HIGH ENERGY BEHAVIOUR OF TOTAL AND ELASTIC CROSS-SECTIONS

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1. INTRODUCTION AND EXPERIMENTAL BACKGROUND

In these lectures I want to concentrate on one particular aspect of high energy behaviour of scattering amplitudes, namely the question of the equality of total cross-sections (Pomeranchuk theorem) and of elastic cross-sections for particle-particle and particle-antiparticle collisions. I already spoke last year, at this school, about the problem of the Pomeranchuk theorem \(^1\). Though since then some theoretical progress has been made it is not yet possible to prove the Pomeranchuk theorem from first principles. I shall give a brief summary of the theoretical situation. On the experimental side you have learnt from Dr. Whetherell that after the last Serpukhov-Allaby-Giacomelli experiment, which shows that the \(K^+p\) cross-section slowly increases in the range of energies 2C GeV/c - 50 GeV/c \(^2\), there is no more evidence for a violation of the Pomeranchuk theorem. On the other hand, it is very difficult to give a convincing experimental proof that total cross-sections do really become equal at infinite energy because of the difficulties of extrapolations.

Anyway, if you look at presently accessible energies, you see that the ratio \(\sigma^-_{(\text{particle})}/\sigma^-_{(\text{antiparticle})}\) is still appreciably different from unity. On the contrary I would like to draw your attention on the fact that elastic cross-sections within limits of experimental errors are already nearly equal. More specifically, if you take energies sufficiently low to have small errors on elastic cross-sections you already observe this phenomenon. For instance \(^3\) at \(s = 20 \text{(GeV)}^2\)

\[
\frac{\sigma^-_{(K^-p)}}{\sigma^-_{(K^+p)}} = 1.27 \pm 0.02
\]
\[
\frac{\sigma^-_{(e^-p)}}{\sigma^-_{(e^+p)}} = 1.07 \pm 0.05
\]

Similarly, at \(s = 30 \text{(GeV)}^2\)

\[
\frac{\sigma^-_{(p\bar{p})}}{\sigma^-_{(p\bar{p})}} = 1.39 \pm 0.02
\]
\[
\frac{\sigma^-_{(e^-\bar{e})}}{\sigma^-_{(e^+\bar{e})}} = 1.00 \pm 0.12
\]

It is argued sometimes that this fact is not significant because of the cross-over phenomena taking place in differential cross-sections. This is not my opinion, and this is what stimulated Henri Cornille and myself to start an investigation on the equality of elastic cross-sections. The existing proofs of this equality were linked with the validity of the ordinary Pomeranchuk theorem and also we found out that a very strong assumption about the phases of the amplitudes was implicitly made in these derivations. We have succeeded partly in that we can show that the ratio \(\sigma^-_{(\text{part})}/\sigma^-_{(\text{anti})} \to 1\) if i) total cross-sections tend to finite limits, ii) elastic cross-sections decrease less fast than \(1/\log s\), a fact which is supported by the recent ISR experiments \(^4\), iii) some technical assumptions not very easy to summarize but which seem acceptable and can be checked a posteriori by examination of differential cross-sections.
2. **THE PROBLEM OF EQUALITY OF TOTAL CROSS-SECTIONS (POMERANCHUK THEOREM)**

Here I remind you a few facts: if total cross-sections tend to finite limits at high energies and if the forward scattering amplitude $F$ grows less fast than $s$ or $E \log E$ where $s$ is the square of the c.m. energy and $E$ the laboratory energy then both particle and antiparticle cross-sections tend to the same limit.

What cannot so far be proved from field theory or anything like that is that $F/E \log E \to 0$. The standard argument is that since

$$\text{Im} F \sim E \frac{\sigma}{2}, \quad \lim \frac{|F|}{(E \log E)} \neq 0$$

implies $\lim |\text{Re} F|/E \log E \neq 0$ and hence $\text{Re} F/\text{Im} F \to \infty$. So, if you believe that high energy scattering is dominantly imaginary you have a proof of the Pomeranchuk theorem. This, however, is just an act of faith.

