ON THE INCOHERENT SPACE CHARGE
LIMIT FOR ELLIPTIC BEAMS

by

W. Hardt

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1. INTRODUCTION

Lloyd Smith has pointed out the difference between a shift of the betatron frequency caused by external forces and by internal forces (as space charge) because the latter depend on the beam diameter. In the case of external forces, the envelope oscillates precisely at twice the betatron frequency, but this is not true for the case of internal forces. For the criterion for a significant instability of the beam is that the beam envelope oscillates at an harmonic of the revolution frequency, because this harmonic can be excited by a gradient error. Thus in the case of space charge, the resonance does not occur at a half integral value of the betatron frequency but at some other value which is to be calculated.

Lloyd Smith has treated the case of a round beam with a working point at the diagonal of the diamond and, in particular a gradient error of equal phase and amplitude for the vertical and horizontal motion (symmetric mode). For this case, he found the instability at $\Delta Q_{sc} = 2\Delta Q$, i.e., the resonance occurs if the betatron frequency shift induced by the space charge forces is twice as large as the distance of the undisturbed betatron frequency from the half integral value. In the notation to be used in this paper, this result is expressed by

$$S_+ (1,1) = 2 . \quad (1.1)$$

Lloyd Smith has also stated that in the case of an antisymmetric mode (gradient errors of equal amplitude but in antiphase for the two transverse directions) resonance occurs if $\Delta Q_{sc} = \frac{1}{2} \Delta Q$, which is now expressed by

$$S_- (1,1) = \frac{1}{2} . \quad (1.2)$$
If the beam is not round but elliptic, its width is normally larger than its height and the Q-shift is therefore expected to be more serious in the vertical direction. The ratio of the average width \(2a_0\) to its average height \(2b_0\) will be denoted by \(\alpha = a_0/b_0\). The working point of the undisturbed machine may lie outside the diagonal indicated by the parameter \(\lambda\)

\[
\Delta Q_y = \lambda \Delta Q_x
\]  

where the index \(x\) relates to the horizontal direction and the index \(y\) to the vertical. Also for \(\alpha \neq 1\) and \(\lambda \neq 1\), again two factors \(S_+(\alpha, \lambda)\) and \(S_-(\alpha, \lambda)\) are expected. By convention the vertical direction is preferred; this relates the vertical space-charge induced Q-shift \(\Delta Q_{sc,y}\) to the distance of the undisturbed vertical betatron frequency from the next (lower) half integral value \(\Delta Q_y\) by

\[
\Delta Q_{sc,y} = S_\pm(\alpha, \lambda) \Delta Q_y.
\]  

The corresponding factor \(T_\pm\) for the horizontal motion

\[
\Delta Q_{sc,x} = T_\pm(\alpha, \lambda) \Delta Q_x
\]  

is related in a simple way to \(S_\pm\) by

\[
\omega T_\pm = \lambda S_\pm
\]  

as it can easily be verified from this paper. Thus it is sufficient to calculate \(S_\pm(\alpha, \lambda)\) only.

The rest of this paper is devoted to the mathematical treatment of obtaining the expression:

\[
S_\pm(\alpha, \lambda) = \frac{1}{2(1+\alpha)} \left[ 3 + 2\alpha + \frac{3}{\lambda}(2 + 3\alpha) \pm \sqrt{\left( \frac{3 + 2\alpha}{\lambda} \left( 2 + 3\alpha \right) \right)^2 + 4\alpha^2} \right]
\]
2. THE ENVELOPE EQUATION IN ONE DIMENSION AND A FEW IMPLICATIONS

Because the frequency of envelope oscillations has to be studied, it is convenient to start with the envelope equation [Eq. (3.2) of Courant and Snyder²];

\[ W'' + KW - \frac{1}{W^3} = 0. \]  \hspace{1cm} (2.1)

\( W \) is related to the amplitude functions \( \beta \), the beam envelope (half width) "a", and the emittance \( \varepsilon \) by

\[ W^2 = \beta c = \frac{e^2}{\lambda/\pi}. \]  \hspace{1cm} (2.2)

Thus, changing to the beam radius itself, Eq. (2.1) becomes

\[ a'' + ka - \frac{(A/\pi)^2}{a^3} = 0. \]  \hspace{1cm} (2.3)

It is assumed that the solution \( a_0 \) of Eq. (2.3) is known for \( K_0(s) \). On referring to this undisturbed solution with \( a = \rho a_0 \) and \( K = K_0 + \Delta K \), Eq. (2.3) becomes

\[ \rho'' + 2\rho' \frac{a_0'}{a_0} + \frac{(A/\pi)^2}{a_0^4} \left( \rho - \frac{1}{\rho^3} \right) = -\Delta K \rho. \]  \hspace{1cm} (2.4)

