EQUAl TIME COMMUTATOR IN A SOLUBLE MODEL

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A B S T R A C T

An equal time commutator in the Lee model is examined. It is found that a related sum rule does not have the canonical value. On the other hand, in the corresponding zero energy theorem the canonical value is correct.

66/1071/5 - TH. 693
2 August 1966
1. - **INTRODUCTION**

In quantum field theory, as is well known \(^1\), the canonical equal time commutators must be used with caution. Some aspects of the situation are illustrated here in the context of a trivial soluble model.

2. - **MODEL**

The model is that of T.D. Lee \(^2\). There are fixed fermions \(N\) and \(P\), and a mobile boson \(\Theta\). The Hamiltonian is

\[
H = T + V
\]

\[
T = m_N \overline{N} N + m_P \overline{P} P + \int \frac{d^3k}{(2\pi)^3} \overline{\Theta(k)} \Theta(k) k
\]

\[
V = g \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \left( \overline{\Theta(k)} \overline{P} N + \Theta(k) \overline{N} P \right)
\]

where \(N, P\) and \(\Theta(k)\) are absorption operators, and \(\overline{N}, \overline{P}\) and \(\overline{\Theta(k)}\) their Hermitian conjugates. We will be especially interested in the limit \(\Lambda \to \infty\). The canonical commutation rules are

\[
\{N, \overline{N}\} = \{P, \overline{P}\} = 1 \quad \left[ \overline{\Theta(k)}, \overline{\Theta(k')} \right] = (2\pi)^3 \delta(k - k')
\]

\[
\{N, P\} = \{N, \overline{P}\} = \left[ N, \overline{\Theta(k)} \right] = \left[ N, \Theta(k) \right] = 0
\]

\[
\left[ P, \overline{\Theta(k)} \right] = \left[ P, \Theta(k) \right] = \left[ \overline{\Theta(k)}, \Theta(k') \right] = 0
\]
Note that the vacuum, defined by

\[ \mathcal{N}\ket{0} = \mathcal{P}\ket{0} = \Theta(\mathcal{R})\ket{0} = 0 \]

and the one \( \mathcal{P} \) state

\[ \ket{\mathcal{P}} = \mathcal{P}\ket{0} \]

are eigenstates of \( H \). On the contrary, the one bare \( \mathcal{N} \) state

\[ \ket{\mathcal{N}} = \mathcal{N}\ket{0} \]

is not an energy eigenstate.

We define "charges" (non of them conserved)

\[ Q^+ = \mathcal{P}\mathcal{N} \quad Q^- = \mathcal{N}\mathcal{P} \quad Q^0 = \mathcal{P}\mathcal{P} - \mathcal{N}\mathcal{N} \]

and note the commutator

\[ [Q^+, Q^-] = Q^0 \quad (1) \]

which will be our main concern. The corresponding Heisenberg operators are

\[ Q^+(t) = e^{iHt} Q^+ e^{-iHt}, \text{ etc} \]

Consider the effect of adding to the Hamiltonian (or subtracting from the Lagrangian) the time dependent perturbation

\[ \alpha(t) Q^+(t) + \beta(t) Q^-(t) \]
The induced perturbation of the $S$ matrix element connecting states $a$ and $b$ will be of the form

$$\Delta S_{ba} = -i \int dt \, \tilde{K}_{b\alpha}^+ (t) \alpha(t) - i \int dt \, \tilde{K}_{b\alpha}^- (t) \beta(t)$$

$$+ (-i)^2 \int dt \, dt' \, \tilde{K}_{b\alpha}^{++} (t-t') \alpha(t) \beta(t') + \cdots$$

In a covariant field theory the propagators $\tilde{K}$, because of their relation to the invariant $S$ matrix, would have well-defined Lorentz covariance (and perhaps gauge covariance) properties. We will consider for $a$ and $b$ only the one $P$ state, and will be interested only in the propagator $\tilde{K}_{PP}^{+-}$, which will be denoted simply by $\tilde{K}$. By the usual arguments,

$$\tilde{K}(t-t') = \langle P | T (Q^+(t), Q^-(t')) | P \rangle$$

where we have a time ordered product of Heisenberg operators.

The Fourier transform of $K$ is readily determined

$$K(\omega) = \int dt \, e^{i \omega t} \langle P | T (Q^+(t), Q^-(0)) | P \rangle$$

$$= \int_0^{\infty} dt \, e^{i \omega t + i m_P t} \langle P | Q^+ e^{-i H t} Q^- | P \rangle$$

$$= \langle P | Q^+ \frac{i}{\omega + m_P - H} Q^- | P \rangle$$

$$= \langle N | i (\omega + m_P - H)^{-1} | N \rangle$$

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Expanding,
\[
(\omega + m_p - H)^{-1} = (\omega + m_p - T)^{-1} + (\omega + m_p - T)^{-1} V (\omega + m_p - T)^{-1} + (\omega + m_p - T)^{-1} V (\omega + m_p - T)^{-1}
\]
we note, using as intermediate states the eigenstates of $T$, that only the even orders contribute to the matrix element of interest. Resumming
\[
-i K(\omega) = \left\{ \omega + m_p - m_N - \langle N | V \frac{1}{\omega + m_p - T} V | N \rangle \right\}^{-1}
\]
\[
= \left\{ \omega + m_p - m_N - \frac{g^2}{4 \pi^2} \int \frac{dk}{k(\omega - k)} \frac{1}{k} \right\}^{-1}
\]
If $m_N$ is low enough this expression will have a pole at a negative value of $\omega$, corresponding to a stable physical $N$ state. Suppose this pole is at $\omega = -\Delta$. Then
\[
m_p - m_N = \Delta - \frac{g^2}{4 \pi^2} \int_{0}^{\Lambda} dk \frac{k}{\Delta + k}
\]
Thus
\[
-i K(\omega) = \left\{ \omega + \Delta + \frac{g^2}{4 \pi^2} \int_{0}^{\Lambda} dk \left( \frac{\omega}{k - \omega} + \frac{\Delta}{k + \Delta} \right) \right\}^{-1}
\]
We are especially interested in $\Lambda \to \infty$, $\Delta$ remaining fixed. To avoid trivial infinities we take
\[
\frac{g^2}{4 \pi^2} = \frac{f^2}{\log \Lambda}
\]
and hold \( r^2 \) constant. Then denoting dependence on \( \Lambda \) by a subscript we have

\[
\lim_{\Lambda \to \infty} K_{\Lambda}(\omega) = K_{\infty}(\omega)
\]

\[
K_{\infty}(\omega) = i\left\{(\omega + \Delta)(1 + f^2)\right\}^{-1}
\]

Note that

\[
\lim_{\omega \to \infty} (-i\omega K_{\infty}(\omega)) = (1 + f^2)^{-1}
\]

On the other hand for fixed finite \( \Lambda \)

\[
\lim_{\omega \to \infty} (-i\omega K_{\Lambda}(\omega)) = 1
\]

It is the difference of these quantities that produces the effects of interest.

3. - PROPAGATOR DISCONTINUITY

Consider the discontinuity in \( K(t) \) at \( t = 0 \)

\[
D = \lim_{\varepsilon \to 0} \left( \tilde{K}(\varepsilon) - \tilde{K}(-\varepsilon) \right)
\]

\[
= \lim_{\varepsilon \to 0} \left\langle \rho\left( Q^+(-\varepsilon) Q^-(\varepsilon) - Q^-(-\varepsilon) Q^+(\varepsilon) \right) \right\rangle
\]

(7)
One might expect that this limit would have the value

$$
\langle p | (Q^+(0) Q^-(0) - Q^-(0) Q^+(0)) | p \rangle = 1
$$

(8)

To verify this we have to invert the Fourier transform $K(\omega)$ of $	ilde{K}(t)$. The discontinuity at $t = 0$ is simply determined by the behaviour of $K(\omega)$ at infinity

$$
D = \lim_{\omega \to \infty} \left(-i\omega K(\omega)\right)
$$

Thus

$$
D_\Lambda = 1
$$

$$
D_\infty = (1 + f^2)^{-1}
$$

What happens is that, as well as the discontinuous variation dictated by the canonical commutator, there is an additional variation of $\tilde{K}(t)$ near $t = 0$ which is fast for large $\Lambda$ and becomes discontinuous in the limit.

We will refer to (7) as the matrix element of the effective commutator and to (8) as that of the canonical commutator. For finite $\Lambda$ they are the same, but for infinite $\Lambda$ they are not.
4. - SUM RULE

Introducing as intermediate states in (8) the complete set of eigenstates $|n\rangle$ of $H$, with energies $E_n$, we would be led to expect the sum rule

$$\int d\omega \varrho(\omega) = 1$$

where

$$\varrho(\omega) = \sum_n \delta(\omega + m_p - E_n) \left[ |\langle p | Q^+ | n \rangle|^2 - |\langle p | Q^- | n \rangle|^2 \right]$$

To test this we note that $\varrho$ can be determined from $K$, which has the representation

$$-i K(\omega) = \int d\omega' \frac{\varrho(\omega')}{\omega - \omega'}, \quad (9)$$

From (6)

$$\int d\omega \varrho_\Lambda(\omega) = 1 \quad (10)$$

as expected. However, from (4)

$$\varrho_\infty(\omega) = (1 + f^2)^{-1} \delta(\omega + \Delta)$$

and

$$\int d\omega \varrho_\infty(\omega) = (1 + f^2)^{-1} \quad (11)$$

To envisage what happens, write

$$\varrho_\Lambda = \varrho_\infty + \varrho_\Lambda$$

(12)
where
\[ \int d\omega \Gamma^\Lambda(\omega) = 1 - \int d\omega \gamma^\infty(\omega) \]
For \( \Lambda \to \infty, \Gamma \to 0 \) everywhere, and so does not contribute to the integral of the limit, as distinct from the limit of the integral.

5. - PROPAGATOR IDENTITY AND ZERO ENERGY THEOREM

By partial integration in (2) we obtain the identity
\[ -i\omega K^\Lambda(\omega) = L^\Lambda(\omega) + \langle p | [Q^+, Q^-] | p \rangle \]
\[ = L^\Lambda(\omega) + 1 \]
(13)

where
\[ L(\omega) = \int dt e^{i\omega t} \langle p | T(\{Q^+(t), Q(0)\}) | p \rangle \]
In particular we have a zero energy theorem
\[ L^\Lambda(0) = -1 \]
(14)

Passing to the limit
\[ -i\omega K^\infty(\omega) = L^\infty(\omega) + 1 \]
\[ L^\infty(0) = -1 \]
(15)
Note that here the canonical value of the commutator, $\mathbf{i}$, is not replaced by the "effective" value, $(1 + \beta^2)^{-1}$.

In analogy with (9), $L(\omega)$ has the representation (for finite $\Lambda$)

$$-\mathbf{i} L_\Lambda(\omega) = \int d\omega' \frac{-\mathbf{i} \omega' \rho(\omega')}{\omega - \omega'}$$  \hspace{1cm} (16)

whence

$$L_\Lambda(\omega) = -\int d\omega' \rho(\omega')$$

From the zero energy theorem (14) we then recover the sum rule (10). If however we assumed that (16) remains true at $\Lambda = \infty$, the theorem (15) would be in conflict with the sum (11). In fact the unsubtracted dispersion relation (16) fails at $\Lambda = \infty$. Using (12) in (16),

$$-\mathbf{i} L_\Lambda(\omega) = \int d\omega' \frac{-\mathbf{i} \omega' \rho_\infty(\omega')}{\omega - \omega'} + \int d\omega' \frac{-\mathbf{i} \omega' \rho(\omega')}{\omega - \omega'}$$

The weight function $\rho$ spreads to increasingly large $\omega'$ as $\Lambda \to \infty$, so that $\omega$ becomes negligible in the second integral. Thus the limiting form of (16) is

$$-\mathbf{i} L_\infty(\omega) = \int d\omega' \frac{-\mathbf{i} \omega' \rho_\infty(\omega')}{\omega - \omega'} + \mathbf{i} \left(1 - \frac{1}{1 + \beta^2}\right)$$

The situation above is the same as for the vacuum polarization tensor in quantum electrodynamics. There also it is known (from gauge invariance) that a certain propagator identity involves
the canonical commutator, while a related sum rule involves an effective commutator. The polarization tensor is (in second order)

\[ \Pi_{\mu\nu}(k) = \int d^4x e^{ikx} \langle 0 | \mathcal{T}(\vec{J}_\mu(x), \vec{J}_\nu(0)) | 0 \rangle \]

In analogy with (13), bearing in mind that the current here has zero divergence, we have

\[ ik_{\mu} \Pi_{\mu\nu}(k) = \langle 0 | \left[ \int d^4x e^{ikx} \vec{J}_\nu(\vec{x},0), \vec{J}_\mu(0) \right] | 0 \rangle \]

The canonical value of the right hand side, zero, is that required by the gauge invariance. On the other hand 1)

\[ \sum_n \left\{ \langle 0 | J_0 | n \rangle \langle n | \vec{\nabla} \cdot \vec{J} | 0 \rangle - \langle 0 | \vec{\nabla} \times \vec{J} | n \rangle \times n | J_0 | 0 \rangle \right\} \neq 0 \]

i.e., the zero canonical commutator of \( J_0 \) with \( \vec{\nabla} \cdot \vec{J} \) would give the wrong sum. In this case there are some peculiar features because a gauge and Lorentz invariant regularization requires an indefinite metric 3).
6. - MUTILATED PROPAGATORS

It is clear from the above that the quantity $\tilde{L}(t)$ develops a $\delta$ function at $t = 0$ as $A \to \infty$. There is no difficulty in handling $\tilde{L}_\infty(t)$ as a distribution, and in this sense it has the Fourier transform $L_\infty(\omega)$ used above. However one could consider the modified definition

$$L'_\infty(\omega) = \lim_{\varepsilon \to 0} \int dt \, L_\infty(t) e^{i\omega t},$$

in which the singular region is excluded from the integration. Partial integration now gives

$$L'_\infty(\omega) = -i \omega K'_\infty(\omega) - \lim_{\varepsilon \to 0} \langle p | e^{i\omega \varepsilon} Q^+(\varepsilon) Q(0) - e^{-i\omega \varepsilon} Q(0) Q^-(\varepsilon) | p \rangle.$$

In particular

$$L'_\infty(\omega) = - \lim_{\varepsilon \to 0} \langle p | Q^+(\varepsilon) Q(0) - Q(0) Q^-(\varepsilon) | p \rangle = - (1 + f^2)^{-1}.$$

So for the new object it is the effective commutator which appears in the zero energy theorem.

The quantity $L'_\infty(\omega)$ differs from $L_\infty(\omega)$ by the contribution of the delta function at $t = 0$:

$$L'_\infty(\omega) = L_\infty(\omega) - L_\infty(\infty).$$
Thus we have here an illustration of the doctrine of Bjorken 4), that
the effective commutators are relevant for propagators whose high
frequency parts have been removed. As observed by Bjorken, the propa-
gators so mutilated are no longer immediately related to the covariant
S matrix, and any Lorentz and gauge covariant properties are in general
spoiled.

7. — TRANSLATION SYMMETRY

For simplicity the model was formulated with fermions fixed
at the origin. The translation symmetry of the original Lee model is
restored simply by allowing the (still immobile) fermions to appear
anywhere. This is effected by making the following substitutions in
the Hamiltonian :

\[
\begin{align*}
\bar{N}N & \rightarrow \int d\vec{x} \quad \bar{N}(\vec{x}) \cdot N(\vec{x}) \\
\bar{P}P & \rightarrow \int d\vec{x} \quad \bar{P}(\vec{x}) \cdot P(\vec{x}) \\
\Theta(\vec{k})\bar{N}P & \rightarrow \int d\vec{x} \quad e^{i\vec{k} \cdot \vec{x}} \Theta(\vec{k}) \cdot \bar{N}(\vec{x}) \cdot P(\vec{x}) \\
\bar{\Theta}(\vec{k})\bar{P}N & \rightarrow \int d\vec{x} \quad e^{-i\vec{k} \cdot \vec{x}} \bar{\Theta}(\vec{k}) \cdot \bar{P}(\vec{x}) \cdot N(\vec{x})
\end{align*}
\]

where

\[
\{\bar{N}(\vec{x}), N(\vec{y})\} = \delta(\vec{x} - \vec{y})
\]

etc.

We can now define "densities"

\[
\begin{align*}
q_1^+ (\vec{x}) &= \bar{P}(\vec{x}) N(\vec{x}) \\
q_1^- (\vec{x}) &= \bar{N}(\vec{x}) P(\vec{x}) \\
q_0^0 (\vec{x}) &= \bar{P}(\vec{x}) P(\vec{x}) - \bar{N}(\vec{x}) N(\vec{x})
\end{align*}
\]
with the canonical commutator

\[ [q^+(x), q^-(y)] = q^0(x) \delta(x - y) \]

Because fermions at different places are in no way coupled the delta function persists trivially in the time ordered product and in the effective commutator - evaluated always in the one fermion subspace. The relation of the new propagator to the old is

\[
\langle p, \gamma | T(q^+(x, t), q^-(x', t')) | p, \gamma \rangle =
\]

\[
\delta(\gamma - x) \delta(\gamma - x') \delta(x' - y') \langle p | T(q^+(t), q^-(t')) | p \rangle
\]

where on the left we specify position of initial and final fermions. If we specify instead momenta \(\vec{p}\) and \(\vec{p}'\):

\[
\langle p, \vec{p} | T(q^+(x, t), q^-(x', t')) | p, \vec{p}' \rangle =
\]

\[
\delta(x - x') \langle p | T(q^+(t), q^-(t')) | p \rangle
\]

Thus the old results for the \(Q's\) can be turned into results for the \(q's\).

The only advantage of the new language is that it is superficially closer to that of realistic field theories. It is to be noted that the difference between canonical and effective commutators is again a factor - and not for example a "gradient term". Thus this complication of the algebra of densities persists in the algebra of their space integrals}
8. - CONCLUSION

This simple model exemplifies a number of ideas that may have wider validity.

1°) The interesting results arise in a situation \((\Lambda \to \infty)\) that cannot be set up within the usual canonical formalism, but only as a limiting case. It may be proper to regard Lorentz invariant renormalizable field theories as limits of regulated theories.

2°) Sum rules connected with canonical commutators may fail. In the present case the failure has nothing to do with gradient terms, suggesting that such failure may occur even for space integrals when these are not conserved.

3°) In propagator identities the canonical commutators may or may not be appropriate, depending on the precise definition of Fourier transforms of time ordered products \(^6\). If these are defined as limits of the corresponding quantities for finite cut-off, the canonical values apply. Non canonical values of sum rules are then associated with subtractions in dispersion relations.

For valuable discussion I am indebted to N. Kroll, T.D. Lee, J.W. Moffat, L. Van Hove and M. Nauenberg.
REFERENCES

1) T. Gato and T. Imamura - Prog. Theor. Phys. 14, 396 (1955);
   J. Schwinger - Phys. Rev. Letters 3, 296 (1959);
   S. Okubo - Trieste Preprint;
   K. Johnson and F. Low - M. I. T. Preprint.

2) T. D. Lee - Phys. Rev. 95, 1329 (1954);
   See also: G. Källén and W. Pauli - Kgl. Danske Videnskab. Selskab
            30, 7 (1955). In the present application the renormalized
            coupling constant, even more so than the bare coupling
            constant, goes to zero in the limit, and there is no ghost
            trouble.

3) W. Pauli and F. Villars - Revs. Modern Phys. 21, 434 (1949);
   See also:
   S. N. Gupta - Proc. Phys. Soc. A66, 129 (1953);


5) See also in this connection: K. Johnson and F. Low, Ref. 1).

6) In connection with such ambiguities see for example:
   M. C. Polivanov in "High energy physics and elementary
   particles", International Atomic Energy Agency, Vienna
   p. 121 (1965).