If you take the position that a violation of the Pomeranchuk theorem may occur you can still make a number of interesting predictions:

1) dominance of the real part of the forward amplitude for $E \to \infty$ which can be detected in principle by Coulomb interference

$$F_p(E) \sim \frac{1}{2} \frac{M}{\pi^2} (\sigma_p - \sigma_A) \equiv \text{Coul} E$$

(1)

where $M$ is the target mass, $E$ the lab energy, $\sigma_p$ and $\sigma_A$ the particle and antiparticle limit cross-sections;

2) equality of differential cross-sections near the forward direction as remarked first by Kinoshita 5)

$$\frac{d\sigma_p}{dt} / \frac{d\sigma_A}{dt} (s, t = -\frac{t}{(2\log E)^2}) \to 1$$

(2)

for $0 \leq t \leq t_0$;

3) accumulation of zeros of the scattering amplitude in the complex $t$ plane near the negative $t$ axis for small $t$ 6)

4) quantitative constraints on the violation due to isospin invariance, unitarity, and analyticity 7):

$$\left| \lim \sigma_{\pi^+ + p} - \sigma_{\pi^- + p} \right| \leq \frac{\pi^{3/2}}{\sqrt{2} \sqrt{m_\pi}} \sqrt{\lim \sigma_{\text{exch}}}$$

(3)

where $\sigma_{\text{exch}}$ is the integrated charge exchange elastic cross-section.
Let me remark that several people thought that iii) i.e., the presence of zeros would lead to physically observable effects such as oscillations of the differential cross-sections. It was Okun and Auberson who pointed out first that this was wrong. The interest of iii), in my opinion, is more theoretical: it may be that from some other source, which I do not know of course, we could exclude the existence of these zeros and therefore prove the Pomeranchuk theorem.

For more details I refer you first of all to my lectures of last year in Erice and to the article by Auberson, Kinoshita and myself which recently appeared in Physical Review.

3. **EQUALITY OF DIFFERENTIAL CROSS-SECTIONS, FIXED t DISPERSION RELATIONS AND THE PHASE PROBLEM**

I already mentioned in the previous section that Kinoshita had proven for the Pomeranchuk violating case

$$\frac{d\sigma^P}{dt}(s, t = \frac{t}{(\log s)^2}) / \frac{d\sigma^A}{dt}(s, t = \frac{t}{(\log s)^2}) \to 1$$

This includes in particular \( t = 0 \)

$$\frac{d\sigma^P}{dt}(s, t = 0) / \frac{d\sigma^A}{dt}(s, t = 0) \to 1 \quad (4)$$

Now, this relation is also valid when the Pomeranchuk theorem is valid. Equation (4), as we shall see, is valid whenever

$$\frac{d\sigma^P}{dt}(s, t = 0) / \frac{d\sigma^A}{dt}(s, t = 0)$$

has a limit (generalized limit including a priori the possibility of having 0 or infinity).

For fixed momentum transfer \( t \leq 0 \), the situation is not so clear. Probably the first "proof" of the equality of \( (d\sigma^P/dt)(s, t) \) and \( (d\sigma^A/dt)(s, t) \) is due to Van Hove 8). Van Hove postulates for the particles and antiparticles amplitudes the form

$$F_P(s, t) \sim s F_P(t)$$

$$F_A(s, t) \sim s F_A(t) \quad (5)$$
Now

\[ F_A(s,t) = F_P(u,t) \]

where

\[ u = 2 \left( M_A^2 + M_B^2 \right) - t - s \]

\( M_A \) and \( M_B \) are the masses of the particles. By turning around infinity one can continue from the particle-particle amplitude to the complex conjugate of the particle-antiparticle amplitude.

If we apply the Phragmén-Lindelöf theorem to \( (F(s,t))/s \) and compare the limits given by (5) of this quantity for

\[ s \to +\infty + i \epsilon \]

and

\[ s \to -\infty + i \epsilon \]

we find

\[ F_P(t) = -\left[ F_A(t) \right]^* \]

(6)

Therefore, we get

1)

\[ \Im \, F_P(t) = \Im \, F_A(t) \]

(7)

which, for \( t = 0 \), with the help of the optical theorem gives back the Fomeranchuk theorem.
\[ |f^P(t)| = |f^A(t)| \]  
(8)

and, therefore

\[ \lim \frac{d\omega^P}{dt}(s,t) = \lim \frac{d\omega^A}{dt}(s,t) \]  
(9)

The first question is to see how far one can go in relaxing the assumption (5). Not long ago assumption (5) would have been rejected on the basis of the shrinking of the diffraction peak. (It is also in contradiction with analyticity and unitarity if we insist on having this behaviour not only for real \( t \) but also for complex \( t \), according to Gribov, Oehme and others.)

