

\[
F^a_1(0) = 1 \quad F^a_2(0) = \kappa_a .
\]

In the representation we used, the explicit form of the Dirac spinor \( u(p) \) describing the nucleon (for completeness see Appendix B) reads

\[
u_s(p) = \sqrt{\frac{E + m_N}{2m_N}} \left( \begin{array}{c} \chi_s \\ \frac{\chi_s}{E + m_N} \end{array} \right) ,
\]

with \( \chi_s \) two-components Pauli spinors

\[
\chi_{s=+1/2} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) , \quad \chi_{s=-1/2} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) ,
\]

and the normalization

\[
\bar{\nu}_s(p)\nu_{s'}(p) = u^\dagger_s(p)\gamma_0 u_{s'} = \delta_{ss'} .
\]

In order to take into account other possible normalizations which people could use studying the same subject in a different framework we will always write:

\[
u_s(p) = \mathcal{N} \left( \begin{array}{c} \chi_s \\ \frac{\chi_s}{E + m_N} \end{array} \right) .
\]

#### 2.2.1 Non-relativistic reduction

By adopting a non-relativistic approach we choose to work in the Breit frame and we treat the nucleons in Heavy Baryon approximation. Making use of
the heavy-mass decomposition

\[ p_1^\mu \rightarrow \nu^\mu + r_1^\mu \quad p_2^\mu \rightarrow \nu^\mu + r_2^\mu \]  

(2.24)

and defining

\[ u_\nu(r) = P_v^\dagger u(p) = \frac{1}{2} (1 + \gamma^\nu) u(p) \]  

(2.25)

we then find

\[ \langle N(p_2)|V_{\mu}\rangle N(p_1) = \frac{1}{\mathcal{N}_1 \mathcal{N}_2} \pi_\nu(r_2) \left[ \bar{G}_E(q^2) v_\mu + \frac{1}{m_N} \bar{G}_M(q^2) [S_\mu, S_\nu] q^\nu \right] u_\nu(r_1) \times \eta^\dagger \frac{\gamma^i}{2} \eta, \]  

(2.26)

where

\[ \mathcal{N}_i = \sqrt{\frac{E_i + m_N}{2m_N}}. \]

Assuming \( v^\mu = (1, 0, 0, 0) \) the factors \( \bar{G}_E(q^2), \bar{G}_M(q^2) \) in the matrix element are nothing but the isovector component of the electric and magnetic Sachs Form Factors \( G_E(q^2), G_M(q^2) \), whose relation to \( F_1^v(q^2), F_2^v(q^2) \), as reported in the first paragraph, reads

\[ G_E^v(q^2) = F_1^v(q^2) + \frac{q^2}{4m_N^2} F_2^v(q^2), \]  

(2.27)

\[ G_M^v(q^2) = F_1^v(q^2) + F_2^v(q^2). \]  

(2.28)

### 2.2.2 Chiral diagrams matching results

The aim is to study the Form Factors of the nucleon from the point of view of HBChPT. The calculations are performed up to order \( \mathcal{O}(p^3) \). All the formalism required to this purpose has been introduced in the first chapter and is also collected in Appendix C. The building blocks providing all the tools we need are the effective Lagrangians \( \mathcal{L}^{(2)}_{\pi\pi} \) and \( \mathcal{L}^{(1),(2)}_{\pi N} \).

The first step consists in selecting the loop diagrams up to the third chiral
order and calculating them, for example with the help of Feynman rules. The vertices and relative Feynman rules used in this work can be found in Appendix C. As an example we will apply the power counting scheme of HBChPT to the following graph, which represents a nucleon probed by a virtual photon interacting with the pion cloud. Given the assumed definitions and the formula (1.34) we have:

\[ N_L = 1, I_M = 2, I_B = 1, N_{21}^M = 1, N_1^B = 2 \]

and \[ D = 4 \cdot 1 - 2 \cdot 2 - 1 + 2 \cdot 1 \cdot 1 + 2 \cdot 1 = 3. \]

Among the one-loop graphs selected by the power counting scheme the non-vanishing ones are depicted in Fig.2.2

We underline the fact that a diagram can give a null contribution not only because of the presence of vanishing vertices, but also because the evaluation of the whole integral could give a null result. This is the case for example of the following diagram:

1 pion-1 photon vertex from \( \mathcal{L}_{1N}^{(1)} \):

\[ \frac{i\epsilon g_{\mu}^{a}}{\not{p}_x} \cdot S e^{a_{\nu} b} S_{\nu}^{-b} \cdot \epsilon. \]

The vertex is non-zero but the loop vanishes for parity reasons.
Figure 2.3: One-loop contribution to the nucleon self-energy to order $q^3$.

The expression of the non-zero loop diagrams are reported in Appendix E. Our interest turns to their $q^2$ dependence: we can observe that the only two graphs containing such a contribution are $a)$ and $d)$, where the photon interacts exclusively with the pion cloud. We point out that together with the explicit $q^2$ dependence, there is also an implicit one due to the definition $\tilde{m} = m^2 - q^2 x (1-x)$ (with $x \in [0,1]$) introduced by the Feynman parameter in the evaluation of the integrals).

Another term to take into account is the self energy $\Sigma(\omega)$ of the nucleon, with $\omega = \nu \cdot r$ (for details see the expression (C.5) of the nucleon propagator). The only diagram of Fig.2.3 which contributes to the self energy is $e)$. The tadpole diagram $f)$ involves indeed an integration over an odd power of the loop momentum giving a vanishing result. Other pieces which contribute to the self energy come from the tree-level terms of $L^{(2)N}$. Neglecting all the structures connected to photons and pions and working in the isospin limit ($m_u = m_d$), the only non-zero piece is $c_1 Tr \chi_+$. Taking into account the renormalization scheme and introducing the appropriate counterterms, the mass shift $\Sigma(\omega)$ takes the form given in Appendix E [4].

The self energy is responsible for a shift of the nucleon mass, so that the full nucleon propagator has a pole at the physical nucleon mass $m_N = m_0 + \Sigma(\omega)$. The $Z$ factor is the nucleon wave function renormalization and is given by the residue of the propagator at the pole. Its $O(p^3)$ expression reads

$$Z_N = 1 + \Sigma'(0),$$

so that [4]

$$Z_N^{(3)} = 1 - \frac{1}{(4\pi F_p)^2} \left\{ \frac{3}{2} g_A^2 m^2 + 8 m^2 B_{20}(\lambda) + \frac{9}{2} g_A^2 m^2 \log \left[ \frac{m}{\lambda} \right] \right\}$$

(2.30)

where all the constants were first introduced in chapter 1 (Eq. (1.11), (1.26))
and with the low energy constant [14]

\[ B_{20} = B_{20}^r(\lambda) + \beta_{20} 16\pi^2 L, \quad \beta_{20} = -\frac{9}{16} g_A^2. \]  

(2.31)

The need for renormalization leads to the introduction of another low energy constant, in order to delete all the infinities taking origin from the diagrams \(a) - d\). It shows the structure [14]

\[ B_{10} = B_{10}^r(\lambda) + \beta_{10} 16\pi^2 L, \quad \beta_{10} = -\frac{1}{6} - \frac{5}{6} g_A^2 \]  

(2.32)

and as we will see its value can be derived from empirical considerations.

All the terms contributing to \(G_E^u(q^2), G_M^u(q^2)\) have been collected. Summing up all the pieces we then find the expression of the electric and magnetic isovector Form Factors [4]

\[
G_E^u(q^2) = 1 + \kappa_v \frac{q^2}{4M_N^2} + \frac{1}{(4\pi F_\pi)^2} \left\{ q^2 \left( -\frac{2}{3} g_A^2 - 2B_{10}^{(r)} \right) + q^2 \left( -\frac{5}{3} g_A^2 - \frac{1}{3} \right) \log \left[ \frac{m_\pi}{\lambda} \right] ight. \\
+ \left. \int_0^1 dx \left[ m_\pi^2 (3g_A^2 + 1) - q^2 x (1 - x) \left( 5g_A^2 + 1 \right) \right] \log \left[ \frac{m_\pi^2}{m_\pi} \right] \right\} 
\]  

(2.33)

\[
G_M^u(q^2) = 1 + \kappa_v - g_A^2 \frac{4\pi M_N}{(4\pi F_\pi)^2} \int_0^1 dx \left\{ \sqrt{m^2 - m_\pi} \right\}
\]  

(2.34)

Remembering the relation between \(F_i(q^2)\) and \(G_i(q^2)\), we are now able to extract the isovector Form Factors. Since we are working at order \(O(p^3)\), the relation (2.7), (2.8) reduce to [3]

\[
G_E^u(q^2) = F_1^u(q^2) + \frac{q^2}{m_N^2} F_2^u(0) + O(1/m_N^3) 
\]  

(2.35)

\[
G_M^u(q^2) = \frac{1}{m_N} F_1^u(0) + \frac{1}{m_N} F_2^u(q^2) + O(1/m_N^3) 
\]  

(2.36)

and the Dirac and Pauli Form Factors evaluated in the framework of HBChPT
Finally read [4]

\[ F_1^u(q^2) = 1 + \frac{1}{(4\pi F_\pi)^2} \left\{ q^2 \left( \frac{2}{3}g_A^2 - 2B_{10}^{(r)} \right) + q^2 \left( \frac{5}{3}g_A^2 - \frac{1}{3} \right) \log \left[ \frac{m_\pi}{\lambda} \right] \right. \]

\[ + \int_0^1 dx \left[ m_\pi^2 \left( 3g_A^2 + 1 \right) - q^2 x(1-x) \left( 5g_A^2 + 1 \right) \right] \log \left[ \frac{\tilde{m}^2}{m_\pi^2} \right] \}

(2.37)

\[ F_2^u(q^2) = \kappa_u \left\{ 1 - g_A^2 \frac{4\pi M_N}{(4\pi F_\pi)^2} \int_0^1 dx \left[ \sqrt{\tilde{m}^2 - m_\pi^2} \right] \right. \}

(2.38)

The results for \(|q|^2 \leq 0.4\text{GeV}^2\) are plotted in Fig.2.4 and Fig.2.5, together with the empirical parametrization of [24] obtained thanks to the best fit to the available set of Form Factors data.

