ABSTRACT

In this dissertation, we focus on aspects of String Theory and General Relativity. The study of objects of various spatial dimensions, or branes, have become an intrinsic part of modern String Theory. Here, we focus on three concrete topics concerning the physics of branes: supertubes, quantum D-brane polarization, and the stability of black branes. We devote the final part of this work to the study of fast travel in 3+1 dimensional spherically symmetric configurations.
SUPERTUBES, BLACK STRINGS AND
D-BRANE SYSTEMS

by

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Chapter 1

Introduction

The failures of field theory to provide finite short-distance interactions that include gravity is often taken to indicate the need for new physics at very high energy. However, the distances at which gravity becomes relevant (of the order of the Plank length $10^{-33}$ cm) are too short to be probed in our current particle accelerators. This leaves us with purely theoretical premises to build a theory of quantum gravity, and therefore, plenty of room to consider radically different approaches.

String theory is one of the most popular candidates for a theory of quantum gravity. In this theory, some of the fundamental objects are strings, whose different vibration modes may represent different types of elementary particles. Among its immediate successes is that all versions of the theory include a particle of zero mass and spin 2. Thus, it contains gravity. Another remarkable features is that the theory naturally possesses a short-distance regulator which seems to prevent the appearance of divergences. On the other hand, Lorentz invariance and supersymmetry in string theory would unfortunately require the number of spacetime dimensions to be 10. However, this apparent inconsistency with observation can be resolved considering solutions with 4 large dimensions and 6 small curved directions.

Among the most interesting aspects of string theory is the existence of not only
strings, but also of other extended objects of spatial dimensions bigger than one ("branes"). These objects play a central role in modern stringy physics, being at the core of the proposed dualities [1]-[4] between gravity and field theories. The particular branes called Dp-branes will be the main subject of this dissertation.

We will devote this chapter to discuss the main elements of the physics of branes. Our intention is to present a summary on the topic that can be used as a reference for later chapters. Although each chapter contains an introduction to the specific problem under study, the following review will establish a connection among the different aspects of D-branes covered by our work.

Before proceeding with the introduction to D-branes, we pause to mention that chapter 5 departs from the preceding ones to treat a subject entirely within General Relativity and in the usual 3+1 dimensions. The main focus of chapter 5 will be the study of spacetimes satisfying positive energy conditions but which allow signals to propagate between two spatial points "faster" than would be allowed in Minkowski space, where we restrict the discussion to spherically symmetric spacetimes in order to make these notions precise. A reader uninterested in the physics of branes may wish to skip to chapter 5.

1.1 Dp-branes

In perturbative string theory\(^1\), Dp-branes appear as the result of imposing Dirichlet boundary conditions at the ends of the open strings\(^2\). This condition restricts the open string endpoints to lie in hyperplanes: the Dp-branes, where \(p\) is the number of spatial dimensions. These hyperplanes are themselves dynamical objects that react

\(^1\)The perturbative expansion of string theory is defined as a sum over all the topologies of 2-dimensional string worldsheet, where the topology of the worldsheet determines the order of each term in the expansion.

\(^2\)We could also impose Neumann boundary conditions which allow the open string endpoints to move freely.
to the presence of the gravitational field and other background fields. In fact, the
very open strings mentioned above naturally describe their fluctuations. In partic-
ular, the low energy fluctuations are described by the massless open string states,
which separate to produce a U(1) gauge field $A_a$ living on the brane and scalars $X^\mu$
representing the transverse oscillations of the brane embedded in spacetime. To study
the low energy dynamics of the D-brane, we must then write down the interactions
among the worldvolume fields $X^\mu$ and $A_a$, and with the massless closed string fields
corresponding to the background, whose low energy dynamics is governed by the su-
pergravity action to be presented later. The couplings with the background metric
$G_{\mu\nu}$, the antisymmetric NS-NS tensor $B_{\mu\nu}$ (NS stands for Neveu-Schwarz) and the
dilaton $\Phi$ are given by the Dirac-Born-Infeld (DBI) effective action, which takes the
form

$$S_{D_p} = -T_p \int d^{p+1}\xi e^{-\Phi} [ -det( G_{ab} + B_{ab} + F_{ab}) ]^{1/2}, \quad (1.1.1)$$

where $T_p$ is the Dp-brane tension, and $\xi^a, a = 0, ... p$ are the coordinates on the brane. Here

$$G_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi)), \quad B_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}(X(\xi)), \quad (1.1.2)$$

are the induced metric and the antisymmetric tensor on the brane, and $F_{ab} = \partial_a A_b - \partial_b A_a$ is the worldvolume field strength. Since $A_a$ only enters through its field strength,
the action is automatically U(1) gauge invariant. For N coincident D-branes, the
abelian U(1) symmetry is enhanced to a nonabelian U(N) symmetry. The extra
massless modes needed to fill out the U(N) representations are obtained from strings
stretching between the N coincident D-branes. Hence, the dynamics is governed by
a non-abelian U(N) field theory, and additional traces over the gauge indices appear
in the action.

The massless closed string states (i.e., those not attached to any branes) of the
type I and II superstring theories also include antisymmetric tensors of rank $q = 0, ... 9,$
the so-called Ramond-Ramond (R-R) forms $C_{(q)}$. It is natural to expect the Dp-brane

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to act as \( \delta \)-function source coupled to the R-R \((p + 1)\)-form in the same way that a point particle acts as an electrically charged source for the vector field \( A_{(1)} \) via the coupling
\[
e \int A_{(1)} = e \int A_\mu dX^\mu = e \int A_\mu \frac{dX^\mu}{d\tau} d\tau.
\]
In the case of a R-R \((p + 1)\)-form, the integral is over the brane worldvolume, which is a subspace of \( p \) spatial dimensions and one time,
\[
\mu_p \int_{p+1} C_{(p+1)},
\]
and \( \mu_p \) is the electric charge. Generalizations of the coupling to arbitrary configurations of R-R forms are given by the Chern-Simons terms
\[
\mu_p \int_{p+1} \exp(F_{(2)} + B_{(2)}) \wedge \sum_q C_{(q)},
\]
where the integral picks out precisely the terms that are proportional to the volume form of the D\( p \)-brane worldvolume.

**Supersymmetry and \( \kappa \)-symmetry**

D-branes preserve some of the vacuum spacetime supersymmetries, and therefore, they are BPS states\(^3\). As discussed in the previous paragraph, they carry conserved charges associated with the antisymmetric R-R forms which are the central charges of the supersymmetry algebra. The supersymmetric extension of the D\( p \)-action can be obtained replacing the bosonic coordinates \( X^\mu \) in the above action with the supercoordinates \( Z^M = (X^\mu, \theta^\alpha) \), and the various bosonic background fields with the corresponding superfields of which they are the leading component in a \( \theta \)-expansion. The index \( A \) of the supervielbein \( E^A_M \) decomposes under the action of the \( d=10 \) Lorentz group as \( A = (\mu, \alpha) \), where \( \mu \) is a \( d=10 \) vector index and \( \alpha \) is a \( d=10 \) spinor.

\(^3\)They saturate the Bogomolnyi-Prasad-Sommerfield bound \( M \geq \sum |Q_i| \), where \( M \) is the mass of the state, and \( Q_i \) are the appropriately normalized charges. Thus, \( T_p = \mu_p \).
index. In chapter 2 and 3, we will respectively focus on the D2 branes and the D4 branes of type IIA string theory whose spinors are in the 32-component Majorana representation. The induced metric takes the form

$$G_{ab} = E_a^\mu E_b^\nu \eta_{\mu\nu} ,$$  \hspace{1cm} (1.1.6)

where

$$E_a^A = \partial_a Z^M E_M^A ,$$  \hspace{1cm} (1.1.7)

$\eta_{\mu\nu}$ is the Minkowski metric and $a, b = 0, \ldots, p$ are the worldvolume vector indexes. The R-R forms and B are taken to be forms in superspace,

$$C^{(r)} = \frac{1}{r!} dZ^{M_1} \ldots dZ^{M_r} C_{M_r \ldots M_1} ,$$  \hspace{1cm} (1.1.8)

Note that the components are written in reverse order since this is the usual convention for superspace forms.

Besides being supersymmetry invariant, the covariant super D$p$-brane action also exhibits a fermionic gauge symmetry on the worldvolume called $\kappa$-symmetry. The combined linear action of $\kappa$-symmetry and of supersymmetry on the spacetime spinor $\theta$ is given by

$$\delta \theta = (1 + \Gamma) \kappa + \epsilon ,$$  \hspace{1cm} (1.1.9)

where $\kappa$ and $\epsilon$ are the corresponding spinor parameters of each symmetry transformations and $\Gamma$ is a hermitian traceless matrix satisfying $\Gamma^2 = 1$; although we will not present here the explicit expression for $\Gamma$ as it can be found elsewhere (see e.g. [5]). One of the crucial properties of $\kappa$ symmetry is that it combines with spacetime supersymmetry to produce a global worldvolume supersymmetry. To see this, let us gauge-fix the $\kappa$-symmetry using the condition

$$P \theta = 0 ,$$  \hspace{1cm} (1.1.10)

where $P$ is a projector, $P^2 = P$, and thus $\lambda \equiv (1 - P)\theta$ are the remaining non-vanishing components of $\theta$. In order to preserve the gauge choice, we must also
impose $P \delta \theta = 0$. Therefore, with the freedom to choose arbitrary $\kappa$ now removed, (1.1.9) becomes a global worldvolume supersymmetry transformation for $\lambda$

$$\delta \lambda = (1 - P) \delta \theta = \delta \theta . \quad (1.1.11)$$

We will be interested in bosonic configurations ($\theta = 0$) that are invariant under a fraction of spacetime supersymmetry. After fixing the gauge, the nontrivial Killing spinor $\epsilon_{\text{unbr}}$ associated with the unbroken supersymmetries satisfy

$$\delta \theta = (1 + \Gamma) \kappa + \epsilon_{\text{unbr}} = 0 , \quad (1.1.12)$$

which in turns implies the condition

$$(1 - \Gamma) \epsilon_{\text{unbr}} = 0 . \quad (1.1.13)$$

In chapter 2, we will be concerned with the symmetries of the D2-brane action. We will study supersymmetric configurations called supertubes [6, 7, 8] where the conditions on the unbroken supersymmetries correspond to those found for a system of D0-branes and fundamental (F1) strings\(^4\). This fact supports the interpretation that, although supertubes are effectively described by a (2+1)-dimensional action, they actually represent D0-F1 states ”blown up” by rotation. We will present independent evidence in chapter 2 showing that such states are in fact bound, but we leave the details for later.

In chapter 3, we follow the formalism used in [9], where, besides the superspace coordinates $Z^M$, the tangent space vector $y^A = (y^a, y^\alpha)$ is also introduced. The component $y^\alpha$ is a fermion and consequently, also obeys $\kappa$-symmetry rules. We will consider bosonic backgrounds for which $\theta = 0$, but we will keep $y^\alpha$ (labelled $\psi$ in chapter 3) as fermionic field living on the brane.

\(^4\)The fundamental (F1) string is the original quantized string which is used to formulate the usual string perturbation theory. The F1-string is different from the D1-brane.
Charges, bound states and brane polarization

We have noted that the Chern-Simons terms (1.1.5) allow the coupling of R-R potentials with rank lower than the volume-form rank (i.e., lower than \( p + 1 \)). Taking into account that the coupling between a \( q \)-form gauge field and its current is of the form \( \int_{p+1} C(q) \wedge J(p+1-q) \), we see that the current \( J(p+1-q) \) can be induced in terms of fluxes of other forms. Thus, in the presence of the appropriate fluxes, the D\( p \)-brane carries D\( q \)-brane charge, where \( q < p \), and the resulting configuration may be interpreted as a bound state of branes of different dimensions. Supertubes are examples of configurations carrying D0-brane and F1-string charges, which are effectively described by the D2-brane action. As we already anticipated, one of the outcomes of our computations in chapter 2 will be that supertubes precisely correspond to D0-F1 bound states. Due to the unbroken supersymmetry, they have exactly zero binding energy.

In this fashion, supertubes are an example of the so-called D-brane polarization effect (see [10, 11]), where a bound state of low-dimension branes may, when polarized, be effectively described as a single brane of higher dimension. In [11], it was shown that the non-abelian generalization of the Chern-Simons action for a system of N D\( p \)-branes gives rise to couplings with R-R forms of rank \( n > p + 2 \). In particular, the author of [11] described the case when N D0-branes in a constant background R-R 4-form field strength expand into the spherical bound state of a D2-brane and N D0-branes. Other examples of brane polarization induced by external fields are those associated with giant gravitons [12], instanton studies [13] and brane inflation [14]. In the case of supertubes, polarization occurs in the absence of external forces, i.e., the background fields are trivial, and the existence of angular momentum is what prevents the expanded configurations from collapsing.

To our knowledge, however, all of the standard applications can be viewed as classical polarization effects, perhaps with quantum corrections. Consider, however,
Dp-branes in the supergravity fields generated by D(p \pm 4)-branes. In this context the unpolarized Dp-branes are known to saturate the BPS bound. Thus, there can be no lower energy classical configuration and the classical ground state remains unpolarized\(^5\). However, this argument leaves open the possible distortion of quantum fluctuations around the classical ground state and associated quantum polarization effects. The investigation of such quantum effects was initiated in [15] for the case \(p=0\). In chapter 3, we will reverse the setting to study the quantum fluctuations of a D4-brane in the background of N D0-branes and our results will indicate a non-vanishing effect.

1.2 Supergravity \(p\)-branes

The low energy dynamics of massless closed string modes is described by the supergravity action. In ten dimensions, there are two distinct closed string theories that contain two spacetime supersymmetries: Type IIA and Type IIB, and they differ from one another in the relative chiralities of their left-moving and right-moving closed string degrees of freedom. The massless spectra of both theories contain the graviton \(G_{\mu\nu}\), the antisymmetric NS-NS tensor \(B_{\mu\nu}\), and the dilaton \(\Phi\), but Type IIA theory includes R-R q-forms \(C_{(q)}\) of even \(q\), while \(q\) is odd for Type IIB theory. In chapter 4, we will be interested in type IIB theory, for which the bosonic part of the action is

\[
S_{IIB} = \frac{1}{2\kappa_0^2} \int d^{10}x (G)^{-1/2} e^{-2\Phi} \left( R + 4 \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{2} |H_{(3)}|^2 \right) - \frac{1}{4\kappa_0} \int d^{10}x (G)^{-1/2} \left( |F_{(1)}|^2 + |\tilde{F}_{(3)}|^2 + \frac{1}{2} |\tilde{F}_{(5)}|^2 \right) - \frac{1}{4\kappa_0^2} \int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \right). \tag{1.2.1}
\]

\(^5\)There could, however, be several degenerate ground states so long as one remains unpolarized.
where $\kappa_0 = g_s^{-1}(8\pi G_N)^{1/2}$, $g_s$ is the string coupling and $G_N$ is the Newton’s constant.

In the above action we have defined the field strengths to be $H_{(3)} = dB_{(2)}$, $F_{(q)} = dC_{(q-1)}$ and

$$\tilde{F}_{(3)} = F_{(3)} - C_{(0)} \wedge H_{(3)} ,$$

$$\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2} C_{(2)} \wedge H_{(3)} + \frac{1}{2} B_{(2)} \wedge F_{(3)} .$$  \hspace{1cm} (1.2.2)

In order to generate a standard Einstein-Hilbert term, we introduce a new metric

$$G_{E\mu\nu} = e^{-\frac{2\Phi}{3}} G_{\mu\nu} ,$$  \hspace{1cm} (1.2.3)

where $G_{E\mu\nu}$ is called the Einstein metric. Thus, the action in the Einstein frame is

$$S_{IIB} = \frac{1}{2\kappa_0^2} \int d^{10}x (G_E)^{-1/2} \left( R_E - \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} e^{-\Phi} |H_{(3)}|^2 \right)$$

$$- \frac{1}{4\kappa_0^2} \int d^{10}x (G_E)^{-1/2} \left( e^{2\Phi} |F_{(1)}|^2 + e^\Phi |\tilde{F}_{(3)}|^2 + \frac{1}{2} |\tilde{F}_{(5)}|^2 \right)$$

$$- \frac{1}{4\kappa_0^2} \int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} .$$  \hspace{1cm} (1.2.4)

The supergravity equations of motion admit a variety of static solutions. In particular, the equations of motion of the R-R forms allow to define several conserved charges in analogy with the electric charge computed using the Gauss’ law of the Maxwell theory. For a $C_{(p+1)}$ form in $d$ dimensions, the associated charge is measured by an integral over a closed surface of dimension $d - 1 - (p + 1)$ which encloses a $(p + 1)$-dimensional source (see [16] for a review on n-form gauge fields). By enclosing we mean that, if one shrinks the surface to zero size, the closed surface will inevitably encounter the $(p + 1)$-dimensional worldvolume of the source. In (1.1.4), we have already anticipated the coupling of $C_{(p+1)}$ to a $\delta$-function source extended in $p + 1$ directions. Thus, supergravity solutions charged under $C_{(p+1)}$ are extended objects and so are called $p$-branes, where $p$ is again the number of spatial directions.

As in 3+1 gravity, $p$-branes can contain horizons and thus form black $p$-branes. As usual, the area of the event horizon is associated with the entropy of the black
brane and is given the name of Bekenstein entropy. In [17], it was pointed out that the Dp-branes of perturbative string theory and static (or stationary) supergravity solutions with R-R charge (including black p-branes) should represent descriptions of the same states of string theory but in different regimes of the theory’s coupling gs.

The above observation became the root of the stringy counting of black hole entropy [18] and the related gauge/gravity duality ideas [1]-[4]. While the U(N) gauge theory that describes the low energy behavior of D-branes is perturbatively valid for gsN << 1, the gravitational description in terms of a black p-brane is valid when the curvature is small compared to the string scale, which translates into 1 << gsN (see, e.g. [4]). In [18], the Bekenstein entropy of a BPS two-charge black hole was precisely matched (for large charges) with the logarithm of the density of states of the field theory living on the worldvolume of the dual D-brane. Due to supersymmetry, the number of BPS states remains unchanged when the value of the string coupling gs varies. Thus, string theory has succeeded in providing a statistical mechanical account in terms of microstates for the thermodynamical entropy of a black hole. In chapter 2, we will comment on a recent interesting proposal [19]-[25] regarding black hole entropy, in which the microstates of a field theory, which are localized on the D-branes, are mapped to extended non-singular geometries that exist in supergravity regime.

Outline of the Dissertation

This dissertation is based on the publications [26], [27], [28], [29]. Each chapter covers one the above papers as they deal with independent and self-contained questions. Also, the use of notation is conveniently adapted to the needs in each separate chapter.

In Chapter 2, the quantum states of the supertube are counted by directly quan-
tizing the linearized DBI action. Our results will show that supertubes represent the
generic D0-F1 bound state. In Chapter 3, the low energy effective field theory of
D4-branes coupled to supergravity fields is used investigate quantum effects for D4-
branes in the D0 supergravity background. The computed effect is divergent in a field
theory approximation, but is expected to be cut-off naturally by stringy corrections.
In chapter 4, we will look at the rotating black string. The study of the parameter
dependance of the Bekenstein entropy will lead us propose the existence of an insta-
bility for the black string that is spinning on its symmetry axis. We will see that the
entropy increases when part of the rotation is carried by helical-like oscillations of the
string.

Chapter 5 is somewhat disconnected from rest of the text. In the context of
General Relativity, we will consider the relation of fast travel and the restriction to
positive energies. We will prove a theorem to the effect that, as viewed from infinity,
signals always propagate faster in Minkowski space than in any other spherically
symmetric spacetime. We then begin an investigation of certain related but more
local questions by studying particular families of spacetimes in detail.
Chapter 2

Counting Supertubes

In this chapter, we will be concerned with the D0-F1 supertube of [6, 7, 8], which is an example of brane polarization. Supertubes have the special distinction that the polarized states are BPS and arise without the application of an external field. In this case, stabilization against collapse is achieved by means of the angular momentum generated by the worldvolume fields.

The supertubes of interest here are solutions of type II A string theory that carry D0 and F1 charge and have the supersymmetries expected of such configurations. These are the original supertubes of [6], though many related configurations can be obtained through duality transformations (e.g. it is dual to the D1-D5 system). The charges are arrayed around a tube of topology $S^1 \times \mathbb{R}$ in space, where the $\mathbb{R}$ represents a translation symmetry of the system and the direction along which the fundamental strings are aligned. Interestingly, the $S^1$ can be an arbitrary curve [30] (see also [31, 32, 33, 34] for earlier and related results) in the space of symmetry orbits; all such configurations are static. We assume the curve is closed and also compactify the $\mathbb{R}$ direction as we are interested in cases with finite charge.

Mateos and Townsend [6] showed that the supertube can be described using the Dirac-Born-Infeld effective action of a D2-brane. The D2-worldvolume is then the
above-mentioned $S^1 \times \mathbb{R}$ tube ($\times$ time). Because the $S^1$ is a closed curve, the configuration has no net D2 charge. However, if the U(1) electric and magnetic fields ($E$ and $B$) are switched on, the configuration gains both a net D0 and a net F1 charge. Supertubes arise when the electric field reaches $E = 1$ in string units (with $2\pi\alpha' = 1$) and when $B$ is nowhere vanishing. The static nature of the supertube can be understood as a balance between the D2-brane tension and the Poynting angular momentum from the simultaneous presence of both electric and magnetic fields [6, 30].

Again, it is natural to conjecture [6] that supertubes are D0-F1 bound states. Because they would be marginal such states saturating the same BPS bound as a system of F1 strings and D0 branes, it is nontrivial to verify that they are in fact bound. However, we will demonstrate this in section 2.2.2 through an explicit quantization of the system in which the spectrum of BPS states is shown to be discrete$^1$.

A much stronger conjecture is that *almost all* D0-F1 bound states are supertubes for large $Q_{D0}, Q_{F1}, J$. This would be of great interest, as supertubes would then provide a *geometric* description of these bound states; states of the supertube are directly labelled by the shape of the $S^1$ cross-section and by the magnetic field as a function of location on this $S^1$.

Mathur et al [19] have provided evidence that this conjecture is correct by computing the Bekenstein entropy in the dual D1-D5 system. In their method, they define the location where all the D1-D5 configurations, which look the same asymptotically, start differing from one another. They take the area of this location to define a Bekenstein entropy and interpret the result as a count of different metrics having the same macroscopic parameters. Their result is consistent with $S \sim \sqrt{Q_{D0}Q_{F1} - J}$ (after dualizing back to the D0-F1 system). They find agreement with the entropy of all

$^1$We will use a linearized description in which the DeWit-Hoppe-Nicolai continuum of membrane states [35] does not arise. This is consistent [1] with our intent to study a single bound state, and not the second-quantized theory of supertubes.
D0-F1 microstates, which can be computed from the fact that the system is also dual to the fundamental string with right-moving momentum, whose entropy is in turn given by the Cardy formula [36].

Still, an explicit quantization and counting of supertubes has remained lacking. The main point of our work below is to verify this conjecture by using a linearization of the D2-brane effective action to directly count quantum supertube states.

The structure of this chapter is as follows. We begin with an overview of the D0-F1 supertubes. Then, we then proceed to linearize the D2 effective action and the conserved quantities about the round supertube configuration (in which the $S^1$ is a circle). However, we will momentarily interrupt the linearization program to present general relations satisfied by the gauge-fixed system. The spectrum of states is then computed in section 2.2.2, whence it is straightforward to count the states in section 2.3 and to establish that our results are valid when $Q_{D0}Q_{F1} \gg Q_{D0}Q_{F1} - J$ and $Q_{D0}Q_{F1} - J \gg 1$. As stated above, our counting verifies that supertubes are marginal bound states with an entropy given by $S = 2\pi \sqrt{2(Q_{D0}Q_{F1} - J)}$. Finally, we close in section 5.4 with a summary and a discussion of the implications for the further conjectures of Mathur et al ([19, 20, 21, 22, 23, 24, 25]) relating to three-charge black holes.

### 2.1 Supertube review

Our starting point will be the D2 Born-Infeld effective action:

$$S_{D2} = -T_{D2} \int d^3 \xi \sqrt{-\det(g + \mathcal{F})} - T_{D2} \int C_{(1)} \wedge \mathcal{F},$$

(2.1.1)

where $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$ and we have included the Chern-Simons term representing the coupling to a background Ramond-Ramond vector potential $C_{(1)}$ and a Neveu-Schwarz two-form potential $B_{\mu\nu}$. $T_{D2}$ is the D2-brane tension.
Supertubes are static solutions of (2.1.1) in a Minkowski background

\[ ds^2 = -dT^2 + dZ^2 + d\vec{X}d\vec{X}, \]

(2.1.2)

where \( \vec{X} = \{X^i\} \) are the cartesian coordinates in \( E_8 \), and \( B_{\mu\nu} = 0 \) and \( C_{[1]} = 0 \).