An immediate generalization was made by Logunov and collaborators \(^9\) who postulated

\[ F^P(s,t) = P^P(t) s^{\alpha^P(t)} (\log s)^{P^P(t)} \log(\log s)^{\gamma^P(t)} \] 
\[ F^A(s,t) = P^A(t) s^{\alpha^A(t)} (\log s)^{P^A(t)} \log(\log s)^{\gamma^A(t)} \] 
(10)

when \( \alpha^A, \rho^A, \beta^A, \gamma^A, \rho^P, \gamma^P \) are real.

Then you prove that \( \alpha^P = \alpha^A \), \( \rho^P = \rho^A \), \( \beta^P = \beta^A \) and finally

\[ \left[ F^A(t) \right]^* = F^P(t) \exp(i \pi \alpha^P(t)) \]

In fact you could also introduce many other things such as

\[ \exp \left[ \alpha^P \log(s) \right] \]

which is in between a power of \( s \) and a power of \( \log s \), and the argument would still work.

Gervais and Yndurain \(^10\) have considered a more general class of amplitudes such that for \( \lambda \) real

\[ \lim_{s \to \infty} \frac{F^P(ds,t)}{F^P(s,t)} = \lambda^{\alpha^P} \] 
(11)
with \( \alpha_p \) real. Such a class contains (10) and (5) and possesses again the property that
\[
\lim_{s \to \infty} \left| \frac{f^P(s,t)}{f^A(s,t)} \right| = 1
\]
if it exists.

A somewhat more general situation for which \( \lim|f^P/f^A| = 1 \), is found in my article \(^{11}\) on the Pomeranchuk theorem:

if \( \text{Im} f^P \) and \( \text{Im} f^A \) have a constant sign beyond a certain energy and
\[ \lim|f^P/f^A| \]
exists, then \( \lim|f^P/f^A| = 1 \).

In this case we see that the assumption made implies that beyond a certain energy the phase of the amplitude stays between \( n\pi \) and \( (n+1)\pi \), and the phase is indeed the crucial quantity.

Will the equality hold if the phase grows as \( s \to \infty \)? Not always as has been shown by Bia\'Zas et al. \(^{12}\). Their example is

\[
F = s^{\alpha_1 + i\alpha_2}
\]

Here the phase is \( \alpha_2 \log s \), and for \( s \to +\infty \) we have
\[
|F| = s^{\alpha_1}
\]

for \( s \to \infty \)
\[
|F| = \exp(-\pi \alpha_2) \quad |s|^{\alpha_1}
\]

Therefore, the ratio of the moduli goes to \( \exp -\pi \alpha_2 \) and not to unity.

This is a rather alarming situation a priori because the amplitude we have described just corresponds to the exchange of what is now called a complex Regge pole (without no signature). Now what is remarkable is that Bia\'Zas et al., just found a limiting case by intuition, because what Cornille and I established recently is the following theorem.

Let \( F(s,t) \) be such that for fixed \( t \), \( s)s_o \) and \( s <-s_o \) it has no zeros (this is to have a well defined phase). If the phase \( \phi(s,t) \) of \( F(s,t) \) is such that \( |\phi(s,t)/\log s| \to 0 \) for \( s \to \infty \) and \( s \to -\infty \), and if \( \lim|F(s,t)/F(-s,t)| \) exists, then \( \lim|F(s,t)/F(-s,t)| = 1 \).
I shall not try to give here the proof of this theorem. The proof is much easier if one makes the more restrictive assumption $|S(s)| < C$ for $s \to \pm \infty$, because then one can factor out the zeros of $P$ in the form of a polynomial and then work with the logarithm of the reduced amplitude.