The accordance between the data and our results is acceptable but not really satisfying. A better theory could be obtained adding a new degree of freedom, the \(\Delta(1232)\) [3]. The \(\Delta(1232)\) is indeed a nucleon resonance lying near the ground state and it is also strongly coupled to the pion-nucleon-photon system. Moreover, working in the limit of an infinite number of colours, the nucleon and the delta resonance are mass-degenerate.

The present study is only performed to leading-one-loop-order, the agree-
Figure 2.5: Pauli form factor $F_2^\nu(q^2)$ from $\mathcal{O}(p^3)$ HBChPT calculation (solid line) in comparison with the empirical parametrization of [16] (dashed line).

The values of the physical constants used for ChPT results are taken from [3] and are reported in Appendix A. In particular, $B_{10}$ is connected to the phenomenological value of the isovector Dirac radius $r_1^\nu$. Remembering its definition (2.16), we then find [5]:

$$
(r_1^\nu)^2 = 6 \frac{dF_2^\nu(q^2)}{dq^2} \bigg|_{q^2=0}
= -\frac{1}{(4\pi F_\pi)^2} \left\{ 1 + 7g_A^2 + (10g_A^2 + 2)\log\left(\frac{m_\pi}{\lambda}\right) \right\} - \frac{12B_{10}^{(r)}(\lambda)}{(4\pi F_\pi)^2}. \quad (2.39)
$$

In order to reproduce the empirical value $(r_1^\nu)^2 = 0.585 \text{ fm}^2$ [24], we have to set $B_{10}^{(r)}(1\text{GeV}) = 0.072$.

In analogy to the isovector form factor, an analogous discussion can be raised for the isoscalar component too. For more details see [3]. However, this thesis focuses on the isovector form factors, as Lattice QCD at present can only produce reliable results in this channel, but not for isoscalar quantities.
2.3 GPDs and Form Factors

Another way to introduce Form Factors is by means of a new class of functions called Generalized Parton Distributions (GPDs).

2.3.1 Definition

As their name suggests, GPDs can be seen as the generalization of the well known Parton Distributions, first introduced by Feynman as phenomenological quantities describing the properties of the nucleon manifest in high-energy scattering. They appear when studying processes such as deep inelastic scattering; they do not depend on the process, but they universally characterize the hadron. Working in a frame where the nucleon has a very large momentum (Infinite Momentum Frame), parton distributions give the probability of finding a parton carrying a fraction $x$ of the longitudinal momentum of the nucleon. Therefore, their probabilistic interpretation provides the distribution of longitudinal momentum and polarization exhibited by quarks, antiquarks and gluons in a fast moving hadron (see Fig.2.6).

PDs are however essential but not sufficient tools to investigate the structure of the nucleon. For example, they give no information about the distribution of partons in the plane transverse to the direction in which the hadron is moving and they do not explain how the orbital momentum of the partons contributes to the total spin of the nucleon. Generalized Parton Distributions try to fill up this lack of information towards a complete and satisfying representation of the nucleonic system.

In order to get an intuitive understanding of what GPDs describe and how they are a generalization of PDs, we will apply once again the case of inclusive deep inelastic scattering $e\,p \rightarrow e\,X$. Applying factorization, the dynamic of the process can be split up into a hard partonic subprocess, accessible in perturbation theory, and a soft one incorporated by parton distribution. Thanks to the optical theorem, the $\gamma^*\,p$ cross section is related to the imaginary part of the forward Compton amplitude $\gamma^*\,p \rightarrow \gamma^*\,p$, whose leading order diagram is depicted in Fig.2.7a. The lower blob in the handbag diagram offers a representation of the idea underlying PDs. If we consider the case of finite momentum transfer to the target with the final photon in
Figure 2.6: Probabilistic interpretation of parton distributions in the infinite momentum frame. \( f(x) \) is the probability of finding a parton with fractional momentum \( x \) in the fast moving nucleon at a given transverse resolution \( \delta z_\perp = 1/Q \) [17]

...
the amplitude for the process that a quark is taken out of the nucleon with a certain longitudinal momentum fraction and then inserted back into the nucleon with a four-momentum transfer.

The definition of GPDs in terms of mathematical tools is usually given in light-cone coordinates, providing a straightforward derivation of their properties and physical interpretation. A generic four-vector $v$ written in this system of coordinates takes the form

$$v = (v^+, v^-, v)$$

with

$$v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3), \quad v = (v^1, v^2).$$

We make use of the following notation:

$$\vec{p} = \frac{p + p'}{2}, \quad \Delta = p' - p, \quad t = \Delta^2$$

and we write the fraction of longitudinal momentum $\xi$, also called skewedness as

$$\xi = \frac{p^+ - p'^+}{p^++p'^+},$$

whose physical region is the interval $[-1, 1]$. We then define the generalized

Figure 2.7: a) Handbag diagram for the forward Compton amplitude $\gamma^*p \rightarrow \gamma^* p$ . b) Handbag diagram for DVCS in the region $\xi < x < 1$. [12]
quark distribution \[12\]

\[
F^q = \frac{1}{2} \int \frac{dz^- e^{i\zeta z^-}}{2\pi} \frac{d\zeta^+}{\zeta^+} \frac{d\zeta}{\zeta} \bigg| \langle p'| \bar{\pi} (-\frac{1}{2} z^+)^+ q(\frac{1}{2} z^-)|p \rangle \bigg|_{z^+=0, z=0}
= \frac{1}{2p^+} \left[ H^q(x, \xi, t)\pi(p')^+u(p) + E^q(x, \xi, t)\pi(p')^+ \frac{i\sigma^{+\alpha} \Delta^\alpha}{2m} u(p) \right],
\]

with \(H^q(x, \xi, t), E^q(x, \xi, t)\) GPDs for the quark sector (analogous definition can be introduced also for gluons, involving axial structures). We note that the GPDs depends on the invariant \(t\), the skewedness \(\xi\) and the kinematical variable \(x\), which is defined in the interval \([-1, +1]\) and whose value with respect to \(\xi\) selects the kind of process which is studied. Since in all known processes where GPDs may be measured, one has \(\xi \geq 0\), in the following we will assume that \(\xi\) is not negative. Three different regions (see Fig.2.8) can be sketched out:

- for \(x \in [\xi, 1]\) both momentum fractions \(x + \xi, x - \xi\) are positive and the distribution represents the emission and reabsorption of a quark. This is what happens for example in Compton scattering;

- for \(x \in [-\xi, \xi]\), \(x + \xi\) is positive while \(x - \xi\) is negative. The second momentum fraction can be interpreted as belonging to an antiquark with momentum fraction \(\xi - x\) and also emitted from the initial proton. This process can be found in pion electroproduction;

- for \([-1, -\xi]\), both the two momentum fraction are negative. This situation happens in processes involving emission and reabsorption of antiquarks with momentum fraction \(\xi - x\) and \(-\xi - x\).
2.3.2 Basic properties and relation to Form Factors

The off-forward parton distributions own several properties and can be applied in many different contexts. To this thesis’s purpose, the most relevant properties of GPDs can be found working in the limit $\xi = 0$ and analysing the so called Mellin moments in $x$.

The connection between GPDs and PDs is made explicit in the forward limit corresponding to $\xi = t = 0$. In this situation $\Delta = 0$, $p' = p$, the definition of GPDs (2.42) then reduces to the more usual expression of parton distribution and we have the following reduction formula:

$$H^q(x, 0, 0) = q(x).$$

(2.43)

In this framework $H^q$ is nothing but the ordinary quark density $q(x)$. For the distribution $E^q$ an analogous relation does not exist, because, as can be seen from the expression (2.42), they are multiplied with a factor proportional to $\Delta$, thus vanishing in this limit. This means that information about this distribution is accessible only in processes involving a finite momentum transfer.

If we assume the transfer of purely transverse momentum (case $\xi = 0$ but $t \neq 0$, so that $\Delta = (0, \overline{\Delta_\perp})$), we discover that GPDs regain a probabilistic interpretation [12]. This becomes clear when going from the momentum to the impact parameter space. We recall that the impact parameter $b_\perp$ is defined as the $\perp$ distance from the center of longitudinal momentum in the IMF. Burkardt has shown that $H^q$, only term surviving in this situation, can be interpreted as Fourier transform of impact parameter dependent PDs [7]:

$$H^q(x, 0, -\overline{\Delta_\perp}^2) = \int d^2r_\perp f(x, b_\perp)e^{-\overline{\Delta}_\perp \cdot b_\perp}.$$  

(2.44)

As the electric Form Factor can be identified with the Fourier transform of a charge distribution in the position space, GPDs for $\xi = 0$ can be identified with Fourier transforms of the function $f(x, b_\perp)$, which gives the probability of finding a parton of momentum fraction $x$ at the impact parameter $b_\perp$, as depicted in Fig.2.9. This implies further knowledge about the spatial distribution of the parton inside the hadron. A new kinematic dimension has
been added, allowing for a deeper insight into the structure of the nucleon.

The connection between Form Factors and GPDs becomes evident when integrating the matrix element (2.42) over $x$. This can be intuitively seen by comparing the formula (2.42) with the matrix element of the quark current (2.5). If we look at $\bar{q}(-\frac{1}{2}z)\gamma^\nu q(\frac{1}{2}z)$ as a generalization of the quark current $\bar{q}\gamma_\mu q$, we can find a kind of parallelism between the structure containing $H^q, E^q$ and those including the Form Factors $F_1, F_2$. From the analytical point of view, integrating over $x$ we have:

\[
\frac{1}{2} \int_{-1}^{+1} dx \int \frac{dz^-}{2\pi} e^{iz^+ z^-} \langle p' | \bar{q}(-\frac{1}{2}z)\gamma^\nu q(\frac{1}{2}z) | p \rangle \bigg|_{z^+ = 0, z_\perp = 0} = \frac{1}{2} \int \frac{dz^-}{2\pi} \delta(p^+ z^-) \langle p' | \bar{q}(-\frac{1}{2}z)\gamma^\nu q(\frac{1}{2}z) | p \rangle \bigg|_{z^+ = 0, z_\perp = 0} = \langle p' | \bar{q}(0)\gamma^\nu q(0) | p \rangle .
\]

Keeping in mind (2.42) and (2.5) we can finally make the identification of

Figure 2.9: Probabilistic interpretation of generalized parton distributions at $\xi = 0$ in the infinite momentum frame (see in the text and Fig.2.1 and Fig.2.6 for further explanations).[17]
the $0^{th}$ moments of GPDs with the usual $F_1, F_2$ Form Factors:

$$\int_{-1}^{1} dx \, H^q(x, \xi, t) = F^q_1(t),$$  \hfill (2.46)

$$\int_{-1}^{1} dx \, E^q(x, \xi, t) = F^q_2(t).$$  \hfill (2.47)

The GPD $H^q$ is therefore related to the charge Form Factor, while the GPD $E^q$ to the magnetic one, appearing in quark helicity spin flip processes.