The static gauge is defined by setting the worldvolume coordinates \( \xi^a = (t, z, \sigma) \) to be \( t = T, \ z = Z, \) and \( \tan \sigma = X^1/X^2 \). Thus, \( z \) represents the coordinate along the length of the tube, while the \( S^1 \) cross-section is an arbitrary curve \( \vec{X}(\sigma) \). It is convenient to introduce the radius \( R(t, z, \sigma) \) in the \( X^1X^2 \) plane defined by \( R^2 = (X^1)^2 + (X^2)^2 \).

The worldvolume gauge field \( F_{\mu\nu} \) is taken to be time and \( z \) independent, with the electric component \( E \equiv F_{tz} \) along the \( z \) direction and the magnetic component \( B \equiv F_{z\sigma} \),

\[ F = E dt \wedge dz + B dz \wedge d\sigma. \]

(2.1.3)

The equations of motion imply that \( E \) is also independent of \( \sigma \), but allow \( B \) to be an arbitrary function of this coordinate. The invariant electric flux \( F_{tz} \) acts as a source of F1 charge along the \( z \) direction, while the Chern-Simons coupling of \( C_{(1)0} \) with the magnetic field \( B \) induces a dissolved D0 charge on the D2-brane. In this background, the expressions for the charges are

\[
Q_{D0} = \frac{T_{D2}}{T_{D0}} \int d\sigma B,
\]

(2.1.4)

\[
Q_{F1} = \frac{1}{T_{F1}} \int d\sigma \Pi_z = \frac{T_{D2}}{T_{F1}} \int d\sigma \frac{E|\partial_\sigma X|^2}{\sqrt{(1 - E^2)|\partial_\sigma X|^2 + B^2}},
\]

(2.1.5)

where \( T_{D0}, T_{F1} \) represent the tensions of the appropriate branes and we have normalized \( Q_{D0}, Q_{F1} \) so that they take integer values.

The electromagnetic field generates an angular momentum \( J \) defined by

\[ J = \sqrt{\frac{1}{2} L_{ij} L^{ij}}, \]

(2.1.6)
where the 2-form $L_{ij}$ is, as usual, given by the integral of the antisymmetric product of the coordinates and the linear momentum density $P_i = \partial L_{D2}/\partial \partial_t X^i$,

$$L_{ij} = \int d\sigma X_i P_j - X_j P_i.$$  \hspace{1cm} (2.1.7)

In the present setting, $L_{ij}$ takes the form

$$L_{ij} = T_{D2} \int d\sigma EB(X_i \partial_\sigma X_j - X_j \partial_\sigma X_i) \sqrt{(1 - E^2)|\partial_\sigma X|^2 + B^2}.$$  \hspace{1cm} (2.1.8)

This configuration preserves 1/4 of the IIA Minkowski vacuum supersymmetries. The condition (1.1.13) of chapter 1 for unbroken supersymmetries is satisfied provided that (see, e.g.[30]) the electric field $E$ is set to 1, $B$ is nowhere vanishing and the Killing spinor $\epsilon_{\text{unbr}}$ solves the supersymmetry conditions corresponding to a system of F1 strings aligned along the z direction and D0-branes homogeneously distributed along the z axis. Under the above restrictions, (super)tubes of arbitrary shape $\vec{X}(\sigma)$ and arbitrary $B(\sigma)$ are supersymmetric, and therefore, saturate the BPS bound

$$P^0 \geq T_{D0}|Q_{D0}| + T_{F1}|Q_{F1}|.$$  \hspace{1cm} (2.1.9)

Here, the energy $P^0$ is obtained from

$$P^0 = \int dz d\sigma T^{00}(X(\xi)) = T_{D2} \int dz d\sigma \frac{B^2 + |\partial_\sigma X|^2}{\sqrt{(1 - E^2)|\partial_\sigma X|^2 + B^2}},$$  \hspace{1cm} (2.1.10)

where $T^{00} = \frac{2}{\sqrt{-G}} \frac{\partial L}{\partial G_{00}}$ is the stress-energy tensor on the D2-brane. Setting $E = 1$ straightforwardly leads to the equality relation in (2.1.9). Note that, as the $S^1$ are closed curves, the D2-brane charge vanishes and only the F1 and D0 charges enter the BPS bound.

Supersymmetry guaranties that supertubes are stable solutions. This stability can be understood as a balance between the D2-brane tension and the angular momentum [6, 30]. As explained in [30], this balance can be achieved supersymmetrically because the preserved charges that enter the BPS bound (2.1.9) are independent of the angular momentum and are the same as those of a supersymmetric collection of F1's and D0's.
In [30], a bound for the angular momentum was derived,

\[ J \leq |Q_{D0}Q_{F1}|. \]  

(2.1.11)

These authors showed that equality in the above relation is uniquely achieved by a circular cross section and constant \( B \).

**Round supertube**

The circular cross-section solution, which we will simply call the round supertube, was the first of these configurations discovered by Mateos and Townsend [6]. As pointed out above, it uniquely saturates the angular momentum bound (2.1.11) on supertubes. The D2 fields take the form

\[ R_{\text{round}}(t, z, \sigma) = R, \]  
\[ X_{\text{round}}^i(t, z, \sigma) = 0 \text{ for } i = 3, 4, 5, 6, 7, 8, \]  
\[ (F_{tz})_{\text{round}} = \pm 1, \ (F_{z\sigma})_{\text{round}} = B, \]  

(2.1.12)  
(2.1.13)  
(2.1.14)

where, from here on, \( R \) and \( B \) are constants that determine the charges and angular momentum of the round supertube about which we expand below.

In order to make these charges finite, let us periodically identify the system under \( z \rightarrow z + L_z \), so that the supertube at any time is an \( S^1 \times S^1 \) embedded in \( S^1 \times \mathbb{R}^9 \). The total charges and angular momentum of the round tube are then

\[ Q_{D0}^{\text{round}} = \frac{2\pi L_z T_{D2}}{T_{D0}} B, \]  
\[ Q_{F1}^{\text{round}} = \text{sgn}(EB) \frac{2\pi T_{D2} R^2}{T_{F1} B}, \]  
\[ J^{\text{round}} = \text{sgn}(EB) 2\pi L_z T_{D2} R^2. \]  

(2.1.15)  
(2.1.16)  
(2.1.17)

Note in particular that since \( T_{D0}T_{F1} = 2\pi T_{D2} \) (see, e.g. [37]), we have \( J^{\text{round}} = Q_{D0}^{\text{round}} Q_{F1}^{\text{round}} \).
2.2 Linearization

Our task is to find a description in which the states can be counted. To this end we expand the gauge-fixed action and all relevant quantities to quadratic order in fields, taking the round supertube (for which the $S^1$ is an isometry direction) as the base point of the expansion. Note [7, 30] that, as this is the unique configuration saturating the angular momentum bound, it will have certain nice properties reminiscent of vacuum states. Let us denote the deviations from the round solution by

$$R = R_{\text{round}} + r, \quad X^i = X^i_{\text{round}} + \eta^i,$$ (2.2.1)

$$A = A_{\text{round}} + a, \quad F_{tz} = E_{\text{round}} + e_z, \quad F_{z\sigma} = B_{\text{round}} + b \quad \text{and} \quad F_{t\sigma} = e_{\sigma}. \quad (2.2.2)$$

It is then straightforward but tedious to expand the quantities of interest to quadratic order in $\eta, a$. The detailed results of the expansions are useful for the next section, but are not particularly enlightening in themselves. We will not burden the reader with such formulae here, reserving them instead for Appendix A.

Before starting the quantization of supertubes, we discuss some general properties of the gauge fixed system. Note that the gauge fixed lagrangian is invariant under $t$ and $\sigma$ translations and therefore, the generators of these transformations are conserved quantities of the system. These are respectively the Hamiltonian $H = p\dot{q} - L$ and the canonical generator of $\sigma$-translations $P^\text{can}_\sigma = p\dot{q}'$. Since each arbitrarily shaped supertube has four associated gauge-independent charges ($Q_{F1}, Q_{D0}, P^0, J$), the conserved $H$ and $P^\text{can}_\sigma$ must be expressed as combinations of them. In the next section we give a general argument showing that our gauge choice and properties of the Dirac-Born-Infeld action act together to guarantee that $H$ and $P^\text{can}_\sigma$ take the form

$$H = P^0 - |Q_{D0}|T_{D0} - |Q_{F1}|T_{F1}L_z,$$ (2.2.3)

$$P^\text{can}_\sigma = Q_{F1}Q_{D0} - J.$$ (2.2.4)

From the above expression we recognize that $H$ is not the total energy of the system.
Instead, our Hamiltonian measures the extend to which a state is excited above the BPS bound (2.1.9).

2.2.1 Time and $\sigma$ translation generators

In this section we show how the important relations (2.2.3) and (4.1.7) follow directly from general considerations of symmetries and our gauge fixing scheme. As a result, they will represent a useful check of the detailed calculations of section 2.2.2 below.

It will be helpful to distinguish here between the full Dirac-Born-Infeld Lagrangian of (2.1.1), which we denote $L$, and the quadratic gauge fixed Lagrangian ($L_{gf}^{(2)}$) explicitly displayed in (A.0.2). We remind the reader that $L_{gf}^{(2)}$ is obtained from $L$ in two stages, first gauge fixing $L$ to form $L_{gf}$, and then taking the quadratic term which yields $L_{gf}^{(2)}$. In particular, note that passing to $L_{gf}^{(2)}$ discards the constant term corresponding to evaluating $L$ on our background, as this term is of order zero in our perturbations.

In fact, we argue in somewhat more generality below. Let us consider the Lagrangian $\tilde{L}_{gf}$ which differs from $L_{gf}$ only by subtracting the background value, while retaining all of the higher terms:

$$\tilde{L}_{gf} := L_{gf} - L|_{Background} = L_{gf}^{(2)} + \text{higher order terms.} \quad (2.2.5)$$

We begin by noting that invariance under $t$ and $\sigma$ reparametrizations implies two important identities for $L$, which we may call the Hamiltonian and momentum constraints:

$$\sum_{\mu} \frac{\partial L}{\partial (\partial_t X^{\mu})} \partial_t X^{\mu} + \sum_i \frac{\partial L}{\partial (\partial_t A_i)} F_{ii} = L, \quad (2.2.6)$$

$$\sum_{\mu} \frac{\partial L}{\partial (\partial_\sigma X^{\mu})} \partial_\sigma X^{\mu} + \sum_i \frac{\partial L}{\partial (\partial_t A_i)} F_{\sigma i} = 0. \quad (2.2.7)$$
We now use the first of these results to identify $H$ in terms of $P^0$, $Q_{F1}$ and $Q_{D0}$.

The Hamiltonian $H$ is by definition

$$H = \int dz d\sigma \left( \sum_\mu \frac{\partial L_{gf}}{\partial (\partial_t X^\mu)} \partial_t X^\mu - \frac{\partial L_{gf}}{\partial (\partial_0 X^0)} \right),$$

where we have used (2.2.5) and the fact that the only time-dependent background field not completely fixed by the gauge condition is $A_z$, whose time derivative is $E = \pm 1$ and whose conjugate momentum defined from (2.1.1) is $\Pi_z$.

Now, $L_{gf}$ is obtained from $L$ by imposing the requirements $X^0 = t$, $X^9 = z$, $X^1 = R(t, z, \sigma) \cos \sigma$, $X^2 = R(t, z, \sigma) \sin \sigma$, and $A_0 = 0$. We denote this process by $|_{gf}$, e.g. $L_{gf} = L|_{gf}$. Expressing $H$ in terms of $L$, we find

$$H = \int dz d\sigma \left( \sum_\mu \frac{\partial L}{\partial (\partial_t X^\mu)} \partial_t X^\mu - \frac{\partial L}{\partial (\partial_0 X^0)} \right) |_{gf},$$

where in the last line we have used the Hamiltonian constraint (2.2.6).

Finally, the general form of the Dirac-Born-Infeld action implies the relation

$$\frac{\partial L}{\partial G_{00}}|_{G=\eta} = -\frac{1}{2} \sum_a \frac{\partial L}{\partial (\partial_a X^0)} \partial_a X^0, \quad (2.2.9)$$

where $|_{G=\eta}$ denotes that we evaluate the expression (after taking any derivatives) for the special case where $G_{ab}$ is the Minkowski metric. After gauge fixing this becomes

$$T_{00}|_{gf,G=\eta} = 2 \frac{\partial L}{\partial G_{00}}|_{gf,G=\eta} = -\frac{\partial L}{\partial (\partial_0 X^0)}|_{gf,G=\eta}. \quad (2.2.10)$$

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Using this together with \( L_{\text{Background}} = -B \text{sgn}(B) \) and the definition of the charges (2.1.4), (2.1.5) we find

\[
H = \int dzd\sigma (T_{00} - \text{sgn}(E)\Pi_z - B \text{sgn}(B))|_{gf} = P^0 - |Q_{F1}|L_z T_{F1} - |Q_{D0}|T_{D0}, \tag{2.2.11}
\]

where in the final step we have used the fact that the integrated magnetic flux is a topological invariant and so is always given by its value in the round tube background. Again we emphasize that the validity of (2.2.11) is in no way restricted to the linear approximation. We will primarily study the case \( \text{sgn}(E) = \text{sgn}(B) = 1 \) for which \( Q_{F1}, Q_{D0} > 0 \).

Now, we apply the analogous reasoning to \( P^\text{can}_\sigma \), which by definition takes the form

\[
P_\sigma = \int dzd\sigma \frac{\partial L_{gf}}{\partial \dot{\eta}_\mu} \partial_{\dot{\eta}^\mu} + \frac{\partial L_{gf}}{\partial \dot{a}_i} (\partial_{\dot{a}} a_i - \partial_i a_\sigma) \tag{2.2.12}
\]

\[
= \int dzd\sigma \sum_{i=3}^8 \frac{\partial L_{gf}}{\partial \dot{t}_i} \partial_{\dot{t}_i} X^1 + \frac{\partial L_{gf}}{\partial \dot{R}} \partial_{\dot{R}} X^2 + \frac{\partial L_{gf}}{\partial \dot{A}_z} (\partial_{\dot{A}} A_z - \partial_z A_\sigma + B).
\]

Let us now compute,

\[
\frac{\partial L}{\partial \dot{t}_i} X^1|_{gf} + \frac{\partial L}{\partial \dot{r}} X^2|_{gf} = \partial_{\dot{\sigma}} R (\cos \sigma \frac{\partial L}{\partial \dot{t}_i} X^1|_{gf} + \sin \sigma \frac{\partial L}{\partial \dot{t}_i} X^2|_{gf}) + (R \cos \sigma \frac{\partial L}{\partial \dot{t}_i} X^1|_{gf} - R \sin \sigma \frac{\partial L}{\partial \dot{r}} X^2|_{gf}) = \partial_{\dot{\sigma}} R \frac{\partial L_{gf}}{\partial \dot{R}} + L_{12}, \tag{2.2.13}
\]

where \( L_{12} \) is the angular momentum density in the \( X^1X^2 \) plane. Substituting the above expression in (2.2.12),

\[
P_\sigma = \int dzd\sigma \left( \frac{\partial L}{\partial \dot{t}_i} X^1 + \frac{\partial L}{\partial \dot{A}_z} F_{\sigma z} \right)|_{gf} + \Pi_z B - L_{12}, \tag{2.2.14}
\]

and using the identity (2.2.7), one arrives at the relation

\[
P^\text{can}_\sigma = Q_{F1}Q_{D0} - J. \tag{2.2.15}
\]
2.2.2 The spectrum of states

We now use the results of section 3.1 to find the spectrum of states for our linearized system. In fact, we can simplify the treatment somewhat by realizing that momentum in the $z$ direction breaks supersymmetry. Since we are interested in BPS states, we may thus restrict attention to modes independent of $z$. The action for such modes is given in (A.0.2), but the resulting equations of motion are:

\[ \frac{R^2 + B^2}{B} \partial_t^2 r + \text{sgn}(E)2(\partial_t \partial_\sigma r - \frac{R}{B} \partial_\sigma z) = 0 \]  \hspace{1cm} (2.2.16)
\[ \frac{R^2(R^2 + B^2)}{B^3} \partial_t^2 a_z + \text{sgn}(E)(\frac{2R^2}{B^2} \partial_\sigma a_z + \frac{2R}{B} \partial_r r) = 0, \]  \hspace{1cm} (2.2.17)
\[ \frac{R^2 + B^2}{B} \partial_t^2 \eta^i + \text{sgn}(E)2\partial_r \partial_\sigma \eta^i = 0, \]  \hspace{1cm} (2.2.18)
\[ \frac{1}{B} \partial_t^2 a_\sigma = 0. \]  \hspace{1cm} (2.2.19)

Note in particular that these equations are identically satisfied when all time derivatives vanish, so that all static configurations are allowed.

Without loss of generality, we will choose $\text{sgn}(E) = \text{sgn}(B) = 1$. We must also consider the Gauss Law constraint which due to gauge fixing no longer follows from our action. However, for $z$-independent modes in our gauge this is just $\partial_\sigma a_\sigma = 0$ at this order.

We now proceed to compute the mode expansions that define the relevant creation and annihilation operators. First, we note that each transverse degree of freedom $\eta^i$ (for $i \in \{3, \ldots, 8\}$) decouples from all other fields and has a solution of the form

\[ \eta^i = \frac{1}{\sqrt{4\pi L_z T_{D2}}} \sum_{k_\sigma \neq 0} \frac{a^i_{k_\sigma}}{\sqrt{|k_\sigma|}} e^{ik_\sigma \sigma} + \frac{b^i_{k_\sigma}}{\sqrt{|k_\sigma|}} e^{ik_\sigma \sigma}, \]  \hspace{1cm} (2.2.20)

where the normalizations have been chosen with foresight to simplify expressions to come. The relevant frequencies are

\[ \omega_\alpha(k_\sigma) = -\frac{2Bk_\sigma}{R^2 + B^2}, \quad \text{and} \quad \omega_b(k_\sigma) = 0. \]  \hspace{1cm} (2.2.21)
On the other hand, the radial and Maxwell degrees of freedom are coupled. Their solutions take the slightly more complicated form

\[ r = \frac{1}{2\sqrt{2\pi L^2 T}} \sum_{k_\sigma \neq \pm 1} \frac{a_{k_\sigma}^\pm}{\sqrt{|-k_\sigma \pm 1|}} e^{i\omega_{a_\sigma}^\pm t + ik_\sigma \sigma} + \frac{b_{k_\sigma}^\pm}{\sqrt{|-k_\sigma \pm 1|}} e^{ik_\sigma \sigma}, \quad (2.2.22) \]

\[ a_z = \frac{\pm i B}{2R\sqrt{2\pi L^2 T}} \sum_{k_\sigma \neq \pm 1} \frac{a_{k_\sigma}^\pm}{\sqrt{|-k_\sigma \pm 1|}} e^{i\omega_{a_\sigma}^\pm t + ik_\sigma \sigma} + \frac{b_{k_\sigma}^\pm}{\sqrt{|-k_\sigma \pm 1|}} e^{ik_\sigma \sigma}, \quad (2.2.23) \]

\[ a_\sigma = (\text{const}_1) t + \text{const}_2, \quad (2.2.24) \]

with the similar but slightly more complicated frequencies

\[ \omega_{a_\sigma}^{\pm}(k_\sigma) = \frac{2B}{R^2 + B^2}(-k_\sigma \pm 1), \quad \text{and} \quad (2.2.25) \]

\[ \omega_{b_\sigma}^{\pm}(k_\sigma) = 0. \quad (2.2.26) \]

The \( a_\sigma \) degree of freedom will not be of further interest below.

Note in particular that \( \omega_{a_\sigma}^{\pm}(k_\sigma) \) vanishes when \( k = \pm 1 \). These zero modes represent the translation symmetries in the \( X^1 \) and \( X^2 \) direction. After quantization, such modes become analogues of coordinates for the free non-relativistic particle. The same is true of the \( n' \) modes with \( k_\sigma = 0 \), associated with translations in \( X^i \) for \( i \in \{3, ..., 8\} \).

A careful treatment shows that their velocities appear in the Hamiltonian \( H \), so that these modes are not annihilated by \( H \) even though they have zero frequency. In particular, these modes are not BPS. We will not concern ourselves with the detailed treatment of these zero modes here – the expressions below should be understood as correct only up to terms involving such modes.

In addition, we have \( \omega_{b_\sigma}^{\pm}(k_\sigma) = \omega_{b_\sigma}(k_\sigma) = 0 \) for all \( k_\sigma \). This is just the linearized description of the known result [30] that the supertube allows arbitrary static deformations of its cross-section and magnetic field, so long as translation invariance in the \( z \)-direction is preserved. Although they have zero frequency, we will see below that such modes are not described by free particle degrees of freedom. Instead, the coefficients \( a_{k_\sigma}^\pm, a_{k_\sigma}^i \) and \( b_{k_\sigma}^\pm, b_{k_\sigma}^i \) are standard creation and annihilation operators which
create or annihilate excitations of the round supertube. As a result, their vanishing frequency means that these modes are annihilated by the linearized Hamiltonain $H$. Since $H$ encodes the BPS condition, it is clear that any $k_z = 0$ excitation of the $b$-modes preserves the BPS-bound.

From the action (2.1.1) and the solutions (2.2.22), the canonical momenta $\pi_z$ (conjugate to $a_z$) and $P_r, P_i$ take the form

$$P_r = -\frac{i}{2} \sqrt{\frac{L_z T_{D^2}}{2\pi}} \sum_{k_\sigma \neq \pm 1} \frac{-k_\sigma \pm 1}{\sqrt{|-k_\sigma \pm 1|}} (a^\pm e^{i\omega t \pm ik_\sigma} - b^\pm e^{i\sigma_\sigma}), \quad (2.2.27)$$

$$\pi_z = \pm \frac{R}{2B} \sqrt{\frac{L_z T_{D^2}}{2\pi}} \sum_{k_\sigma \neq \pm 1} \frac{-k_\sigma \pm 1}{\sqrt{|-k_\sigma \pm 1|}} (a^\pm e^{i\omega t \pm ik_\sigma} - b^\pm e^{i\sigma_\sigma}), \quad (2.2.28)$$

$$P_i = \sqrt{\frac{L_z T_{D^2}}{4\pi}} \sum_{k_\sigma \neq 0} \frac{-k_\sigma}{\sqrt{|k_\sigma|}} (a^i e^{i\omega t \pm ik_\sigma} - b^i e^{i\sigma_\sigma}), \quad (2.2.29)$$

As explained in the Appendix A, the canonical momentum $\pi_z$ differs by a linear term from the F1 charge density $\Pi_z$ as a consequence of an integration by parts performed in the action (A.0.2). The electric charge is not affected by this transformation so it remains as the integral of $\Pi_z$ and in particular, it has a mode expansion different from the integral of (2.2.28).

A straightforward but lengthy calculation from the canonical commutation relations $[p(t, \sigma), q(t, \sigma')] = -i\delta(\sigma - \sigma')$ shows that the $a$’s and $b$’s satisfy

$$[a^+_k, a^-_{k'}] = -\delta_{k_\sigma + k'_\sigma} \text{sgn}(k_\sigma - 1), \quad (2.2.30)$$

$$[b^+_k, b^-_{k'}] = \delta_{k_\sigma + k'_\sigma} \text{sgn}(k_\sigma - 1), \quad (2.2.31)$$

$$[a^i_k, a^i_{k'}] = -\delta_{k_\sigma + k'_\sigma} \text{sgn}(k_\sigma), \quad (2.2.32)$$

$$[b^i_k, b^i_{k'}] = \delta_{k_\sigma + k'_\sigma} \text{sgn}(k_\sigma), \quad (2.2.33)$$
while the remaining commutators vanish. In addition, the reality conditions require
\[
(a_{k\sigma}^+)^\dagger = a_{-k\sigma}, \quad (b_{k\sigma}^+)^\dagger = b_{-k\sigma},
\]
\[
(a_{k\sigma}^-)^\dagger = a_{-k\sigma}, \quad (b_{k\sigma}^-)^\dagger = b_{-k\sigma}.
\]
Thus we may identify \((a_{k\sigma}^+, b_{-k\sigma}^-)\) for \(k\sigma > 1\) and \((a_{-k\sigma}^-, b_{k\sigma}^+)\) for \(k\sigma < 1\) as creation operators and their adjoints as annihilation operators. Similarly, \((a_{k\sigma}^i, b_{-k\sigma}^-)\) for \(k\sigma > 0\) are the creation operators for the \(\eta\)-modes. In particular, for \(k_z = 0\) the BPS \((b)\) modes carry negative angular momentum around the cylinder while the non-BPS \((a)\) modes carry positive angular momentum. This is in accord with the result of [30] that the round supertube is the unique BPS state of maximal angular momentum. As a result, the round state acts like a vacuum state relative to the set of BPS excitations.