The second term of the left-hand side disappears on changing the independent variable \( s \) to the angle \( \phi \) by

\[ \rho' = \frac{d\rho}{ds} = \frac{1}{Q_0\beta_0} \frac{d\rho}{d\phi} = \frac{1}{Q_0\beta_0} \rho \quad \text{and} \quad \rho'' = \frac{1}{Q_0^2\beta_0^2} \left( \rho' \frac{d\beta_0}{d\phi} + \rho - \frac{1}{\rho} \right). \]

With \( \phi \) as the independent variable, Eq. (2.4) becomes

\[ \frac{1}{Q_0^2} \dot{\beta} + \left( \rho - \frac{1}{\rho} \right) = -\Delta K \beta^2 \rho. \]  \hspace{1cm} (2.5)

Before dealing with the specific space-charge problem, some implications of Eq. (2.5) will be studied.

The disturbance \( \Delta K \) causes two effects:

i) it will affect the behaviour of the envelope by virtue of Eq. (2.5);

ii) it will shift the betatron frequency and the envelope frequency.
The shift of the betatron frequency might be calculated by the expression

$$
\Delta \omega^{\text{dis}} = \frac{1}{4\pi} \int \Delta K_{\theta} d\theta = \frac{R}{2} \left( \Delta K_{\theta}^2 \right) = \frac{Q_0}{4\pi} \int \Delta K_{\theta}^2 d\theta = \frac{Q_0}{2} \left( \Delta K_{\theta}^2 \right)
$$

(Eq. (4.31) of Ref. 2).

The index "dis" means disturbance and will be replaced by "sc" in the particular case of space charge. Equation (2.5) can be written by means of Eq. (2.6)

$$\frac{1}{Q_0} \frac{d}{dt} \rho + \left( \rho - \frac{1}{\rho^3} \right) = - \frac{2\Delta \omega^{\text{dis}}}{Q_0} \rho + \text{oscillating terms.}
$$

The oscillating terms are periodic with the machine lattice and would not appear in "smooth approximation". But even with a better approximation they can be disregarded on averaging under the assumption that the numbers of machine periods is remote from twice the $Q$-value (frequency of envelope oscillations). This assumption should be met for any accelerator.

Because in the following only $\Delta \omega^{\text{dis}} < 0$ will be of interest, the symbol $q = - (\Delta \omega^{\text{dis}}/Q_0)$ is introduced ($q > 0$). Equation (2.7) is still wrong in the case of space charge; namely, it will be shown in the fourth paragraph that the right-hand term has another power in $\rho$. In order to be more general, the following equation will be studied for any power in

$$\frac{1}{Q_0} \frac{d}{dt} \rho + \left( \rho - \frac{1}{\rho^3} \right) = 2q \rho^m,
$$

(Eq. 2.8)

Equation (2.8) has a constant solution given by

$$\rho_0^m - 1 = 2q \rho_0^{m+3}
$$

or approximately $\rho_0 - 1 \approx \frac{q}{2}$ for $\rho_0 - 1 \ll 1$. This constant solution belongs to a matched beam which does not undergo free oscillations. But even starting with such a beam, envelope oscillations could be excited by a gradient error if the frequency for envelope oscillations...
becomes an integer. In order to show this, the new variable $\sigma$ is introduced referring to the matched solution by $\rho = \sigma \rho_0$. Then Eq. (2.9) becomes

$$
\frac{1}{\rho_0^2} \rho + \sigma - \frac{1}{\sigma^2 \rho_0} - \sigma^n \left( 1 - \frac{1}{\rho_0^2} \right) = 0 \quad (2.10)
$$

Because Eq. (2.10) is non-linear, the frequency for envelope oscillations might be defined by referring to small amplitudes as is also usual in other non-linear problems. Thus Eq. (2.10) might now be linearized by putting $\sigma = 1 = \eta$

$$
\frac{1}{Q_0} \eta + 4 \left( 1 - \frac{3m}{2} q \right) \eta = 0 \quad (2.11)
$$

the frequency for envelope oscillations being given by

$$
2Q_0 \left( 1 - \frac{3m}{2} q \right)^{1/2} \approx 2Q_0 \left( 1 - \frac{3m}{4} q \right). \quad (2.12)
$$

The nearest integer "$n" for the envelope oscillations corresponds to a half integer for the betatron frequency and is introduced by putting $2Q_0 = n + 2 \Delta q = n(1 + p)$. Inserting this into Eq. (2.12) for $q = (4/3 + m)p$ or $\Delta q_{\text{dis}} = (4/3 + m)\Delta q$ or in the notation of Eq. (1.4) for

$$
S(1, 1) = \frac{4}{3 + m}. \quad (2.13)
$$

In the fourth paragraph it will become clear that $m = 1$ corresponds to outer forces causing the Q-shift, whilst $m = 0$ and $m = -1$ correspond to space-charge forces causing the Q-shift acting on a round beam. In particular, $m = -1$ belongs to the symmetric mode [see Eq. (1.1)] and $m = 0$ belongs to the antisymmetric mode [see Eq. (1.2)].