A new device useful to go further with the study of the hadron structure has been presented. GPDs are an exclusive tool which can be exploited in many different ways. I reported the basic properties which are useful to the present work, finding out another mode to introduce the Form Factors.
Chapter 3

Generalized Form Factors to $\mathcal{O}(p^4)$

The well known Form Factors can be identified with the 0th $x$-moment of GPDs. The study of higher Mellin moments of GPDs suggests the introduction of new structures called Generalized Form Factors, obtained by matrix elements of local twist-two operators. Starting from these considerations we will take the same path used for Form Factors. Performing a non-relativistic reduction we will determine the analogous of the Sachs factors and try to extract them from calculations of HBChPT up to order $\mathcal{O}(p^4)$. Limiting our study to the forward case ($q^2 = 0$), we will analyse the possible connections to phenomenology with reference to recent lattice results.

3.1 Definition

The definition of Generalized Form Factors comes from the following relation, which connects $x$-moments of GPDs to derivative operators between the quark fields (as in chapter 2, we restrict the research to the quark sector)[12]:

$$ (\bar{\nu}^+)^{(n+1)} \int dx \; x^n \int \frac{dz^-}{2\pi} e^{ix\bar{\nu}^+ z^-} \left[ \bar{\nu}^{-1/2} z^+ \gamma^+ q(\frac{1}{2}z^-) \right]_{z^+ = 0, z^- = 0} = (i \frac{d}{dz^-})^n \left[ \bar{\nu}^{-1/2} z^+ \gamma^+ q(\frac{1}{2}z^-) \right]_{z^- = 0} = \bar{\nu}(0) \gamma^+ (i \overset{\rightarrow}{\partial}_\mu)^n q(0) $$

(3.1)
with \( \overrightarrow{\partial}^\mu = \frac{1}{2}(\overrightarrow{\partial}^\mu - \overrightarrow{\partial}^\mu) \).

Since we are requiring local gauge invariance, we will make use of the same expression with the replacement \( \partial^\mu \rightarrow D^\mu \), with \( D \) denoting the covariant derivative introduced in Eq.(1.1). In the following we will refer to Mellin moments, which for a generic function \( f(x) \) are defined as \( f_n(x) = \int_{-1}^1 dx x^n f(x) \). (The zeroth moment in \( x \) will then correspond to the first Mellin moment and so on).

We now ask for an irreducible representation of the Lorentz group on the right side of Eq.(3.1). To this purpose we symmetrize in all indices and we make the result traceless on all pairs of indices by subtracting all possible contractions between any two elements (obtained by multiplication with \( g^{\mu_i \nu_j} \)). We finally conclude that higher Mellin moments in \( x \) are Form Factors of the local twist-two operator \([12]\)

\[
O^{(n)}_{\mu, \mu_1 \ldots \mu_n} = \overrightarrow{\gamma}(i \overrightarrow{D}_{\mu_1}) \ldots (i \overrightarrow{D}_{\mu_n}) q - traces , \quad (3.2)
\]

where symmetrization on all uncontracted Lorentz indices is denoted by \( \{ \} \).

Developing the theory for the \( n^{th} \) moment, the most general decomposition of the matrix element reads \([12]\)

\[
\langle p' | O^{(n)}_{\mu, \mu_1 \ldots \mu_n} | p \rangle = \sum_{i=0, \text{even}}^n A^q_{n+1,i}(t) \overrightarrow{\pi}(p') \gamma_i u(p) \Delta_{(\mu_1 \ldots \Delta_{\mu_i}} \overrightarrow{\pi}_{\mu_{i+1} \ldots \mu_n)} \\
- \sum_{i=0, \text{even}}^n B^q_{n+1,i}(t) \frac{i}{2m_N} \overrightarrow{\pi}(p') \Delta^\alpha \sigma_{\alpha \mu_{(\mu_1} \ldots \Delta_{\mu_i}} \overrightarrow{\pi}_{\mu_{i+1} \ldots \mu_n)} u(p) \\
+ \frac{1}{m} \overrightarrow{\pi}(p') u(p) mod(n, 2) C^q_{n+1,i}(t) \Delta_{(\mu \Delta_{\mu_1} \ldots \Delta_{\mu_n})} , \quad (3.3)
\]

where \( t = \Delta^2 = (p' - p)^2 \) is the squared momentum transfer, \( \overrightarrow{\pi} = \frac{1}{2}(p' + p) \) and \( mod(n, 2) \) is 1 and 0 for odd and even \( n \) respectively. We note that time reversal invariance implies the presence of only even powers of the vector \( \Delta^\alpha \). Recalling the definition of the skewness given in Eq.(2.41) we find that the light cone component \( \Delta^+ \) of the momentum transfer is related to the light cone component \( \overrightarrow{p}^+ \) of the mean proton momentum by \( \Delta^+ = -2\xi \overrightarrow{p}^+ \).
Keeping in mind this relation, together with Eq.(3.1) and the Form Factor decomposition (3.3), we can write the $n^{th}$ moments of GPDs as [12]

$$
\int_{-1}^{1} dx \, x^n \, H^q(x, \xi, t) = \sum_{i=0, \text{even}}^{n} \frac{(2\xi)^i}{2} A^q_{n+1,i}(t) + mod(n, 2)(2\xi)^{n+1}C^q_{n+1,i}(t),
$$

(3.4)

$$
\int_{-1}^{1} dx \, x \, E^q(x, \xi, t) = \sum_{i=0, \text{even}}^{n} \frac{(2\xi)^i}{2} B^q_{n+1,i}(t) - mod(n, 2)(2\xi)^{n+1}C^q_{n+1,i}(t).
$$

(3.5)

We now limit our interest to the first moment of GPDs; by the substitution $n = 1$, the matrix element of the operator $O^q_{\{\mu\nu\}}$ takes the form

$$
\langle p' | O^q_{\{\mu\nu\}} | p \rangle \equiv \frac{i}{2} \langle p' | \overline{\gamma}_\nu (\mu \overline{D}_\mu) q | p \rangle
= A^q_{2,0}(\Delta^2) \, \overline{u}(p') \gamma_\nu (\mu \overline{D}_\mu) u(p)
- B^q_{2,0}(\Delta^2) \frac{i}{2m_N} \, \overline{u}(p') \Delta^\alpha \sigma_\alpha (\mu \overline{D}_\mu) u(p)
+ C^q_{2,0}(\Delta^2) \frac{1}{m_N} \, \overline{u}(p') \Delta (\mu \Delta_\nu). 
$$

(3.6)

and the first moments of GPDs thus read

$$
\int_{-1}^{1} dx \, x \, H^q(x, \xi, t) = A^q_{2,0}(t) + \xi^2 C^q_{2,0}(t),
$$

(3.7)

$$
\int_{-1}^{1} dx \, x \, E^q(x, \xi, t) = B^q_{2,0}(t) - \xi^2 C^q_{2,0}(t).
$$

(3.8)

Focusing on their momentum dependence, we can first observe that in the forward limit $\xi = t = 0$ the second term of the right-hand side equation vanishes. Remembering the connection between GPDs and PDs embodied by the formula (2.43), we find that the first moment of $H^q$ is nothing but the first moment of the unpolarized parton distribution:

$$
A^q_{2,0}(0) = \langle x_q \rangle = \int_{0}^{1} dx \, x (q,(x) + q,(x)),
$$

(3.9)

where $q,(x)$ and $q,(x)$ are the quark distribution with parallel and antiparallel
spin to the spin of the nucleon respectively. This relation represents an important link to phenomenology and to lattice calculations. The empirical value of \( \langle x_q \rangle \) is indeed known and lattice data are also available. As we will see in the next paragraphs, since lattice and chiral theories cover different energy regions, an acceptable and satisfying linkage between the two theories is hard to achieve.

By analogy with Form Factors, an expansion of the Generalized Form Factors with respect to the squared momentum \( t \) suggests the introduction of corresponding radii (here explicitly written for \( A_{2,0} \) but the same statement can obviously be extended to \( B_{2,0} \) and \( C_{2,0} \)):

\[
A_{2,0}^q(t) = A_{2,0}^q(0) \left[ 1 + \frac{1}{6} (r_{A,2}^2) t + O(t^2) \right] \tag{3.10}
\]

and consequently

\[
\langle r_{A,2}^2 \rangle = \frac{6}{A_{2,0}^q(0)} \left. \frac{dA_{2,0}^q(t)}{dt} \right|_{t=0} \tag{3.11}
\]

The slopes/radii of the Generalized Form Factors provide new insight into the structure of the nucleon, especially when they can be compared with the (experimentally) known radii of the electromagnetic Form Factors discussed in chapter 2.