Finally, we wish to express the charges in terms of the creation and annihilation operators \(a_{k\sigma}^\pm, a_{k\sigma}^i\) and \(b_{k\sigma}^\pm, b_{k\sigma}^i\). Once again, the procedure is straightforward but lengthy. The resulting expressions are:
\[
H = \sum_{k\sigma > 1} \omega_a^- (-k\sigma) a_{k\sigma}^+ a_{-k\sigma}^- + \sum_{k\sigma < 1} \omega_a^+(k\sigma) a_{-k\sigma}^- a_{k\sigma}^+ + \sum_{k\sigma > 0} \omega_a(k\sigma) a_{k\sigma}^i a_{-k\sigma}^i (2.2.36)
\]
\[
J = J^{\text{round}} + \sqrt{2\pi L_z T_{D2}} \frac{R}{B} (b_0^+ + b_0^-)
\]
\[
+ \sum_{k\sigma > 1} \left[ \frac{2R^2 k\sigma}{R^2 + B^2} + \frac{2B}{R^2 + B^2} + \frac{1}{2(k\sigma - 1)} \right] a_{k\sigma}^+ a_{-k\sigma}^-
\]
\[
- \sum_{k\sigma < 1} \left[ \frac{2R^2 k\sigma}{R^2 + B^2} + \frac{2B}{R^2 + B^2} + \frac{1}{2(k\sigma - 1)} \right] a_{-k\sigma}^- a_{k\sigma}^+
\]
\[
- \sum_{k\sigma > 1} \frac{b_{-k\sigma}^- b_{k\sigma}^+}{2(k\sigma - 1)} + \sum_{k\sigma < 1} \frac{b_{k\sigma}^+ b_{-k\sigma}^-}{2(k\sigma - 1)}
\]
\[
+ \sum_{k\sigma \neq \pm 1} \frac{b_{k\sigma}^+ b_{-k\sigma}^-}{4\sqrt{|1 - k\sigma^2|}} + \sum_{k\sigma \neq \pm 1} \frac{b_{-k\sigma}^- b_{k\sigma}^+}{4\sqrt{|1 - k\sigma^2|}} + \sum_{k\sigma > 0} \frac{2R^2 k\sigma}{R^2 + B^2} a_{k\sigma}^i a_{-k\sigma}^i (2.2.37)
\]
\[
Q_{D0} = Q_{D0}^{\text{round}} = \frac{2\pi L_z T_{D2}}{T_{D0}} B,
\]
\[
(2.2.38)
\]

\(^2\)With the understanding that “excitations” lower the angular momentum instead of raising it.
Here we have chosen to emphasize the Hamiltonian $H$ instead of the energy $P^0$, though the latter is easily recovered through the relation (2.2.3). Since we have not explicitly included Fermions, normal ordering has been used to obtain a finite result for (2.2.36). We have also chosen to express the charge $Q_{F1}$ in terms of $Q_{D0}$ and the angular momentum, as one sees that the combination $\Delta = Q_{F1}Q_{D0} - J$ defined above takes a fairly simple form in terms of the creation and annihilation operators. This result is in accord with the arguments of section 2.2.1 since $\Delta$ is equal to the canonical generator of $\sigma$-translations $P^\sigma_{\text{can}}$.

### 2.3 Counting States

Let us now fix $H = 0$, $Q_{D0}$, and the quantity $\Delta := Q_{F1}Q_{D0} - J$ (but not $Q_{F1}$ or $J$ individually). We see from (2.2.39) that when restricted to BPS states (those with $\omega = 0$), the operator $Q_{F1}Q_{D0} - J$ takes the form of the energy of a system of 8 right-moving 1+1 massless scalars. Furthermore, the argument in (2.2.1) shows that this follows from general considerations, and thus that the Fermionic contributions suppressed here must take the corresponding form. Thus, the entire system is a 1+1 right-moving conformal field theory with central charge $c = 12$. Note that fixing $Q_{D0}$ places no restrictions on such effective right-moving fields, as $Q_{D0}$ is given by the magnetic flux, a topological invariant. Thus, we now apply the Cardy formula \cite{36} to

\[ \Delta := Q_{F1}Q_{D0} - J \]
\[ = \sum_{k_\sigma > 1} k_\sigma (b_+_{k_\sigma} b_+_{k_\sigma} - a^+_{k_\sigma} a^-_{k_\sigma}) - \sum_{k_\sigma < 1} k_\sigma (b^-_{k_\sigma} b^-_{k_\sigma} - a^-_{k_\sigma} a^+_{k_\sigma}) \]
\[ + \sum_{k_\sigma > 0} k_\sigma (b^j_{-k_\sigma} b^j_{-k_\sigma} - a^i_{k_\sigma} a^j_{-k_\sigma}) \]
\[ = P^\sigma_{\text{can}}. \]
find the degeneracy of states at level $\Delta = k$, for $k \gg 1$, 

$$d(k, c) \sim \exp \left(2\pi \sqrt{\frac{kc}{6}}\right), \quad (2.3.1)$$

which leads to the entropy

$$S = \ln d(\Delta, 12) = 2\pi \sqrt{2(Q_{D0}Q_{F1} - J)}. \quad (2.3.2)$$

What remains is to argue that the entropy depends on the charges only through the combination $Q_{D0}Q_{F1} - J$, and to tie up a loose end having to do with the quantization of charge and angular momentum. The latter issue arises from a careful inspection of (2.2.37), which shows that $J$ (and thus $Q_{F1}$) has a linear term which necessarily leads to a continuous spectrum. That the spectrum of $Q_{F1}$ is continuous is an artifact of our not yet imposing that the gauge group $U(1)$ is compact. To do so, we must quotient the configuration space of the connection by an appropriate translation. It turns out to be convenient to deal with both these issues simultaneously.

To do so, let us recall that the above quotient compactifies the configuration space of the zero mode $(a_z)_{k=0, k_z=0} = \frac{T_{D0}}{2\pi R_L T_{D0}} \int dz d\sigma \, a_z$, where we have chosen the normalization to be such that $(a_z)_{k=0, k_z=0}$ is compactified with period $2\pi$. Thus, while $(a_z)_{k=0, k_z=0}$ will no longer be a well-defined operator, the exponentiated operator $e^{in(a_z)_{k=0, k_z=0}}$ will be well-defined for any integer $n$.

It is useful to consider only the time independent part of this zero mode:

$$(a_z)_{k=0, k_z=0, \omega=0} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt (a_z)_{k=0, k_z=0}, \quad (2.3.3)$$

which depends only on the time independent (and BPS) $b$-modes. Note that the exponential $e^{in(a_z)_{k=0, k_z=0, \omega=0}}$ is again well-defined for any integer $n$.

\footnote{The angular momentum $J$ also has continuous spectrum, but it is a familiar result that quantization of $J$ imposes the Dirac quantization condition on the product of electric and magnetic charge. Note that a proper description of magnetic charge again requires compactification of the gauge group.}

\footnote{It is also gauge invariant. Invariance under small diffeomorphisms is manifest from the integrations...}
Now, since $\Pi_z$ is the canonical conjugate to $a_z$ defined by the action (2.1.1), conjugation of $Q_{F1}$ by $e^{in(a_z)k=0,k_z=0,\omega=0}$ will simply add $n$ units of charge:

$$e^{-in(a_z)k=0,k_z=0,\omega=0}Q_{F1}e^{in(a_z)k=0,k_z=0,\omega=0} = Q_{F1} + n.$$ (2.3.4)

But we see explicitly that $e^{in(a_z)k=0,k_z=0,\omega=0}$ commutes with the expression (2.2.39) for $Q_{D0}Q_{F1} - J$. Furthermore, since $e^{in(a_z)k=0,k_z=0,\omega=0}$ is time-independent, it must commute with $H$ and so maps BPS states to BPS states. There is thus a unitary (and, in particular, bijective) map acting within the class of BPS states that changes $Q_{F1}$, but leaves $Q_{D0}Q_{F1} - J$ invariant. It follows that the number of states in each set of fixed $Q_{F1}$ can depend on the charges only through the combination $Q_{D0}Q_{F1} - J$ and thus that, when all charges are fixed, the entropy is indeed $2\pi\sqrt{2(Q_{D0}Q_{F1} - J)}$ to leading order in the charges.

### 2.3.1 Limits of Validity

We have now attained our main goal and verified the conjectured form of the entropy within the domain of our linearized treatment. It is important, however, to characterize the size of this domain. After all, our use of Cardy’s formula required $\Delta \gg 1$, and one might worry that this constraint might be in conflict with our linear treatment.

We need to estimate the size of some higher order correction to our calculations. However, since supertubes are exact solutions to the Born-Infeld action [6, 30], there are no corrections to the solutions at this level. Furthermore, it has been argued [38] that such supertube solutions receive no corrections from higher derivative terms in the D2 effective action\(^5\). Furthermore, the action vanishes when evaluated on the world-volume. Invariance under large diffeomorphisms may be checked, but in the end is essentially equivalent to the fact (2.3.4) that the operator translates $Q_{F1}$ by an integer. We thank David Gross for raising this issue.

\(^5\)One may note that T-dualizing the $O(F^4)$ higher derivative terms obtained in [39] would appear to lead to such corrections. However, since $E = 1$ for the supertube one cannot expect the correct behavior to be obtained by considering corrections at any finite order in $F$. Thus [38] and [39] are not in conflict. We thank Iosef Bena for this observation.
supertube configurations. Thus, we will not obtain useful error estimates from the action or equations of motion.

On the other hand, our charges do receive corrections from the higher order terms: even for supertubes, the expression (2.1.5) is not quadratic. Thus we may estimate our errors by comparing contributions to $Q_{F1}$ from different orders. Rather than calculate the third order term, we will simply compare the second-order contribution with the zero-order term. (Note that the linear term gives only a rather trivial shift of the background and, in particular, is independent of $\Delta$.)

From $\Delta = Q_{F1}Q_{D0} - J$, we see that there are in fact two types of quadratic contributions to $Q_{F1}$: those that appear in $J$ and $\Delta$ itself. Restricting $\Delta$ to be small requires merely $\Delta \ll Q_{F1}Q_{D0}$.

Let us now consider the quadratic terms in $J$. We are interested only in the BPS modes, so we need only include those terms built from $b_{\sigma}^\pm$. Examination of (2.2.36) shows that typical matrix elements of such terms are of rough size $\sum_{k\sigma \geq 1} N_{k\sigma}/k\sigma$, where $N_{k\sigma}$ is the number operator associated with each mode. Since $k\sigma$ takes values in the positive integers, such terms are always smaller than $\Delta$ and impose no further restriction.

### 2.4 Discussion

We have seen above that the set of supertube states is given by a discrete spectrum and therefore that supertubes represent marginal bound states of D0 branes and F1 strings. In the linearized system, these BPS states correspond to an infinite number of zero-frequency modes. This result is perhaps most easily explained by noting [38] that $\sigma$ is a null direction with respect to the (inverse) open-string metric (defined in [40]) on the supertube. Thus, our shape degrees of freedom are more similar to excitations of a 1+1 massless field than to those of the more familiar sort of zero
mode.

To leading order in the charges, the entropy of supertube states is given by $S = 2\pi \sqrt{2(Q_{D0}Q_{F1} - J)}$. This is identical to the leading-order entropy of all D0-F1 bound microstates. Note that an advantage to counting states in which all three of $Q_{D0}$, $Q_{F1}$, and $J$ are fixed and not only $Q_{D0}$, $Q_{F1}$, is that restricting $Q_{D0}Q_{F1} - J \ll Q_{F1}Q_{D0}$ allows us to treat the system perturbatively.

Thus, our results support the conjecture that supertubes provide an effective description of generic D0-F1 bound states. It would be interesting to extend this analysis by applying similar techniques to the supergravity solutions [7] of D0-F1 supertubes or to the dual D1-D5 solutions [41], or perhaps by studying the linearization around other (less symmetric) Born-Infeld supertube configurations. In addition, it would be of interest to relate our entropy calculations to the entropy of the two-charge black rings of Emparan and Elvang [42].

We note that results for the multiply wound case where $\tan(X^1/X^2) = \sigma/n$ may also be of interest. Such results are easily obtained from those above by applying the methods of section 2.2.1 and noting that the only change is the replacement $J \to nJ$ as the tube now rotates $n$ times in the $X^1X^2$ plane under $\sigma \to \sigma + 2\pi$. Thus, the entropy of small fluctuations about the round tube with $n$ wrappings is given by $S = 2\pi \sqrt{2(Q_{D0}Q_{F1} - nJ)}$. For fixed $Q_{D0}, Q_{F1}, J$ we see that the entropy is greatest for the case $n = 1$.

The results above are of use for understanding the two-charge system, but similar studies for the related three-charge systems could have implications for black holes and thus be of much greater interest. In particular, Mathur et al [19, 20, 21, 22] have conjectured that similar results hold for such three-charge systems: that almost all such bound states can be described in terms of extended horizon-free configurations in which the entropy is readily apparent, for example with the distinct states being labelled by the shape of the object and the values of associated worldvolume fields.
If this were so, it would leave no room for black holes as a distinct class of states. What Mathur et al wish to conjecture is that black holes represent only an effective statistical average over collections of more fundamental states; see [19, 20, 21] for details.

In the attempt to find classical geometric descriptions of 3-charge bound states contributing to the entropy \( S = 2\pi\sqrt{Q_1 Q_2 Q_3} \), several families of BPS solutions have been constructed. In [23, 24, 25], the gravity duals of various subsets of such microstates were found to be smooth geometries with no horizons. Other families of solutions were presented in [43, 44, 45, 46], but they include regular solutions as well as solutions with pathologies. However, unlike the situation with the 2-charge system, a geometric description of generic 3-charge bound states has not been proposed yet. Were all such solutions found, it would be exciting to check whether quantization of their phase space could lead to a matching with the above value of the entropy, providing then evidence either for or against Mathur’s new picture of black holes.
Quantum polarizations of D4 branes

Quantum polarizations of D-branes were first explored in [15], which considered a bound state of \( N \) test \( D0 \)-branes placed in the supergravity background generated by a collection of parallel \( D4 \)-branes. This configuration saturates the BPS bound, and therefore any classical polarization effect would have to somehow lift the system from its ground state. Instead, we considered quantum distortions of the ground state that did not affect the BPS relation. It was argued under such conditions that, to lowest order in the weak field limit, the width of the \( D0 \)-bound state changes by an amount proportional to \( R_0(gN)^{1/3}f^2 \), where \( R_0 \) is the unperturbed width of the bound state and \( f \) is a dimensionless measure of the Ramond-Ramond field strength at the \( D0 \)-branes. This result was shown to match the corresponding distortion of the near-\( D0 \) supergravity solution in the manner expected from gauge/gravity duality.

Here we study the opposite limit and consider a test \( D4 \)-brane placed in the supergravity background generated by a collection of \( D0 \)-branes. The advantage of this context is that one may find effects even for the abelian theory associated with a single brane. Instead of examining the size of the brane, we compute the induced density of \( D0 \)-brane charge, \( \langle \rho_{D0} \rangle \), defined by the coupling of the \( D4 \)-brane to the Ramond-Ramond 1-form \( C^{(1)} \). In the approximation of interest, this charge density is
proportional to $F \wedge F$ plus a quadratic fermion term, where $F$ is the (abelian) Yang-Mills fields on the $D4$-brane. Because $\rho_{D0}$ contains quadratic terms, the expectation value $\langle \rho_{D0} \rangle$ is sensitive to quantum fluctuations. Note, however, that this is indeed a polarization effect as the integral of $\rho_{D0}$ must vanish.

The calculations below are performed using the low energy effective field theory for the $D4$-brane, including the couplings of world-volume Fermions to bosonic supergravity backgrounds found in [47, 9]. We find that $\langle \rho_{D0} \rangle$ does not vanish in the supergravity background generated by $D0$-branes. Instead, it diverges in our field theory treatment. This is somewhat surprising given the supersymmetry of our setting\(^1\), but appears not to contradict any known results. In a full string-theoretic treatment one naturally expects that this divergence will be cutoff at the string scale.

We begin in section 3.1 below with a short review of the results of [9] and a precise statement of our setup. The field-theoretic calculation of $\langle \rho_{D0} \rangle$ is then presented in section 3.2. As is clear from the above description, our calculation begs a full string-theoretic treatment. While such a calculation is not undertaken in this work, it is interesting to assume that a full string treatment cuts off our divergences at the string scale but leaves them non-vanishing, and to consider the implications. We discuss such implications in section 5.4, showing that such a term has the right form to arise from a 1-loop (annulus) string diagram. In particular, our polarization effect would require the $D4$-brane effective action to have a 1-loop term of the form $\int d^5x |p(dC^{(1)})|^2$, where $p(dC^{(1)})$ denotes the pull-back of the bulk 2-form field strength $dC$ to the brane and the notation $|p(dC^{(1)})|^2 = [p(dC^{(1)})]^{IJ}[p(dC^{(1)})]_{IJ}$, where the contraction is performed using the induced metric on the brane. We will use $I, J$ to denote worldvolume indices and $A, B$ to denote spacetime directions. It is convenient to mention here that we use analogous notation $\hat{I}, \hat{J}$ and $\hat{A}, \hat{B}$ for tangent space

\(^1\)In particular, as shown in [9] our field theory retains an explicit invariance under an 8-supercharge supersymmetry algebra when coupled to the $D0$ background.
directions, and similarly $i, j$ and $a, b$ for world-volume and spacetime *spatial* directions (i.e., orthogonal to the $D0$ worldlines) and $\hat{i}, \hat{j}, \hat{a}, \hat{b}$ for the corresponding tangent space directions.

### 3.1 Preliminaries

Recall that our goal is to study deformations of the $D4$-brane ground state when placed in the supergravity field generated by a collection of $D0$-branes. We shall therefore take the $D4$-brane as a test object whose back-reaction on the supergravity fields can be ignored. One expects this approximation to be valid in the limit where the string coupling $g$ is taken to zero but the number of $D0$-branes is increased so that the supergravity background remains fixed.

#### 3.1.1 The charge density operator, $\rho_{D0}$

Since we wish to compute the charge density which couples to the Ramond-Ramond vector potential $C_A$, and since this charge is defined by varying the $D4$-brane action with respect to $C_A$, we will need the general coupling of the $D4$-brane to this field. The coupling of the bosonic $D4$-brane fields is familiar, but the Fermion couplings are more complicated. The complete set of such couplings was calculated in [9] to quadratic order in Fermions. This will suffice for our purposes as we have already stated that we will take $g$ small, and so may work perturbatively in world-volume fields.

The lowest order effect is thus given by the quadratic truncation of the $D4$-brane effective action, which is just the $N = 4 \ U(1)$ theory coupled to our background (3.1.6). In particular, the Fermion terms we require will be second order in Fermions and will involve no coupling to the world-volume gauge field $F_{ij}$. Thus we will use a
truncated effective action of the form

\[ S_{D_4}^{\text{trunc}} = S_{D_4}^{(0)} + S_{D_4}^{(2) \text{trunc}}, \]
\[ S_{D_4}^{(0)} = -T_{D_4} \int d^5 \xi e^{-\phi} \sqrt{-g} + T_{D_4} \int C e^{-\mathcal{F}}, \] (3.1.1)

where \( S_{D_4}^{(2) \text{trunc}} \) will contain the appropriate quadratic Fermion terms, \( g \) is the induced metric, \( \mathcal{F}_{IJ} = F_{IJ} + B_{IJ} \) is the sum of the \( U(1) \) world-volume field strength \( F_{IJ} \) and the Neveu-Schwarz potential \( B_{IJ} \), \( C = \sum_n C^{(n)} \) is a formal sum of the IIA Ramond-Ramond potentials and the integral \( \int C e^{-\mathcal{F}} \) picks out the form of rank 5 to integrate.

We will also use \( X^A(\xi) \) to denote the embedding of the brane in spacetime.

The quadratic Fermion term is written in terms of a real Majorana Fermion \( \psi \), which lives in the 32-component representation of the Clifford algebra

\[ \{\Gamma^A, \Gamma^B\} = 2\eta^{AB}. \] (3.1.2)

The conjugate spinor \( \bar{\psi} \) is defined by \( \bar{\psi}_\beta = \psi^\alpha C_{\alpha\beta} \), where \( C \) is the anti-symmetric charge-conjugation matrix which we take to be \( C_{\alpha\beta} \equiv \Gamma^\dagger_\alpha \beta \). Following [9], we use the notation \( \Gamma^\dagger = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \) for the ten-dimensional chirality operator. We also use the notation \( \Gamma_{D_4} = \frac{1}{5! \sqrt{-g}} \epsilon^{IJKLM} \Gamma_{IJJKLM} \Gamma^\dagger \) for an interesting world-volume chirality operator, where \( \epsilon \) denotes the Levi-Civita tensor density (which takes value \( \pm 1, 0 \) for any metric). Finally, we will use the notation \( \Gamma_{I_1...I_n} = \Gamma_{[I_1...I_n]} \) denoting antisymmetrization with weight one; e.g. \( \Gamma_{01} = \frac{1}{2}(\Gamma_0 \Gamma_1 - \Gamma_1 \Gamma_0) = \Gamma_0 \Gamma_1 \).

We may also drop any couplings of Fermions to the background Neveu-Schwarz two-form \( B_{AB} \) (though these are non-trivial and were computed in [47, 9]) since it will vanish in the background generated by \( D0 \)-branes and we will not need to vary it. With this understanding the truncated quadratic Fermion action \( S_{D_4}^{(2) \text{trunc}} \) may be seen from [9] to be

\[ S_{D_4}^{(2) \text{trunc}} = \frac{i T_{D_4}}{2} \int d^5 \xi e^{-\phi} \sqrt{-g} \bar{\psi}(1 - \Gamma_{D_4} (\Gamma^I D_I - \Delta)) \psi, \] (3.1.3)
where

\[
D_A = \partial_A + \frac{1}{4} \omega_{ABC} \Gamma^{BC} + \frac{1}{8} e^\phi \left( \frac{1}{2!} F^{(2)}_{BC} \Gamma^B \Gamma^C + \frac{1}{4!} F^{(4)}_{BCDE} \Gamma^B \Gamma^C \Gamma^D \Gamma^E \right) \quad \text{and}
\]

\[
\Delta = \frac{1}{2} \Gamma^A \partial_A \phi + \frac{1}{8} e^\phi \left( \frac{3}{2!} F^{(2)}_{BC} \Gamma^B \Gamma^C + \frac{1}{4!} F^{(4)}_{BCDE} \Gamma^B \Gamma^C \Gamma^D \Gamma^E \right).
\]

Here \( \omega \) is the spin connection of the spacetime metric and we have chosen to denote bulk Ramond-Ramond fields by bold-face \( F^{(n)} = dC^{(n-1)} + \text{Wess} - \text{Zumino terms} \) in order to distinguish them from the \( U(1) \) world-volume field \( F \) on the \( D4 \)-brane. The superscript \( (n) \) denotes the rank of the form.

It is now straightforward to vary the action (3.1.1) and obtain the current \( J^A \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{trunc}}}{\delta C^A_{(1)}} \) that couples to \( C^A_{(1)} \). The result is

\[
J^A = \frac{T_{D4}}{8 \sqrt{-g}} \frac{\partial X^A}{\partial \xi^M} \frac{\delta X^M}{\delta C^A_{(1)}} F_{IJ} F_{KL} + i \frac{T_{D4}}{8 \sqrt{-g}} \partial_B \left( \sqrt{-g} \bar{\psi} \left( 1 - \Gamma_{D4} \right) \left( -\Gamma^{BA} + 2 \Gamma^I \partial_I (X^B X^A) \right) \Gamma^B \psi \right).
\]

(3.1.5)

We will in particular be interested in the charge density \( \rho_{D0} \equiv J^0 \), where \( 0 \) denotes the direction along the world-lines of the \( D0 \)-branes that generate the background of interest, as all other components of \( J^A \) will vanish by symmetry in our background.