I think, however, that it is important to be able to prove that Bialas got the limiting case. In fact you can construct a family of examples of the form

$$F(s, t) = s f(t) \exp \left[ \log (s_0 - s)^\gamma - \log (t_0 - t)^\gamma \right]$$  \hspace{1cm} (15)

which for $s \to \infty$ gives

$$F \sim s f(t) \exp \left[ - \frac{\pi^2 \gamma}{2} \left( (\gamma - 1) \log s \right)^{\gamma - 2} \right]$$

The phase is therefore

$$\pi \gamma \log s^{\gamma - 1},$$

for $s \to \infty$, the modulus is

$$s f(t) \exp \left[ \pm \pi^2 \gamma (\gamma - 1) \log s^{\gamma - 2} \right]$$

for $s \to \pm \infty$; for $\gamma < 2$ we see that the ratio of the moduli function goes to unity. For $\gamma = 2$ it is $\exp(\pm \pi^2)$, but then the phase grows like $\log s$.

For the proof of the theorem I have to send to a future, non-existing publication by Cornille and myself.

4. CONTROLLING THE PHASE BY POSITIVITY

Now that we know what assumption on the phase is needed to prove that the ratio of the moduli goes to unity, we shall try to see what information we can get on this phase from positivity. Of course, to do this we have to have a well-defined phase i.e., $|P(s, t)| \neq 0$ for $s > s_0$, $s < -s_0$, $t$ physical. This assumption, however, can be checked a posteriori by looking at measurements of differential cross-sections and analogous things for the spin case while an assumption on the phase, like that implied by Eqs. (8) or (10) cannot be checked experimentally, except in a "gedanken" experiment with interference between two scattering centres.
The first obvious thing is that an elastic forward amplitude has a positive imaginary part from the optical theorem. Therefore \( F(s,t=0) \) satisfies the conditions of the theorem and

\[
\lim_{\Delta t \to 0} \frac{d\sigma^F(s,t=0)}{dt} = 1
\]

(if it exists).

This in itself is a consequence of positivity, but now we are going to use positivity in a much more refined way. Let me remind you that the absorptive part of scattering amplitude can be written as

\[
A(s,t) = \sum (2\ell + 1) \text{Im} f_\ell(s) P_\ell \left( 1 + \frac{t}{2k^2} \right)
\]

(15)

where \( P_\ell \) are Legendre polynomials, \( k \) is the c.m. momentum. Positivity is expressed by

\[
\text{Im} f_\ell \geq 0
\]

(16)

Here we shall show that (15) and (16) imply a property of Hölder continuity of the absorptive part with respect to \( t \) in the physical region, and that this Hölder continuity can be controlled by the knowledge of the forward absorptive part, i.e., of the total cross-section.

Consider the difference

\[
A(s,t_1) - A(s,t_2) = \sum (2\ell + 1) \text{Im} f_\ell(s) \left[ \frac{P_\ell(1 + \frac{t_1}{2k^2})}{P_\ell(1 + \frac{t_2}{2k^2})} \right]
\]

Suppose that you succeed in finding a bound on

\[
\left| \frac{P_\ell(1 + \frac{t_1}{2k^2})}{P_\ell(1 + \frac{t_2}{2k^2})} \right|
\]

which does not depend on \( \ell \). Then you succeed in finding a bound on \( A(s,t_1) - A(s,t_2) \) in terms of \( \sum (2\ell + 1) \text{Im} f_\ell \) which is the forward absorptive part. Such a bound exists, it is
\[ |P_{k}(1+\frac{t_1}{2k^2})-P_{k}(1+\frac{t_2}{2k^2})| \leq \frac{2}{\pi} \sqrt{\frac{|t_1-t_2|}{|t_1-t_2|^2}} \sqrt{\frac{|t_1-t_2|}{|t_1-t_2|^2}} \]  

for \(-2k^2 < t_1, t_2 \leq 0\).