3. THE Q-SHIFT INDUCED BY SPACE-CHARGE FORCES

In order to generalize these considerations for an elliptic beam, Eq. (2.5) must be written separately for the two transverse directions and the right-hand terms must be related to the space-charge forces. These forces act on particles homogeneously distributed over
the elliptic beam cross-section and are given by 3)

\[
\begin{align*}
 f_x &= \frac{e^2 N \epsilon}{2 \pi^2 R \epsilon_0 B \sigma^2 (1 + b/a)^2} x^2 \leq a^2 \left( 1 - \frac{y^2}{b^2} \right) \\
 f_y &= \frac{e^2 N \epsilon}{2 \pi^2 R \epsilon_0 B b^2 (1 + a/b)^2} y^2 \leq b^2 \left( 1 - \frac{x^2}{a^2} \right) 
\end{align*}
\]

(3.1)

where \( N \) = total numbers of particles
\( R \) = average radius
\( B \) = bunching factor
\( a, b \) = horizontal, vertical half dimensions of the beam
(contributions from phase oscillations neglected).

The forces are related to \( \Delta K \) by

\[
\begin{align*}
 \Delta K_x &= - \frac{f_x}{eU_p \beta^* Y} \\
 \Delta K_y &= - \frac{f_y}{eU_p \beta^* Y}
\end{align*}
\]

(3.2)

with \( eU_p \), being the proton rest energy.

From Eqs. (2.6), (3.1) and (3.2) follow the expressions

\[
\begin{align*}
 \Delta Q_{sc,x} &= - \frac{eN}{4\pi\epsilon_0 B \sigma^2 Y} U \frac{1}{p_x} \left( \frac{1}{1 - \frac{y}{b}} \right) \\
 \Delta Q_{sc,y} &= - \frac{eN}{4\pi\epsilon_0 B \sigma^2 Y} U \frac{1}{p_y} \left( \frac{1}{1 - \frac{x}{a}} \right)
\end{align*}
\]

(3.3)

where \( A_x = \pi a \sigma_x \) and \( A_y = \pi b \sigma_y \). Equations (3.3) are adequate to study the influence of a wriggle on the Q-shift. In practical cases, this influence is so small that the averaging process can be replaced by putting
where $a_0, b_0$ belong to $\beta_{0x} = R/Q_{0x}$ and $\beta_{0y} = L/Q_{0y}$. Combining Eqs. (3.1), (3.2), (3.3), and (3.4), Eq. (2.5) now becomes for the two transverse directions

$$\begin{align*}
\frac{1}{Q_{0x}} \rho_x + \left( \rho_x - \frac{1}{\rho_x} \right) &= \frac{2q(1+\alpha)}{a(\sqrt{\rho_x} + \rho_y)} \\
\frac{1}{Q_{0y}} \rho_y + \left( \rho_y - \frac{1}{\rho_y} \right) &= \frac{2q(1+\alpha)}{a\rho_x + \rho_y}
\end{align*}
$$

(3.5)

where $q = \Delta^2 sc, y/Q_{0y}$ and $a = a_0/b_0$.

4. **The Treatment of the Space-Charge Problem in Two Co-ordinates**

Equations (3.5) correspond already to Eq. (2.8) of the one-dimensional case. Thus the next steps are analogous to the procedure of getting the next equations (2.9) until (2.13). The constant solutions of Eq. (3.5) are approximately given by $\rho_{0x} - 1 \approx q/2a$ and $\rho_{0y} - 1 \approx q/2$. In order to refer to these constant solutions, the new variables $\sigma_x$ and $\sigma_y$ are introduced by

$$\rho_x = \sigma_x \rho_{0x} \quad \text{and} \quad \rho_y = \sigma_y \rho_{0y}.$$

After linearization with $\sigma_x - 1 = \eta_x$ and $\sigma_y - 1 = \eta_y$, the equations corresponding to Eq. (2.11) are obtained

$$\begin{align*}
\frac{1}{Q_{0x}} \eta_x + 4 \left[ 1 - \frac{3}{2a} \left( \frac{\alpha}{1+\alpha} \right) \right] \eta_x + \frac{2q}{a(1+\alpha)} \eta_y &= 0 \\
\frac{1}{Q_{0y}} \eta_y + 4 \left[ 1 - \frac{3}{2a} \left( \frac{1}{1+\alpha} \right) \right] \eta_y + \frac{2q}{a(1+\alpha)} \eta_x &= 0
\end{align*}
$$

(4.1)
The proof shows that in the case of a round beam \( (a = 1) \), Eq. (4.1) becomes Eq. (2.9) with \( m = -1 \), assuming \( \eta_x = \eta_y \) (symmetric mode), and that Eq. (2.9) with \( m = 0 \) is obtained assuming \( \eta_x = -\eta_y \) (antisymmetric mode). Thus the statement at the end of the second paragraph is confirmed.