### 3.1.2 Specifics of the \( D0 \)-background

Since we consider Fermions below, we will work in terms of the vielbien \( e^\hat{A}_A \). The direction picked out by the \( D0 \)-worldline is clearly special and corresponds to \( A = 0 \). We will use the symbol \( a = \{1, ..., 9\} \) to indicate one of the directions transverse to the zero-branes. Thus, the supergravity background is

\[
ds^2 = e^\hat{A} e^\hat{B} \eta_{\hat{A} \hat{B}}, \quad \text{with} \quad e^\hat{A} = H^{1/4} \delta^\hat{a} dx^b, \quad e^\hat{0} = H^{-1/4} dt,
\]

(3.1.6)

with all other fields vanishing. The function \( H \) is a harmonic function on the nine-dimensional space defined by \( x^1, ..., x^9 \) and sourced by the distribution of \( D0 \)-branes.
We will proceed without assuming any particular form for $H$, but for the case of $N_0$ $D0$-branes at the origin $H$ takes the familiar form $H = 1 + 60 \pi^2 \frac{g_\ell N_0}{r^6}$, where $r^2 = \sum_a x^a x^a$.

The particular form of (3.1.6) allows a dramatic simplification of the effective action (3.1.1). Following the discussion in section 5 of [9], it is useful to also impose static gauge $\xi^I = x^I$ for $I = 0, 1, 2, 3, 4$ and to impose the $\kappa$-symmetry gauge

$$\bar{\psi} \frac{1}{2} (1 - \Gamma_{D4}) = \bar{\psi}. \quad (3.1.7)$$

Thus, from now on we take $\psi$ to be a constrained fermion satisfying (3.1.7) so that it has only 16 independent components, though the $\Gamma$s are $32 \times 32$ matrices.

Finally, at this stage we use our weak coupling approximation to truncate the action by dropping all remaining terms beyond quadratic order in world-volume fields (including interactions between the Fermions and the scalars $X^p$), as such terms give sub-leading contributions in the $g_s \rightarrow 0$ limit. With this understanding the action (3.1.1) in the background (3.1.6) becomes

$$S_{D4}^{\text{trunc}} = S_{D4}^{(0) \text{trunc}} + S_{D4}^{(2) \text{trunc}}, \quad \text{with}$$

$$S_{D4}^{(0) \text{trunc}} = - T_{D4} \int d^5 x - T_{D4} \int d^5 x \left\{ \frac{1}{4} F^{IJ} F_{IJ} + \frac{1}{2} \partial^I X^p \partial_I X^q g_{pq} + \frac{1}{8} \Theta \tilde{\epsilon}^{ijkl} F_{ij} F_{kl} \right\},$$

$$S_{D4}^{(2) \text{trunc}} = iT_{D4} \int d^5 x \bar{\psi} \left[ \Gamma^I \partial_I - \frac{1}{8} \partial_I \ln H \Gamma^I (1 + 2 \Gamma_0 \Gamma^0) \right] \psi. \quad (3.1.8)$$

Here we have used indices $I, J$ to denote spacetime directions $\{0,1,2,3,4\}$ on the brane, lower case $i,j$ to denote space directions $\{1,2,3,4\}$ on the brane, and indices $p,q$ to denote directions $\{5,6,7,8,9\}$ transverse to the brane. Below, we will also use $\hat{I}, \hat{i}, \hat{p}$ to denote the corresponding tangent space directions. We have also introduced $\Theta = (H^{-1} - 1)$, the moduli metric $g_{pq} = H^{1/2} \delta_{pq}$ and the worldvolume metric $g_{IJ}$

$$g_{IJ} = \begin{pmatrix} - H^{-1/2} & 0 \\ 0 & H^{1/2} \delta_{ij} \end{pmatrix}, \quad (3.1.9)$$
which is used to raise and lower the indices $I, J, i, j$, and the Levi-Civita tensor density $\tilde{\epsilon}^{0ijkl}$ whose non-zero entries are $\pm 1$.

The supersymmetries of the action (A.0.2) and their algebra were also derived in [9]. For completeness, we repeat them here. They are

$$
\delta_\varepsilon \psi = \left( \frac{1}{4} F' IJ \Gamma IJ \Gamma \dot{\phi} + \frac{i}{2} \partial_I X^p \Gamma^I \Gamma_p \right) \varepsilon , \\
\delta_\varepsilon A_I = i \bar{\varepsilon} \Gamma_I \Gamma \dot{\phi} \psi , \\
\delta_\varepsilon X^p = i \bar{\varepsilon} \Gamma^p \psi ,
$$

where $\varepsilon = H^{-1/8} \varepsilon^{(0)}$ and $\varepsilon^{(0)}$ is any a constant spinor satisfying

$$
\frac{1}{2} (1 + \Gamma_0 \Gamma \dot{\phi}) \varepsilon^{(0)} = 0 \quad \text{and,} \\
\frac{1}{2} (1 + i \Gamma_{01234} \Gamma \dot{\phi}) \varepsilon^{(0)} = 0.
$$

Note that the two projectors commute, so that $1/4$ of the 32 supersymmetries survives.

From [9], the commutator of two such supersymmetry transformations corresponding to $\varepsilon^1, \varepsilon^2$ acting on a bosonic field ($X$ or $A$) is

$$
[\delta_{\varepsilon^1}, \delta_{\varepsilon^2}] = \left( -i \bar{\varepsilon}^2 \Gamma^0 \varepsilon^1 \right) \partial_0 - Q \left[ i \bar{\varepsilon}^2 \Gamma^0 A_0 \varepsilon^1 \right],
$$

where $Q$ is the generator of gauge transformations; i.e. $Q[\Lambda]X = Q[\Lambda] \psi = 0$, but $Q[\Lambda]A_i = \partial_i \Lambda$. In reaching the above form we have used the fact that, since $\Gamma_{1234} \varepsilon = \varepsilon$, one has $-i \bar{\varepsilon}^2 \Gamma^I \varepsilon^1 = 0$. Note that the factors of $H$ in the first term cancel so that it represents a constant time translation, which is indeed a symmetry of the action (A.0.2).

We may also use the $\kappa$-symmetry condition (3.1.7) and the identity that $\bar{\psi} \Gamma^{ABCDE} \psi = 0$ for any Majorana spinor $\psi$ to simplify the expression (3.1.5) for the current which couples to $C_I$. The result is

$$
J^I = \frac{T_{D4}}{8 \sqrt{-g}} \tilde{\epsilon}^{IJKLM} F_{JK} F_{LM} + \frac{i T_{D4}}{4 \sqrt{-g}} \partial_J \left( \sqrt{-g} \bar{\psi} \Gamma^J \Gamma \dot{\phi} \psi \right),
$$

38
where one is pleased to note that all derivatives transverse to the brane have disappeared. Note that since the scalars $X^p$ do not appear in (3.1.13) and are decoupled from all other fields, they are irrelevant to our calculation and will not appear in any discussion below.

### 3.2 The induced $D0$ charge density

We are now nearly ready to compute the expectation value $\langle \rho_{D0} \rangle = \langle J^0 \rangle$ in our background. In order to properly take into account the deformation of the ground state, it is useful to compute $\langle \rho_{D0} \rangle$ in the corresponding Euclidean signature background and then to analytically continue back to Lorentz signature.

We find it easiest to keep track of the relevant signs and factors of $i$ by proceeding exactly as stated above; that is, by analytically continuing the background and making no changes in the coordinates. That is, we take the Lorentzian action (3.1.1) to define a function $S_L(X, F, \psi; b)$, where $b$ is the supergravity background and simply substitute the Euclidean background $b_E$ defined by (3.1.6) with the replacements

$$e^{\hat{\theta}} = -iH^{-1/4}dt \quad \text{and} \quad C = -i(H^{-1} - 1)dt.$$  \hfill (3.2.1)

In particular, the metric still has the form $ds^2 = e^{\hat{A}}e^{\hat{B}}\eta_{\hat{A}\hat{B}}$ with $\eta_{\hat{A}\hat{B}}$ the Minkowski metric. The Levi-Civita tensor density $\tilde{\epsilon}^{IJKLM}$ of course remains $\pm 1$ or 0, but $\sqrt{-g} := det(e)$ changes by the above factor of $-i$. We also follow the standard convention of introducing another factor of $-i$ in the Euclidean action, which we define for any background $b$ as $S_E(X, F, \psi; b) = -iS_L(X, F, \psi; b)$.

Evaluating the Euclidean action $S_E(X, F, \psi; b_E)$ on the background $b_E$ of (3.2.1) yields:

$$S_{ED4}^{\text{trunc}} = S_{D4}^{(0)} + S_{D4}^{(2) \text{trunc}},$$
\[ S_{ED4}^{(0)} = T_{D4} \int d^5 x + T_{D4} \int d^5 x \left\{ \frac{1}{4} F^{IJ} F_{IJ} + \frac{1}{2} \partial^I X^m \partial_I X^m g_{mn} - \frac{1}{8} \Theta \epsilon^{ijkl} F_{ij} F_{kl} \right\} . \] (3.2.2)

To display the Fermion action it is useful to first clarify our definition of the analytic continuation. We take \( \Gamma^A \) to be independent of the background, with \( \Gamma^A \) defined in terms of \( e^A \) and \( \hat{\Gamma}^A \). Thus, \( \Gamma^0 \) depends on the background; it is anti-Hermitian in a Lorentzian background and Hermitian in a Euclidean one. On the other hand, \( \Gamma_0^E \) is always anti-Hermitian. However, we find it convenient to define \( \Gamma_0^E := i \Gamma^0 \) and \( \Gamma_{0,E} := -i \Gamma_0^E \), from which we see that \( \Gamma_0^E = \Gamma_{0,E} \). Note that we have:

\[ \Gamma_{D4} = i \Gamma_{0,E} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \hat{\Gamma}^\phi, \quad \text{and} \quad \Gamma^\phi = -i \Gamma_{0,E} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma^\phi. \] (3.2.3)

We also introduce \( \bar{\psi}_E = -i \bar{\psi} \). With such understandings the quadratic Fermion action is

\[ S_{ED4}^{(2) \text{trunc}} = -T_{D4} \int d^5 x \bar{\psi}_E \left[ \Gamma^I \partial_I - \frac{1}{8} \partial_i \ln H \Gamma^i (1 + i 2 \Gamma_{0,E} \Gamma^\phi) \right] \psi , \] (3.2.5)

where \( \psi \) continues to satisfy

\[ \frac{1}{2} \bar{\psi}_E (1 - \Gamma_{D4}) = \bar{\psi}_E. \] (3.2.6)

Similarly, we define a Euclidean current \( J_0^E(X, F, \psi; b) \) on any background \( b \) through \( J_0^E(X, F, \psi; b) := -i J_0^E(X, F, \psi; b_E) \). Evaluating this current on \( b_E \) yields

\[ J_0^E = \frac{T_{D4}}{8 H^{3/4}} \epsilon^{ijkl} F_{ij} F_{kl} + i \frac{T_{D4}}{4 H^{3/4}} \partial_k \left( H^{3/4} \bar{\psi}_E \Gamma^\phi \Gamma_0^E \hat{\Gamma}^\phi \psi \right) . \] (3.2.7)

It will be convenient to restrict attention to weak supergravity fields so that we may treat the system perturbatively. This amounts to the condition \( \delta H \equiv H - 1 \ll 1 \), so that we may approximate \( \partial_i \ln H \approx \partial_i H \) and \( \Theta = H^{-1} - 1 \approx 1 - H \).
Because $J^0_E$ contains products of operators at coincident points, the individual terms are likely to be divergent. Our strategy will be to point-split each term along some displacement $\delta$ in the Euclidean time direction and then add the contributions from each term together, analyzing the limit $\delta \to 0$. The purely Bosonic part $J^0_{bE}$ of the current (3.2.7) will be studied in subsection 3.2.1 below, while the part $J^0_{fE}$ quadratic in Fermions will be studied in subsection 3.2.2. We will then collect the terms and study the coincidence limit in subsection 3.2.3. The reader may wonder what happens to these divergences in the trivial background $H = 1$. As we will see below, it turns out that the index and $\Gamma$-matrix structure of (3.2.2) and (3.2.7) cause both contributions to $J^0_E$ to vanish identically for $H = 1$, even at finite point-splitting parameter $\delta$.

### 3.2.1 The Bosonic part of the Euclidean Current

Let us now consider the point-split bosonic contribution,

$$
\langle J^0_{bE}(x, y) \rangle = \frac{T_{D^4}}{8} \epsilon^{ijkl} \langle F_{ij}(x)F_{kl}(y) \rangle (1 + O(\delta H)) = -\frac{T_{D^4}}{2} \epsilon^{ijkl} \partial_i x \partial_j y \langle A_k(x)A_l(y) \rangle (1 + O(\delta H)),
$$

where we have written this result in terms of the two-point function of the world-volume connection $A_J$ that leads to the field strength $F_{IJ}$. We have also explicitly indicated the two arguments $x, y$ of the point-split current. The subscripts $x, y$ on indices indicate the points at which the corresponding derivatives act. The two point function $\langle A_k(x)A_l(y) \rangle$ may be computed from the equation of motion for $F$, which may be written

$$
\partial_I F^{IJ} = -\frac{1}{2} \epsilon^{ijkl} F_{kl} \partial_i H + O(\delta H^2).
$$

The two-point function satisfies this same equation, but with an additional delta-function source.
As we will solve the problem perturbatively, we wish to express (3.2.9) in the form

\[ \delta^{JL} \delta^{IK} \partial_I F_{KL} = L^{JI} A_J, \]  

(3.2.10)

where \( L^{JI} \) is a linear differential operator that is also linear in \( \delta H \). Since every term in (3.2.9) contains two derivatives (which act either on \( A^K \) or on \( H \)), each term in \( L^{JI} \) must contain two derivatives as well (which act either on \( H \) or on the argument of \( L^{JI} \)). The perturbative solution for the two-point function will then be

\[ T_{D4}\langle A_K(x)A_L(y)\rangle = G_{KL}(x, y) - \int d^5 z G_{KI}(x, z)L^{IJ}G_{JL}(z, y), \]  

(3.2.11)

where \( G_{KL}(x, y) \) is the flat-space two-point function (corresponding to \( H = 1 \)).

From (3.2.9) there are two possible sources of corrections to the flat-space two-point function \( \langle A_k(x)A_l(y)\rangle_0 \). The first is from the metric factors used to raise the indices in \( F^{IJ} \) on the left-hand side of (3.2.9), the second is from the explicit source term on the right-hand side. At lowest order in \( \delta H \) the full correction term is the sum of these two independent sets of corrections.

Let us consider the first set of corrections, working in the Euclidean version of flat-space Lorentz gauge: \( \delta^{IJ} \partial_I A_J = 0 \); i.e., in a gauge that preserves all symmetries and in which \( G_{KL} = (\delta_{KL} - \partial_K \partial_L)G \), where \( \partial^2 = \delta^{IJ} \partial_I \partial_J \) is the flat Euclidean Laplacian and \( G \) is the scalar Green’s function satisfying

\[ \partial^2 G(x, y) = -\delta(x, y). \]  

(3.2.12)

Consider in particular the contribution of such corrections to the factor \( \partial_k \partial_j \langle A_k(x)A_l(y)\rangle \) appearing in (3.2.8). Note that (3.2.8) contracts this with \( \epsilon^{ijkl} \), so that we may neglect any terms proportional to the flat-space metric (on any pair of indices). Thus, non-trivial terms can arise only when each index is generated by the action of a derivative \( (\partial_i, \partial_j, \partial_k, \partial_l) \) on one of the Green’s functions or on \( H \). Since \( \partial_i, \partial_j \) are explicit derivatives and \( L^{JI} \) contains two additional derivatives, there are indeed four
derivatives in each such correction term. However, each of these four derivatives must act on \( G(x, z) \), \( G(z, y) \), or \( H(z) \). Thus, some two of these derivatives act on the same function and, when antisymmetrized by contraction with \( \epsilon^{ijkl} \), cause the result to vanish. Thus, we may neglect all factors of \( H \) in the metric and replace \( L_{JL} \) by

\[
L^{JL}_{\text{right}} = -\epsilon^{0i jkl} (\partial_i H) \partial_k.
\]

Similarly, the anti-symmetry of \( \epsilon^{ijkl} \) implies that the zero-order contribution to \( J_{bE}^0 \) vanishes for all \( x, y \). Since we wish to compute \( \epsilon^{ijkl} \partial_x \partial_y \langle A_i(x) A_l(y) \rangle \) (i.e., a correlator of field strengths), it is also clear that we may simply replace \( G_{IJ} \) by \( \delta_{IJ} \), dropping the longitudinal correction term \(-\partial_k \partial^k G\), as this term will again lead to commutators of coordinate derivatives. Thus, we have

\[
\langle J_{bE}^0(x, y) \rangle = \frac{1}{2} \epsilon^{0ijkl} \epsilon^{0ijk'} l' \delta_{kk'} \delta_{ll'} \int d^5 z \partial_i G(x, z) [\partial_j H(z)] \partial_j' \partial_j G(z, y)
\]

\[
= - (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) \int d^5 z \partial_i G(x, z) [\partial_j H(z)] \partial_j G(z, y),
\]

(3.2.14)

where all derivatives are with respect to the \( z^i \) coordinates. We will postpone detailed analysis of the limit \( x \to y \) until after computation of the fermion contribution \( J_{fE}^0 \), to which we now turn.

### 3.2.2 The Fermionic part of the Euclidean Current

Our approach to the Fermionic contribution \( J_{fE}^0 \) will proceed in parallel with our calculation of the bosonic term \( J_{bE}^0 \) above. We wish to consider the Fermionic term

\[
\langle J_{fE}^0(x, y) \rangle = \frac{i T_{D4}}{4} \partial_i \left( H^{3/4} \langle \bar{\psi}_E(x) \psi(y) \rangle_{\alpha}^\beta (\Gamma^E \Gamma^\phi)^{a} \right)
\]

(3.2.15)

where \( \partial_i = \partial_x + \partial_y \) acts on functions of both \( x \) and \( y \) and where we have explicitly displayed the spinor indices \( \alpha, \beta \). It will not matter at which point the factor \( H^{3/4} \) is evaluated as we will shortly see that to leading order we may replace this factor with 1.
The two-point function is again determined by the equation of motion, which for
the spinor $\psi$ is just

$$\frac{1}{8} \partial_I \ln H \Gamma^I (1 + i2\Gamma_E \hat{\phi}) \psi,$$

(3.2.16)

where $\partial_I = \Gamma^I \partial_I$ and the subscript indicates the implicit dependence on $H$. Again, we wish to express (3.2.16) as a linear perturbation of the flat-space result:

$$\hat{\varphi} \psi = L \psi,$$

(3.2.17)

where $\hat{\varphi} = \hat{\varphi}_0 = \Gamma^0 \partial_0 + \Gamma^j \partial_j$ and $L$ is linear in $\delta H$. Here it is useful to introduce the notation

$$\Gamma^I_E = \begin{cases} \Gamma^0_E & \text{for } I = 0 \\ \Gamma^j & \text{for } I = j, \end{cases}$$

(3.2.18)

so that we may write $\hat{\varphi}_0 = \Gamma^I_E \partial_I$. The two-point function is then

$$2T_D^4 \langle \psi_E(x) \psi(y) \rangle^\alpha_{\beta} = G^\alpha_{\beta}(x, y) - \int d^5 z \ G^\alpha_{\gamma}(x, z) L^\gamma G^\beta_{\alpha}(z, y),$$

(3.2.19)

where $G^\alpha_{\beta}(x, y) = (P \hat{\varphi} G(x, y))^\alpha_{\beta}$ where $G(x, y)$ is again the scalar Green’s function, the derivatives act on the first argument, and $P = \frac{1 - \Gamma_D^4}{2}$ is the projection onto spinors satisfying the constraint (3.1.7). Note that the 2 on the left-hand side of (3.2.19) is a result of our unconventional normalization of the action for Majorana Fermions.

As in the bosonic case, we may consider two sorts of contributions to $L$: those from the left-hand side of (3.2.16) and those from the right-hand side. Contributions from the left-hand side yield $L^\alpha_{\beta} = \frac{1}{4} (1 - H)(\Gamma^0_E \partial_0 - \Gamma^j \partial_j)$. Note that $(L^\alpha_{\beta})^\gamma G^\gamma_{\alpha}(z, y) = \frac{1}{4} (1 - H)(\eta^{IJ} \partial_I \partial_J G(1 - P))_{\alpha} + 2 \partial_0 \partial_j G(\Gamma^0_E \partial^\gamma)^{\gamma}_{\alpha}).$ As a result, the corresponding contributions to (3.2.15) involve traces of the matrices $P \Gamma^J_E \Gamma^0_E \hat{\phi}$ and $P \Gamma^J_E \Gamma^0_E P \Gamma^k \Gamma^i \Gamma^0_E \hat{\phi} = P \Gamma^J_E \Gamma^k \Gamma^i \hat{\phi}$, where we have used $P^2 = P$. But we have

$$(1 - \Gamma_D^4) \Gamma^0_E \Gamma^i \Gamma^j \Gamma^\phi = \Gamma^\phi (1 - \Gamma_D^4) \Gamma^0_E \Gamma^j \Gamma^i,$$

(3.2.20)

so it is in fact sufficient to average the left- and right-hand sides of (3.2.20) and, using cyclicity of the trace, to compute the trace of $\Gamma^0_E \Gamma^j \Gamma^i \Gamma^\phi$. However, this operator
anti-commutes with any $\Gamma^k$ for which $k \neq i, I, J$ and so must have vanishing trace. Thus it is sufficient to replace $L$ by $L_{\text{right}} = \frac{1}{8} \partial_t \ln H \Gamma^I_E (1 + 2 \Gamma_E \hat{0} \Gamma^\phi)$. 

Similar $\Gamma$-matrix algebra shows that $G_{\alpha}^{\beta}(x, y) (\Gamma^I_E \Gamma^J_E \Gamma^\phi)^{\alpha}_{\beta} = 0$ so that the zero-order contribution to $\langle J^I_{fE} \rangle$ vanishes identically at any $x, y$. The full expectation value is therefore

$$\langle J^I_{fE}(x, y) \rangle = \frac{1}{64} \text{Tr} [P \Gamma^I_E \Gamma^k (-i + 2 \Gamma_E \hat{0} \Gamma^\phi) P \Gamma^J_E \Gamma^l \Gamma^0_E \Gamma^\phi]$$

$$\times (\partial_l x + \partial_k y) \int d^5 z \partial_I G(x, z) [\partial_k H(z)] \partial_J G(z, y) + O(\delta H^2) \text{ (3.2.21)}$$

where all derivatives inside the integral are performed with respect to $z$. Note that to this order we may replace each remaining $\Gamma^I$ by its flat-space counterpart. We find

$$\text{Tr} [P \Gamma^I_E \Gamma^k (-i + 2 \Gamma_E \hat{0} \Gamma^\phi) P \Gamma^J_E \Gamma^l \Gamma^0_E \Gamma^\phi] = 16 (-2[\delta^{Il} \delta^{kJ} + \delta^{Ik} \delta^{Jl} - \delta^{IJ} \delta^{kl} + 2 \delta^{0l} \delta^{0J} \delta^{kl}])$$

$$- \epsilon^{0IJkl} + O(\delta H^2). \text{ (3.2.22)}$$

However, the term involving $\epsilon^{0IJkl}$ will contribute a term to $\langle J^I_{fE} \rangle$ proportional to the commutator of two derivatives, which of course vanishes. Thus the fermionic contribution is

$$\langle J^I_{fE}(x, y) \rangle = \frac{1}{2} (2 \delta^{Il} \delta^{kJ} - \delta^{IJ} \delta^{kl} + 2 \delta^{0l} \delta^{0J} \delta^{kl}) \int d^5 z \partial_I G(x, z) [\partial_l \partial_k H(z)] \partial_J G(z, y) + O(\delta H^2) \text{, (3.2.23)}$$

where we have used the fact that $\partial_k \partial_l = \partial_l \partial_k$ to simplify the factor involving Kronecker delta’s, and we continue with the convention that all derivatives inside the integral are with respect to $z$. We note that this expression is structurally quite similar to the bosonic contribution (3.2.14), except that a different combination of derivatives is involved as well as a different overall coefficient. In particular, Euclidean time derivatives of $G$ do appear in the fermion contribution, while they were absent in (3.2.14). We also note that, in comparison with the bosonic contribution, the Fermionic contribution weights the term where $l, k$ are contracted with $I, J$ by an
extra factor of two relative to the term where \( l, k \) are contracted together. As we will shortly see below, these features will prevent the bosonic and fermionic divergences from canceling.

### 3.2.3 The coincidence limit

Having obtained the expressions (3.2.14) and (3.2.23), we now turn to an exploration of the coincidence limit \( x \to y \). To this end, it will be convenient to reparametrize the problem in terms of the average location \( \Delta^+_I = (x^I + y^I)/2 \) and the difference \( \Delta_-^I = (x^I - y^I)/2 \) of the two points. We will take the separation to be purely in the Euclidean time direction, so that \( \Delta_-^0 = 0 \). We will be most interested in the singular contributions to (3.2.14) and (3.2.23), which result from the region where \( z \) is close to either \( x \) or \( y \). As a result, it is convenient to parametrize \( z \) as \( z^I = \Delta^+_I + \eta^I \).