Therefore, we get the following Hölder condition of the absorptive part of an elastic amplitude \(13\)

\[ |A(s,t_0) - A(s,t_2)| \leq \sqrt{\frac{2}{\pi}} \sqrt{\frac{|t_1-t_2|}{|t_1-t_2|^2}} A(s,0) \]  

(18)

This Hölder condition will help us in finding a bound on the phase of \(A(s,t)\) if we know in addition something on the modulus of \(F(s,t)\) which can be obtained from a measurement of differential cross-sections.

Suppose that for \(-T < t \leq 0\) we have

\[ \left| \frac{F(s,t)}{A(s,0)} \right| > R \]  

(19)

Consider an interval \(t_1, t_2\) such that the factor appearing in the right-hand side of (18) satisfies the inequality

\[ \sqrt{\frac{2}{\pi}} \sqrt{\frac{|t_1-t_2|}{|t_1-t_2|^2}} < \frac{R}{2} \]

Then, in this interval, for \(t_1 < t_3 < t_2\) we have

\[ |A(s,t_3) - A(s,t_1)| < \frac{1}{2} \text{ minimum of } |F| \]

Then two things can happen.

a) \[ |A(s,t_1)| \leq \frac{1}{2} \text{ minimum of } |F| \]

then we get immediately

\[ |A(s,t_3)| < \text{ minimum of } |F| \]

for all \(t_1 < t_3 < t_2\) and hence \(\text{Re } F\) cannot vanish in \(t_1 < t < t_2\).
\[ |A(s, t_i)| > \frac{1}{2} \text{ minimum of } |F| \]

Then \(|A(s, t_2)| > 0\) in the whole interval and \(\text{Im } F\) cannot vanish.

In both cases, the phase variation of \(F\) from \(t_1\) to \(t_2\) is less than \(\pi\).

Now, given an arbitrary interval \(T_1 \rightarrow T_2\) we can split in intervals
\[ T_1, t_2, t_3, \ldots, t_n, t_{n+1}, T_2 \]
such that
\[
\sqrt{\frac{2}{\pi}} \sqrt{\frac{|t_{n+1} - t_n|}{|t_{n+1} \cdot t_n|}} = \frac{R}{2} \tag{20}
\]

The number of intervals will be
\[
N = \frac{\log |T_1/T_2|}{\log R} \tag{21}
\]
where \(R\) is a solution of
\[
\sqrt{\frac{2}{\pi}} \sqrt{\frac{R-1}{R}} = \frac{R}{2} \tag{22}
\]

so that the phase variation from \(T_1\) to \(T_2\) is \(N\pi\). Therefore, if \(|F(s, t)/A(s, 0)|\)
has a lower bound \(\neq 0\) for \(|T| \leq t \leq 0\) as \(s \to \infty\), the phase variation from \(t = T_1\) to
\(t = T_2\) remains bounded when \(s \to \infty\). On the other hand, the phase variation between
\(t = T_1\) and \(t = 1/(\log s)^2\) is less than \(\log(|T_1|/|\log s|^2)\) i.e., \(\log \log s\) and the
phase variation from \(t = 1/(\log s)^2\) to \(t = 0\) is finite, as it is easy to show from
the analyticity properties of the scattering amplitude. We conclude therefore that if
\[ |F(s, t)| > R A(s, 0), |\phi(s, t)| < \log \log s + |\phi(s, 0)| \]

but \(\phi(s, 0)\) is bounded, and hence
\[ |\phi(s, t)| < \log \log s \tag{23} \]

and, hence, the proof of the previous section applies and if
\[ \frac{d\phi}{dt} / \frac{dA}{dt} \]
has a limit, this limit is unity. This settles at least one case:
if \((\sigma^P)/(dt) = g^P(t)\) and \((\sigma^A)/(dt) = g^A(t)\), where \(g^A\) and \(g^P\) do not vanish. The condition \(|\rho(s,t)/A(s,0)| > R\) is satisfied and, therefore, \(g^A(t) = g^P(t)\). Notice that no a priori assumption has been made about the phase.