In order to find the dangerous \( q \)-values, it is convenient to put

\[
2q_{ox} = n(1 + \rho_x) \\
2q_{oy} = n(1 + \rho_y) \\
\eta = -n^2.
\]

The result can be written in matrix form

\[
\begin{pmatrix}
4p_x - \frac{q}{\alpha} \frac{3 + 2\alpha}{1 + \alpha} & \frac{q}{\alpha(1 + \alpha)} \\
\frac{q}{\alpha(1 + \alpha)} & 4p_y - \frac{q}{\alpha} \frac{3 + 2\alpha}{1 + \alpha}
\end{pmatrix}
\begin{pmatrix}
\eta_x \\
\eta_y
\end{pmatrix} = 0. 
\tag{4.2}
\]

In the presence of gradient errors Eq. (4.2) becomes an inhomogeneous system because these errors appear at the right-hand side, in particular the dangerous \( n^{th} \) Fourier component. This system can easily be solved, and the solutions have a denominator essentially given by the determinant of the matrix. The determinant is quadratic in \( q \), the roots indicating the dangerous values which are

\[
a_\pm = \frac{1}{3(1 + \alpha)} \left\{ (2 + 3\alpha)p_x + \\
(3 + 2\alpha)p_y \pm \sqrt{\left[ (2 + 3\alpha)p_x - (3 + 2\alpha)p_y \right]^2 + 4\alpha^2 p_x p_y} \right\}. 
\tag{4.3}
\]
Substituting \( q_+ = S_+ p_y \) and \( p_y = \lambda p_{\gamma} \), this can be written as

\[ \text{equation (1.7)}. \]

Some properties of \( S_+ \) and \( T_+ \) are

\[
\frac{\lambda}{a} S_+(a,\lambda) = S_+\left(\frac{1}{a},\lambda\right) = T_+(a,\lambda) = \frac{\Delta Q_{sc,x}}{\Delta Q_{ox}}
\]

\[ S_+(a,\lambda) T_-(a,\lambda) = S_-(a,\lambda) T_+(a,\lambda) = \frac{\pi}{3} \]

\[ S_+(a,\alpha) = S_+(0,\lambda) = T_+(\alpha,\lambda) = 2 \]

\[ S_-(a,\alpha) = S_-(\omega,\lambda) = T_-(0,\lambda) = \frac{\pi}{3}. \]

In the numerical evaluation of Fig. 1, the undisturbed working point has been kept fixed. Thus for various values of \( \lambda \) the horizontal borders of the diamond move accordingly, as indicated. Generally \( S_-( \text{ or } T_-) \) has to be taken for space-charge estimations relating to the nearest resonance. The aim of this paper is only to point out the location where a significant instability might occur. It is much more difficult to predict the magnitude of the resonance amplitude. The non-linear behaviour will prevent the envelope from becoming infinite. Lloyd Smith has shown the maximum amplitude in the particular case of \( S_+(1,1) \). It seems that even for very small (vanishing in practice) gradient errors, a blow-up of about 10\% is unavoidable. A description of the whole phenomenon is more complicated, for various reasons.

i) The assumption of an homogeneous particle density inside the beam is certainly wrong, but simplifies the analytical treatment enormously.

ii) In calculating \( S_+ \), \( q \ll 1 \) and \( p \ll 1 \) has been assumed, in order to neglect terms quadratic in \( q \). Thus the calculation is less precise for low \( Q \) machines.
iii) The bunching factor in Eqs. (3.1) and (3.3) denotes only the minimum value of longitudinal average particle density over local particle density. Higher bunching factors would belong to other transverse slices through a bunch with a tendency to move the instabilities further away.

iv) If crossing of the "instability lines" is intended, the amplitude will also depend on the speed at which crossing occurs. A similar complication is that the speed of phase oscillations is also involved.

v) Image forces are neglected in Eq. (3.1). It might be worth while, and not very difficult, to extend the theory in this direction.

**

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REFERENCES


\[ \frac{S_{\pm}(\alpha, \lambda)}{\alpha} = \frac{T_{\pm}(\alpha, \lambda)}{\lambda} \]

**Fig. 1**

Horizontal borders of the diamond

Separatrix between \( S_+ \) and \( S_- \)