### 3.2 Non relativistic reduction

Since we are working within a non-relativistic framework, we need to perform a non-relativistic reduction of the matrix element (3.6). As usual, working in the Breit frame and adopting the heavy mass decomposition, we arrive at the formula:

\[
\langle p' \mid O_{\mu \nu}^q \mid p \rangle = \frac{1}{N_1N_2} \bar{u}_p r_2 \left\{ G_1(t)(v_\mu v_\nu + v_\nu v_\mu - \frac{1}{2} g_{\mu \nu}) + G_2(t)([S_\mu, S_\nu] \Delta^\alpha v_\nu + v_\mu [S_\nu, S_\alpha] \Delta^\alpha) + G_3(t)(\Delta_\mu \Delta_\nu + \Delta_\nu \Delta_\mu - \frac{1}{2} g_{\mu \nu} \Delta^2) \right\} u_p(r_1) \times \frac{\Gamma^a}{2} \eta^a \tag{3.12}
\]
where the three structures $G_i(t)$ are combinations of the isovector $A_{2,0}^v(t), B_{2,0}^v(t), C_{2,0}^v(t)$ as follows:

$$G_1(t) = \langle \bar{u} u \rangle \sqrt{m_N^2 - \frac{t}{4}} \left[ A_{2,0}^v(t) + \frac{t}{4m_N^2} B_{2,0}^v(t) \right], \quad (3.13)$$

$$G_2(t) = \langle \bar{u} u \rangle \sqrt{m_N^2 - \frac{t}{4}} \left[ \frac{1}{m_N} (A_{2,0}^v(t) + B_{2,0}^v(t)) \right], \quad (3.14)$$

$$G_3(t) = \langle \bar{u} u \rangle \left[ \frac{1}{m_N} C_{2,0}^v(t) \right]. \quad (3.15)$$

We prefer not to specify the normalization and we write $\langle \bar{u} u \rangle$ at a general level. The relation between $\langle \bar{u} u \rangle$ and $N_1N_2$ comes straightforward by recalling the definition of the nucleon spinor given in Eq.(2.23). In order to have dimensionless Form Factors, the prefactor $\sqrt{m_N^2 - t/4}$, which in the forward limit ($t = 0$) reduces to the nucleon mass, is factorized out. Distinguishing isovector from isoscalar (denoted as usual by the superscript $(s)$) components, therefore we have six new Generalized Form Factors in the $n = 1$ sector.

### 3.3 Chiral diagrams to $O(p^4)$

The goal is now the study of Generalized Form Factors from the point of view of HBChPT. From the chiral calculation of the matrix element (3.12) the just introduced $G_i(t)$ can be extracted and from them the structures $A_{2,0}^v(t), B_{2,0}^v(t), C_{2,0}^v(t)$. Unlike the Form Factor case, the external source is not simply a vector any more but represented by a tensor field $V^{\mu\nu}$ counting as two in the power scheme. By applying the power counting scheme of HBChPT (see Eq.(1.34)), we find that the Generalized Form Factors at the leading-one-loop level are not $O(p^3)$ like the electromagnetic Form Factors but they just appear at the fourth chiral order. Keeping in mind these preliminary statements the questioned diagrams, contributing to the nucleon matrix element up to $O(p^4)$, are those depicted in Fig.3.1, where the full circle denotes the insertion of the external source and $e$ represents the $Z$ factor. We observe that to the $O(p^4)$ no diagrams involving couplings of
Figure 3.1: Loop diagrams contributing to the nucleon matrix element up to order $O(p^4)$. The solid and dashed lines denote nucleon and pion, in order. The full circle represent the external tensor field.
the external tensor field to mesons are included. Diagrams like \( a \) or \( d \) of Fig.2.2 take origin from the Lagrangian \( \mathcal{L}_{\pi\pi}^4 \), consequently they only start to contribute at the next to leading order \( \mathcal{O}(p^5) \). Since the present work is only concerned with the results at leading-one-loop order, in the following discussion we will need to consider only those diagrams which include the interaction of the tensor field with baryons.

A consideration about the expression of \( G_i(t) \) is now necessary: since we choose to work including contributions up to the fourth order, the equations (3.13), (3.14), (3.15) reduce to

\[
G_1(t) = \langle \bar{u}u \rangle \sqrt{m_N^2 - \frac{t}{4} \left[A_{2,0}^u(t) + \frac{t}{4m_N^2}B_{2,0}^u(t = 0) + \mathcal{O}(p^5)\right]} , \quad (3.16)
\]

\[
G_2(t) = \langle \bar{u}u \rangle \sqrt{m_N^2 - \frac{t}{4} \left[\frac{1}{m_N}(A_{2,0}^u(0) + B_{2,0}^u(t = 0)) + \mathcal{O}(p^5)\right]} , \quad (3.17)
\]

\[
G_3(t) = \langle \bar{u}u \rangle \left[\frac{1}{m_N}C_{2,0}^u(t = 0) + \mathcal{O}(p^5)\right] . \quad (3.18)
\]

Eqs. (3.16 – 3.18) are some of the central results of this thesis. They tell us how one can connect the non-relativistic Form Factors \( G_1(t), G_2(t), G_3(t) \) obtained from the HBChPT calculation to the sought after Generalized Form Factors \( A_{2,0}^u(t), B_{2,0}^u(t), C_{2,0}^u(t) \), valid to \( \mathcal{O}(p^4) \). In particular we observe that to this order only \( A_{2,0}^u \) acquires \( t \)-dependence, while \( B_{2,0}^u \) and \( C_{2,0}^u \) only receive contribution at \( t = 0 \). For a complete study of the \( t \)-dependence of the Generalized \( n = 1 \) Form Factors it will be then necessary to extend the calculation presented here to higher orders in the chiral expansion [13].

3.3.1 Mathematical tools

First of all, we need to construct the effective Lagrangians describing the interaction vertices appearing in the loop diagrams \( a \) – \( d \) of Fig.3.1. The evaluation of the wave function renormalization is straightforward: the self energy of the nucleon is already known \( (\Sigma(\omega)) \) so that, in comparison with the Form Factors, the only difference lies on the external field insertion; instead of the electromagnetic field represented by the photon \( (\nu_\mu = -e\frac{\gamma_\mu}{2}A_\mu) \) we have a generic tensor field \( V^{\mu\nu} \) and thus also a different coupling. The second
order Lagrangian solving this purpose takes the form
\[
\tilde{\mathcal{L}}_{\pi N}^{(2)} = m_0 \tilde{c}_8 \overline{N}_v \left( v^\mu v_\nu + v^\nu v_\mu - \frac{2}{d} g^{\mu\nu} \right) V_{\mu\nu} N_v + m_0 \tilde{c}_9 \overline{N}_v \left( v^\mu v_\nu + v^\nu v_\mu - \frac{2}{d} g^{\mu\nu} \right) V_{\mu\nu}^{(s)} N_v,
\]
(3.19)

where the tilde (\(\sim\)) distinguishes the present Lagrangian from the one introduced in the first chapter (\(\mathcal{L}_{\pi N}^{(2)}\)) (see Eq.C.10) and contributing to the self energy. The constants \(\tilde{c}_8, \tilde{c}_9\) indeed start from eight in order to continue the enumeration appearing in \(\mathcal{L}_{\pi N}^{(2)}\). They are dimensionless because the unit of energy represented by \(m_0\) (the nucleon mass in the chiral limit) is factored out. We note that the \(G_i(t)\) structures are given in terms of the physical nucleon mass \(m_N\), which at the order we are working can be identified with \(m_0\) without loss of consistency in the theory. The tensor \(V^{\mu\nu}\) is split up into its vectorial and scalar components (denoted by the superscript \((s)\)) as follows:

\[
V_{\mu\nu} = \tilde{v}_{\mu\nu} \frac{1}{2} \left( u^\dagger \tau^a u + u u^a u^\dagger \right),
\]
(3.20)

\[
V_{\mu\nu}^{(s)} = \tilde{v}_{\mu\nu} I \frac{1}{2},
\]
(3.21)

where \(u\) represents as usual the pions \((u^2(x) = U(x))\), whose exponential parametrization has been defined in Eq.(1.13). As we will see in the next paragraph, all the vertices playing a role in the fourth order graphs arise from \(\tilde{\mathcal{L}}_{\pi N}^{(2)}\).

Going further to higher order Lagrangians, the most general next to leading one reads
\[
\mathcal{L}_{\pi N}^{(3)} = i B_{40} m_0 \overline{N}_v \left[ S_{\{\mu}, S_{\nu]} \left[ D^{\mu}, V^{\mu\alpha} \right] V_{\alpha\nu} \right] N_v + i B_{41} m_0 \overline{N}_v \left[ S_{\{\mu}, S_{\nu]} \left[ \partial^{\mu}, V^{\mu\alpha(s)} \right] V_{\alpha\nu} \right] N_v,
\]
(3.22)

where \(B_{40}\) and \(B_{41}\) are dimensionless low energy constants. The analysis of these couplings shows that they provide the leading contributions to both the isoscalar and the isovector Form Factor \(B_{2,0}\) at \(t = 0\). The matrix \(\sigma_{\mu\nu}\) connected to the commutator involving the Pauli Lubansky operator \(S_\mu\), together with the momentum \(q^\nu\) derived from the derivative \(D^\nu\) reproduce
indeed the structures coupled to $B_{2,0}^v$.

At last, the fourth order Lagrangian written as usual with respect to all symmetries of the system and involving all the possible Lorentz invariant structures takes the form

$$\mathcal{L}^{(4)}_{\pi N} = m_0 S_{40} \overline{N}_v \left( u^\mu v^\nu + v_\nu v_\mu - \frac{2}{d} g_{\mu\nu} \right) \text{Tr}(\chi_+^s) V_{\mu\nu} N_v$$

$$+ m_0 S_{41} \overline{N}_v \left( u^\mu v^\nu + v_\nu v_\mu - \frac{2}{d} g_{\mu\nu} \right) \text{Tr}(\chi_+^s) V_{\mu\nu}^s N_v$$

$$+ S_{42} \overline{N}_v \left( [D_\nu, [D_\mu, V^{\mu\nu}]] + [D_\mu, [D_\nu, V^{\mu\nu}]] - \frac{2}{d} [D^{2}, g^{\mu\nu} V_{\mu\nu}] \right) N_v$$

$$+ S_{43} \overline{N}_v \left( [D_\nu, [D_\mu, V^{\mu\nu(s)}]] + [D_\mu, [D_\nu, V^{\mu\nu(s)}]] - \frac{2}{d} [\partial^{2}, g^{\mu\nu} V_{\mu\nu(s)}] \right) N_v$$

$$+ m_0 S_{44} \overline{N}_v \left( u^\mu v^\nu + v_\nu v_\mu - \frac{2}{d} g_{\mu\nu} \right) [D^\alpha, [D_\alpha, V^{\mu\nu}]] N_v$$

$$+ m_0 S_{45} \overline{N}_v \left( u^\mu v^\nu + v_\nu v_\mu - \frac{2}{d} g_{\mu\nu} \right) [\partial^\alpha, [\partial_\alpha, V^{\mu\nu(s)}]] N_v.$$

(3.23)

We observe that the built up Lagrangian is actually of the fourth order: focusing for example on the last structure, since the tensor field counts as two and each derivative as one, the valuation gives $1 + 1 + 2 = 4$. The pieces $S_{ij}$ are again unknown dimensionless low-energy constants. The first two lines provide quark mass insertions for $A_{2,0}^v(t)$. The structure $\text{Tr}(\chi_+)$ was first met when studying the second order pion-nucleon Lagrangian $\mathcal{L}^{(2)}_{\pi N}$ Eq.(C.10); for the definition we refer to Appendix C. The third line exhibits two derivatives acting on the tensor and it is therefore associated with the form factor $C_{2,0}$ while the last one contributes to the radii of $A_{2,0}^v(t)$.

All the possible Lagrangian up to $\mathcal{O}(p^4)$ have been introduced, providing us with all the formalism required for systematic calculation. We can now go further and calculate the matrix element (3.12) in the framework of HBChPT.

### 3.3.2 Forward limit: matching results

In this section we analyse the special case $\xi = t = 0$ corresponding to the forward limit. Applying the usual Feynman rules to the Lagrangian $\mathcal{L}^{(2)}_{\pi N}$,
we derive the pertinent vertices and we can calculate the loop functions we
need (for the explicit form of the vertices see Appendix C). We found out that
when integrating over the pion momentum, the diagrams \(a\) and \(b\) vanish
because the calculation involves the scalar product \(S \cdot \nu\) whose value is zero.
The results for the non-zero loop diagrams are reported in Appendix F. We
can observe that the forward limit exclusively contributes to the isovector
component of the Generalized Form Factors. In particular, we know from
(3.12) that the only structure surviving as \(t\) (and thus \(\Delta\)) goes to zero is
\(G_1(0)\), which from (3.16) and (3.9) reduces to
\[
G_1(0) = \langle \bar{u}u \rangle m_N \left[ A_{2;0}^v(0) + \mathcal{O}(p^5) \right]
= \langle \bar{u}u \rangle m_N \langle x \rangle_{u-d},
\]
where we underline again that the Form Factors are dimensionless defined
and the normalization is left arbitrary. The performed calculations finally
lead to the \(\langle x \rangle_{u-d}\) chiral result
\[
\langle x \rangle_{u-d} = \tilde{c}_8 \left\{ 1 - \frac{(3g_A^2 + 1)}{4\pi^2} m_{\pi}^2 \log \left[ \frac{m_{\pi}^2}{\lambda^2} \right] - \frac{2}{(4\pi^2)g_A^2 m_{\pi}^2} \right\} + 4m_{\pi}^2 S_{40}^{(r)},
\]
where the finite counterterm \(S_{40}^{(r)}(\lambda)\) is scale dependent and belongs to the
following constant \(S_{40}\) arising from \(\mathcal{L}_{N}^{(4)}\):
\[
S_{40} = S_{40}^{(r)}(\lambda) + \beta_{40} \frac{16\pi^2 L}{16}, \quad \beta_{40} = \frac{\tilde{c}_8}{(4\pi^2)g_A^2} \left( \frac{9}{16} g_A^2 + \frac{1}{2} \right).
\]
The computation of the leading non-analytic contributions to the matrix
element (3.2) for \(n = 1\) in the nucleon within the frame of HBChPT can
be found in more pieces of literature \([8], [1]\). Both results given in these
publications both show the same first two terms and, in particular, they
reproduce the same logarithm of Eq.(3.25). The new aspect of our work is the
use of Lagrangians with tensor fields with the introduction of counterterms
(the low energy constant \(S_{40}\) needed for a finite, ”scale independent result”. This
formalism allows us to go systematically to higher order calculation
towards an even more complete theory.
Now we want to study the $m_{\pi}^2$ dependence of $\langle x \rangle_{u-d}$ exploiting all the available phenomenological values appearing in this expression. Data for $\langle x \rangle_{u-d}$ are conveniently accessible from the web from each of the major collaborations (CTEQ, GRV, MRS, for more details see [23]). Unfortunately, it is difficult to provide quantitative estimates of systematic and statistical errors. In Table A.1 provides the average value for $\langle x \rangle_{u-d}$ taken from [23] and obtained by SESAM collaboration using each of the world unpolarized or polarized data sets. The phenomenological value assumed for the momentum fraction is therefore 0.154(3), where the maximum difference between values for all the relevant data sets is reported in brackets. Starting from Eq.(3.26) we define $c \equiv 4S_{40}^{(r)}$, we vary it from $-3$ to $+3$ and we fix the value of $\bar{c}_8$ requiring that $\langle x \rangle_{u-d}$ evaluated at the physical pion mass takes the empirical value accessible from experiments of high energy scattering. By this way we obtain the plot in Fig. 3.2, where the square represents the phenomenological value $\langle x \rangle_{u-d} = 0.154$ [23] corresponding to $m_{\pi} = 140\text{MeV}$. The triangles denote lattice data, the first and the last ones are quenched while the second one comes from full QCD data [23],[15]. The plot shows three different lines;
the dotted ones quickly go to infinite and correspond to the extreme case $c = \pm 3$. At the moment $a$ and $c$ are not very well constrained because there are not enough lattice data at small pion mass.

The plot is not surprisingly quite rough. Since we are dealing with a leading-one-loop HBChPT calculation, we can not expect that it is valid for pion masses much larger than the physical pion mass. By including more structures than just leading-one-loops we hope to extend the validity of the chiral calculations up to up to $m_\pi = 600$ MeV, which for example was possible in the chiral extrapolations of nucleon form factors [18]. The results achieved by the present research, including only leading-one-loops contributions, are therefore fine and exactly as expected. A better and more satisfying comparison between our chiral results and lattice calculations requires new lattice data points at lower pion masses. In Fig.3.3 a new plot is represented where the chiral extrapolation curve reveals a kind of upward trend. The values for $\tilde{c}^\theta$ and $S_{40}^{(r)}$ for the curve shown in Fig.3.3 are reported in Table.3.1, while the phenomenological values and constants used for all the plots represented can be found in Table.A.1. The value of the renormalization scale $\lambda$ is fixed to 1 GeV. With reasonable values of the coupling constants (Table 3.1) the
extrapolation curve may go upward towards the data. This suggests that chiral physics does bring the large lattice results down to the result at the physical point. Nevertheless the work performed till now is not enough for a meaningful and quantitative comparison to lattice data [13].

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{c}^8$</td>
<td>0.121591</td>
</tr>
<tr>
<td>$S_{40}^{(r)}$</td>
<td>- 0.025 GeV$^{-2}$</td>
</tr>
</tbody>
</table>

Table 3.1: Values for $\tilde{c}^8$ and $S_{40}^{(r)}$ for the curve shown in Fig.3.3.

As for now, there is no full analytical theory for $< x >_{u-d}$ joining the chiral regime at which we work with the heavy quark regime of available lattice calculations. Moreover, a linear extrapolation of the lattice data for $< x >_{u-d}$ overestimates the experimental values by some 50%. This means that important physics is omitted from lattice calculations and extrapolation. In order to compare lattice results to the experimentally measured moments, while preserving the correct behavior of $< x >_{u-d}$ in the regime near the chiral limit, both references [23] and [10] start from the following leading non-analytic behavior taken from [1], [8] (and also consistent with our result of Eq.(3.25)):

$$< x^n >_{u-d} = a_n \left[ 1 - \frac{(3g_A^2 + 1)m^2}{(4\pi F_\pi)^2} \log \frac{m^2}{m^2 + \mu^2} \right] + b_n m^2 $$

(3.27)

and they fit it with the form

$$< x^n >_{u-d} = a_n \left[ 1 - \frac{(3g_A^2 + 1)m^2}{(4\pi F_\pi)^2} \log \left( \frac{m^2}{m^2 + \mu^2} \right) \right] + b_n m^2 $$

(3.28)

The mass $\mu$ is a phenomenological cutoff introduced in the non-analytic term with the aim of caring for the omitted physics explained in term of production of pion clouds, whose contribution plays an influent role when quark masses sufficiently become light. By fixing the cutoff $\mu$, the extrapolation formula of Eq.(3.28), containing the leading chiral behavior, is consistent both with the lattice measurements and the experimental data. Yet, an observation is needed; the introduction of such a cutoff is not required by first principles
but it is forced into the extrapolation formula in order to fit the experimental measurement. The introduction of the extra $m_\pi^2$ term in the denominator of the logarithm, as well as the modification of the scale dependent structure are made according to model assumptions without any evident physical reason. Our Eq.(3.25) is instead the correct $O(p^4)$ HBChPT result for $< x >_{u-d}$, where the scale dependence is canceled by the low energy constant $S_{40}$.

Thanks to the introduced counterterms coming from $\mathcal{L}^{(4)}_{\pi N}$, Eq.(3.25) is complete and derived without appealing to arbitrary extra terms.

### 3.3.3 $O(p^4)$ results for the $n=1$ Generalized Isovector Form Factors.

Putting together every information derived in the previous sections, we can finally write down the contribution of the introduced counterterms to the isovector Generalized Form Factors $A_{v}^{n=1}(t)$, $B_{v}^{n=1}(0)$ and $C_{v}^{n=1}(0)$. As already observed, to the order $O(p^4)$ only $A_{v}^{n=1}(t)$ acquires $t-$dependence, while $B_{v}^{n=1}(0)$ and $C_{v}^{n=1}(0)$ only receive contribution at $t = 0$.

Starting from the analysis of $A_{v}^{n=1}(t)$, we find out what terms make up the expansion

$$A_{v}^{n=1}(t) = A_{v}^{n=1}(0) \left[ 1 + \frac{1}{6} (r_{A,2}^2) t + O(t^2) \right].$$

The first term is the momentum fraction $< x >_{u-d}$ of Eq.(3.25), containing the counterterm $S_{40}$ coming from the first line of $\mathcal{L}^{(4)}_{\pi N}$ (Eq.(3.23)). The radius $(r_{A,2}^2)$, coefficient of the $t-$dependent term, takes also origin from $\mathcal{L}^{(4)}_{\pi N}$, from the piece containing $S_{44}$ as counterterm. The Generalized Form Factors takes then the form:

$$A_{v}^{n=1}(t) = \hat{c}_8 \left\{ 1 - \frac{3g_A^2 + 1}{(4\pi F_\pi)^2} m_\pi^2 \log \left[ \frac{m_\pi^2}{\Lambda^2} \right] - \frac{2}{(4\pi F_\pi)^2 g_A^2 m_\pi^2} \right\}$$

$$+ 4m_\pi^2 S_{40}^{(r)} + S_{44} t + O(p^5).$$

The first line of the Lagrangian $\mathcal{L}^{(3)}_{\pi N}$ (Eq.(3.22)) contributes to the Form Factor $B_{v}^{n=1}(0)$, which therefore will be proportional to the counterterm $B_{40}$:

$$B_{v}^{n=1}(0) = B_{40} m_N + O(p^5).$$
The counterterm $B_{40}$ has unit of energy$^{-1}$, so that the Form Factor is dimensionless. At last, the piece of $\mathcal{L}_{\pi N}^{(4)}$ with the counterterm $S_{42}$ goes into the third Generalized Form Factor, $C_{2,0}^{v}(0)$, which reads

$$C_{2,0}^{v}(0) = S_{42} m_N + \mathcal{O}(p^5), \quad (3.32)$$

where the counterterm $S_{42}$ has again the unit of inverse of energy.

The remaining counterterms $c^9$, $B_{41}$, $S_{41}$, $S_{43}$ and $S_{45}$ of the built up Lagrangians contribute only to the isoscalar component of the Generalized Form Factors and are therefore not discussed in the present work.
Conclusions and Final Outlook

By making use of the formalism of Heavy Baryon Chiral Perturbation Theory we have reproduced the well-known results for the isovector Dirac and Pauli Form Factors. In chapter 2 we have presented a new class of functions whose analysis gives rise to the definition of the so called Generalized Form Factors. We have then applied the acquired knowledge in HBChPT to a leading-one-loop calculation of the $n=1$ isovector Generalized Form Factors of the nucleon. One of the main results of the present work is the derivation of the non-relativistic Form Factors $G_i(t)$, which can be obtained from HBChPT calculations and are connected to the sought Generalized Form Factors by Eq.(3.16) – (3.18). Since the Generalized Form Factors start to contribute at the fourth chiral order, we have constructed the needed effective Lagrangians up to $O(p^4)$ and we have derived from them the vertices to compute the one-loop diagrams acting at this order. The $O(p^4)$ results for $A_{2,0}^u(t)$, $B_{2,0}^u(t)$ and $C_{2,0}^u(t)$ are discussed in the paragraph 3.3.3, with particular attention to the role played by the counterterms appearing in the just introduced Lagrangians. The calculation of $A_{2,0}^u(t)$ in the forward limit case has provided the expression of the momentum fraction $<x>_u d$. Our result of Eq.(3.25) is equivalent to those shown in previous literature ([29], [1]), but the introduction of explicit counterterms coming from the new Lagrangians has paved the way to systematic calculations beyond leading-one-loop order. As clearly suggested by Fig.(3.3) such calculation are necessary for an improvement of the theory.
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Appendix A

Notation and useful formulae

\(a_\mu\) \quad \text{Lorentz vector} \ a

\(\vec{a}\) \quad \text{space-like three-vector}

\(m_N\) \quad \text{nucleon mass}

\(m_\pi\) \quad \text{pion mass}

\(F_\pi\) \text{ pion decay constant}

\(g_A\) \quad \text{axial coupling constant}

\(v_\mu\) \quad \text{external vector field}

\(v_\mu = -e\frac{\gamma_3}{2}A_\mu\) \quad \text{(for the isovector photon)}

\(a_\mu\) \quad \text{axial vector field}

\(s\) \quad \text{scalar field}

\(p\) \quad \text{pseudoscalar field}

\(u^2(x) = U(x) = \exp\left[\frac{i\pi\cdot\pi}{F_\pi}\right]\) \quad \text{matrix field representing the pion}

\(\nabla_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu)\) \quad \text{mesonic covariant derivative}

\(u_\mu \equiv i(u^\dagger \nabla_\mu u - u \nabla_\mu u^\dagger)\) \quad \text{axial vielbein}
\[ \Psi = \begin{pmatrix} p \\ n \end{pmatrix} \]

\[ D_\mu = \partial_\mu + \Gamma_\mu \]

\[ \Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{i}{2}(v_\mu + a_\mu)u - \frac{i}{2}u(v_\mu - a_\mu)u^\dagger \]

\[ v_\mu \]

\[ P_{\pm,L,R} \]

\[ N_v, h_v \]

\[ d \]

\[ \lambda \]

\[ \gamma_E \]

\[ L = \frac{\lambda^{d-4}}{4\pi^2} \left[ \frac{1}{d-4} + \frac{1}{2}(\gamma_E - 1 - \ln A\pi) \right] \]

\[ \tau^a = \vec{\sigma} \]

\[ \eta \]

\[ \tilde{m}^2 = m_\pi^2 - q^2 x(1 - x) \]

\[ q = \Delta = p' - p \]

\[ q^2 = -Q^2 = \Delta^2 = t \]

\[ \vec{p} = \frac{p + p'}{2} \]

\[ \xi \]
### Table A.1: Phenomenological values and ChPT constants.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(r_1^2)$</td>
<td>0.585 fm$^2$ [24]</td>
</tr>
<tr>
<td>$m_\pi$</td>
<td>140 MeV [3]</td>
</tr>
<tr>
<td>$F_\pi$</td>
<td>92.5 MeV [3]</td>
</tr>
<tr>
<td>$g_A$</td>
<td>1.26 [3]</td>
</tr>
<tr>
<td>$&lt; x &gt;_{u-d}$</td>
<td>0.154(3) [23]</td>
</tr>
<tr>
<td>$\kappa_v$</td>
<td>3.71 [3]</td>
</tr>
<tr>
<td>$B_{10}^{(r)} (\lambda = 1$ GeV)</td>
<td>0.072</td>
</tr>
<tr>
<td>$\beta_{10}$</td>
<td>$-\frac{1}{6} - \frac{2}{3} g_A^2$ [14][3]</td>
</tr>
<tr>
<td>$\beta_{20}$</td>
<td>$-\frac{9}{16} g_A^2$ [14][3]</td>
</tr>
<tr>
<td>$\beta_{40}$</td>
<td>$\frac{8}{(4\pi F)^2} \left( \frac{9}{16} g_A^2 + \frac{1}{2} \right)$</td>
</tr>
</tbody>
</table>

Here are some useful formulae:

**Conversion factor:**

$$\text{GeV}^{-2}(\hbar c)^2 = 0.0389 \text{ fm}^2 = 0.389 \text{ mbarn} \quad (A.1)$$

**Gordon Identity:**

$$\bar{u}(p_2) \gamma^\mu u(p_1) = \bar{u}(p_2) \left[ \frac{p_1^\mu}{2m} + \frac{i\sigma^\mu q_\nu}{2m} \right] u(p) \quad (A.2)$$

**Identities for sigma matrices:**

$$\sigma_a \sigma_b = \delta_{ab} + i \varepsilon_{abc} \sigma^c \quad (A.3)$$

$$(\sigma \cdot \vec{q}) (\sigma \cdot \vec{p}) = (\vec{q} \cdot \vec{p}) + i \sigma \cdot (\vec{q} \times \vec{p}) \quad (A.4)$$

$$(\vec{\sigma} \cdot \vec{q}) \vec{\sigma} = \vec{q} + i (\vec{q} \times \vec{\sigma}) \quad (A.5)$$
\vec{\sigma} \cdot \vec{q} (\vec{\sigma} \times \vec{q}) = i \vec{q}^2 - i \vec{q} \cdot (\vec{\sigma} \cdot \vec{q}) \tag{A.6}

\vec{\sigma} \cdot \vec{q} \vec{\sigma} \cdot \vec{q} = \vec{q}^2 \tag{A.7}
Appendix B

Dirac Algebra

Metric tensor

\[ g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

Dirac representation of $\gamma_\mu$ matrices:

\[ \gamma^\mu = (\gamma^0, \vec{\gamma}) \]

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \]

With $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ Pauli matrices:

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Definition:

\[ \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \]

\[ \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \]
Useful properties:

\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}, \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu, \quad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^k)^\dagger = -\gamma^k (k = 1, 2, 3) \]

\[ \sigma^\dagger_{\mu\nu} = \sigma^{\mu\nu}, \quad \gamma_5^\dagger = \gamma_5 \]

\[ [\sigma_i, \sigma_j] = 2i\varepsilon^{ijk}\sigma_k \quad \varepsilon^{ijk} = \text{totally antisymmetric} \quad \varepsilon^{123} = 1 \]

\[ \{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad tr(\sigma_i\sigma_j) = 2\delta_{ij} \]

Pauli Lubanski spin operator:

\[ S^\mu_v = \frac{i}{2}\gamma_5\sigma^{\mu\nu}u_\nu \]

Relative properties:

\[ S_v \cdot u = 0, \quad S^2_v = \frac{1 - d}{4} \]

\[ \{S^\mu_v, S^\nu_v\} = \frac{1}{2}(u^\mu u^\nu - g^{\mu\nu}), \quad [S^\mu_v, S^\nu_v] = i\varepsilon^{\mu\nu\rho\sigma}u_\rho S^\sigma_v \]

In the representation I choose the spinor \( u_s(p) \) obeying the Dirac equation \((\slashed{p} - m)u_s(p) = \pi_s(p)(\slashed{p} - m) = 0\) takes the form:

\[ u_s(p) = \sqrt{\frac{E + m}{2m}} \left( \frac{\chi_s}{\slashed{p}} \right), \quad \chi_s = \pm 1 \]

(B.1)

with \( \chi_s \) two-components Pauli spinors

\[ \chi_{s=+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{s=-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

(B.2)

and normalization

\[ \pi_s(p)u_{s'}(p) = u^\dagger_s(p)\gamma_0u_{s'} = \delta_{ss'} \]

(B.3)
Appendix C

Lagrangians and Feynman Rules

The calculations have been performed making use of the following HBChPT Lagrangians. Pertinent Feynman rules are also given.

\[ \mathcal{L}^{(2)}_{\pi\pi} = \frac{1}{4} F_\pi^2 \{ Tr[\nabla_\mu U_1 \nabla^\mu U_2 + \chi^\dagger U_3 + \chi U_4] \} . \tag{C.1} \]

Vertices from \( \mathcal{L}^{(2)}_{\pi\pi} \)

pion propagator:

\[ \frac{i\delta^{ab}}{l^2 - m_\pi^2 + i0} \tag{C.2} \]

2 pions, photon (\( l_1 \) in, \( l_2 \) out):

\[ e\epsilon^{abc} \cdot (l_1 + l_2) \tag{C.3} \]
\[ \mathcal{L}^{(1)}_{\pi N} = \overline{N_v}(i\gamma \cdot D + g_A S \cdot u)N_v, \quad (C.4) \]

**Vertices from \( \mathcal{L}^{(1)}_{\pi N} \)**

- **nucleon propagator:**
  \[ \frac{i}{v \cdot (r) + i0} \quad (C.5) \]

- **1 pion (l out):**
  \[ \frac{g_A S \cdot l \tau^a}{F_\pi} \quad (C.6) \]

- **photon:**
  \[ \frac{ie(1 + \tau^3) \epsilon \cdot v}{2} \quad (C.7) \]

- **1 pion, 1 photon:**
  \[ \frac{ieg_A}{F_\pi} \epsilon \cdot S \epsilon^{a3b} \tau^b \quad (C.8) \]

- **2 pions:**
  \[ \frac{1}{4F_\pi^2} v \cdot (l + l + q) \epsilon^{abe} \tau^e \quad (C.9) \]

- **2 pions, 1 photon:**
  \[ \frac{ie}{4F_\pi^2} (\tau^a \delta^{b3} + \tau^b \delta^{a3} - 2\tau^3 \delta^{ab}) \epsilon \cdot v \quad (C.10) \]
\( \mathcal{L}^{(2)}_{\pi N} = \mathcal{N} \left\{ \frac{1}{2m} (v \cdot D)^2 - \frac{1}{2m} D \cdot D - \frac{ig_A}{2m} \{ S \cdot D, v \cdot u \} + c_1 \text{Tr} \chi_+ \right\} \\
+ (c_2 - \frac{g_\pi^2}{8m})(v \cdot u)^2 + c_3 u \cdot u + (c_4 + \frac{1}{4m})[S_\mu, S_\nu]u_\mu u_\nu \\
+ c_5 \text{Tr}(\bar{\chi}_+) - \frac{i}{4m} [S^\mu, S^\nu]( (1 + c_6) f^+_{\mu\nu} + c_7 \text{Tr}(f_{\mu\nu}^+)) \} \mathcal{N} \quad (C.11) \\

with \( g_A, m, c_i \) taken in the chiral limit and:

- \( \tilde{\chi}_+ = \chi_+ - \frac{1}{2} \text{Tr} \chi_+ \) with \( \chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u \) (non vanishing only for \( m_u \neq m_d \)),

- \( f^+_{\mu\nu} = u^\dagger F_{\mu\nu}^R u + u F_{\mu\nu}^L u^\dagger \) with \( F_{\mu\nu}^{RL} \) field strength tensor corresponding to external (isovector) left/right vector sources (isovector photon, W and Z boson),

- \( c_1 \) is a mass renormalization counterterm and is related to the pion-nucleon \( \sigma \) term, \( \sigma_{\pi N} \sim \langle p' \mid m(u\bar{u} + d\bar{d}) \mid p \rangle \),

- \( c_2, c_3, c_4 \) occur in the reaction \( \pi N \rightarrow \pi N \),

- \( c_5 \) can be determined in terms of the strong contribution to the neutron-proton mass difference (case \( m_u \neq m_d \)),

- \( c_6 = \kappa_u, c_7 = \frac{1}{2}(\kappa_u - \kappa_v) \) with \( \kappa \) anomalous magnetic moment of the nucleon (always in the chiral limit).

\[
\tilde{\mathcal{L}}^{(2)}_{\pi N} = m_0 \tilde{c}_8 \mathcal{N}_v \left( v^{\mu} v^{\nu} + v^{\nu} v^{\mu} - \frac{2}{d} g^{\mu\nu} \right) V_{\mu\nu} N_v \\
+ m_0 \tilde{c}_9 \mathcal{N}_v \left( v^{\mu} v^{\nu} + v^{\nu} v^{\mu} - \frac{2}{d} g^{\mu\nu} \right) V^{(s)}_{\mu\nu} N_v \quad (C.12)
\]

**Vertices from \( \tilde{\mathcal{L}}^{(2)}_{\pi N} \)**

tensor field (external source):

\[
m_0 \tilde{c}_8 v^{\mu} v^{\nu} \tau^s \bar{\nu}_{\mu\nu} + m_0 \tilde{c}_8 v^{\nu} v^{\mu} \tau^s \bar{\nu}_{\mu\nu} - \frac{2}{d} m_0 \tilde{c}_8 g^{\mu\nu} \bar{\nu}_{\mu\nu} \quad (C.13)
\]
2 pions, tensor field:

\[ m_0 \, \bar{c}_8 \, \gamma_\mu \gamma_\nu \frac{1}{2F^2_\pi} (\delta^{ab} \tau^b + \delta^{bs} \tau^a - 2\delta^{ab} \tau^s) \bar{\nu}_{\mu \nu} \quad (C.14) \]

1 pion, tensor field:

\[ -m_0 \, \bar{c}_8 \, \epsilon^{\mu \nu \alpha \beta} \bar{\nu}_{\alpha \beta} \epsilon^{asc \tau^c} \quad (C.15) \]

\[ \mathcal{L}_{\pi N}^{(3)} = \quad B_{40} \, m_0 \, \N_v \, [S_{\{\mu, S_\nu\}} \, [D^\mu, V^{\mu \alpha} \, \nu_\alpha \} \, N_v \\
+ B_{41} \, m_0 \, \N_v \, [S_{\{\mu, S_\nu\}} \, [\partial_\mu, V^{\mu \alpha(s)} \, \nu_\alpha \} \, N_v \, \quad (C.16) \]

\[ \mathcal{L}_{\pi N}^{(4)} = \quad m_0 \, S_{40} \, \N_v \, (\gamma^\nu \gamma_\nu V_\mu + \nu_\mu \gamma_\nu V_\mu - 2 \frac{d}{d} g_{\mu \nu}) \, Tr(\chi) \, V_\mu \, N_v \\
+ m_0 \, S_{41} \, \N_v \, (\gamma^\nu \gamma_\nu V_\mu + \nu_\mu \gamma_\nu V_\mu - 2 \frac{d}{d} g_{\mu \nu}) \, Tr(\chi) \, V_\mu^{(s)} \, N_v \\
+ S_{42} \, \N_v \, \left( [D_\nu, [D_\mu, V^\mu]] + [D_\mu, [D_\nu, V^\mu]] - \frac{2}{d} [D^\mu, g^\mu \nu V_\mu] \right) \, N_v \\
+ S_{43} \, \N_v \, \left( [\partial_\nu, [\partial_\mu, V^{\mu(s)}]] + [\partial_\mu, [\partial_\nu, V^{\mu(s)}]] - \frac{2}{d} [\partial_\mu, g^{\mu \nu} V_\mu^{(s)}] \right) \, N_v \\
+ m_0 \, S_{44} \, \N_v \, (\gamma^\nu \gamma_\nu V_\mu + \nu_\mu \gamma_\nu V_\mu - 2 \frac{d}{d} g_{\mu \nu}) \, [D^\alpha, [D_\alpha, V^\mu]] \, N_v \\
+ m_0 \, S_{45} \, \N_v \, (\gamma^\nu \gamma_\nu V_\mu + \nu_\mu \gamma_\nu V_\mu - 2 \frac{d}{d} g_{\mu \nu}) \, [\partial_\mu, [\partial_\alpha, V^{\mu(s)}]] \, N_v \, \quad (C.17) \]
Appendix D

Loop functions

Here is a list of practical loop functions used in my calculations (d = space-time dimension).

\[
\frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(m_\pi^2 - l^2)} \equiv \Delta_\pi = m_{\pi}^{d-2}(4\pi)^{-d/2}\Gamma(1 - \frac{d}{2}) \tag{D.1}
\]

\[
\Delta_\pi = 2m_\pi^2 \left( L + \frac{1}{16\pi^2} \ln \frac{m_\pi}{\lambda} \right) + \mathcal{O}(d - 4) \tag{D.2}
\]

\[
L = \frac{\lambda^{d-4}}{16\pi^2} \left[ \frac{1}{d - 4} + \frac{1}{2}(\gamma_E - 1 - \ln 4\pi) \right] \tag{D.3}
\]

\[
\frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{l_\mu l_\nu}{(m_\pi^2 - l^2)^2} = \frac{(-)}{(4\pi)^{d/2}} \frac{g_{\mu\nu}}{2} \Gamma(- \frac{d}{2} + 1)m_\pi^{d+1} \tag{D.4}
\]

\[
\frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{l_\mu l_\nu}{(m_\pi^2 - l^2)^2} \text{ with } \{1, l_\mu, l_\mu l_\nu\} = \{J_0(\omega), \nu_\mu J_1(\omega), g_{\mu\nu} J_2(\omega) + \nu_\mu \nu_\nu J_3(\omega)\} \tag{D.5}
\]

\[
J_0(\omega) = -4L\omega + \frac{\omega}{8\pi^2} \left( 1 - 2\ln \frac{m_\pi}{\lambda} \right) - \frac{1}{4\pi^2} \sqrt{m_\pi^2 - \omega^2} \arccos \frac{-\omega}{m_\pi} + \mathcal{O}(d - 4) \tag{D.6}
\]

\[
J_1(\omega) = \omega J_0(\omega) + \Delta_\pi, \quad J_2(\omega) = \frac{1}{d - 1} \left[ (m_\pi^2 - \omega^2) J_0(\omega) - \omega \Delta_\pi \right] \tag{D.7}
\]

\[
J_3(\omega) = \omega J_1(\omega) - J_2(\omega) \tag{D.8}
\]
\[
\frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{\{1, l_\mu, l_\nu\}}{(v \cdot l)(v \cdot l - \omega)(m_\pi^2 - l^2)} = \{\Gamma_0(\omega), v_\mu \Gamma_1(\omega), g_{\mu\nu} \Gamma_2(\omega) + v_\mu v_\nu \Gamma_3(\omega)\}
\]

thanks to the identity
\[
\frac{1}{v \cdot l(v \cdot l - \omega)} = \frac{1}{\omega} \left( \frac{1}{v \cdot l - \omega} - \frac{1}{v \cdot l} \right)
\]

\[
\Gamma_i(\omega) = \frac{1}{\omega} [J_i(\omega) - J_i(0)], \quad (i = 0, 1, 2, 3)
\]

\[
\frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{\{1, l_\mu, l_\nu\}}{(v \cdot l - \omega)^2(m_\pi^2 - l^2)} = \{G_0(\omega), v_\mu G_1(\omega), g_{\mu\nu} G_2(\omega) + v_\mu v_\nu G_3(\omega)\}
\]

thanks to the identity
\[
\frac{1}{(v \cdot l - \omega)^2} = \frac{\partial}{\partial \omega} \left( \frac{1}{v \cdot l - \omega} \right)
\]

\[
G_i(\omega) = \frac{\partial}{\partial \omega} J_i(\omega), \quad (i = 0, 1, 2, 3)
\]

Feynman parameter [25]:
\[
\frac{1}{ab} = \int_0^1 dz \frac{1}{[az + b(1 - z)]^2}
\]
Appendix E

Loop diagrams of the vector Form Factors to $O(p^3)$
\[
E_{1a} = \int \frac{dl^4}{(2\pi)^4} \overline{u}(r_2) \left[ \frac{-g_\alpha S \cdot (l + q) r^a}{F_\pi} + \frac{i}{v \cdot (r - l) + i0^+} \left( \frac{i \delta^{ab}}{(l + q)^2 - m^2_\pi + i0^+} \right) \right] u(r_1) \\
= \sum_{i} g_A^2 \frac{2 m_\pi^2}{(4\pi F_\pi)^2} \eta^i \left\{ \overline{u}(r_2) \epsilon \cdot (l + q) \left[ \left( 6 m_\pi^2 - \frac{5}{3} q^2 \right) \left( 16\pi^2 L + \log \frac{m_\pi}{\lambda} \right) \right] \\
+ 2 m_\pi^2 - \frac{2}{3} q^2 + \int_0^1 dx \left( 3 m_\pi^2 - 5 q^2 x(1 - x) \right) \log \left[ \frac{m_\pi^2}{m_\pi^2} \right] \right\} \\
- \overline{u}(r_2)[S_\mu, S_\nu] e^a_{\mu} q^a_{\nu} u(r_1) \int_0^1 dx 4\pi \sqrt{m^2} \right\} 
\]

(E.1)

\[
E_{1b} = \int \frac{dl^4}{(2\pi)^4} \overline{u}(r_2) \left[ \frac{-g_\alpha S \cdot l^b}{F_\pi} + \frac{i}{v \cdot (r - l + q) + i0^+} \left( \frac{i e(1 + \tau^3) \epsilon \cdot v}{2} \right) \right] u(r_1) \\
= \sum_{i} g_A^2 \frac{2 m_\pi^2}{(4\pi F_\pi)^2} \eta^i \left\{ \eta^i \frac{1}{2} \eta \overline{u}(r_2) \epsilon \cdot v u(r_1) \left[ \frac{3}{2} m_\pi^2 + \frac{9}{2} m_\pi^2 \left( 16\pi^2 L + \log \frac{m_\pi}{\lambda} \right) \right] \right\} \\
- \frac{\tau^i}{2} \eta \left( r_2 \epsilon \cdot v u(r_1) \right) \left[ \frac{m_\pi^2}{2} + \frac{3}{2} m_\pi^2 \left( 16\pi^2 L + \log \frac{m_\pi}{\lambda} \right) \right] \right\} 
\]

(E.2)

\[
E_{1c} = \int \frac{dl^4}{(2\pi)^4} \overline{u}(r_2) \left[ \frac{i e}{4 F_\pi^2} \left( \tau^a \delta^{b3} + \tau^b \delta^{a3} - 2 \tau^3 \delta^{ab} \right) \epsilon \cdot v \right] \frac{i \delta^{ab}}{l^2 - m^2_\pi + i0^+} u(r_1) \\
= \overline{u}(r_2) \frac{i e}{4 F_\pi^2} \left( -2 \tau^i \right) \epsilon \cdot v \Delta_\pi u(r_1) \\
= i \frac{-e}{F_\pi^2} \eta^i \frac{1}{2} \eta \overline{u}(r_2) \epsilon \cdot v u(r_2) \left[ 2 m_\pi^2 \left( L + \frac{1}{16\pi^2} \log \frac{m_\pi}{\lambda} \right) \right] \\
= i \frac{-e}{(4\pi F_\pi)^2} \eta^i \frac{1}{2} \eta \overline{u}(r_2) \epsilon \cdot v u(r_1) \left\{ 2 m_\pi^2 \left( 16\pi^2 L + \log \frac{m_\pi}{\lambda} \right) \right\} 
\]

(E.3)
\[ E_{1d} = \int \frac{dl^4}{(2\pi)^4} \bar{\pi}(r_2) \frac{1}{4F_\pi^2} \nu \cdot (l + l + q) \varepsilon_{df}^a \varepsilon_{bf}^f \frac{i\delta^{ab}}{(l + q)^2 - m_\pi^2 + i\theta^+} \varepsilon \cdot (l + l + q) \]

\[ \frac{i\delta^{dc}}{l^2 - m_\pi^2 + i\theta^+} \rho(r_1) \]

\[ = i \frac{1}{(4\pi F_\pi^2)^2} \eta^+ \frac{\tau^i}{2} \eta \bar{\pi}(r_2) \varepsilon \cdot \nu \rho(r_1) \left\{ (2m_\pi^2 - \frac{1}{3}q^2)(16\pi^2 L + \log \frac{m_\pi}{\lambda}) \right. \]

\[ + \int_0^1 dx \frac{\tilde{m}^2}{m_\pi^2} \log \frac{\tilde{m}^2}{m_\pi^2} \right\} \]  

(E.4)

with \( \tilde{m}^2 = m_\pi^2 - q^2 x(1 - x) \)

\[ \Sigma(\omega) = \frac{3g_A^2}{4F_\pi^2} \left\{ 2L(2\omega^2 - 3m_\pi^2) = \frac{\omega}{8\pi^2} (2\omega^2 - 3m_\pi^2) \ln \frac{m_\pi}{\lambda} + \frac{\omega}{8\pi^2} (m_\pi^2 - \omega^2) \right. \]

\[ - \frac{1}{4\pi^2} (m_\pi^2 - \omega^2)^{3/2} \arccos \frac{-\omega}{m_\pi} \} - 4m_\pi^2 \left( c_1 + \frac{B_{20} \omega}{8\pi^2 F_\pi^2} \right) + \frac{B_{15} \omega^3}{(4\pi F_\pi)^2} \ldots \]  

(E.5)

\[ Z_N^{(3)} = \left\{ 1 - \frac{1}{(4\pi F_\pi)^2} \left[ \frac{3}{2} g_A m_\pi^2 + 8m_\pi^2 B_{20}(\lambda) + \frac{9}{2} g_A m_\pi^2 \log \left[ \frac{m_\pi}{\lambda} \right] \right] \right\} \]  

(E.6)
Appendix F

Loop diagrams of the Generalized Form Factors to $\mathcal{O}(p^4)$
\[ F_{1c} = \int \frac{d^4l}{(2\pi)^4} \overline{u}(r_2) \left( -\frac{g_A S \cdot l^{\tau b}}{F_\pi} \frac{i\delta^{ab}}{l^2 - m_\pi^2 + i0^+} + \frac{i}{v \cdot (r - l) + i0^+} A^{(2)} v_\mu v_\nu \tau^s \bar{\nu}^{\mu\nu} \right) \]

\[ \overline{u}(r_2) \tau^s u(r_1) \left\{ \frac{1}{4} m_\pi^2 \log \left( \frac{m_\pi}{\lambda} \right) + \frac{m_\pi^2}{12} + 6\pi^2 m_\pi^2 L \right\} \]

(F.1)

\[ F_{1d} = \int \frac{d^4l}{(2\pi)^4} \overline{u}(r_2) \frac{1}{2} \frac{i\delta^{ab}}{(l + q)^2 - m_\pi^2 + i0^+} A^{(2)} v_\mu v_\nu \frac{1}{2F_\pi^2} (\delta^{ab} \tau^b + \delta^{ba} \tau^a - 2\delta^{ab} \tau^s) \bar{\nu}^{\mu\nu} u(r_1) \]

\[ \overline{u}(r_2) \tau^s u(r_1) \left\{ \frac{1}{(4\pi F_\pi)^2} m_\pi^2 \log \left( \frac{m_\pi}{\lambda^2} \right) + 32\pi^2 m_\pi^2 L \right\} \]

(F.2)

\[ F_{1e} = A^{(2)} v_\mu v_\nu \overline{u}(r_2) \tau^s \bar{\nu}^{\mu\nu} u(r_1) \left\{ 1 - \frac{1}{(4\pi F_\pi)^2} \left[ \frac{3}{2} g_A^2 m_\pi^2 + 8m_\pi^2 B_{20}(\lambda) \right] m_\pi^2 \log \left( \frac{m_\pi}{\lambda} \right) + \frac{9}{2} g_A^2 m_\pi^2 \log \left( \frac{m_\pi}{\lambda} \right) \right\} \]

(F.3)

\[ S_{cde} = A^{(2)} v_\mu v_\nu \overline{u}(r_2) \tau^s \bar{\nu}^{\mu\nu} u(r_1) \left\{ 1 - \frac{(3g_A^2 + 1)}{(4\pi F_\pi)^2} m_\pi^2 \log \left( \frac{m_\pi}{\lambda^2} \right) - \frac{2}{(4\pi F_\pi)^2} g_A^2 m_\pi^2 \right\} + 4m_\pi^2 \epsilon_4^{(r)} \]

(F.4)

with \( A^{(2)} = m_0 \tilde{c}_8 \)
Bibliography


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   http://Schwinger.harvard.edu/~georgi/review.pdf.