More generally, following the same lines one can introduce

\[ R(s, t) = \min_{t \leq t' \leq 0} \frac{|\Phi(s, t')|}{A(s, 0)} \]

Following the same lines of thought one can therefore consider the case where \(R(s, t) \rightarrow 0\) as \(s \rightarrow \infty\). In that case \(\rho\) in Eq. (22) behaves like

\[ \rho \propto 1 + \frac{\pi R^2}{4} \]

and the phase variation is less than

\[ \frac{4}{\pi R^2} \left| \log \frac{T_2}{T_1} \right| \]

So, as long as

\[ R(s, t) \gg \frac{\sqrt{\log s}}{\sqrt{\log e}} \]

the phase variation from \(T_1\) to 0 is less than \(\log s\) and

\[ \lim_{t \rightarrow 0} \frac{d\rho^P}{dt} / \frac{d\rho^A}{dt} \rightarrow 1 \]

At the present time we cannot settle the case

\[ \frac{d\rho^P}{dt} = \frac{1}{\log e} g^P A(t) \]

which would give after integration

\[ \delta_{elastic} = const / \log s \]

This is unfortunate but hopefully will be settled soon.

5. THE SMALL \(t\) BEHAVIOUR

There are two problems. First of all, if it happens that \(\sigma^P_t / \sigma^A_t\) then we know that

\[ \rho^A(s, t) \sim \pm s \log s \cdot f(t \log s^2) \]

(26)
where \( f \) is a non-trivial entire function (at least for a sequence of energies \( \epsilon \)). Then a non-zero contribution to the elastic cross-section comes from \( t \sim \text{const}(\log \epsilon)^2 \) and we must have control on it. The Kinoshita relation

\[
\frac{d \sigma^P}{dt}(s, t = -\frac{\epsilon}{(2g_s)^2}) \sqrt{\frac{d \sigma^A}{dt}(s, t = -\frac{\epsilon}{(2g_s)^2})} \to 1
\]

is not sufficient for our purpose since it holds only for \( 0 < \epsilon < \epsilon_0 \). What we have done is this. By Schwarz inequality we get:

\[
\int_{-T}^{0} \left( \frac{d\sigma^P}{dt} - \frac{d\sigma^A}{dt} \right) dt \leq \int_{-T}^{0} \left( \frac{d\sigma^P}{dt} + \frac{d\sigma^A}{dt} \right) dt + \int_{-T}^{0} \left( \frac{d\sigma^P}{dt} \right)^2 dt + \int_{-T}^{0} \left( \frac{d\sigma^A}{dt} \right)^2 dt
\]

where \( + \) and \( - \) refer to even and odd signature amplitudes. And hence

\[
\left[ \int_{-T}^{0} \left( \frac{d\sigma^P}{dt} - \frac{d\sigma^A}{dt} \right) dt \right]^2 \leq \left[ \int_{-T}^{0} \left( \frac{d\sigma^P}{dt} \right)^2 dt + \int_{-T}^{0} \left( \frac{d\sigma^A}{dt} \right)^2 dt \right] \times \left( \sigma_{el}^P + \sigma_{el}^A \right)
\]

(27)

Now, looking at dispersion relations for the even amplitude we can prove that in the average

\[
\langle |\text{Re} f|^2 \rangle \leq s \left( \sigma_{el}^P + \sigma_{el}^A \right)
\]

More specifically, we have obtained:

\[
\int_{0}^{s} (\text{Re} f)^2 ds \leq s^3 \left( \sigma_{el}^P + \sigma_{el}^A \right)^2
\]

Therefore, in the average

\[
\langle \int_{-T}^{0} \left| \frac{d\sigma^P}{dt} - \frac{d\sigma^A}{dt} \right| dt \rangle \leq t \left( \sigma_{el}^P + \sigma_{el}^A \right) \sqrt{\sigma_{el}^P - \sigma_{el}^A}
\]

(28)
This equation shows that by taking $T$ small enough we can make the contribution of

$$\int_T^0 \left( \frac{d\sigma^P}{dt} - \frac{d\sigma^A}{dt} \right) dt$$

to the difference between integrated elastic cross-sections $\sigma^P_{el} - \sigma^A_{el}$ as small as we wish. Unfortunately, (28) is not sufficient in the case where elastic cross-sections go to zero, i.e., the bound is too large as compared to the individual values of cross-sections. So we have to use another trick which is to consider

$$\frac{d\sigma^P}{dt}(s, t(s)) \quad \text{and} \quad \frac{d\sigma^A}{dt}(s, t(s))$$

where $t(s)$ is a function going to zero in a smooth way. In practice, one has to consider all possible analytic monotoneous functions between $t(s) = \text{const}$ and $t(s) = 1/(\log s)^2$ and write dispersion relations for $P(s, t(s))$. This is a bit complicated but can be done. Crossing is awkward, but in the end one finds again that if

$$\frac{d\sigma^P}{dt}(s, t(s)) / \frac{d\sigma^A}{dt}(s, t(s))$$

has a limit this limit is unity, if the phases grow less fast than log $s$.