Recall that the explicit form of the scalar Green’s function is

\[
G(x, y) = \frac{1}{3V(S^4)} \frac{1}{|x^I - y^I|^3},
\]

where \( V(S^4) \) is the volume of a unit \( S^4 \) and \( |x - y| \) is the Euclidean length of the 5-vector \( x - y \). Using this result, we may write our results in the form

\[
\langle J^0 E(x, y) \rangle = -\frac{1}{|\Delta_-|^3} \frac{1}{V(S^4)^2} \int d^5\eta \frac{(\eta^I - \Delta^I_+/|\Delta_-|)(\eta^J + \Delta^J_-/|\Delta_-|)}{|\eta - \hat{x}^0|^5 |\eta + \hat{x}^0|^5} A_{IJ}(\Delta_+ + |\Delta_0^0|\eta^I),
\]

where \( \hat{x}^0 \) is a unit vector in the positive \( x^0 \) direction. Note in particular that \( |\Delta_-| = |\Delta_0^0| \), the absolute value of the time component of \( \Delta_- \). In the above expression,

\[
A_{IJ}(z) = A_{IJ, \text{base}}(z) + A_{IJ, \text{fermi}}(z) \quad \text{with,}
\]

\[
A_{IJ, \text{base}}(z) = (\delta^k_I \delta^l_J - \delta_{IJ} \delta^{lk} + \delta^k_I \delta_J^0 \delta^0_l \delta_{I0} \delta_{J0}) \partial_k \partial_l H(z),
\]

\[
A_{IJ, \text{fermi}}(z) = -\frac{1}{2} (2 \delta^k_I \delta^l_J - \delta_{IJ} \delta^{kl} + 2 \delta^0_I \delta^0_J \delta^{kl}) \partial_k \partial_l H(z). \quad (3.2.25)
\]

One may now expand \( A_{IJ}(z) \) about the point \( \Delta_+ \) to obtain a power series in \( |\Delta_-| \). Since any odd parity integrand will integrate to zero, only even terms in this expansion will contribute. Furthermore, terms of order \( z^4 \) or higher in \( A_{IJ}(z) \) will give
vanishing contribution in the limit $\Delta_- \to 0$. Thus, the only relevant terms involve $A_{IJ}(\Delta_+)$ and $\partial_K \partial_L A_{IJ}(\Delta_+)$. The complete list of relevant integrals is provided in Appendix B, and quickly yields the result

$$\langle J^0_{bE}(x, y) \rangle = \frac{1}{12} \frac{V(S^3)}{[V(S^4)]^2} \left( \frac{1}{|\Delta_-|^3} \partial^2 H(\Delta_+) + \frac{1}{3} |\Delta_-|^2 (\partial^2 H(\Delta_+)) \right) + O(\Delta_-),$$

$$\langle J^0_{fE}(x, y) \rangle = -\frac{1}{18} \frac{V(S^3)}{[V(S^4)]^2} |\Delta_-|^3 \partial^2 H(\Delta_+) + O(\Delta_-),$$

(3.2.26)

where $\partial^2 = \delta^i_j \partial_i \partial_j$ is the Laplacian in the 1,2,3,4 directions. It is interesting that the $O(\Delta^{-1})$ Fermion contribution vanishes.

Finally, we should continue the result back to Lorentzian spacetime. To do so, let us first compute the Lorentzian current in the Euclidean background $b_E$ defined by

$$(3.2.1):$$

$$J^0_{L}(b_E) = iJ^0_{E}(b_E) = \frac{i}{36} \frac{V(S^3)}{[V(S^4)]^2 |\Delta_-|^3} \partial^2 C^0(b_E)(\Delta_+)(1 + O(\Delta_+)),$$

(3.2.27)

where we have used (3.2.14) to see that the factor of $H$ above came only from $C^0(b_E) = g^{00}(b_E)C_0(b_E) = -i(H^{-1} - 1) = i\delta H + O(\delta H^2)$, where the $g^{00}$ results from the contraction of the two Levi-Civita symbols. Thus, we may analytically continue to a Lorentzian background to find

$$\langle J^0_L \rangle = \frac{1}{36} \frac{V(S^3)}{[V(S^4)]^2 |\Delta_-|^3} \partial^2 C^0(b_E)(\Delta_+)(1 + O(\Delta_+)).$$

(3.2.28)

### 3.3 Discussion

The calculations above find that, when a $D4$-brane probe is placed in the supergravity background generated by $D0$-branes, the expectation value of the point-split $D0$-brane charge density $\langle \rho_{D0}(x, y) \rangle$ is non-zero at leading order. Furthermore, our low-energy field theory calculation gives a divergent result in the coincidence limit $x \to y$. 47
However, we note that the divergences are proportional to $\partial^2\perp H$ or $(\partial^2\perp)^2 H$. Thus, as one would expect from charge conservation, they yield zero total induced $D0$-charge when integrated over the $D4$-brane. We also note that this charge density vanishes in the limit $H \to 1$ where the $D0$ source is infinitely far away.

Such calculations clearly beg a fully string theoretic treatment. It is natural to expect such stringy calculations to merely cut off our field-theoretic divergence at the string scale, $\Delta^0_\perp \sim \ell_s$, yielding a finite induced Lorentz-signature charge density of the form

$$\langle \rho_{D0}(x) \rangle = \frac{1}{36} \frac{V(S^3)}{[V(S^4)]^2 |\Delta|^{-3}} \partial^2\perp H(\Delta_+)(1 + O(\Delta_+^3)) \sim a \ell_s^{-3} \partial^2\perp H(x)(1 + O(\ell_s^2)), \tag{3.3.1}$$

where $a$ is an unknown coefficient of order 1 which one naively expects to be positive$^2$.

In particular, (3.3.1) indicates a non-vanishing quantum polarization of the $D4$-brane by the $D0$-background, although such an effect does not occur classically at any order in $\alpha'$.

The effect arises because the boson and fermion contributions fail to cancel, though they do have opposite signs. We note that the sign of the final effect is the natural one expected of a polarizable medium, which follows from the natural tendency of an applied electric field to separate (in this case, virtual) charges.

It also interesting to ask what a term of the form (3.3.1) would imply for the quantum-corrected low-energy $D4$-brane effective action$^3$. That is, we may ask what term in an action $S_{D4}^{qc}$ would, when treated classically, yield an induced charge density of this form. The charge density is by definition the variation of $S_{D4}^{qc}$ with respect to the background Ramond-Ramond field $C^{(1)}$. Thus, a charge density linear in the background fields could in principle arise from a quadratic term involving two powers.

---

$^2$ Of course, it is possible that this quantum induced effect could be cancelled by some intrinsic $c$-number $O(g)$ correction to the $D0$ charge density on a $D4$-brane. We thank Allen Adams for raising this possibility.

$^3$ We thank Joe Polchinski for raising this question.
of $C^{(1)}$, or from a term involving one power of $C^{(1)}$ and one power of the metric or dilaton. However, there are no Lorentz-invariant quadratic couplings of a 1-form to a metric or scalar, so the coupling must be quadratic in $C^{(1)}$. The Lorentz invariant such term that leads to (3.3.1) is

$$-rac{a}{4}(2\pi)^4 T_{D4} \int d^5 x \left( g_s \ell_s^2 F_{IJ}^{(2)} F_{IJ}^{(2)} \right),$$

where we have used $T_{D4} = \frac{1}{(2\pi)^4 \ell_s^5 g_s}$ to write this term using the familiar normalizations of the $D4$-effective action in order to make clear that it does indeed have the form of a first order correction in $g_s$; i.e., a one-loop (annulus) string correction. Note that $F_{IJ}^{(2)}$ is the pull-back of the bulk Ramond-Ramond two form field strength to the brane. Thus, we expect the quantum-corrected $D4$ action to contain pull-backs of bulk kinetic terms.

We note, however, that such terms are known from [48] not to arise for type II branes at order $g_s^0$ at any order in $\alpha'$, though Einstein-Hilbert terms on the brane do arise as $\alpha'$ correction to branes in bosonic string theory [50]. Returning to the type II context, one may expect that, in order for terms (3.3.2) to reside in a supersymmetric effective action or to follow from a covariant term in the M5-brane effective action, an Einstein-Hilbert term for the world-volume metric would also be required. As pointed out in [49], such a term could have interesting cosmological implications in braneworld scenarios. However, this term appears not to arise [51] for type II branes.

It is not clear to us how this tension is resolved, though it may be that the quantum polarization term is cancelled by an explicit $O(g)$ term as suggested above in footnote

---

4 Note that that quantum corrections of our form arise only from corrections to the Green's functions that follow from couplings of bulk fields to world-volume fields in the classical $D4$-effective action. Since $C^{(1)}$ appears in this action only through its pull-back, corresponding terms induced in the quantum-corrected effective action must also involve only the pull-back of $C^{(1)}$.

5 We thank Savdeep Sethi for raising this latter question.

6 In particular, after our original posting of this work on the arxiv, the author of [51] shared with us his unpublished calculations which explicitly show that the coefficient of the Einstein-Hilbert term vanishes. One is tempted to believe that supersymmetry lies behind the vanishing of this coefficient, but no argument for this seems to be known.
Chapter 4

Gyrating strings: a new instability of black strings?

Black holes remain one of the most intriguing objects in general relativity, and are known for their simplicity and stability. As we saw in chapter 1, gravity in more than 3+1 dimensions can produce extended black objects such as strings and p-branes. In many ways, these extended black objects behave like their black hole cousins of 3+1 Einstein gravity. Indeed, the most familiar examples of black strings and branes have translational symmetries, and dimensional reduction along these symmetries yields black hole solutions to lower dimensional theories.

However, properties of black branes can sometimes differ significantly from those of the associated black holes. The Gregory-Laflamme instability [52, 53, 54] is a classic example of such behavior. In [52] it was shown that certain black strings are dynamically unstable to a breaking of translational (but not rotational) symmetry along the string. The ultimate fate of this instability remains a matter of investigation, but conjectures [55] as to this fate have led to the discovery of inhomogeneous black strings[58] (though these solutions cannot be the endpoint of the Gregory-Laflamme instability, at least in $d \leq 13$ dimensions [56]; see also [57]) and related work on
The general theory of such instabilities remains to be understood. An oft-discussed conjecture in this context was stated by Gubser and Mitra [60], who proposed that black branes might have dynamical instabilities precisely when they have thermodynamic instabilities, in the sense that the Hessian of second derivatives of the energy with respect to the conserved charges has negative eigenvalues. This conjecture has been proven in certain contexts [61].

Here we argue for an instability which places a new twist on such discussions. We examine the three-charge (D1-D5-P) spinning BPS black brane of type IIB supergravity, which becomes a 5+1 black string when compactified on $T^4$. When compactified along the remaining translation symmetry and dimensionally reduced to 4+1 dimensions, the resulting black hole is dual to that studied in [62] by Breckenridge, Myers, Peet and Vafa (BMPV) and is a rotating version of the black hole whose entropy was counted [18] by Strominger and Vafa using D-brane techniques. As in the works above, we take the direction along the string to be compactified on a circle of length $L$ in order to yield finite charges. The near-extremal solution was studied in [63, 64, 65]. In particular, from the results of [64] one can show that it has no thermodynamic instabilities in the sense of Gubser and Mitra\(^1\). Thus their conjecture predicts dynamical stability\(^2\).

However, this theory contains other BPS black strings carrying the same charges. In particular, strings were described in [67] in which all or part of the angular momentum is carried by *gyrations* of the string as opposed to spin. In such solutions the black string at any instant of time can be thought of as being helical in space, with the helical profile traveling along the string at the speed of light. Such solutions are

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\(^1\)In addition, the microscopic entropy of such objects in string theory was computed in [16] using methods from the correspondence between string theory on Anti-de Sitter (AdS) space and the associated conformal field theory (CFT).

\(^2\)However, as will be discussed later, there is one direction which the system is thermodynamically only marginally stable.
easily generated by applying the technique of Garfinkle and Vachaspati [68] to the spinning strings of [62]. In fact, the shape of such gyrations need not be a helix, but can be much more general. As a result, the space of such BPS black branes is highly degenerate; for fixed charges, one must still specify several functions on a circle in order to determine the solution uniquely. Oscillating versions of similar strings had previously been constructed in [69, 70, 72, 71].

Below we study the general gyrating BMPV string with anti-self dual angular momentum, i.e. with equal amount of rotation in two orthogonal planes. We find that, for large enough angular momentum \( J > J_{\text{crit}} = 3Q_1Q_5/2\sqrt{2} \), it is entropically favorable for the excess angular momentum \( J - J_{\text{crit}} \) to be carried by gyrations. In particular, maximizing the entropy \( S \) over the class of solutions carrying anti-self dual spin angular momentum \( J_{\text{spin}} \) and anti-self dual gyrational angular momentum \( J_{\text{gyro}} \), we find that for \( J > J_{\text{crit}} \) the entropy \( S \) is an increasing function of \( J_{\text{gyro}} \) on the interval \( [0, J - J_{\text{crit}}] \), with positive slope for \( 0 \leq J_{\text{gyro}} < J_{\text{crit}} \). This indicates a first order phase transition and suggests that a small perturbation of a non-gyrating black string could grow to become large; i.e., it suggests an instability of the black string\(^3\) in the same way that super-cooled water is unstable to the formation of ice and super-heated water is unstable to boiling.

Some useful details of the solutions are described in section 4.1 below, as they were suppressed in [67]. Section 4.2.1 then assembles the argument and comments on the importance of small deviations from extremality. We also discuss two possible scenarios for the final fate of this instability. Section 5.4 contrasts our situation with that associated with super-radiance, observes that an analogous argument holds for the D-brane bound states associated with our black strings in weakly coupled string

\(^3\)The discussion in [67] explicitly focussed on the case where the gyrations are small enough that the object could be well described via dimensional reduction to a black hole. In such cases, it was shown that the effect of gyrations on the black string entropy is negligible. However, the gyrating string has significantly larger entropy only when \( J - J_{\text{crit}} \) is of order \( J \), and in this case the oscillations are necessarily large.
theory, and makes further final comments.

4.1 The gyrating black string

Consider a D1-D5-P black brane solution which is asymptotically $R^5 \times S^1 \times T^4$. We will think of the $T^4$ as being small so that the solution is effectively a black string in 5+1 dimensions. For a translationally invariant brane, the ten-dimensional type IIB supergravity solution is

$$ds^2 = \left(1 + \frac{r_o^2}{r^2}\right)^{-1} \left( -du dv + \frac{p}{r^2} du^2 + \frac{2\gamma}{r^2} (\sin^2 \theta d\varphi - \cos^2 \theta d\psi) du \right)$$

$$+ \left(1 + \frac{r_o^2}{r^2}\right) dx_i dx^i + dy_a dy^a. \quad (4.1.1)$$

The index $a$ runs over the four directions of the $T^4$ and $i$ runs over the four space directions transverse to the branes. These latter 4 directions are associated with an $S^3$ labelled by constant values of $r^2 = \sum_i x^i x^i$. The angles $\theta, \varphi, \psi$ label this $S^3$, while the coordinates $z,t$ label the worldvolume directions of the string. Here we have chosen the special case where D1- and D5-brane charges are tuned to achieve constant dilaton and four-torus volume, so that (4.1.1) is the metric in either the string or Einstein frame. The three form $H$ has associated charges

$$Q_1 = \frac{2V}{(2\pi)^6 g} \int e^\phi \ast H = \frac{V r_0^2}{(2\pi)^4 g}, \quad Q_5 = \frac{1}{4\pi^2 g} \int H = \frac{r_0^2}{g}, \quad (4.1.2)$$

where $g$ is the string coupling and $V$ is the volume of the four-torus. All other fields are set to zero. This solution has a null Killing vector field $\partial/\partial v$, so one may attempt to add travelling waves via the method of [68]. It turns out that there are many interesting such waves for this system, which were studied extensively in [72, 73, 67] following similar work of [69, 70, 71] for other systems.
We are interested here in the class of waves briefly discussed in [67] which can be viewed as *gyrations* of the brane itself in the $x^i$ directions. Such solutions differ from (4.1.1) only by the addition of a term proportional to $\ddot{h}_i(u)x^i du^2$, where $h_i$ are arbitrary functions of $u$:

$$ds^2 = \left(1 + \frac{r^2}{r^2_o}\right)^{-1} \left[-dudv + \frac{p}{r^2} du^2 + \frac{2\gamma}{r^4} (x^1 dx^2 - x^2 dx^1 - x^3 dx^4 + x^4 dx^3) du - 2\dot{h}_i(u)x^i du^2\right] + \left(1 + \frac{r^2}{r^2_o}\right) dx_i dx^i + dy_i dy^i.$$  \hspace{1cm} (4.1.3)

After the change of coordinates

$$v' = v + 2\dot{h}_i x^i + \int^{u} \dot{h}^2 du,$$  \hspace{1cm} (4.1.4)

$$x'^i = x^i + h^i,$$  \hspace{1cm} (4.1.5)

asymptotic flatness of the metric becomes manifest:

$$ds^2 = \left(1 + \frac{r^2}{r^2_o}|\vec{x} - \vec{h}|^2\right)^{-1} \left[-dudv + \frac{p}{|\vec{x} - \vec{h}|^2} du^2 + \frac{2\gamma}{|\vec{x} - \vec{h}|^4} (\Delta^1 d\Delta^2 - \Delta^2 d\Delta^1 - \Delta^3 d\Delta^4 + \Delta^4 d\Delta^3) du + \left(2 + \frac{r^2}{|\vec{x} - \vec{h}|^2}\right) \frac{r^2_o}{|\vec{x} - \vec{h}|^2} (\dot{h}^2 du^2 - 2\dot{h}_i dx^i du)\right] + \left(1 + \frac{r^2}{|\vec{x} - \vec{h}|^2}\right) dx_i dx^i + dy_i dy^i,$$  \hspace{1cm} (4.1.6)

where $\Delta^i = x^i - h^i$. Note that the black string horizon is located at $x^i = h^i(u)$ in the new coordinates, suggesting that the string is indeed oscillating in the $x^i$ directions.
Following [67], we have in mind circular oscillations of the type associated with a net angular momentum which we refer to as \textit{gyrations} of the string in the $x^i$ directions.

The conserved linear and angular momenta of the gyrating black string can be read off from the asymptotic form of (4.1.6). The results are as follows:

\begin{align*}
P &= \frac{Lp}{\kappa^2} + P_{gyro}, \quad P_{gyro} = \frac{2r_0^2}{\kappa^2} \int_0^L \dot{h}^2 du, \quad (4.1.7) \\
J_\phi &= J_{12} = J_{12}^{gyro} + \frac{L\gamma}{\kappa^2}, \quad (4.1.8) \\
J_\psi &= J_{34} = J_{34}^{gyro} - \frac{L\gamma}{\kappa^2}, \quad (4.1.9) \\
J_{ij}^{gyro} &= \frac{2r_0^2}{\kappa^2} \int_0^L (h_i \dot{h}_j - h_j \dot{h}_i) du. \quad (4.1.10)
\end{align*}

Here $\kappa^2 = (2\pi)^5 g^2 / V$. Note that the expressions for $P_{gyro}$ and $J_{ij}^{gyro}$ are identical to those of a material string with tension $2r_0^2 / \kappa^2$.

Now, for any string the gyrational angular momentum is bounded by a linear function of gyrational momentum. This result may be derived in several ways. For example, one might note that the expressions for $P_{gyro}$ and $J_{gyro}^{gyro}$ have the same form of those in chapter 2 for the angular momentum (1.2.8) and charge (1.2.5) of a supertube. Applying the same arguments of [30] to derive the bound (1.2.11), here yields $|J_{12}| \leq \frac{P_{gyro}L}{2\pi}$. Another argument notes that quantizing the string will yield vector particles, each carrying spin $\pm 1$. Thus, the angular momentum must be bounded by the number $N_{gyro} = P_{gyro}L / 2\pi$ of associated momentum quanta.

However, requiring our angular momentum to be anti-self dual is not compatible with saturating this bound. To understand the additional constraint from anti-self duality, consider expression (4.1.10) for $J_{gyro}^{gyro}$ for the case where only a single wave number $k = 2\pi n / L$ is excited so that $h_i$ takes the form

\begin{equation}
h_i = A_i \cos(2\pi n / L) + B_i \sin(2\pi n / L). \quad (4.1.11)
\end{equation}
The resulting gyrational momentum and angular momentum are

\[ J_{ij}^{\text{gyro}} = \frac{4\pi n r_0^2}{\kappa^2} (A_i B_j - B_i A_j), \quad P^{\text{gyro}} = \frac{4\pi^2 n^2 r_0^2}{L \kappa^2} \sum_i (A_i A_i + B_i B_i). \quad (4.1.12) \]

In particular, the angular momentum lies in the plane defined by the vectors \( A_i \) and \( B_j \). Thus, to obtain an anti-self dual \( J_{ij} \) requires excitations in at least two momentum modes and, if achieved with only two modes, requires the associated planes to be orthogonal.

Next, we note that \( P^{\text{gyro}} \) in (4.1.12) is proportional to \( n^2 \) while \( J_{ij}^{\text{gyro}} \) is proportional only to \( n \). Thus, if we wish to maximize \( J^{\text{gyro}} \) for a given \( P^{\text{gyro}} \), it is clear that we wish to use the lowest modes possible. Thus, we do best if we use only the \( n = 1 \) and \( n = 2 \) modes; say, with the fundamental mode carrying angular momentum \( J_{12} \) in the 12 plane while the \( n = 2 \) mode carries angular momentum \( J_{34} \) in the 34 plane. For anti-self duality we require \( J_{12} = -J_{34} \). It is also clear from (4.1.12) that we wish to choose the vectors \( A_i \) and \( B_i \) for any given mode to be orthogonal but of the same magnitude. But then the extra power of \( n \) in \( P^{\text{gyro}} \) implies that the \( n = 2 \) mode carries twice the gyrational momentum as the \( n = 1 \) mode. Thus, the largest anti-self dual \( J^{\text{gyro}} \) is obtained by placing \( N_{\text{gyro}} \) excitations in the fundamental \((k = 1)\) mode of the string and using them to carry \( J_{12} = N_{\text{gyro}} / 3 \), while also placing \( N_{\text{gyro}} / 3 \) excitations in the \( k = 2 \) mode and using it to carry \( J_{34} = N_{\text{gyro}} / 3 \) and \( 2N_{\text{gyro}} / 3 \) momentum quanta. This configuration has \( J_{\text{gyro}} = \sqrt{J_{12}^2 + J_{34}^2} = \frac{2}{3} N_{\text{gyro}} \) units of angular momentum. Other configurations with the same \( J_{\text{gyro}}, P_{\text{gyro}} \) may be obtained through rotations that preserve \( J_{ij}^{\text{gyro}} \), but there are no configurations with greater anti-self dual angular momentum.

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4.2 Entropy and Instability

In [67] it was shown that adding such gyrations does not affect the induced metric on the horizon, and in particular that the area of the horizon does not change\footnote{However, a mild null shock-wave singularity does form along the horizon. The metric is $C^0$ at the horizon (so that, in particular, its area is well-defined) but it is not $C^1$. One expects that any amount of excitation above the BPS bound will smooth out this singularity, though the resulting solution is unlikely to be stationary. See the appendix of [73] for details of the extreme solutions.}. Therefore, the entropy of the gyrating string has the same form as that of the spinning D1-D5-P string [62], but with the total momentum replaced by $P - P_{gyro}$ and the total angular momentum replaced by $J - J_{gyro}$. In terms of the number of momentum quanta $N = P L / 2 \pi$ and $N_{gyro} = P_{gyro} L / 2 \pi$, the result is:

$$S = 2 \pi \sqrt{Q_1 Q_5 (N - N_{gyro}) - (J - J_{gyro})^2}. \quad (4.2.1)$$

Given a gyrating black string with $|J_{gyro}| \leq \frac{\sqrt{2}}{3} N_{gyro}$, we can always decrease $N_{gyro}$ to obtain a solution with larger entropy. Thus maximally entropic strings saturate this bound to yield:

$$S = 2 \pi \sqrt{Q_1 Q_5 (N - \frac{3}{\sqrt{2}} |J_{gyro}|) - (J - J_{gyro})^2}. \quad (4.2.2)$$

When the entropy is now maximized over $J_{gyro}$, the absolute value in (4.2.2) leads to two distinct behaviors. For $J < J_{crit} := \frac{3 Q_1 Q_5}{2 \sqrt{2}}$, the maximum is at $J_{gyro} = 0$. However, for $J > J_{crit}$, entropy is maximized for $J_{gyro} = J - J_{crit}$ and thus $J_{spin} = J_{crit}$.

We now use the above observations to argue for a new type of black string instability. Namely, we suggest that certain non-gyrating spinning black strings are unstable to the development of gyration for $J > J_{crit}$. Now, one does not expect BPS solutions to have a linear instability\footnote{For static solutions, the BPS bound and the results of [74] work together to forbid linear instabilities. However, we know of no general theorems for the stationary case.}. Indeed, we have seen that they are marginally stable to developing gyration, as any amount of gyration leads to a stationary solution. That is, the parameter $J_{gyro}$ effectively labels a moduli space of BPS solutions.
However, a generic perturbation of the spinning string will result in motion through this moduli space (as well as some amount of excitation of the string off of the moduli space). The important observation is that near $J_{\text{gyro}} = 0$ motion in the direction of increasing $J_{\text{gyro}}$ is entropically favored over motion in the direction of decreasing $J_{\text{gyro}}$. In fact, we have seen that the entropy is maximized at $J_{\text{gyro}} = J - J_{\text{crit}}$, so that this value is entropically stable. As one may expect\(^6\) this entropic stability to be enforced dynamically, we conjecture that the gyrating string with $J_{\text{gyro}} = J - J_{\text{crit}}$ is \textit{dynamically} stable, perhaps due to higher order dynamical effects beyond the linear level. Similarly, we conjecture that the spinning non-gyrating string with $J > J_{\text{crit}}$ is dynamically \textit{unstable}, perhaps due to higher order effects.

Instead of considering BPS objects, one might consider strings with energies slightly in excess of the BPS bound. For a nearly BPS object with $J$ substantially greater than $J_{\text{crit}}$, one expects a similar entropy formula and a similar instability. In particular, if some form of continuity holds then we have that:

1. Near BPS solutions can also be labeled by a parameter $J_{\text{gyro}}$. Since one expects non-BPS gyrating strings to radiate, these solutions are unlikely to be stationary. Their gyrating phase will be transient. However, this means only that $J_{\text{gyro}}$ will refer to the gyrational angular momentum at some particular moment of time.

2. The derivative $\frac{\partial S}{\partial J_{\text{gyro}}}$ will be positive at $J_{\text{gyro}} = 0$, where the derivative is taken with all conserved charges held fixed.

Thus, one expects a near-BPS non-gyrating string with $J > J_{\text{crit}}$ to be unstable to transfer of angular momentum from spin to gyration.

It is interesting to speculate as to the final state into which this string decays. There are two natural possibilities. The first is that the string sheds its excess angular

\(^6\)Were the horizons completely smooth, this would be enforced by the area theorem [75, 76].
momentum through classical radiation and eventually becomes a *stable* non-gyrating string with $J \leq J_{\text{crit}}$. The second is that the dominant effect is shedding of excess energy and that the final state is a gyrating BPS string. One might expect that either final state can arise and that the outcome depends on the particular values of the parameters. Note, however, that if we were to place the unstable string in a small reflecting cavity, this would prevent the loss of significant amounts of either $E$ or $J$, so that one would expect decay into a stable non-BPS gyrating black string. Due to the small domain and simple boundary conditions, this might be a particularly interesting arena for numerical simulations. It would also lead to a clear signal: an equilibrium state that is far from being rotationally invariant.

### 4.3 Discussion

We have argued for an instability of the D1-D5-P system near BPS black strings with $J > J_{\text{crit}} = \frac{3Q_1Q_5}{4\sqrt{2}}$. Note that such strings exist only when the number $N$ of momentum quanta exceeds a certain bound: $N \geq \frac{9}{8}Q_1Q_5$. Thus, like the original Gregory-Laflamme instability, the instability arises only for sufficiently long strings.

Now, the thermodynamics noted above might also be taken to suggest an instability to simply radiating angular momentum (and momentum) to infinity. In fact, this latter sort of potential instability could in principle occur at a smaller value of $J$, since gravitational waves can carry more $J$ for a given amount of $P$. It is natural to take guidance from the study of 3+1 dimensional Kerr black holes, where one finds a similar thermodynamics: Kerr black holes have an entropy that decreases with increasing $J$. Thus, radiation of a small amount of energy is allowed if it carries a large angular momentum. In the case of Kerr one finds no instability for massless fields, but merely super-radiance: a given incident wave undergoes a finite amount of amplification and then disperses to infinity. In contrast, a true linear instability
does result [108, 78] if the black hole is surrounded by a large mirror (or is placed in a large AdS space [79]) so that the wave is continually redirected toward the black hole.

However, an instability for Kerr can arise for massive fields, which can be bound to the black hole by the gravitational potential. Results are known for minimally coupled scalar fields [80, 81, 82, 83]. In our context, one may expect radiation modes with momentum in the z-direction (i.e., Kaluza-Klein modes) to behave similarly, and our gyrational mode is much like such a bound state. Indeed, in the non-BPS case it is not clear to us to what extent it can be meaningfully distinguished from such bound states. But while a study of such bound states for non-BPS strings is difficult, it is clear the gyrational mode is the unique such bound state in the BPS limit. Since this limit will be important below, we focus on the gyrational mode.

We have seen that gyrations cannot be re-absorbed into the black string since \( \frac{\partial S}{\partial J_{\text{gyro}}} > 0 \). Thus, such gyrations can decay only through radiation to infinity. Whether or not a linear instability occurs will then be determined by a competition between two effects: the amplification of the traveling wave and the tendency to radiate the gyrational traveling wave to infinity. Both are expected to vanish in the BPS limit. However, one expects the amplification to increase with \( J - J_{\text{crit}} \), while there is no reason for this parameter to affect the rate at which the gyrational traveling wave is radiated to infinity. Thus, one expects that, at least by tuning parameters so that \( J - J_{\text{crit}} \) is large while the string remains nearly BPS, one can indeed produce an instability.

Our argument is highly suggestive, but clearly falls short of a proof. We have also described two possible final states. The system clearly calls for more detailed investigation and may yield a variety of interesting phenomena. In addition to those mentioned above, it is may also be fruitful to investigate relations between gyrating black strings and black tubes or black rings [84, 85, 86].
Supposing now that an instability (of any type discussed above) does occur, let us briefly reflect on the broader implications. An interesting attempt to understand the general nature of black string instabilities is encoded in the Gubser-Mitra conjecture of [60]. Quoting from [60], this is the conjecture that

...for a black brane with translational symmetry, a Gregory-Laflamme instability exists precisely when the brane is thermodynamically unstable. Here, by Gregory-Laflamme instability we mean a tachyonic mode in small perturbations of the horizon; and by thermodynamically unstable we mean that the Hessian matrix of second derivatives of the mass with respect to the entropy and the conserved charges or angular momenta has a negative eigenvalue.

This conjecture has been proven for a certain class of black strings [61], but our system appears to be a counter-example to the conjecture holding in complete generality. Let us consider a slightly non-BPS spinning string with $J > J_{\text{crit}}$. We choose the non-BPS case as, with asymptotically flat boundary conditions, we expect gyrating strings to radiate so that non-BPS non-gyrating strings will form an isolated family in the space of stationary solutions. Thus we may cleanly talk about “the entropy of the black string with fixed conserved charges and angular momenta.” Nearly-BPS objects are generally thermodynamically stable. The details of the non-extremal solutions can be found in [64], and show that no instabilities are present near extremality. Our discussion above suggests a dynamical instability, and thus a counter-example to the above conjecture. However, it should be noted that, even at a finite distance from extremality, one finds an interesting conspiracy that forces the Hessian to have a single zero eigenvalue. Thus, it is possible that the conjecture may be preserved if non-extremal strings have a marginally stable mode leading to gyration, but no linear instabilities.
Another argument for a counter-example to Gubser-Mitra was given in [87] by exhibiting a marginally stable mode in a thermodynamically stable but near-BPS black string. It is interesting to rephrase our results in the same terms: on general grounds, one may expect that some Kaluza-Klein modes around rotating black strings are unstable as they should act like massive fields around Kerr black holes. However, in general we expect such black strings to be thermodynamically unstable. It is only in the BPS limit that one expects thermodynamic stability. The gyrational mode of the BPS black string studied here allows us to see that a small number of marginal bound states persist for this string in the extremal limit, suggesting the presence of stable, marginal, and unstable modes within any neighborhood of the BPS limit, and in particular in the thermodynamically stable regime.

As a final comment, the reader may wish to return to the discussion of [62] and ask how the entropy of gyrating BPS black strings is to be understood from string theory. The answer is that gyration of a D-brane bound state is described by the U(1) “center-of-mass” degrees of freedom, since the D-branes gyrate collectively in a way that does not excite the relative motion degrees of freedom. The counting of states for the spinning black string given in [62] does not include the effect of this degree of freedom as it is known to carry little entropy (see, e.g., [88]). Thus, momentum that goes into exciting gyration of the bound state produces no entropy. But this is just what was observed in the black string entropy in equation (4.2.1). The point is that the U(1) degrees of freedom can carry angular momentum, and can do so more ‘cheaply’ than can the collective modes. Thus, allowing the U(1) degrees of freedom carry linear momentum $P_{\text{gyro}}$ and angular momentum $J_{\text{gyro}}$ leads directly to the entropy (4.2.1) for the gyrating D-brane bound state. In particular, this means that for certain values of the global charges D-brane bound states are also unstable to gyration in the presence of interactions (e.g., via closed-string exchange) between the center-of-mass U(1) and other degrees of freedom.
Chapter 5

Fast travel in spherically symmetric spacetimes

It is a familiar fact that the presence of gravitational fields affects the time a signal takes to travel between two spatial points. One version of this effect, the delay of light rays passing near gravitational sources, was first noticed by Shapiro, and has been the subject of many precision tests of general relativity [89]. It is then natural to investigate under which conditions time advance can be achieved. For example, a negative mass source turns the Shapiro time delay into the desired advance of nearby light rays with respect to those farther away from the mass.

Few authors have published examples of spacetimes where time advance takes place. Perhaps the most widely known example is the Alcubierre’s bubble [90], but other similar constructions exist (see Krasnikov’s tube in [91]). The fast signals in these examples remain inside the local light cones and therefore no causality violation occurs. To describe the advance, the time a signal takes to propagate between two locations \(x\) and \(y\) is compared to the time of flight of a signal connecting the ’same’ locations in Minkowski spacetime, and the former is found to be shorter. This comparison brings about the issue of how to map points of two different spacetimes and
hence, how to correctly define time advance (or, by the same token, time delay). The
diffeomorphism invariance of general relativity is well-known to make such notions
extremely difficult to define. In the above mentioned examples, this is resolved by
considering the fast spacetime to differ from Minkowski only in a localized region that
does not contain \( x \) and \( y \). However, the issue remains in more general contexts.

To build a hyperfast communication system based on the above modification of
space, our civilization would need to create exotic matter that is so far not found in
nature. In the absence of a negative cosmological constant, observational evidence
supports the assumption that physical systems only manifest positive energy densities
\(^1\),\(^2\). This assumption on matter is known as the weak energy condition, or the null
energy condition if the observer follows a null path. Adding the requirement that the
energy always flows in casual trajectories constitutes the dominant energy condition.
The stress-energy tensors associated with the fast solutions of Alcubierre [90] and
Krasnikov [91] (as well as with wormhole solutions, see e.g. [92]) correspond to
‘exotic’ types matter that violate some of these energy conditions. A more general
result proven by Olum establishes that for spacetimes that are Minkowski outside a
localized region occurrence of time advance as defined above implies the violation of
the weak energy condition somewhere in the modified region.

There are substantial theorems to the effect that fast travel is not possible with-
out violating some positive energy condition. Such theorems include the results of
Hawking [101] on the formation of closed timelike curves and those of Olum [102],
Visser, Bassett, and Liberati [103], and Gao and Wald [104] which relate more di-
rectly to ‘fast travel’. These theorems can be quite powerful and each involves a
somewhat different concept of ‘fast travel’. In [102], Olum proposes a definition of

\(^1\)Although the Casimir effect shows that this assumption is not supported by quantum systems,
it is not clear how to implement this effect in order to produce time advance

\(^2\)See e.g. [94, 95, 96, 97, 98] for a summary of the current understanding of the limitations
on negative energy fluxes from quantum field theoretic effects and [100] for some discussion of the
relationship between the ‘negative energies’ of stringy orientifolds and the weak energy condition.
fast (superluminal) travel applicable to general spacetimes that does not require comparison with a reference metric. He derives a rather more abstract theorem showing that without violations of the null energy condition, a certain de-focusing property cannot arise. This property is expected to be related to localized regions of fast travel, and in particular the idea that there is a ‘fastest’ path to follow. Gao and Wald [104] give arguments against the possibility of devising time advance for sufficiently long null geodesics, but provide no result within any given region of spacetime.

In contrast, the approach followed by Visser, Bassett and Liberati [103] refers to a Minkowski background to define time differences. Working with linearized Einstein gravity, they attempt to show that the null energy condition forces the light cones to contract with respect to the light cones of Minkowski spacetime, an effect that would rule out the Shapiro time advance of weak gravitational fields. As these authors point out, generalizing of their computing scheme to strong gravitational fields is already technically complicated. But even in the weak field case, their results turned out to be highly gauge dependent and could be reversed by a suitable gauge choice, as shown by the authors of [104].

Here, we focus on spherically symmetric static spacetimes, whose main features we summarize in 5.1. In this particular context, we are able to design a setting in which the relative time advance between two spacetimes is well-defined. We study the effect of imposing the various energy conditions (whose rigorous definitions are also found in 5.1) on the propagation time of a light signal. It does not come as a surprise that these energy conditions again limit the amount of achievable time advance. This is illustrated through the examples of section 5.3, though the same examples show that an advance can happen. In section 5.2, we derive a bound on the time of flight, but we fail to discover the fastest spacetime saturating this bound.
5.1 Review on spherically symmetric static spacetimes

The metric for a spherically symmetric and static spacetime is commonly presented in the Schwarzschild coordinates \((t, r, \theta, \phi)\) taking the general form (see, e.g. [93]):

\[
\begin{align*}
\text{ds}^2 &= - f(r) dt^2 + h(r) dr^2 + r^2 (d\theta + \sin^2 \theta d\phi)^2. \\
&= f(r) dt^2 + h(r) dr^2 + P_r(r) dr^2 + P_\theta(r) d\theta^2 + P_\phi(r) d\phi^2. 
\end{align*}
\]

(5.1.1)

The invariance of the metric under spatial rotations is expressed through the isometry group \(\text{SO}(3)\), and the orbits of this group are points on two-spheres labelled by the spherical coordinates \((\theta, \phi)\). All metric functions are independent of \((\theta, \phi)\). These orbits correspond to closed surfaces uniquely parameterized by their area \(A\), which we use to define the area-radius coordinate \(r\) through \(r = (A/4\pi)^{1/2}\), as can be seen from (5.1.1). The spacelike hypersurfaces are labelled by the coordinates \((r, \theta, \phi)\) and are orthogonal to the orbits of the timelike Killing vector field \((\partial/\partial t)^a\), in accordance with the definition of a static spacetime. Here, \(r\) must satisfy \(\nabla_a r = 0\) in order to be a good coordinate.

In these coordinates, the stress-energy tensor takes the diagonal form

\[
T_{\mu}^{\nu} = \text{diag}(-\rho, P_r, P_\theta, P_\phi),
\]

(5.1.2)

where, by spherical symmetry, the angular pressures are related by \(P_\phi = P_\theta\). A non-vanishing value of off-diagonal components \(T_{\mu0}\) would imply the presence of energy currents (or fluxes), in contradiction with the staticity condition. Also, for spherically symmetric configuration all forces on matter are radially directed or homogeneously lie on the two-spheres, setting the rest of off-diagonal components of \(T_{\mu\nu}\) equal to zero.

Apart from geometric restrictions imposed by symmetry, one could in principle consider an arbitrary stress-tensor. However, it is generally believed that the energy
density of matter, as measured by an observer whose 4-velocity is $t^\mu$, is nonnegative, i.e.,

$$T_{\mu\nu}t^\mu t^\nu \geq 0.$$  

(5.1.3)

This assumption is known as the \textit{weak energy condition}. The limiting case in which $t^\mu$ becomes a null vector is contained in the above condition, and it is called the \textit{null energy condition}. It seems natural to also expect that the speed of the energy flow of matter is always less than the speed of light. This translates into the requirement that $T^\mu_{\nu}t^\nu$, if non-zero, should be a future directed timelike vector, and it is referred to as the \textit{dominant energy condition}. In our context this imposes

$$\rho \geq |P_r|, \quad \rho \geq |P_\theta|. \quad (5.1.4)$$

We are now ready to write the Einstein equations, $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, for the unknown metric components $g_{tt} = -f$ and $g_{rr} = h$, with sources given by (5.1.2). These equations have a more compact form when written in terms of the spherically symmetric mass function $m(r)$, which is related to the metric function $h(r)$ by $h = (1-2m/r)^{-1}$. We obtain three independent equations,

$$\partial_r m = 4\pi r^2 \rho,$$

(5.1.5)

$$\frac{\partial_r f}{2f} = \frac{m + 4\pi r^3 P_r}{r(r-2m)},$$

(5.1.6)

and the stress-energy conservation equation $\nabla^\mu T_{\mu\nu} = 0$ in a spherically symmetric background,

$$\partial_r P_r = - \left( \frac{\partial_r f}{2f} + \frac{2}{r} \right) P_r - \frac{\partial_r f}{2f} \rho + \frac{2}{r} P_\theta.$$  

(5.1.7)

Notice that the physical interpretation of $m(r)$ as the mass of the configuration becomes clear from integrating (5.1.5), where $\rho$ corresponds to the energy density.

We will be interested in a particular type of spherically symmetric configurations which are vacuum outside some sphere of area $4\pi R^2$. This choice of spacetimes
will introduce a concrete framework to study the question of time advance, but we postpone this discussion to section 5.2.1. According to *Birkhoff’s theorem*, the metric in such an exterior region is always given by the Schwarzschild solution

\[ f(r) = \left(1 - \frac{2M}{r}\right), \quad h(r) = \left(1 - \frac{2M}{r}\right)^{-1}, \quad (5.1.8) \]

with some mass \( M \leq \frac{R}{2} \). This mass represents the ADM mass measured in asymptotically flat infinity.

In this setting, one may allow the spacetime to have a discontinuity in the extrinsic curvature across the separation sphere at \( r = R \), which will indicate the presence of a shell of matter at that position. This matching of the discontinuity across a given hypersurface \( \Sigma \) to the presence of delta-function layer of matter is known as the *Israel junction condition*. The surface stress-energy tensor on \( \Sigma \) is defined by the integral [110]

\[ S^\mu_\nu = \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} T^\mu_\nu \, dn = \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} G^\mu_\nu \, dn, \quad (5.1.9) \]

where \( n \) is the proper distance measured along the normal to the hypersurface. The metric components along \( \Sigma \) remain continuous. In the case of spherically symmetric metrics (5.1.1), \( \Sigma \) must be an orbit of the symmetry, so that \( dn = \sqrt{h} \, dr \) and

\[ 8\pi S^t_t = \left[ -\frac{2}{r \sqrt{h}} \right]^+, \quad S^r_r = 0, \quad (5.1.10) \]
\[ 8\pi S^\theta_\theta = \left[ \frac{1}{\sqrt{h}} \left( \frac{f'}{2f} + \frac{1}{r} \right) \right]^+, \quad (5.1.11) \]

where the brackets indicate the jump across \( \Sigma \) in the value of the enclosed function. For a sphere at \( r = R \), the exterior is given by the Schwarzschild solution (5.1.8), and the surface tensor is

\[ 8\pi S^t_t = -\frac{2}{R} \left( \sqrt{1 - \frac{2M}{R}} - \frac{1}{\sqrt{h}} \right), \quad S^r_r = 0, \quad (5.1.12) \]
\[ 8\pi S^\theta_\theta = \frac{1 - \frac{M}{R}}{R \sqrt{1 - 2\frac{M}{R}}} - \frac{1}{\sqrt{h}} \left( \frac{f'}{2f} + \frac{1}{R} \right). \]
Here, the metric components and their derivatives are evaluated by taking the limit $r \to R$ from below. We will continue to use this convention: any discontinuous function evaluated at $R$ is to be understood as the limit $r \to R$ from below.

The surface stress-energy will also be subjected to the energy conditions. In particular, we notice that a non-negative surface energy density $S^t_t$ requires $M \geq m(R)$, i.e., that the mass contained in the region $r < R$ is less than or equal to the total mass $M$ of the spacetime.

5.2 A bound on fast spacetimes

In this section we prove a theorem showing that, as viewed from infinity, no signal connecting two static worldlines $(r, \theta, \phi) = (r_1, \theta_1, \phi_1)$ and $(r, \theta, \phi) = (r_2, \theta_2, \phi_2)$ in a static spherically symmetric spacetime is faster than a signal connecting the corresponding worldlines in Minkowski space. We begin with the following Lemma:

**Lemma 1:** Consider an asymptotically flat spherically symmetric spacetime which is static for $r > r_0$, satisfies $m(r_0) \geq 0$, and satisfies the weak energy condition. In such a spacetime, no clock with $r > r_0$ runs faster than a clock at infinity. That is, if the Killing time $T$ is normalized at infinity, the proper time $\tau$ of any static clock increases no faster than $T$.

From (5.1.8), we see that our Lemma is just the statement that the metric function $f$ satisfies $f < 1$. To prove this result, let us make use of the Einstein equations given above. Consider the density profile $\rho(r)$ and the pressure profile $P_r(r)$ in our spacetime. Since asymptotic flatness requires $f = 1$ at infinity, $f(r)$ is determined by integrating (5.1.6) inward from infinity. As a result, for a fixed density profile and fixed $m(r_0)$, reducing the radial pressure at any $r$ increases $f$ at every smaller value of $r$. Now recall that the given spacetime must satisfy the weak energy condition,
which implies a lower bound on the pressure \( \rho \geq -P_r \). Thus, choosing a new pressure profile \( \tilde{P}_r = -\rho \) that saturates the above bound, will generate an \( \tilde{f} \) with the largest allowed value, i.e., \( \tilde{f}(r) > f(r) \) at each \( r \). Note that our new spacetime is described by the same function \( h(r) \) as the original. Now, since \( h = (1 - 2m/r)^{-1} \), we have

\[
\frac{\partial_r h}{2h} = \frac{-m + 4\pi r^3 \rho}{r(r - 2m)}.
\]

(5.2.1)

Since \( \tilde{P}_r = -\rho \), comparison with (5.1.6) shows that we have \( \partial_r \ln h = -\partial_r \ln \tilde{f} \); i.e., \( \tilde{f}h = constant \). Evaluating this in the asymptotic region we find \( \tilde{f} = 1/h \). But, using the timelike vector \( (\partial/\partial t)^a \) in (5.1.3) yields \( \rho \geq 0 \) so that \( m \geq 0 \) from (5.1.5) and \( h = (1 - 2m/r)^{-1} > 1 \). Thus \( f < \tilde{f} < 1 \), proving Lemma 1. In fact, our result is somewhat stronger as we only used \( \rho \geq -P_r \) (and not the entire weak energy condition).

With the aid of Lemma 1, it is now easy to prove the following theorem:

**Theorem 1.** Consider a smooth, spherically symmetric, spacetime satisfying the weak energy condition and static for \( r > r_0 \) and \( m(r_0) \geq 0 \). Suppose the Killing time to be normalized at infinity and consider two orbits \( x \) and \( y \) of the time translation symmetry lying on symmetry spheres with areas \( 4\pi R_x^2 \) and \( 4\pi R_y^2 \) and separated by an angle \( \theta \) on the spheres. Then, as viewed from infinity, no signal staying within the static region can be sent between \( x \) and \( y \) faster than one could be sent if \( x \) and \( y \) lay on the corresponding sized spheres in Minkowski space with the same angular separation; i.e., the Killing time \( T \) required satisfies

\[
T \geq \sqrt{R_x^2 + R_y^2 - 2R_xR_y\cos \theta}.
\]
It is clear that the fastest signal must follow a null geodesic with \( \phi = \text{const} \). Thus, using Lemma 1 and \( h(r) > 1 \) we find that the signaling time satisfies

\[
T = \int ds \sqrt{\frac{h}{f} \dot{r}^2(s) + \frac{r^2}{f} \dot{\theta}^2(s)} \geq \int ds \sqrt{\dot{r}^2(s) + r^2 \dot{\theta}^2(s)} \geq \sqrt{R_x^2 + R_y^2 - 2R_x R_y \cos \theta}.
\]

(5.2.2)

where the last term corresponds to the shortest travel time \( T^{\text{Mink}} \) for a signal in Minkowski spacetime. Thus, if we identify spheres of the same area as representing the same set of locations in two spherically symmetric static spacetimes, we can in some sense show that Minkowski space is the ‘fastest’ within such class of spacetimes. However, this result has much of the ‘asymptotic’ flavor that we wished to avoid. In particular, the notion of how ‘fast’ the spacetime is has been referred to the observer at infinity.

5.2.1 Establishing a concrete framework

Let us now consider spacetimes which are precisely Schwarzschild outside some sphere of area \( 4\pi R^2 \) and of total mass \( M \). The above theorem implies that the signaling time between two orbits \( x \) and \( y \) on that sphere satisfies \( T \geq 2R \sin \theta/2 \) where \( \theta \) is the angular separation of \( x \) and \( y \). But, in terms of the proper time \( \tau R = T/\sqrt{1 - 2M/R} \) measured by an observer at \( R \), this is

\[
\tau_R \geq 2R (\sin \theta/2) \sqrt{1 - 2M/R} = \tau_R^{\text{Mink}} \sqrt{1 - 2M/R},
\]

(5.2.3)

where \( \sqrt{1 - 2M/R} \leq 1 \). Thus, we see that the perspective of observers on the shell is quite different since (5.2.3) does not necessarily imply that Minkowski space is the ‘fastest’ in the sense associated with the proper time \( \tau \).

It is therefore useful to study the situation in more detail. For the kind of spacetimes under consideration (vacuum outside the sphere at \( R \) and with total mass \( M \)), Birkhoff’s theorem provides a setting in which one feels confident that the sphere at
$R$ is in some sense ‘the same sphere’ regardless of how the interior is filled. With this in mind, we ask the question: what interior solution satisfying the dominant energy condition allows a causal signal to propagate between two given orbits $x$ and $y$ (at $R$) of the time translation Killing field in the smallest amount of time $\tau_R$, and what is this shortest signaling time?

The next section is devoted to explore the above question. We begin below by investigating a number of examples. While we will not succeed in identifying a ‘fastest’ spacetime, we will learn much about the problem, and find some interesting interaction with the energy conditions.

5.3 Some examples of ‘faster’ spacetimes

5.3.1 Preliminaries

As discussed in the section (5.2.1), we would like to find an interior matter distribution that produces a maximum time advance for a light ray emitted by external observers. We explore this question studying three families of spacetimes in detail. All of these families satisfy the dominant energy condition, which is our primary regime of interest. The first one is a Minkowski interior patched to the Schwarzschild exterior via a thin shell. The other two families correspond to various ways of saturating the energy conditions. Although the last two cases will prove to be faster than the first one, perturbative analysis show that there exist other spacetimes which are faster yet.

In each example, we will follow a similar strategy. The quantity we will compute is the time a null signal takes to travel between two points $x$ and $y$. For simplicity, we take the points $x$ and $y$ to lie on the poles of the sphere at $r = R$. The fastest path connecting them must be a null geodesic with $\phi = \text{const}$ so that the integral

$$T = \int ds \sqrt{\frac{h}{f} r^2(s) + \frac{r^2}{f} \theta^2(s)}$$  \hspace{1cm} (5.3.1)
provides the time of flight as measured by the Killing time $T$ normalized at infinity. We will analyze what restrictions the dominant energy condition imposes on the different family parameters, which will also bound (5.3.1). In order to determine whether a configuration provides the shortest signalling time, we will compute the linearized change in the travel time near such configuration, so we start by presenting below the corresponding linearization method.

**Linearization Strategy**

In order to simplify the presentation later, we first describe how one analyzes the extend to which one can increase the time-advance of a given solution by adding a linear perturbation. The first order variation of (5.3.1) under small perturbations of the interior metric (5.1.1) is simply

$$
\delta T = \int \frac{ds}{2\sqrt{f}\sqrt{h r^2 + r^2 \dot{\theta}^2}} \left[ r^2 \delta h - (h r^2 + r^2 \dot{\theta}^2) \frac{\delta f}{f} \right].
$$

(5.3.2)

However, a more useful form is obtained by assuming a regular origin so that boundary conditions imply $\delta m(0) = 0$. The equations of motion (5.1.5)-(5.1.7) then imply three linear differential equations for the variations of $f$, $h$, $\rho$, $P_r$ and $P_\theta$ which can be used to rewrite (5.3.2) in terms of $\delta \rho$ and $\delta P_r$. From (5.1.5), we get that the variation of $h$ is simply

$$
\delta h = 2 \frac{h^2}{r} \delta m = 8\pi \frac{h^2}{r} \int_0^r dr' r'^2 \delta \rho(r').
$$

(5.3.3)

In order to solve for $\delta f$, we use (5.1.5), (5.1.6) and the definition of $m(r)$ to derive

$$
\frac{1}{rh} \partial_r (\ln fh) = 8\pi (\rho + P_r),
$$

(5.3.4)

from which we get

$$
\frac{\delta f}{f} = -\frac{\delta h}{h} + \frac{\delta h(R)}{h(R)} + 8\pi \int_R^r dr' r'[h(\delta \rho + \delta P_r) + (\rho + P_r)\delta h].
$$

(5.3.5)
For completeness, we write the variation of (5.1.7),
\[ \partial_r (\delta P_r) = -\frac{2}{r} (\delta P_r - \delta P_\theta) - \frac{\partial_r f}{2f} (\delta \rho + \delta P_r) - (\rho + P_r) \partial_r \left( \frac{\delta f}{f} \right) \] (5.3.6)

It is now straightforward to substitute (5.3.3) and (5.3.5) in the variation of \( T \), (5.3.2),
\[ \delta T = \int \frac{ds}{2\sqrt{J}} \frac{2}{\sqrt{h r^2 + r^2 \dot{\theta}^2}} \left[ (2h r^2 + r^2 \dot{\theta}^2) \frac{\delta h}{h} - (h r^2 + r^2 \dot{\theta}^2) \left( \frac{\delta h(R)}{h(R)} + \frac{\delta f(R)}{f(R)} \right) + 8\pi (h r^2 + r^2 \dot{\theta}^2) \int_r^R d\tau' \tau' [h(\delta \rho + \delta P_r) + (\rho + P_r) \delta h] \right]. \] (5.3.7)

We note for future reference that the derivation of (5.3.7) uses the inner boundary condition \( \delta m(0) = 0 \) but does not require any outer boundary condition. All the examples we will study obey the equation of state \( \rho + P_r = 0 \) in the relevant region, so that, in order to maintain (5.1.4), perturbations should satisfy \( \delta \rho + \delta P_r \geq 0 \).

The existence of perturbations that generate negative \( \delta T \) will prove that a particular spacetime under study is not the fastest. To find such a perturbation, we consider variations having \( \delta \rho + \delta P_r = 0 \) in order to eliminate the positive contribution of the last term in (5.3.7). In our applications below, we will also have \( \delta f(R) = 0 \) so that the perturbed \( g_{tt} \) is also continuous. Under these assumptions, the expression for \( \delta T \) reduces to
\[ \delta T = \int \frac{ds}{2} \left( \frac{2h r^2 + r^2 \dot{\theta}^2}{\sqrt{f} \sqrt{h r^2 + r^2 \dot{\theta}^2}} \right) \frac{\delta h}{h} - \frac{1}{2} \int ds \frac{h(\delta h(R))}{f(R) h(R)} \]
\[ = \int \frac{ds}{2} \left( \frac{2h r^2 + r^2 \dot{\theta}^2}{\sqrt{f} \sqrt{h r^2 + r^2 \dot{\theta}^2}} \right) \frac{\delta h}{h} - \frac{T}{2} \frac{\delta h(R)}{h(R)}. \] (5.3.8)

A spacetime with identically vanishing \( \delta T \) would be an excellent candidate for the fastest spacetime. While we have not been able to find such a solution consistent with the positive energy condition, the result (5.3.7) is nevertheless quite useful in showing that the simple cases presented in the subsequent sections do not minimize the travel time (5.3.1).
A closer look at Surface stresses

Surface layers provide a way to satisfy the boundary conditions at $r = R$ by any selected interior configuration. However, the same energy conditions we have imposed on the bulk stresses must be satisfied by the surface stresses, e.g., $S_t^t \geq |S_\theta^\theta|$. This condition will, of course, restrict the possible interior metrics. For example, a non-negative surface energy density $S_t^t$ requires $M \geq m(R)$, i.e., the mass contained in the region $r < R$ must be less than or equal to the total mass $M$ of the spacetime.

We have already argued in the proof of Lemma 1 that shorter times of flight favor the equation of state $P_r = -\rho$. For these solutions, (5.1.6) becomes

$$\frac{\partial_r f}{2f} = \frac{m - rm'}{r(r - 2m)}. \tag{5.3.9}$$

Substituting the above equation into (5.1.12) gives:

$$8\pi S_\theta^\theta = \frac{1 - \frac{M}{R}}{R\sqrt{1 - \frac{2M}{R}}} - \frac{1 - \frac{m(R)}{R}}{R\sqrt{1 - \frac{2m(R)}{R}}} + \frac{m'(R)}{R\sqrt{1 - \frac{2m(R)}{R}}}. \tag{5.3.10}$$

Since the sum of the first two terms is positive for $M \geq m(R)$, and $m' \geq 0$, we find that, for configurations with $P_r = -\rho$, the shell angular pressure $S_\theta^\theta$ is also non-negative. Hence, we only need to impose $S_t^t \geq S_\theta^\theta$, which can be rewritten as

$$\sqrt{\beta\left(\frac{m(R)}{R}\right)} - \sqrt{\beta\left(\frac{M}{R}\right)} \geq \frac{m'(R)}{\sqrt{1 - \frac{2m(R)}{R}}}. \tag{5.3.11}$$

where $\beta\left(\frac{M}{R}\right) = (3 - 5\frac{M}{R})^2/(1 - 2\frac{M}{R})$. The form of $\beta$ restricts the ratio of the asymptotic parameters, $M/R$, to lie in the range $0 \leq \frac{M}{R} \leq \frac{12}{25}$, since only for those values the left hand side of (5.3.11) can be non-negative: for $\frac{12}{25} \leq \frac{M}{R} \leq \frac{1}{2}$, $\beta$ is a monotonically increasing function bigger than $\beta(0) = \beta\left(\frac{12}{25}\right)$ so that (5.3.11) can not be satisfied for $M \geq m(R)$. 

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We now turn our attention to the effect the perturbations of the previous section have on the surface stress-energy. From (5.1.12), it is easy to derive

\[
8\pi \delta S^t_t = -\frac{\delta h}{Rh^{3/2}}
\]
\[
8\pi \delta S^\theta_\theta = \frac{\delta h}{2h^{3/2}} \left( \frac{f'}{2f} + \frac{1}{R} \right) - \frac{1}{h^{1/2}} \frac{\delta f'}{2f}.
\] (5.3.12)

In the examples that follow, it will be enough to satisfy \( \delta S^t_t - \delta S^\theta_\theta > 0 \) in order to preserve the energy conditions on the shell. Using equation (5.1.6) and its linearized version

\[
\delta \left( \frac{f'}{2f} \right) = 1 + 8\pi r^2 P_r \delta m + \frac{4\pi r^2}{r - 2m} \delta P_r,
\] (5.3.13)

together with (5.3.3), we find

\[
8\pi (\delta S^t_t - \delta S^\theta_\theta) = \frac{1}{\sqrt{1 - \frac{2m}{R}}} \left[ \left( \frac{m + 4\pi R^3 P_r}{R^2(R - 2m)} \right) \delta m - \frac{2}{R^2} \delta m + 4\pi R \delta P_r \right].
\] (5.3.14)

As mentioned before, we will only consider configurations for which \( P_r = -\rho = -\frac{m'}{4\pi r^2} \) and variations having \( \delta P_r = -\delta \rho = -\frac{\delta m'}{4\pi r^2} \). Substituting these relations in (5.3.14) we obtained

\[
8\pi (\delta S^t_t - \delta S^\theta_\theta) = \frac{1}{R^2 \sqrt{1 - \frac{2m}{R}}} \left[ \left( \frac{m - m'R}{R - 2m} - 2 \right) \delta m - R \delta m' \right].
\] (5.3.15)

where the variation \( \delta m \) and \( \delta m' \) have to me such that the above expression is positive.

### 5.3.2 The empty shell

Let us begin with the simplest allowed spacetime: a flat region inside the sphere of radius \( R \), and a thin shell of matter at \( r = R \) which generates a Schwarzschild exterior. The metric of the empty interior \( r < R \) is

\[
ds^2 = -(1 - \frac{2M}{R})dt^2 + dr^2 + r^2 d\Omega^2,
\] (5.3.16)
i.e., (5.1.1) with \( f = 1 - 2M/R \) and \( h = 1 \). Here we have chosen the normalization of \( t \) so that \( g_{tt} \) is continuous at \( r = R \) as required by the Israel junction conditions.

From (5.3.16), we see that the surface stress-energy of the shell is

\[
8\pi S_t^t = \frac{-2}{R} \left( \sqrt{1 - \frac{2M}{R}} - 1 \right), \quad 8\pi S_\theta^\theta = \frac{1}{R} \left( \frac{1 - \frac{M}{R}}{\sqrt{1 - 2\frac{M}{R}}} - 1 \right).
\] (5.3.17)

From the previous section, we recall that the dominant energy condition \( S_t^t \geq |S_\theta^\theta| \) imposes \( 0 \leq \frac{M}{R} \leq \frac{12}{25} \).

Since the interior is just Minkowski space, it is clear that the fastest trajectory from one pole of the sphere to the other follows a radial path with \( \dot{\theta} = 0 \). The travel time (5.3.1) for this path is

\[
T_{\text{Empty}} = 2\int_0^R \frac{dr}{\sqrt{1 - \frac{2M}{R}}} = \frac{2R}{\sqrt{1 - \frac{2M}{R}}}. \] (5.3.18)

In terms of the proper time \( \tau_r \) measured by a static observer at \( r = R \) this is just \( \tau_{\text{Empty}}^r = 2R \), which, as could have been anticipated, coincides with the proper time that would be assigned by the analogous observer to the analogous signal in pure Minkowski space.

Although this spacetime is a natural one to study, it is not the fastest. This may be seen by considering the variation (5.3.8) of \( \delta T \) under a perturbation \( \delta \rho (r) = -\delta P_r (r) = -\delta P_\theta (r) = \delta \rho (0) > 0 \), so that \( \delta h (r) = \frac{8\pi}{3} \delta \rho (0) r^2 \). Since the signal takes no time to cross the shell, it is sufficient to apply (5.3.8) at some \( r \) just a bit less than \( R \), so we do not need the explicit form of the perturbation at the shell. The perturbed spacetime clearly satisfies the energy conditions in the interior and, since the original shell at \( r = R \) does not saturate these conditions\(^3\), there is no danger that they will be violated at \( r = R \) for small \( \delta \rho \). For this perturbation one finds \( \delta T = \frac{-8\pi \delta \rho (0) R^3}{9\sqrt{1 - \frac{2M}{R}}} < 0 \) for a radial trajectory, so that the empty shell spacetime is not the fastest.

\(^3\)The case with \( \frac{M}{R} = \frac{12}{25} \) does in fact satisfy \( S_t^t \geq |S_\theta^\theta| \) and requires more care. It may be treated as in section 5.3.3 below.
5.3.3 De Sitter space in a bottle

Since the only constraints in our problem are the energy conditions, one might expect these conditions to be saturated by our hypothetical fastest spacetime. The weak energy condition is saturated by taking \( \rho = -P_\theta = -P_r > 0 \), in which case stress-energy conservation (5.1.7) requires \( \rho(r) \) to be just some constant \( \rho_0 \). In the previous subsection we already found that the travel time is reduced by perturbing our empty shell spacetime in this direction.

Thus, we are now motivated to study a spacetime with a de Sitter interior, i.e., \( T^\mu_\nu = \rho_0 \times \text{diag}(1, -1, -1, -1) \) for \( r < R \). Again, in order to satisfy stress-energy conservation, we need to add a shell at \( r = R \). This shell effectively constitutes a ‘bottle’ whose stresses and gravitational self-attraction keeps the piece of de Sitter space with \( r < R \) from expanding.

The metric for our “De Sitter space in a bottle” takes the form

\[
ds^2 = -(1 - \frac{2M}{R}) \frac{1 - b^2 r^2}{1 - b^2} dt^2 + \frac{1}{1 - b^2} \frac{r^2}{R^2} dr^2 + r^2 d\Omega^2,
\]

(5.3.19)

where \( b^2 = \frac{8\pi}{3} \rho_0 R^2 < 1 \). Here, \( t \) has again been normalized in the interior so that \( g_{tt} \) is continuous across \( r = R \). Comparing (5.3.19) and (5.1.12), one finds the surface stresses to be

\[
8\pi S_t^t = \frac{-2}{R} \left( \sqrt{1 - \frac{2M}{R}} - \sqrt{1 - b^2} \right), \quad 8\pi S_\theta^\theta = \frac{1}{R} \left( \frac{1 - \frac{M}{R}}{\sqrt{1 - \frac{2M}{R}}} - \frac{1 - 2b^2}{\sqrt{1 - b^2}} \right)
\]

(5.3.20)

Both stresses are non-negative for \( m(R) = \frac{R}{2} b^2 \leq M \). Now, the condition \( S_t^t \geq S_\theta^\theta \) restricts the values of \( b \) to be

\[
b^2 \leq b^2_{\text{max}}(\frac{M}{R}) = \frac{3}{4} - \frac{\beta(\frac{M}{R})}{32} - \frac{\beta(\frac{M}{R})}{32} \sqrt{1 + \frac{16}{\beta(\frac{M}{R})}},
\]

(5.3.21)

where \( \beta(\frac{M}{R}) \) was defined below (5.3.11).
As for the empty shell spacetime, the geodesics connecting antipodal orbits \( x \) and \( y \) can only be radially directed\(^4\). Thus, we again have \( \dot{\theta} = 0 \) and the travel time is

\[
\tau_R = T(b) \sqrt{1 - \frac{2M}{R}} = 2 \int_0^R dr \frac{\sqrt{1 - b^2}}{1 - b^2 r^2} = \tau_{\text{Empty}}^R \sqrt{1 - \frac{b^2}{2b}} \ln \frac{1 + b}{1 - b}.
\]

(5.3.22)

The factor multiplying \( \tau_{\text{Empty}}^R = 2R \) is less than 1 for the allowed range of values of \( b \), as shown in [plot \( T(b) \) vs \( b \)]. Therefore, this space is faster than the empty shell spacetime. As (5.3.22) is a monotonically decreasing function of \( b > 0 \), the smallest allowed time (for fixed \( \frac{M}{R} \)) occurs when (5.3.21) is saturated, *i.e.*, when \( S_t^1 = |S_\theta^0| \). In particular, the largest effect occurs for \( \frac{M}{R} = \frac{2}{5} \), when (5.3.21) gives the biggest allowed \( b \), \( b = \frac{19}{32} \left( 1 - \sqrt{\frac{105}{361}} \right) \), and, therefore, the smallest value of (5.3.22), \( \frac{\tau_R}{\tau_{\text{Empty}}^R} \approx 0.987 \).

However, a perturbation analysis again shows that spacetimes outside this class are faster yet. Again we apply (5.3.8) to the region inside the shell. Let us denote the perturbed quantities with tildes. A perturbation \( \delta \rho(r) = -\delta P_r(r) = -\delta P_\theta(r) = \delta \rho(0) > 0 \), corresponding to \( \dot{b} = b_{\text{max}} + \delta b > b_{\text{max}} \) would reduce the time by the amount \( \delta \tau_0 = T(\dot{b}) - T(b_{\text{max}}) < 0 \), but, it would also violate the energy condition \( S_t^1 \geq |S_\theta^0| \). In order to respect the dominant energy condition at the shell, we instead use a sequence of perturbations \( \{\delta_n \rho\} \) of the form

\[
\delta_n \rho(r) = -\delta_n P_r(r) = A_n (r - r_n) + \delta \rho(0) \quad \text{for} \quad R > r > r_n,
\]

\[
\delta_n \rho(r) = -\delta_n P_r(r) = \delta \rho(0) \quad \text{for} \quad r < r_n.
\]

(5.3.23)

which differ from \( \delta \rho(0) \) in the region \( R > r > r_n \). Our goal will be to satisfy the energy conditions for large enough \( n \). Here \( A_n \) are constants and linearized stress-energy conservation (5.3.6) together with the energy condition in the interior requires

\(^4\)Note that a non-radial geodesic would lead to an \( S^1 \) of such geodesics, and thus to a light cone with a caustic at finite affine parameter. As it is readily seen from the conformal diagram (see, *e.g.*, [75]), this does not occur in de Sitter space.
$A_n = -\frac{2}{r}(\delta_n \rho + \delta_n P_\theta) < 0$. Similarly, at the shell the dominant energy condition requires

$$\tilde{S}_t^t - \tilde{S}_\theta^\theta = \delta_n S_t^t - \delta_n S_\theta^\theta = \frac{1}{R^2 \sqrt{1-b_{\text{max}}^2}} \left[ -\frac{2}{1-b_{\text{max}}^2} \delta_n m(R) - R\delta_n^\prime m(R) \right] > 0.$$

(5.3.24)

Let us choose $r_n$ to converge to $R$ and also require each $\delta_n \rho = \delta m(R)$ to yield the same value $\delta m(R)$ for the change in the mass function $m(r)$ evaluated just inside the shell. Note that a fixed value of $\delta m(R) = \frac{8\pi R^2}{3} \delta \rho(0)$ for large $n$, is readily achieved by taking the negative constants $A_n$ to scale with $(R - r_n)^{-2}$. In this case, the second term of $\tilde{S}_t^t - \tilde{S}_\theta^\theta$ is positive and scales with $(R - r_n)^{-1}$, rendering (5.3.24) satisfied for sufficiently large $n$.

It is clear that at each point $r$ in the interior $\delta_n \rho(r)$ converges to $\delta \rho(0)$. Thus, it makes sense to express the variation $\delta_n T$ of $T$ under $\delta_n \rho$ in terms of the variation $\delta_0 T$ obtained by the constant density perturbation associated with simply shifting $b$.

From (5.3.8) we find in the limit

$$\delta_n T \to \delta_0 T + \frac{\delta_0 m(R) - \delta m(R)}{R(1-b^2)} T(b)$$

$$= \delta_0 T + \frac{\delta_0 m(R) - \delta m(R)}{b\sqrt{1-b^2}} \frac{1}{\sqrt{1-2M/R}} \ln \frac{1+b}{1-b},$$

(5.3.26)

where $\delta_0 m(R)$ refers to the change in the mass $m(r)$ evaluated just inside the shell under the constant density perturbation associated with changing the density uniformly by $\delta \rho(0)$.

Since $A_n$ is negative, the second term in (5.3.26) is positive. However, it is clear from the construction of $\delta_n \rho$ that we are free to take $\delta m(R)$ as close as desired to $\delta_0 m(R)$ without changing $\delta b = \frac{8\pi R^2}{3} \delta \rho(0)$. As a result, this second (positive) term can be made negligible in comparison with the first (negative) term. We have therefore
established the existence of small perturbations which preserve the positive energy conditions but reduce the travel time below that of the background “dS in a bottle” spacetime. A similar analysis applies to the empty shell spacetime in the extreme case \( M/R = 12/25 \).

Because we imposed the dominant energy condition, spacetimes in this class were restricted to be much slower than would be guaranteed by Theorem 1. In contrast, note that we can do much better if we enforce only the weak energy condition. This will require \( S^t_t \geq 0 \) and thus \( b^2 \leq 2M/R \), but this is the only requirement. Denoting the bound set by Theorem 1 by \( \tau^\text{bound}_R \) and comparing with (5.3.22) for \( b^2 = 2M/R \), one finds

\[
\frac{\tau_R}{\tau^\text{bound}_R} = \frac{1}{2b} \ln \frac{1 + b}{1 - b}. \tag{5.3.27}
\]

So, for \( b^2 = 2M/R \sim 1 \), we find \( \tau_R \gg \tau^\text{bound}_R \). Nevertheless, \( \tau_R \to 0 \) so that \( \tau_R \ll \tau^\text{Empty}_R \). Thus, this example suggests that the dominant energy condition may be significantly more restrictive that the weak energy condition in searching for fast spacetimes in the context outlined in section 5.2.1.

### 5.3.4 Saturating the dominant energy condition

We now turn to our third example. We saw in the proof of Lemma 1 that it was advantageous to set \( \rho = -P_r \) and take \( \rho \) as large as possible. The same is true with our current boundary conditions. However, stress-energy conservation places bounds on how rapidly \( P_r \) may change. In particular, we can rewrite (5.1.7) as

\[
\partial_r P_r = -\frac{\partial_r f}{2f} (\rho + P_r) + \frac{2}{r} (\rho - P_r) - \frac{2}{r} (\rho - P_\theta). \tag{5.3.28}
\]

Maintaining \( P_r = -\rho \) with a rapidly decreasing \( \rho(r) \) may force \( P_\theta \) to be very large and perhaps to violate the dominant energy condition \( \rho - P_\theta = \frac{\epsilon}{2} \partial_r \rho + 2\rho > 0 \). In
fact, if one has already imposed $P_r = -\rho$, taking $P_\theta = \rho$ allows $P_r$ to decrease at the fastest possible rate as one moves away from the boundary $r = R$.

As a result, we are motivated to consider spacetimes with $\rho = -P_r = P_\theta$. Stress-energy conservation (5.3.28) then requires

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^4,$$  \hspace{1cm} (5.3.29)$$
for constants $\rho_0 \geq 0$ and $0 < r_0 < R$. To evade the divergence at $r = 0$, we excise the region $r < r_0$ and sew in a piece of another spacetime. For lack of a more inspired choice, we once again use a piece of de Sitter space. We demand that $\rho$ is continuous at $r_0$ so that $m$ is $C^1$ and there is no additional shell of mass at this junction.

The mass function is

$$m = \begin{cases} 4\pi \rho_0 r_0^3 & \text{for } 0 \leq r \leq r_0 \\ -4\pi \rho_0 r_0^3 \left(\frac{1}{r^2}\right) + \frac{16\pi}{3} \rho_0 r_0^3 = -\frac{\xi}{r} + a & \text{for } r_0 \leq r < R. \end{cases}$$  \hspace{1cm} (5.3.30)$$
where

$$r_0 = \frac{4c}{3a} \quad \text{and} \quad \frac{4\pi}{3} \rho_0 = \left(\frac{3}{c}\right)^3 \left(\frac{a}{4}\right)^4.$$  \hspace{1cm} (5.3.31)$$
We refer to this case as the “dS/DEC” spacetime due to the saturation of the dominant energy condition for $r > r_0$ and the presence of the de Sitter region for $r < r_0$.

Let us introduce the dimensionless variables

$$\hat{t} = \frac{2t}{T_{Empty}} = \frac{t}{R} \sqrt{1 - \frac{2M}{R}}, \quad \hat{r} = \frac{r}{R}, \quad \hat{a} = \frac{a}{R}, \quad \hat{c} = \frac{c}{R^2}, \quad \hat{m}(\hat{r}) = \frac{m(r)}{R},$$

in terms of which the metric takes the form

$$ds^2 = R^2 \left[-\frac{1 - \frac{2\hat{m}(\hat{r})}{\hat{r}}}{1 - 2\hat{a} + 2\hat{c}} d\hat{t}^2 + \frac{1}{1 - \frac{2\hat{m}(\hat{r})}{\hat{r}}} d\hat{r}^2 + \hat{r}^2 d\Omega^2\right].$$  \hspace{1cm} (5.3.32)$$
Note that for $\hat{c} = \frac{3\hat{a}}{4}$, the de Sitter region fills all the interior $r < r_0 = R$. As a result, we require $\hat{c} \leq \frac{3\hat{a}}{4}$. The value $\hat{r}_{BH} = \hat{a} + \sqrt{\hat{a}^2 - 2\hat{c}}$, which is real for $\hat{c} \leq \frac{\hat{a}^2}{2}$, would correspond to the location of a Killing horizon, i.e.,

$$g_{tt}(\hat{r}_{BH}) \propto 1 - \frac{2\hat{a}}{\hat{r}_{BH}} + \frac{\hat{r}_{BH}^2}{\hat{r}_{BH}^2} = 0.$$
But note that $\hat{c} \leq \frac{\hat{a}^2}{2}$ yields $\hat{r}_0 \leq \frac{2\hat{a}}{3} < \hat{a} < \hat{r}_{BH}$. Thus, to avoid the existence of a horizon\(^5\), we must have $\hat{c} > \frac{\hat{a}^2}{2}$.

We also investigate any further restriction imposed by requiring the shell to satisfy the dominant energy condition. Again using (5.1.12), the relevant stresses are

\[
8\pi S^t_t = -\frac{2}{R} \left( \sqrt{1 - \frac{2M}{R}} - \sqrt{1 - 2\hat{a} + 2\hat{c}} \right),
\]

\[
8\pi S^\theta_\theta = \frac{1}{R} \left( \frac{1 - \frac{M}{R}}{\sqrt{1 - \frac{2M}{R}}} - \frac{1 - \hat{a}}{\sqrt{1 - 2\hat{a} + 2\hat{c}}} \right).
\]

The condition $S^t_t \geq |S^\theta_\theta|$ also constraints the values of $(\hat{a}, \hat{c})$ through

\[
\hat{c} \geq \frac{3}{4} + \frac{5}{4} \hat{a} + \frac{1}{16} \beta\left(\frac{M}{R}\right) + \frac{1}{16} \beta\left(\frac{M}{R}\right) \sqrt{1 + \frac{8(\hat{a} - 1)}{\beta\left(\frac{M}{R}\right)}},
\]

where $\beta\left(\frac{M}{R}\right)$ is again as in (5.3.11) and $0 \leq \frac{M}{R} \leq \frac{12}{25}$. A plot of the allowed regions in the $\hat{a}\hat{c}$ plane for three different values of $\frac{M}{R}$ is shown in figure 1. Curves of the form (5.3.35) move to the right in the $\hat{a}\hat{c}$ plane for increasing $\frac{M}{R} \leq \frac{2}{5}$, and back to the left for $\frac{M}{R} > \frac{2}{5}$.

For a radial trajectory, we can explicitly write down the expression for the time of flight

\[
\hat{T}(\hat{a}, \hat{c}) = \frac{T(\hat{a}, \hat{c})}{T_{Empty}}
\]

\[
= \sqrt{1 - 2\hat{a} + 2\hat{c}} \left[ \frac{4}{\hat{a}} \left( \frac{2\hat{c}}{\hat{a}} \right)^{\frac{3}{2}} \ln \left( \frac{1 + \sqrt{\frac{3\hat{a}^2}{8\hat{c}}}}{1 - \sqrt{\frac{3\hat{a}^2}{8\hat{c}}}} \right) + 1 - \frac{4\hat{c}}{3\hat{a}} + \hat{a} \ln \frac{1 - 2\hat{a} + 2\hat{c}}{\left(\frac{4\hat{c}}{3\hat{a}}\right)^2(1 - \frac{3\hat{a}^2}{8\hat{c}})} - \frac{\sqrt{2\hat{c}(1 - \frac{\hat{a}^2}{\hat{c}})}}{\sqrt{1 - \frac{\hat{a}^2}{\hat{c}}}} \left( \sqrt{2\hat{c}} \sqrt{1 - \frac{\hat{a}^2}{\hat{c}}} - \sqrt{\frac{4\hat{c}}{3\hat{a}}} \left(1 - \frac{3\hat{a}^2}{4\hat{c}}\right) \right) \right].
\]

\(^5\)One could consider spacetimes with a black hole instead of a dS interior, but then there are no radial null geodesics connecting antipodal points on the sphere. We explored the behavior of selected non-radial null geodesics numerically in such a spacetime but in each case found $T > T_{Empty}$. For this reason we chose to concentrate on radial geodesics and on spacetimes that allow them.
Figure 5.1: The allowed region in the \( \hat{a} \hat{c} \) plane for configurations (5.3.32) is given by \( \frac{\hat{a}^2}{2} < \hat{c} \leq \frac{3\hat{a}}{4} \) and condition (5.3.35). The case \( \hat{c} = \frac{3\hat{a}}{4} \) represents the “dS in a bottle” spacetime of section 5.3.3. The dash-dotted line is obtained by setting \( \frac{M}{R} = \frac{2}{5} \) in equation (5.3.35). For other values of \( \frac{M}{R} \), the allowed region becomes smaller, as shown by the dashed line which represents condition (5.3.35) for both \( \frac{M}{R} = \frac{1}{5} < \frac{2}{5} \) and \( \frac{M}{R} = \frac{7}{15} > \frac{2}{5} \). The thin dotted line indicates the points \( (\hat{c}_{\text{min}}, \hat{a}_{\text{min}}) \) where the time (5.3.36) attains its minimum values in the allowed regions for each \( \frac{M}{R} \).
Figure 5.2: The minimum time of flight (5.3.36) in “dS/DEC” configurations as a function of $\left( \frac{M}{R} \right)$ is represented by the lower curve. For comparison, the upper curve shows the minimum time of flight in the (slower) “dS in a bottle” configurations as a function of $\left( \frac{M}{R} \right)$.

This is complicated to study analytically. We have therefore used a simple C++ program to compute the minimum value of $\hat{T}$ for each $\frac{M}{R}$. The results are plotted in figure 2 and show a minimum at $M/R = 2/5$ at a value of approximately 0.939.

Note that $\hat{T}_{\text{min}}(\frac{M}{R})$ decreases monotonically for $0 < \frac{M}{R} < \frac{2}{5}$. Since, in this interval, the allowed region of parameters $(\hat{a}, \hat{c})$ grows monotonically with $\frac{M}{R}$, the minimum of $\hat{T}$ for each $\frac{M}{R}$ must be attained on the boundary of the allowed region. This means that the minimum occurs where $S_t^c \geq |S^\theta_\theta|$ is saturated. A similar analysis applies for $\frac{2}{5} < \frac{M}{R} < \frac{12}{25}$.

Note that the uppermost curve ($\hat{c} = \frac{3a}{4}$) in figure 1 represents the “dS in a bottle” spacetimes. Since it does not cross the middle curve showing the fastest “dS/DEC” spacetimes, we see that “dS in a bottle” is never the fastest case and we have indeed improved upon the results of section 5.3.3.

Perturbing around configurations $(c_{\text{min}}, a_{\text{min}})$ once again shows that the signaling time for this family of spacetimes can be reduced by perturbations outside of the family. Let us begin an observation: we have already shown that the time of flight
would decrease if we were allowed to move farther to the right in figure 1 for the same $M, R$. This corresponds to a perturbation $\delta_0 \rho$ satisfying $\delta_0 \rho + \delta_0 P_r = 0$ in the interior and preserving the dominant energy condition in the interior. However, it leads to a violation of the dominant energy condition at the shell. We therefore follow the strategy used in section 5.3.3 of adapting this initial guess (which we call $\delta_0 \rho, \delta_0 m, \delta_0 T$) to form a sequence of perturbations $(\delta_n \rho, \delta_n m, \delta_n T)$ which preserve the dominant energy condition at the shell for large enough $n$.

This condition requires:

$$\ddot{S}_t - \ddot{S}_\theta = \delta S_t - \delta S_\theta = \frac{1}{R^2 \sqrt{1 - 2a + 2c}} \left[ -R \delta m'(R) - \frac{2 - 5a + 6c}{1 - 2a + 2c} \delta m(R) \right] > 0.$$  \hfill (5.3.37)

Each perturbation $\delta_n \rho$ will be associated with a radius $r_n$ such that $\delta_n \rho = \delta_0 \rho > 0$ for $r < r_n$. We take the $r_n$ to increase with $n$ and to converge to $R$. Choose some $r_1$ and let $\delta_1 \rho$ be any such smooth perturbation which decreases for $r_1 < r < R$. Such a $\delta_1 \rho$ will respect the dominant energy condition in the interior. For later use, we also require that the induced change $\delta_1 m(R)$ in the mass function just below the shell satisfy $\delta_1 m(R) < \delta_0 m(R)$.

We now take $\delta_n \rho$ to induce the same change in the mass just inside the shell for all $n$: $\delta_n m(R) = \delta_1 m(R)$. Therefore, the sequence $\{\delta_n \rho\}$ has the property that $\delta_n \rho = \frac{\delta m}{4\pi r^n}$, which are decreasing functions for $r_n < r < R$, become large and negative at $r = R$ when $n$ becomes large and $r_n \to R$. Then (5.3.37) is clearly satisfied for large $n$.

Since on the other hand $\delta_n \rho(r) \to \delta_0 \rho(r)$ for $r < R$, we find

$$\delta_n T \to \delta_0 T + \left[ \delta_0 m(R) - \delta_1 m(R) \right] \frac{T}{R - 2a + 2c/R}. \hfill (5.3.38)$$

As in section 5.3.3, the first term is negative by construction, and the second term can be chosen to be arbitrarily small. Thus, we have demonstrated the existence of perturbations of the “dS/DEC” spacetime preserving the dominant energy condition
and further reducing the signaling time between antipodal points.

5.4 Discussion

In this work we have investigated the possibility of fast travel in static spherically symmetric spacetimes. We derived a simple theorem to the effect that, when the signaling time is measured by an observer at infinity, a signal propagating through a spacetime satisfying the (timelike) weak energy condition never arrives at its destination sooner than would a corresponding signal in Minkowski space. Spherical symmetry and the static Killing field were essential in identifying a corresponding signal in Minkowski space.

However, we were not satisfied with this result and wished to investigate related questions concerning more local notions of signaling time. For example, it is of interest whether the observers who send and receive the signals find the propagation time to be less or greater than what they would expect based on their Minkowski space intuition. The theorem of section 5.2 does place a lower bound on this signaling time, but it is a bound that is arbitrarily small compared to the naive Minkowski signaling time\(^6\) when the signal propagates near the horizon of a black hole. We also wished to explore the consequences of requiring stronger energy conditions to hold.

For this reason we investigated several families of spacetimes in detail. We were most interested in cases where the dominant energy condition holds. With this restriction, we found that we could indeed construct positive energy spacetimes that improve upon the naive Minkowski time of \(2R\), but only by factors of order one. Our fastest such spacetime improves this result by approximately 6%, Perturbative analysis tell us that spacetimes exist which are faster yet, but of course give us no idea

\(^6\)i.e., a proper time of \(2R\) for a light ray to propagate across a sphere of area \(4\pi R^2\).
of how much faster they might be. There thus remains a sizable gap\(^7\) between the fastest spacetime known to us and the bound we have derived. Discovering how this gap may be closed remains an open issue for future research.

Perhaps the most interesting suggestion from our investigation is that imposing only the weak energy condition may allow much faster spacetimes. In particular, we found in section 5.3.3 that we could construct spacetimes satisfying the weak energy condition which allowed signaling across our sphere in a proper time significantly faster than \(2R\). For \(2M/R \sim 1\) we found that while our signaling time was much larger than the bound of Theorem 1, it could be made arbitrarily short compared to the naive Minkowski bound.

Most of the work to date has considered the (null) weak energy condition because it leads to powerful analysis techniques based on the Raychaudhuri equation and focussing theorems. In our case, we saw that the weak energy condition led directly to our lemma and our theorem in section 5.2. One would expect that both of these results to generalize beyond the spherically symmetric context and to again require only the weak energy condition for their proof.

On the other hand, realistic spacetimes should also satisfy the dominant energy condition\(^8\). Thus, our examples suggest that they should be subject to significantly stronger constraints. If this is indeed the case, new analysis tools more appropriate to the dominant energy condition will need to be constructed before one can conclusively identify the fastest DEC spacetime and the fastest allowed signaling time. We leave this task for future work.

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\(^7\)When \(2M/R \sim 1\). On the other hand, for \(2M/R \ll 1\) the bound is of course close to the naive Minkowski estimate: \(\tau_{\text{bound}} = 2R(1 + O(M/R))\).

\(^8\)Unless one allows a negative cosmological constant.
Chapter 6

Closing remarks

We have worked on three concrete questions in the context of string theory. Each of them are contributions to specific lines of research within the physics of branes. We have looked at superstrings, which represent an interesting class of solutions where some of the worldvolume fields are free to take arbitrary values. As a consequence, superstrings provide a description for a large number of 1/4 BPS microstates of string theory. The study of D-brane polarization has also received much of attention in the literature. The work of chapter 3 illustrates that quantum deformations of the classical ground state of a D-branes may appear as a result of polarization effects. We also contributed to the subject of brane instability by observing that a purely spinning black string must be unstable to transfer part of its spin angular momentum into gyrations.

Working on the topic of ‘fast travel’ in General Relativity has been fun and interesting. The question of whether we may in principle alter the spacetime geometry to take a form that is convenient for communication needs or simply to explore other corners of the universe is a fascinating one. Our investigation revealed that, even within the restrictions of positive energy, a spherically symmetric region could be filled up with a medium that would, in a sense measured by local observer, allow
light to cross that region faster than if it were empty.
Appendix A

Quadratic Expansions

This appendix simply lists the formulae, suppressed in chapter 1, which describe 1) the quadratic expansions of the action for the $z$-independent fields for the D2-action expanded about the round supertube and 2) the charges in terms of the perturbations $\eta^\mu, a_i$.

The action for the $z$-independent modes is

$$S = S_{\text{round}} + S^{(2)} + \text{higher order terms} \quad \text{(A.0.1)}$$

$$S^{(2)} = -\text{sgn}(B) \frac{L_z T_{D2}}{2} \int dt d\sigma \left[ -\frac{R^2 + B^2}{B} \left( (\partial_t r)^2 + |\partial_t \eta|^2 \right) \
- 2\text{sgn}(E) (\partial_t r \partial_\sigma r + \partial_\sigma \eta^i \partial_t \eta^i) \right] - \text{sgn}(E) \frac{2R}{B} (\partial_t a_z r - \partial_\sigma r a_z) \
- \frac{R^2 (R^2 + B_0^2)}{B^3} (\partial_t a_z)^2 - \frac{1}{B} (\partial_t a_\sigma)^2 - \text{sgn}(E) \frac{2R^2}{B^2} \partial_t a_z \partial_\sigma a_z \right]. \quad \text{(A.0.2)}$$

From this action we compute the canonical momenta conjugate to the set fields.
\((\eta, r, a_z, a_\sigma)\),

\[
\mathcal{P}_r = \text{sgn}(B) L_z T D_2 \left( \frac{R^2 + B^2}{B} \partial_t r + \text{sgn}(E) \partial_\sigma r - \frac{R}{B} a_z \right)
\]  
(A.0.3)

\[
\pi_z = \text{sgn}(B) L_z T D_2 \left( \frac{R^2 (R^2 + B^2)}{B^3} \partial_t a_z + \text{sgn}(E) \left( \frac{R^2}{B^2} \partial_\sigma a_z + \frac{R}{B} r \right) \right)
\]  
(A.0.4)

\[
\mathcal{P}_i = \text{sgn}(B) L_z T D_2 \left( \frac{R^2 + B^2}{B} \partial_t \eta_i + \text{sgn}(E) \partial_\sigma \eta_i \right)
\]  
(A.0.5)

\[
\pi_\sigma = \text{sgn}(B) L_z T D_2 \left( \frac{1}{B} \partial_t a_\sigma \right)
\]  
(A.0.6)

In computing (A.0.2) from (2.1.1) we have performed an integration by parts, which induces a canonical transformation designed to make the momenta (A.0.3) take a more symmetric form. As a result, the canonical momentum \(\pi_z\), defined by the action (A.0.2) conjugate to the connection differs by linear terms from the \(\Pi_z\) (2.1.5), defined by (2.1.1). Thus, while the electric charge \(Q_{F1}\) remains the integral of \(\Pi_z\), it is not the integral of \(\pi_z\).

The charges and Hamiltonian take the form

\[
Q_{D0} = Q_{D0}^{\text{round}},
\]  
(A.0.7)

\[
Q_{F1} = \frac{1}{T_{F1}} \int d\sigma \Pi_z
\]

\[
= Q_{F1}^{\text{round}} + \text{sgn}(EB) \frac{T_{D2}}{2T_{F1}} \int d\sigma \left[ \frac{4R}{B} r + 2\text{sgn}(E) \frac{R^2 (R^2 + B^2)}{B^3} e_z \right.
\]

\[
+ \frac{R^2 (R^2 + B^2)}{B^3} (\partial_t r^2 + |\partial_t \eta|^2)
\]

\[
- \frac{R^4}{B^3} (|\partial_t r|^2 + |\partial_t \eta|^2)
\]

\[
+ \frac{2}{B} (r^2 + |\partial_t r|^2 + |\partial_t \eta|^2) + \text{sgn}(E) \frac{2(R^2 + B^2)}{B^2} (\partial_t r \partial_\sigma r + \partial_t \eta^i \partial_\sigma \eta^i)
\]

\[
+ \frac{3R^4 (R^2 + B^2)}{B^5} e_z^2 + \frac{2R^2}{B^3} b^2 + \frac{R^2}{B^3} e_\sigma^2
\]

\[
- \text{sgn}(E) \frac{2R^2 (3R^2 + B^2)}{B^4} e_z b - \frac{4R}{B^2} r b
\]

\[
+ \text{sgn}(E) \frac{4R (2R^2 + B^2)}{B^3} r e_z
\],

(A.0.8)
\[ J = J^{\text{round}} + \text{sgn}(B) \frac{T_{D2}}{2} \int dz d\sigma \left[ \text{sgn}(E)4Rr + 2 \frac{R^2(R^2 + B^2)}{B^2} e_z \right. \\
+ \text{sgn}(E)2r^2 + \text{sgn}(E) \frac{R^2(R^2 + B^2)}{B^2} ((\partial_r r)^2 + |\partial_\eta|^2) \\
- \frac{R^4}{B^2} ((\partial_r r)^2 + |\partial_\eta|^2) + \text{sgn}(E) \frac{3R^4(R^2 + B^2)}{B^4} e_z^2 \\
+ \frac{R^4}{B^2} e_\sigma^2 - \frac{4R^4}{B^3} e^b_z + \frac{4R(2R^2 + B^2)}{B^2} r e_z], \tag{A.0.9} \]

\[ P^0 = P_0^{\text{round}} + \frac{\text{sgn}(B)}{2} T_{D2} \int dz d\sigma \left[ \frac{4R}{B} r + 2 \text{sgn}(E) \frac{R^2(R^2 + B^2)}{B^3} e_z \\
+ \frac{(R^2 + B^2)^2}{B^3} ((\partial_r r)^2 + |\partial_\eta|^2) - \frac{R^2(R^2 - B^2)}{B^3} ((\partial_z r)^2 + |\partial_\eta|^2) \\
+ \frac{2}{B} (r^2 + (\partial_r r)^2 + |\partial_\eta|^2) + \text{sgn}(E) \frac{2(R^2 + B^2)}{B^2} (\partial_r r \partial_\sigma r + \partial_\sigma r \partial_\eta i \partial_\eta i) \\
+ \frac{R^2(3R^2 + B^2)(R^2 + B^2)}{B^5} e_z^2 + \frac{2R^2}{B^3} b^2 + \frac{R^2 + B^2}{B^3} e_\sigma^2 \\
- \text{sgn}(E) \frac{2R^2(3R^2 + B^2)}{B^4} e^b_z + \frac{4R}{B^2} r e_z \\
+ \text{sgn}(E) \frac{4R(2R^2 + B^2)}{B^3} r e_z], \tag{A.0.10} \]

\[ H = P^0 - |Q_{D0}| T_{D0} - |Q_{F1}| T_{F1} L_z, \]

\[ = \text{sgn}(B) \frac{T_{D2}}{2} \int dz d\sigma \left[ \frac{R^2 + B^2}{B} ((\partial_r r)^2 + |\partial_\eta|^2) + \frac{R^2}{B} ((\partial_z r)^2 + |\partial_\eta|^2) \\
+ \frac{R^2(R^2 + B^2)}{B^3} (\partial_t a_z)^2 + \frac{1}{B} (\partial_t a_\sigma)^2 \right]. \tag{A.0.11} \]

Note in particular that \( H \) is not the energy \( P^0 \) that couples to the gravitational field. Instead, \( H \) measures the extent to which a state is excited above the BPS bound. Note also that expressions (A.0.7-A.0.11) are valid even when the fields depend on \( z \).
Appendix B

List of Relevant Integrals

The following is a list of integrals needed to attain the result 3.2.26.

\[
\int d^5\eta \frac{\eta^i\eta^i}{|\eta - \hat{x}^0|^5 |\eta + \hat{x}^0|^5} = \frac{V(S^3)}{36}\delta^{ij} \quad (B.0.1)
\]

\[
\int d^5\eta \frac{(\eta^0 - 1)(\eta^0 + 1)}{|\eta - \hat{x}^0|^5 |\eta + \hat{x}^0|^5} = -\frac{V(S^4)}{18} \quad (B.0.2)
\]

\[
\int d^5\eta \frac{\eta^i\eta^j(\eta^0 - 1)(\eta^0 + 1)}{|\eta - \hat{x}^0|^5 |\eta + \hat{x}^0|^5} = 0 \quad (B.0.3)
\]

\[
\int d^5\eta \frac{\eta^i\eta^j\eta^k\eta^l}{|\eta - \hat{x}^0|^5 |\eta + \hat{x}^0|^5} = \frac{V(S^3)}{54}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \quad (B.0.4)
\]
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