6. PUTTING EVERYTHING TOGETHER

a) $\sigma^P_{el}$ and $\sigma^A_{el}$ have finite non-zero limits

Here we shall make a pseudo-monotony assumption (which can be checked).

Assume

$$\frac{d\sigma}{dt}(s, t) < \frac{d\sigma}{dt}(s, t')$$

(29)

if $t < t' \leq 0 \quad \forall > 1$. Then, either $(d\sigma/dt)(s, t) > 0$ for $t \to \infty$ or $(d\sigma/dt)(s, t) \to 0$.

If it goes to zero, then we should not worry about $(d\sigma/dt)(s, t')$ for $t' \leq t$ since it will not contribute to the integrated cross-section. We also assume that the very large $|t|$ contribution is negligible and therefore we have no convergence problem.

If $(d\sigma/dt)(s, t)$ does not go to zero then, from the pseudo-monotony assumption (29) we see that condition (25) is satisfied and hence the phase is as it should be. Notice also (what we have not said yet) that if $d\sigma^P/dt \to 0$ one can prove that $d\sigma^A/dt$ also goes to zero without assumption on the phase. Hence, $(d\sigma^P/dt)/(d\sigma^A/dt) \to 1$ and this holds for all $t$'s for which $\lim(d\sigma^P/dt) \neq 0$. Therefore, for arbitrary $T < 0$

$$- \int_T^0 \left( \frac{d\sigma^P}{dt} - \frac{d\sigma^A}{dt} \right) dt \to 0.$$
On the other hand \( \int_0^\infty \) contributes as little as we wish even if the Pomeranchuk theorem is violated.

Hence \( \sigma_{ee}^P - \sigma_{ee}^A \to 0 \).

If the elastic cross-section goes to zero less fast than \( \log \log s/\log s \), using the same "monotony" assumption we prove that as far as we have non-zero contribution to the elastic cross-section the phase conditions are fulfilled and hence

\[
\frac{\sigma_{ee}^P}{\sigma_{ee}^A} \to 1
\]

The only dark problems left are:

(i) violently oscillating differential cross-sections \( \) notice that condition (26) still allows some oscillations \( \);

(ii) contributions from very large \( t \) for which you cannot use dispersion relations ;

(iii) the most worrying is the case

\[
\sigma_{ee}^P \sim \frac{\sigma_P}{\log s} \quad \text{and} \quad \sigma_{ee}^A \sim \frac{\sigma_A}{\log s} ;
\]

notice, however, that in the generalized Regge case

\[
F_P \sim \sigma_P(t) s^{\alpha_P(t)} \quad F_A \sim \sigma_A(t) s^{\alpha_A(t)}
\]

where \( \alpha_P \) and \( \alpha_A \) are allowed to be complex, you still get equality of cross-sections because \( \alpha_P = \alpha_A \) and positivity forces you to have

\[
\Im \alpha_P(0) = \Im \alpha_A(0)
\]

\[
\sigma_A^\ast(t) = \exp \left[ -\pi \Im \alpha_P(t) \right] \sigma_P(t)
\]

so in the neighbourhood of \( t = 0 \) \( \sigma_A \) = \( \sigma_P \) and this is the only region contributing to the elastic cross-section.
REFERENCES


3) See, for instance:

   U. Amaldi et al. - Communication to the Amsterdam Conference (1971).


6) In historical order:
   J. Finkelstein - Phys.Rev.Letters 24, 172 (1970);
   R.C. Casella - Phys.Rev.Letters 24, 1463 (1970);


13) See Appendix I of the paper by: