Instantons in Field Theory

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Introduction

It appears that all fundamental interactions in nature are of the gauge type. The modern theory of hadrons - quantum chromodynamics (QCD) - is no exception. It is based on local gauge invariance with respect to the color group SU(3), which is realized by an octuplet of massless gluons.

The idea of gauge invariance, however, is much older and derives from quantum electrodynamics, which was historically the first field model in which successful predictions were obtained. By the end of the forties, theoreticians had already learned how to calculate all observable quantities in electrodynamics in the form of series in $\alpha = 1/137$. The first steps in QCD at the end of the seventies were also made in the framework of perturbation theory. However, it gradually became clear that, in contrast to electrodynamics, quark-gluon physics was not exhausted by perturbation theory. The most interesting phenomena - the confinement of colored objects and the formation of the hadron spectrum - are associated with non-perturbative (i.e., not describable in the framework of perturbation theory) effects, or rather, with the complicated structure of the QCD vacuum, which is filled with fluctuations of the gluon field.

In general, there are solutions of the classical, nonlinear equations of motion that exhibit particle-like behavior that give us powerful insight into the non-perturbative behavior of these theories. Among these solutions are solitons, monopoles and instantons. These solutions cannot be obtained from solutions of the corresponding linear part of the field equations and treating the non-linear part perturbatively.

Instantons are finite-action solutions to the Euclideanized equations of motion. At the classical level, instantons are not very different from static solutions of Minkowskian equations. This is obviously because static solutions involve only the spatial coordinates, i.e. the Euclidean subspace of Minkowskian space-time. Indeed, some of these static solutions can be directly used as instantons for lower dimensional systems. The only difference is that the requirement of finiteness of energy of solitons will be replaced by the requirement of finiteness of Euclidean action in the case of instantons. The name 'instantons' was coined by 'tHooft (1976) because, unlike solitons of Minkowskian systems which are not localised in time, these solutions are localised in $x_4$ (the imaginary time coordinate) as well.
Chapter 1

Instantons in a Double Well.
Euclidean Space

1.1 The Euclidean Space

The transition amplitude between two quantum basis states in the Heisenberg representation is defined as:

\[ \langle x_f, \frac{T}{2} | x_i, -\frac{T}{2} \rangle = \langle x_f | e^{-\frac{i}{\hbar} HT} | x_i \rangle = N \int Dxe^{i S[x]} \]  (1.1)

where \( N \) is normalization constant, and is defined in Eq. (A.14)(appendix A). If we go from states with a definite coordinate to states with a definite energy:

\[ H | n \rangle = E_n | n \rangle \]  (1.2)

using the completeness relations for the quantum states, we have:

\[ \langle x_f | e^{-\frac{i}{\hbar} HT} | x_i \rangle = \sum_n e^{-\frac{i}{\hbar} E_n T} \langle x_f | n \rangle \langle n | x_i \rangle \]  (1.3)

In this way we have obtained a sum of oscillating exponentials. If we are interested in the ground state (in field theory we are always interested in the lowest state, the vacuum), it is much more convenient to transform the oscillating exponentials into decreasing exponentials by doing the substitution \( t \rightarrow -it' \). Then in the limit \( t' \rightarrow \infty \) only a single term survives in the sum and this directly tells us what are the energy \( E_0 \) and the wave function \( \Psi_0 \) of the lowest level \( e^{-E_0 t'} \Psi_0 (x_f) \Psi_0^+ (x_i) \).

If the Lagrangian is \( L = \frac{1}{2} \dot{x}^2 - V(x) \) then the action is given by:

\[ S[x] = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \left[ \frac{1}{2} \dot{x}^2 - V(x) \right] \]  (1.4)
The best way to evaluate the path integral Eq. (1.1) is to go to the Euclidean space, thus by rotating to imaginary time,

\[ t \rightarrow -it' \quad T \rightarrow -iT' \quad (1.5) \]

\[
\frac{dx}{dt} = i \frac{dx}{dt'} \quad (1.6) \\
L = \frac{1}{2}m\dot{x}^2 - V(x) \rightarrow -\frac{1}{2}m\dot{x}^2 - V(x) \quad (1.7) 
\]

The action transforms under this Wick rotation:

\[
S = \int_{-T}^{T} L dt \rightarrow \int_{-T'}^{T'} (-\frac{1}{2}m\dot{x}^2 - V(x))(-i)dt' = \int (\frac{1}{2}m\dot{x}^2 + V(x)) dt' = iS_E \quad (1.8) 
\]

for the transition amplitude we will have:

\[
\langle x_f | e^{\frac{-i}{\hbar} HT} | x_i \rangle = N \int \mathcal{D} x e^{i\frac{S_E}{\hbar}[x]} = N \int \mathcal{D} x e^{i(S_E[x])} \quad (1.9) 
\]

and we obtain in Euclidean space (from now on we will write t,T for the new Euclidean variables):

\[
\langle x_f | e^{-\frac{1}{\hbar} HT} | x_i \rangle = N \int \mathcal{D} x e^{-\frac{i}{\hbar} S_E[x]} \quad (1.10) 
\]

where:

\[
S_E[x] = \int_{-T}^{T} dt \left[ \frac{1}{2}m\dot{x}^2 + V(x) \right] = \int_{-T}^{T} dt \mathcal{L}_E(x, \dot{x}) \quad (1.11) 
\]

### 1.2 Double Well Potential

In this section we will apply the above mentioned formula to the double well potential \( V(x) = g^2 (x^2 - a^2)^2 \). The classical equation which is obtained from the extremum of the action in (1.11) has the form:

\[
\frac{\delta S_E[x]}{\delta x(t)} = 0 \quad \text{or} \quad m\ddot{x} - V'(x) = 0 \quad (1.12) 
\]

The Euclidean equations, therefore, correspond to a particle moving in an inverted potential otherwise also known as a double humped potential. The Euclidean energy associated with such a motion is given by \( E = \frac{1}{2}m\dot{x}^2 - V(x) \).
It is easy to find two elemental solutions to the Euclidean classical equation of motion in Eq. (1.12). Namely, \( x(t) = \pm a \) (trivial solutions) satisfy the classical equation with minimum energy \( E = 0 \). In this case the particle stays at rest on top of one of the hills. In the Minkowskian space this will corresponds to the case where the particle executes small oscillations at the bottom of either of the wells and these small oscillations can be approximated by a harmonic oscillator motion.

There are, however, another interesting solution, one where the particle begins at the top of one hill at time \(-T/2\), and moves to the top of the other hill at time \(T/2\). Let be:

\[
x_{cl}(t) = \pm a \tanh \frac{\omega(t-t_c)}{2}
\]

where \( t_c \) is a constant and we have identified \( m\omega^2 = V''(\pm a) = g^2a^2 \); then, it is straightforward to see that (1.13) are nontrivial solutions of Eq. (1.12) with \( m\ddot{x}_{cl} - V'(x_{cl}) = 0 \). These solutions are analogous to the kink and the anti-kink solution of the \( \phi^4 \) model. Notice that this particle-mechanics example may be thought of as one-dimensional scalar field-theory. Their instantons are just the static solitons of the corresponding field theory in two (1+1) dimensions. This pattern can be follow in many other cases when one is looking for instanton solutions.

We also note that, for these solutions Eq. (1.13):

\[
x_{cl}(t \to -\infty) = \mp a \quad \text{and} \quad x_{cl}(t \to +\infty) = \pm a
\]

As it is seen in Fig.3 and Fig.4 these solutions correspond to the particle starting out on one of the hill tops at \( t \to -\infty \) and then moving over to the other hill top at \( t \to +\infty \). For such solutions we have:

\[
\frac{1}{2}m\dot{x}_{cl}^2 = \frac{1}{2}m \frac{2}{m} V(x_{cl}) = V(x_{cl}) \Rightarrow E = \frac{1}{2}m\dot{x}^2 - V(x) = 0
\]

where \( E \) is the Euclidean energy for our system. As in the case of the trivial solution these also correspond to minimum energy solutions. We can calculate the action corresponding to such a
classical motion:

\[ S_E[x_{cl}] = S_0 = \int_{-\infty}^{\infty} dt \left( \frac{1}{2} m \dot{x}_{cl}^2 + V(x_{cl}) \right) = \int_{-\infty}^{\infty} dt m \dot{x}_{cl}^2 = m \int_{\mp a}^{\pm a} dx_{cl} \dot{x}_{cl} = \]

\[ m \int_{\mp a}^{\pm a} dx_{cl} (\pm \frac{a \omega}{2} \cosh - \frac{\omega(t - t_c)}{2}) = m \int_{\mp a}^{\pm a} dx_{cl} (\mp \frac{\omega}{2a} (x_{cl}^2 - a^2)) = \pm \frac{m \omega}{2a} \left( \frac{1}{3} x_{cl}^3 - a^2 x_{cl} \right) \bigg|_{\mp a}^{\pm a} = \frac{m \omega}{2a} \left( \frac{4a^3}{3} \right) = \frac{2}{3} m a^2 = \frac{2m^2 \omega^3}{3g^2}. \]

The solutions in (1.13) are, therefore, finite action solutions in the Euclidean space and have the shape shown in Fig. 1.2 and Fig. 1.2. They are known respectively as the \textbf{instanton} and the \textbf{anti-instanton} solutions (classical solutions in Euclidean time). If we calculate the Lagrangian for such a solution, then we find:

\[ L_E = \frac{1}{2} m \dot{x}_{cl}^2 + V(x_{cl}) = m \dot{x}_{cl}^2 = \]

\[ \frac{ma^2 \omega^2}{4} \cosh - \frac{\omega(t - t_c)}{2} \]

The graphical representation of this function is shown in Fig.5, we can deduce that the Lagrangian is fairly localized around \( t = t_c \) with a size of about \( \Delta t \sim \frac{1}{\omega} = \frac{\sqrt{m}}{ga} \). We say therefore that instantons are localized solutions in time with a size of about \( \frac{1}{\omega} \). The constant \( t_c \) means the time when the solution reaches the valley of the Euclidean potential and is arbitrary. This is a result of the time translation symmetry in the theory.

Just as we can have a one instanton or one anti-instanton solution, we can also have multi-instanton solutions in such a theory. In this case the particle leaves one top and afterwards comes back to the same top repeating this cycle several times. However, for simplicity, let us calculate the contribution to the transition amplitude coming only from the one instanton or one anti-instanton.
trajectory. From (A.1) and (A.2) (O.I. means One Instanton):
\[ \langle +a| e^{-\frac{i}{\hbar} H_T} | -a \rangle_{O.I.} = N \int \mathcal{D}x \ e^{-\frac{i}{\hbar} S_E[x]} \approx \]
\[ N \int \mathcal{D}\eta \ e^{-\frac{1}{2} \int \int dt_1 dt_2 \eta(t_1) \ \frac{\delta^2 S_E[x]}{\delta x_{cl}(t_1) \delta x_{cl}(t_2)} \eta(t_2) + O(\eta^3)} = \]
\[ \frac{N}{\sqrt{\det[\frac{1}{\hbar} (-m \frac{d^2}{dt^2} + V''(x_{cl}))]}} e^{-\frac{i}{\hbar} S_0} \]
(1.17)

Here we have defined \( x(t) = x_{cl}(t) + \eta(t) \), and for the one instanton case \( x_{cl}(t) = a \tanh \frac{\omega(t-t_c)}{2} \) as we have already seen in Eq. (1.13).

If we look the determinant for the O.I. case in detail, we see that \( \frac{dx_{cl}}{dt} = \frac{a \omega}{2} \cosh^{-2} \frac{\omega(t-t_c)}{2} \) and from Eq. (A.16): \( (-m \frac{d^2}{dt^2} + V''(x_{cl})) \frac{dx_{cl}}{dt} = 0 \), that is the same as the the zero eigenvalue equation Eq. (A.11), rotated to imaginary time. We can define the normalized zero eigenvalue solution of this equation as:
\[ \psi_0(t) = \left( \frac{S_0}{m} \right)^{-\frac{1}{2}} \frac{dx_{cl}}{dt} = \left( \frac{S_0}{m} \right)^{-\frac{1}{2}} \frac{a \omega}{2} \cosh^{-2} \frac{\omega(t-t_c)}{2} \]
(1.18)

Using Eq. (A.13), the determinant in Eq. (1.17) can be obtained from this solution simply as:
\[ \det[\frac{1}{\hbar} (-m \frac{d^2}{dt^2} + V''(x_{cl}))] \propto \psi_0(\frac{T}{2}) \quad T \to \infty \]
(1.19)
and it is seen that:
\[ \lim_{T \to \infty} \psi_0(\frac{T}{2}) \to 0 \]
(1.20)

That means that the determinant vanishes. The reason for this is that \( \psi_0(t) \) happens to be an exact eigenstate of the operator \( (-m \frac{d^2}{dt^2} + V''(x_{cl})) \) with zero eigenvalue. (This means that \( \psi_0(\pm \frac{T}{2}) = 0 \) for \( T \to \infty \)). In this case we say that there is a zero mode in the theory.

### 1.3 Zero Modes and Collective Coordinates

The presence of a zero mode in the theory will indicate a symmetry in the system. To see this, let us recall that the determinant in Eq. (1.17) appeared once that we integrated out the Gaussian fluctuations. The part we have to analyze is:
\[ \int \mathcal{D}\eta \ e^{-\frac{i}{\hbar} \int \int dt_1 dt_2 \eta(t_1) \ \frac{\delta^2 S_E[x]}{\delta x_{cl}(t_1) \delta x_{cl}(t_2)} \eta(t_2)} \]
(1.21)
Since $\psi_0(t)$ represents a zero mode of the operator $\delta^2 S_E[x_{cl}] / \delta x_{cl}(t_1)\delta x_{cl}(t_2)$, if we make a change of the integration variable as:

$$\delta \eta(t) = \epsilon \psi_0(t)$$  \hspace{1cm} (1.22)

where $\epsilon$ is a constant parameter, then the Gaussian does not change. That means that the transformation in Eq. (1.22) defines a symmetry of the quadratic action. If we expand the fluctuations around the classical trajectory in a complete basis of the eigenstates of the operator $\delta^2 S_E[x_{cl}] / \delta x_{cl}(t_1)\delta x_{cl}(t_2)$, then we can write:

$$\eta(t) = \sum_{n \geq 0} c_n \psi_n(t)$$  \hspace{1cm} (1.23)

$$\mathcal{D}\eta(t) = \prod_{n \geq 0} dc_n$$  \hspace{1cm} (1.24)

Substituting this expansion into Eq. (1.21), we obtain:

$$\int \mathcal{D}\eta e^{-\frac{1}{\hbar} \int dt_1 dt_2 \eta(t_1) \frac{\delta^2 S_E[x_{cl}]}{\delta x_{cl}(t_1)\delta x_{cl}(t_2)} \eta(t_2)}$$

$$= \int \prod_{n \geq 0} dc_n e^{-\frac{1}{\hbar} \sum_{n > 0} \lambda_n c_n^2}$$  \hspace{1cm} (1.25)

$$= \int dc_0 \int \prod_{n > 0} dc_n e^{-\frac{1}{\hbar} \sum_{n > 0} \lambda_n c_n^2}$$  \hspace{1cm} (1.26)

where $\lambda_n$ are the eigenvalues corresponding to the eigenstates $\psi_n$. Here we see that the zero mode disappears of the exponent and therefore there is no Gaussian damping for the $dc_0$ integration.

We have seen in Eq. (1.25) that the integral is divergent. The time translation invariance causes this divergence actually the position of the center of the instanton can be arbitrary. In the next lines we will replace the $dc_0$ integration by an integration over the position of the center of the instanton.

Let us expand around the actual trajectory of the instanton:

$$x(t) = x_{cl}(t - t_c) + \eta(t - t_c), \text{ or}$$

$$x(t + t_c) = x_{cl}(t) + \eta(t) = x_{cl}(t) + \sum_{n \geq 0} c_n \psi_n(t)$$  \hspace{1cm} (1.28)

(Since the trajectory is independent of the center of the instanton trajectory, the fluctuations must balance out the $t_c$ dependence) multiplying the last equation with $\psi_0(t)$ and integrating over time,
we obtain:
\[
\int_{T_2}^{T_1} dt x(t + t_c)\psi_0(t) = \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} dt [x_{cl}(t) + \sum_{n \geq 0} c_n \psi_n(t)]\psi_0(t) = \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} dt \left(\frac{S_0}{m} - \frac{1}{2} x_{cl}(t)\right)\frac{dx_{cl}(t)}{dt} + c_0 = c_0
\]

The first term drops out because it has the same value at both the limits. We are interested in large T limits when all these results hold. In this way we see that \( c_0 = c_0(t_c) \) and we can change the \( c_0 \)-integration to an integration over \( t_c \). Let us consider an infinitesimal change in the path in Eq. (1.28) arising from a change in the coefficient of the zero mode namely \( \delta \eta = \delta c_0 \psi_0(t) \) where we assume that \( \delta c_0 \) is infinitesimal.

However, we also note that this is precisely the change in the path, that we would have obtained to leading order, had we translated the center of the instanton as:
\[
t_c \rightarrow t_c + \delta t_c = t_c + \delta c_0 \left(\frac{S_0}{m}\right)^{-\frac{1}{2}}
\]

In this case:
\[
\delta x(t + t_c) = \delta \eta = \delta c_0 \psi_0(t) = \delta c_0 \left(\frac{S_0}{m}\right)^{-\frac{1}{2}} \frac{dx_{cl}(t)}{dt}
\]

From Eq. (1.29), we see that the Jacobian of this transformation to the leading order is:
\[
\frac{dc_0(t_c)}{dt_c} \simeq \left(\frac{S_0}{m}\right)^{\frac{1}{2}}
\]

We now substitute Eq. (1.30) into Eq. (1.25) to obtain:
\[
\int D\eta e^{\frac{i}{\hbar} \int dt_1 dt_2 \eta(t_1) \frac{dx_{cl}(t_1)}{dx_{cl}(t_2)} \eta(t_2)} = \left(\frac{S_0}{m}\right)^{\frac{1}{2}} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} dt_c \prod_{n > 0} d\lambda_n e^{-\frac{1}{\hbar} \sum \lambda_n c_n^2}
\]

Here \( det' \) stands for the value of the determinant of the operator without the zero mode. This new coordinate \( t_c \) is called collective coordinate. Finally we obtain the form of the transition amplitude
in the presence of an instanton:

\[
\langle a|e^{-\frac{i}{\hbar} HT}|-a\rangle_{O.I.} = \frac{N(S_0)^{\frac{1}{2}} e^{-\frac{i}{\hbar} S_0}}{\sqrt{\text{det}'(\frac{1}{i\hbar}(-m \frac{d^2}{dt^2} + V''(x_c(t))))}} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_c
\]  

(1.34)

We emphasize that although we have analyzed only a single specific example with the simplest instanton \(x_{cl}(t) = \pm a \tanh \omega (t - t_c)\), the method of dealing with zero-frequency modes is in fact general. Thus, in the Yang-Mills instantons any invariance will generate a zero-frequency mode, and the integration with respect to the corresponding coefficient must be replaced by integration with respect to some collective variable.

For the evaluation of the determinant \(\text{det}'\) see Coleman [14], we would like to mention that with a similar procedure, we calculate the transition amplitude for multi-instanton contributions. The important point is that we would obtain, that the splitting between the two energy levels is:

\[
\Delta E = E_+ - E_- = 4\sqrt{2m\hbar \omega^2 a e^{-\frac{i}{\hbar} S_0}} \propto \omega^2 e^{-\frac{a^2 \omega^2}{\sqrt{2} \hbar}}
\]  

(1.35)

this quantity cannot be expanded in a series in \(g^2\) and therefore it would be not possible to obtain it by perturbation theory.
Chapter 2

Euclidean Yang-Mills Configurations

2.1 Generalities and Yang-Mills Equations

In this chapter we will only consider the SU(2) Yang-Mills gauge fields $A^a_\mu$, self-coupled to one another and we will not consider for the moment the ‘matter fields’ $\Phi^a$. To simplify the notation we will represent the three vector-fields $A^a_\mu$, $a = 1, 2, 3$, by a matrix-valued vector-field $A_\mu$ defined by:

$$A_\mu(x) := \sum_a g \sigma^a A^a_\mu(x)$$  \hspace{1cm} (2.1)

the values of the field are evaluated at each point $x$, where $x$ denotes the vectorial components $x_1, x_2, x_3, x_4$. Here $g$ is the coupling constant and $\sigma^a$ are the familiar Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (2.2)

These Pauli matrices $\sigma^a$ form the three generators of the two-dimensional representation of the group SU(2), which is the gauge group of our system. This $2 \times 2$ anti-Hermitian matrix $A^a_\mu(x)$ represents for each $\mu$ and at each point $x$, the same information as the three Yang-Mills fields $A^a_\mu(x)$. In the same way we define a matrix-valued tensor field:

$$F_{\mu\nu}(x) := \sum_a g \sigma^a F^a_{\mu\nu}(x)$$  \hspace{1cm} (2.3)

where:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$  \hspace{1cm} (2.4)
The gauge transformations are not formally altered by going to the Euclidean metric. In terms of the matrix field $A_\nu$ they become (with $U \in SU(2)$):

$$A_\mu \rightarrow UA_\mu U^{-1} + U\partial_\mu U^{-1} \quad \text{and} \quad F_{\mu\nu} \rightarrow UF_{\mu\nu}U^{-1}. \quad (2.5)$$

We will often use the identity:

$$\partial_\mu (UU^{-1}) = 0 = (\partial_\mu U)U^{-1} + U\partial_\mu (U^{-1}). \quad (2.6)$$

Another advantage of this matrix notation is that it reveals in an obvious way the parallel to electromagnetism where the gauge group is the abelian U(1). In that case the group elements are $U = e^{i\alpha(x)}$, the fields $A_\mu(x)$ are just numbers instead of matrices, and Eq. (2.4) and Eq. (2.5) reduce to familiar equations in electromagnetism. In this matrix notation, it can easily be seen that this action and the field equation are invariant under the gauge transformations Eq. (2.5).

Returning to the Euclidean Yang-Mills system, we have four matrix-valued fields $A_\mu (\mu = 1, 2, 3, 4)$ at each Euclidean space point $x_\mu (\mu = 1, 2, 3, 4)$. As seen in chapter 1 Eq. (1.8), the Euclidean action is obtained from the Minkowskian Lagrangian by the prescription $S = iS_E, x_4 = -ix'_4$ after discarding the scalar field $\phi^a$, and using the matrix notation for $A_\mu$. This action is:

$$S = -\frac{1}{2g^2} \int d^4x Tr[F_{\mu\nu}F_{\mu\nu}] \quad (2.7)$$

It is important to notice that the coupling constant $g$ is also present into the scale of the field $A_\mu$ see Eq. (2.1), explicitly it appears only in the factor $-1/2g^2$.

**Proposition 2.1.1.** The resulting Euclidean Yang-Mills equation is:

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 \quad (2.8)$$

**Proof.** One requires the action to be stationary with respect to small variations of the gauge fields $A_\mu \rightarrow A_\mu + \delta A_\mu$ and its derivative $\delta(\partial_\nu A_\mu) = \partial_\nu(\delta A_\mu)$ where $\delta A_\mu$ transforms in an adjoint representation of the group $\delta A'_\mu(x) = U(x)\delta A_\mu(x)U^{-1}(x)$. The change of the field strength under the variation of $A_\mu$ can be expressed as follows:

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + D_\mu(\delta A_\nu) - D_\nu(\delta A_\mu)$$

where we have used the definition of covariant derivative $D_\mu \phi = \partial_\mu \phi + [A_\mu, \phi]$ and the variation of the Gauge field action is:

$$\delta \int d^4x \frac{1}{2g^2} Tr[F_{\mu\nu}F_{\mu\nu}] = 2 \int d^4x Tr[F_{\mu\nu}(D_\mu \delta A_\nu)]$$

and using the relation:

$$D_\mu Tr[(F_{\mu\nu} \delta A_\nu)] = Tr[(D_\mu F_{\mu\nu})\delta A_\nu] + Tr[F_{\mu\nu}(D_\mu \delta A_\nu)]$$
As a consequence of the invariance of $\text{Tr}[F_{\mu\nu}\delta A_{\nu}]$ under Gauge transformations we can replace the covariant derivative by the ordinary derivative. By integrating the last equation and neglecting the surface terms at infinity:

$$\delta \int dx^4 \frac{1}{2g^2} \text{Tr}[F_{\mu\nu}F_{\mu\nu}] = -\frac{2}{g^2} \int dx^4 \text{Tr}[(D_\mu F_{\mu\nu})\delta A_{\nu}]$$

\[\square\]

### 2.2 Homotopy Classification

As in the case of the double well instantons, the Yang-Mills instantons are finite-action solutions of Eq. (2.8). We first identify the boundary conditions to be satisfied by any finite-action field configurations including the finite-actions solutions of Eq. (2.8). Based on these boundary conditions, we shall make a homotopy classification. Actual solutions will be obtained only in the next chapter.

We will consider zero-action configurations as a first step towards identifying finite-action configurations. From Eq. (2.7), we see that $S = 0$ if and only if $F_{\mu\nu} = 0$. This allows an infinite set of possibilities for the parent field $A_\mu$. Note that $F_{\mu\nu} = 0$ is a gauge-invariant condition, i.e. not only $A_\mu = 0$ will satisfy it but also any gauge-transformed field obtained from $A_\mu = 0$. Taking into account Eq. (2.5) these fields are given by:

$$A_\mu(x) = U(x)\partial_\mu(U^{-1}(x))$$

These fields are called **pure gauges** and $U(x)$, at each $x$, is any element of the group SU(2) in its $2 \times 2$ representation. It can be directly verified, using Eq. (2.4), that this equation yields $F_{\mu\nu} = 0$. The converse, that $F_{\mu\nu} = 0$ everywhere implies Eq. (2.9), can also be proved. We are interested in finite-action configurations, it is clear from Eq. (2.7) that $F_{\mu\nu}$ must vanish on the boundary of Euclidean four-space, i.e. on the three-dimensional spherical surface $S_3^\infty$ at $r = \infty$ where $r := |x| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}$ is the radius in four dimensions, more precisely $F_{\mu\nu}$ must tend to zero faster than $1/r^2$ as $r \to \infty$. The fact that $F_{\mu\nu} = 0$ on $S_3$ means, based on Eq. (2.9) the following boundary conditions on $A_\mu$:

$$\lim_{r\to\infty} A_\mu = \lim_{r\to\infty} U\partial_\mu U^{-1}$$

where $U$ is some group-element-valued function. Based on Eq. (2.10) we can associate to every finite-action configuration $A_\mu$ a group function $U$, defined on the surface at infinity, $S_3^\infty$. This surface may be parameterised by three variables $\alpha_1, \alpha_2$ and $\alpha_3$, say for example the polar angles in four dimensions. The problem is how to define the radial derivative when $U$ depends only on $\alpha_1, \alpha_2$ and $\alpha_3$, a general $A_\mu$ may have a radial component as well. We can use the gauge transformations to overcome
this difficulty, we will gauge transform $A_\mu$ such that its radial component vanishes everywhere. For example if we have an $A_\mu(x)$ with non zero radial component $A_r(x)$ then we can perform a gauge transformation, using as a gauge function:

$$\tilde{U}(x) = P(exp \int_0^r dr' A_r(x'))$$

The 'path ordering' prefix $P$ means an ordering of the path of integration. This ordering is specified by defining $\tilde{U}$ to be a product or the exponentials $exp(A_r dr')$ over each infinitesimal element of the path, arranged sequentially as we go from the starting point ($x=0$) to the end-point $x$.

$$A'_r(x) = \tilde{U}A_r\tilde{U}^{-1} - (\partial_r \tilde{U}\tilde{U}^{-1})$$

$$= \tilde{U}(A_r - A_r)\tilde{U}^{-1} = 0$$

Given a point $x$, consider a path, which in this case runs from the origin to $x$ along the radius vector. The component of $A_\mu$ parallel to this path is $A_r$. Thus we have a line-integral of 'path ordered' exponentials of $A_\mu$ along this radial path. It is important to note that $A_\mu(x')$ at each $x'$ is a matrix and these matrices at different $x'$ will not commute in general and their ordering has to be defined.

We can write the boundary condition as (since the action is gauge invariant, if $A_\mu$ has finite action, so does $A'_\mu$, with $A'_r \equiv 0$):

$$(A'_\mu(x))_{S^\infty_3} = U(\alpha_1, \alpha_2, \alpha_3)\partial_\mu U^{-1}(\alpha_1, \alpha_2, \alpha_3)$$

(2.11)

where $U(\alpha_1, \alpha_2, \alpha_3)$ has to be defined only on $S^\infty_3$. The infinite different possibilities of choosing the boundary condition, that is to say $U(\alpha_1, \alpha_2, \alpha_3)$, can be related by homotopy considerations. Since the matrices $U$ form a two-dimensional, representation of $SU(2)$, the function $U(\alpha_1, \alpha_2, \alpha_3)$ represents a mapping of $S^\infty_3$ into the group space of $SU(2)$. By definition of $SU(2)$, the matrices $U$ are the set of all $2 \times 2$ unitary unimodular matrices. In order to classify the mappings of $S^\infty_3$ into $SU(2)$, let us study the topology of $SU(2)$. The elements of $SU(2)$ can be written uniquely in the form:

$$U = \sum_{\mu=1}^4 a_\mu s_\mu$$

(2.12)

where $s_4 = I$, is the unit $2 \times 2$ matrix, $s_{1,2,3} = i\sigma_{1,2,3}$, and $a_\mu$ are any four real numbers satisfying $\Sigma_\mu a_\mu a_\mu = 1$. The group is thus parametrised by these four real variables $a_\mu$ subject to this constraint. The group space is therefore the three-dimensional surface of a unit sphere in four dimensions. We will call this surface $S^\text{int}_3$. In this way we see that the function $U(\alpha_1, \alpha_2, \alpha_3)$ is a mapping of $S^\infty_3$ into $S^\text{int}_3$. 
2.2.1 The Pontryagin Index

We can define homotopy classes (a discrete number of classes) in the mappings $S^\infty_3 \to S^{int}_3$, each characterised by an integer $Q$. This integer is often called the Pontryagin index. The important point is that mappings from one class cannot be continuously deformed into a mapping from another class, we can say that they are topologically separated. Any given sector corresponds to a given homotopy class of the $U(\alpha_1, \alpha_2, \alpha_3)$ occurring in the boundary condition Eq. (2.11) for $A_\mu$. A field $A_\mu(x)$ in four-space belonging to a given sector $Q$ cannot be continuously deformed so as to fall into another sector without violating finiteness of action. For, if this were possible, the $U(\alpha_1, \alpha_2, \alpha_3)$ related to the boundary value of $A_\mu(x)$ would have also deformed from one homotopy class to another, which cannot happen by definition. The expression for the Pontryagin index $Q$ is:

$$Q \equiv \int Q(x)d^4x = -\frac{1}{16\pi^2} \int d^4x Tr[\tilde{F}_{\mu\nu}F_{\mu\nu}]$$ (2.13)

where:

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}.$$ (2.14)

and where $Q(x)$ is the topological charge density. Now we will show that the expression Eq. (2.13) gives the winding number i.e. the number of times the group SU(2) is wrapped around the surface at infinity $S^\infty_3$. We will show it stepwise with the following propositions:

**Proposition 2.2.1.** The integrand of the topological charge $Q$ is a total 4-divergence i.e. $Q(x) = \partial_\mu j_\mu$, where:

$$j_\mu = -(1/8\pi^2)\epsilon_{\mu\nu\alpha\beta} Tr[A_\nu(\partial_\alpha A_\beta + \frac{2}{3}A_\alpha A_\beta)]$$ (2.15)

**Proof.** We begin with an identity:

$$D_\mu \tilde{F}_{\mu\nu} \equiv \partial_\mu \tilde{F}_{\mu\nu} + [A_\mu, \tilde{F}_{\mu\nu}] = \epsilon_{\mu\nu\alpha\beta}\{\partial_\mu(\partial_\alpha A_\beta + A_\alpha A_\beta) + [A_\mu, (\partial_\alpha A_\beta + A_\alpha A_\beta)]\} = 0$$ (2.16)

where the last equality follows simply from expanding out the terms and using the antisymmetry of $\epsilon_{\mu\nu\alpha\beta}$. We can write:

$$-16\pi^2 Q(x) = Tr[F_{\mu\nu}\tilde{F}_{\mu\nu}] = Tr[(\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{F}_{\mu\nu} + (A_\mu A_\nu - A_\nu A_\mu)\tilde{F}_{\mu\nu}] =$$

$$= Tr[(\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{F}_{\mu\nu}] = Tr[(\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{F}_{\mu\nu} - A_\mu \partial_\nu \tilde{F}_{\mu\nu}] =$$

$$= Tr[\partial_\mu A_\nu \tilde{F}_{\mu\nu} - \partial_\nu (A_\mu \tilde{F}_{\mu\nu})]$$

where the cyclic property of the trace as well as $D_\mu \tilde{F}_{\mu\nu} = 0$ have been used. And finally by expanding out $\tilde{F}_{\mu\nu}$:

$$-16\pi^2 Q(x) = Tr[\epsilon_{\mu\nu\alpha\beta}(\partial_\mu A_\nu)(\partial_\alpha A_\beta + A_\alpha A_\beta) - \partial_\nu(A_\mu \partial_\alpha A_\beta + A_\mu A_\alpha A_\beta)] =$$

$$= Tr[\epsilon_{\mu\nu\alpha\beta}[2\partial_\mu(A_\nu \partial_\alpha A_\beta + \frac{2}{3}A_\nu A_\alpha A_\beta)]$$
where we have used: $\text{Tr}[\varepsilon_{\mu \nu \alpha \beta} (\partial_{\mu} A_{\nu}) A_{\alpha} A_{\beta}] = \frac{1}{3} \text{Tr}[\varepsilon_{\mu \nu \alpha \beta} (\partial_{\mu} (A_{\nu} A_{\alpha} A_{\beta}))]$ which follows from the cyclicity of trace and the antisymmetry of $\varepsilon_{\mu \nu \alpha \beta}$.

With the help of this proposition we can express $Q$ as an integral over $S^{\infty}_{3}$ as follows:

$$Q = \int Q(x) d^{4}x = \oint_{S^{\infty}_{3}} d\sigma_{\alpha} j_{\alpha}$$

(2.17)

On the surface at infinity $S^{\infty}_{3}$, our finite-action configurations have $F_{\mu \nu} = 0$ and therefore:

$$\varepsilon_{\mu \nu \alpha \beta} \partial_{\alpha} A_{\beta} = -\varepsilon_{\mu \nu \alpha \beta} A_{\alpha} A_{\beta}$$

(2.18)

and using Eq. (2.15) we have:

$$Q = \frac{1}{24\pi^{2}} \oint_{S^{\infty}_{3}} d\mu \varepsilon_{\mu \nu \alpha \beta} \text{Tr}[A_{\nu} A_{\alpha} A_{\beta}]$$

(2.19)

And inserting the asymptotic behavior of the fields $A_{\mu}$ Eq. (2.11):

$$Q = -\frac{1}{24\pi^{2}} \oint_{S^{\infty}_{3}} d\mu \varepsilon_{\mu \nu \alpha \beta} \text{Tr}[(\partial_{\mu} U)U^{-1}(\partial_{\alpha} U)U^{-1}(\partial_{\beta} U)U^{-1}]$$

(2.20)

In Eq. (2.20) we have the integrand in terms of the group-element-valued function $U$ on $S^{\infty}_{3}$. Now we have to discuss how to measure the group volume, and thus we can deduce how many times the group has been wrapped around $S^{\infty}_{3}$ by a given gauge function $U$. We can parametrise $SU(2)$ by three independent variables $\xi_{1}, \xi_{2}$ and $\xi_{3}$. The group measure may be written as:

$$d\mu(U) = \rho(\xi_{1}, \xi_{2}, \xi_{3}) d\xi_{1} d\xi_{2} d\xi_{3}$$

the function $\rho(\xi_{1}, \xi_{2}, \xi_{3})$ called density has to be such that the measure $d\mu(U)$ is invariant under group translations. This would correspond in the case of finite group with the fact that the number of elements cannot change when we multiply them by some given element. For example with $U' = \bar{U} U$ we have $(\xi_{1}, \xi_{2}, \xi_{3}) \rightarrow (\xi'_{1}, \xi'_{2}, \xi'_{3})$ and the density should be such that:

$$d\mu(U') = \rho(\xi'_{1}, \xi'_{2}, \xi'_{3}) d\xi'_{1} d\xi'_{2} d\xi'_{3} = d\mu(U') = \rho(\xi_{1}, \xi_{2}, \xi_{3}) d\xi_{1} d\xi_{2} d\xi_{3}$$

**Proposition 2.2.2.** The density:

$$\rho(\xi_{1}, \xi_{2}, \xi_{3}) = \varepsilon_{ijk} \text{Tr}(U^{-1} \frac{\partial U}{\partial \xi_{i}} U^{-1} \frac{\partial U}{\partial \xi_{j}} U^{-1} \frac{\partial U}{\partial \xi_{k}})$$

(2.21)

satisfies translation invariance.

**Proof.**

$$U(\xi_{1}, \xi_{2}, \xi_{3}) \rightarrow U'(\xi'_{1}, \xi'_{2}, \xi'_{3}) = \bar{U} U(\xi_{1}, \xi_{2}, \xi_{3}) ; \quad U = \bar{U}^{-1} U' ; \quad U^{-1} = (U')^{-1} \bar{U} \quad \text{and}$$

$$\rho(\xi_{1}, \xi_{2}, \xi_{3}) = \varepsilon_{ijk} \text{Tr}((U')^{-1} \bar{U} U^{-1} \frac{\partial U'}{\partial \xi'_{p}} \frac{\partial U^{-1}}{\partial \xi_{i}} U^{-1} \frac{\partial U'^{-1}}{\partial \xi'_{q}} \frac{\partial \xi'^{p}}{\partial \xi_{i}} U^{-1} \frac{\partial \xi_{q}}{\partial \xi_{i}} \frac{\partial \xi_{r}}{\partial \xi_{i}} \frac{\partial \xi_{s}}{\partial \xi_{i}}) =$$

$$= \varepsilon_{ijk} \text{Tr}((U')^{-1} \partial \xi'^{p} \frac{\partial U'}{\partial \xi'_{p}} (U')^{-1} \partial \xi'^{q} \frac{\partial U'^{-1}}{\partial \xi'_{q}} \partial \xi'_{r} U^{-1} \frac{\partial \xi'^{r}}{\partial \xi_{i}} U^{-1} \frac{\partial \xi^{p}}{\partial \xi_{i}} U^{-1} \frac{\partial \xi^{q}}{\partial \xi_{i}} U^{-1} \frac{\partial \xi_{r}}{\partial \xi_{i}} U^{-1} \frac{\partial \xi_{s}}{\partial \xi_{i}}) \quad \text{but}$$

$$\varepsilon_{ijk} | \frac{\partial \xi'^{p}}{\partial \xi_{i}} \frac{\partial \xi'^{q}}{\partial \xi_{j}} \frac{\partial \xi'^{r}}{\partial \xi_{k}} = \varepsilon_{pqr} \text{Det} \frac{\partial \xi'}{\partial \xi} |$$
Proposition 2.2.3. The expression of $Q$ Eq. (2.13) reduces to an integral over the group measure

Proof. We can write Eq. (2.20) as:

\[
Q = -\frac{1}{24\pi^2} \int_{S_3^\infty} d\sigma_{\mu}[\varepsilon_{\mu\nu\alpha\beta}\text{Tr}(U^{-1}\frac{\partial U}{\partial \xi_i}U^{-1}\frac{\partial U}{\partial \xi_j}U^{-1}\frac{\partial U}{\partial \xi_k}) \frac{\partial \xi_i}{\partial x_\alpha} \frac{\partial \xi_j}{\partial x_\beta}]
\] (2.22)

Instead of considering the hypersphere $S_3^\infty$ at infinity, we will consider an infinite hypercube with cartesian coordinates $x_\mu$. The eight sides of this cube are the surfaces $x_\mu = \pm\infty$, for $\mu = 1, 2, 3, 4$. The last equation can be written for example for the hypercube’s surface at $x_4 = -\infty$:

\[
Q = -\frac{1}{24\pi^2} \int dx_1 dx_2 dx_3 \varepsilon_{lmn} \frac{\partial \xi_i}{\partial x_1} \frac{\partial \xi_j}{\partial x_2} \frac{\partial \xi_k}{\partial x_3} d\xi_1 d\xi_2 d\xi_3
\] (2.23)

But we have: $\varepsilon_{lmn} \frac{\partial \xi_i}{\partial x_1} \frac{\partial \xi_j}{\partial x_2} \frac{\partial \xi_k}{\partial x_3} = \varepsilon_{ijk} d\xi_1 d\xi_2 d\xi_3$ and therefore the contribution from the surface $x_4 = -\infty$ is:

\[
-\frac{1}{24\pi^2} \int d\xi_1 d\xi_2 d\xi_3 \varepsilon_{ijk} \text{Tr}(U^{-1}\frac{\partial U}{\partial \xi_i}U^{-1}\frac{\partial U}{\partial \xi_j}U^{-1}\frac{\partial U}{\partial \xi_k})
\] (2.24)

A similar contribution comes from all eight surfaces of the hypercube, and finally we can say taking into account Eq. (2.21) that:

\[
Q \propto \int d\xi_1 d\xi_2 d\xi_3 \rho(\xi_1, \xi_2, \xi_3)
\] (2.25)

\[
Q \propto \int d\mu(U).
\] (2.26)

2.2.2 Valid Gauge Transformations

We still have to solve the fact that the boundary condition Eq. (2.11) is not gauge invariant, in fact:

\[
U\partial_\mu(U^{-1}) \rightarrow U'(U\partial_\mu(U^{-1}))(U')^{-1} + U'\partial_\mu(U')^{-1} = (U'U)\partial_\mu(U'U)^{-1}
\] (2.27)

On the other hand we have just seen that $Tr[\tilde{F}_{\mu\nu}F_{\mu\nu}]$ as well as $Q$ are gauge invariant.

If we could choose $U' = U^{-1}$, so that $A_\mu = 0$ on $S_3^\infty$ then $Q$ could be reduced to zero and that would violate the gauge invariance of $Q$. In Eq. (2.27) we have to distinguish between the behavior of $U$ and of $U'$. The former appears in the boundary condition and therefore it has to be non-singular only at $S_3^\infty$. The latter transforms the field throughout space and has to be non-singular at all $x$.

We can express $U'$ in polar coordinates as $U' = U'(|x|, \alpha_1, \alpha_2, \alpha_3)$, due to the non-singular behavior of $U'$ at all $x$, $U'$ must be a continuous function for all the values of $|x|$, $\alpha_1$, $\alpha_2$ and $\alpha_3$. Therefore at $|x| = 0$, $U'(0, \alpha_1, \alpha_2, \alpha_3)$ must be independent of the $\alpha_i$ in order to be non-singular.
That is, it must be a constant matrix. Any constant SU(2) matrix can be continuously obtained from the identity matrix. $U'$ must be continuously deformable to the identity matrix for all $|x|$. Letting $|x| \to \infty$, the boundary value of $U'$ on $S^\infty_3$ must also be continuously deformable to the identity matrix, i.e. it must lie in the $Q=0$ sector. If $U'$ were equal to $U^{-1}$ on $S^\infty_3$ then $U$ also would belong to $Q=0$. Conversely, when $U$ belongs to $Q \neq 0$, it cannot be continued all the way down to $|x| = 0$ without encountering a singularity.

From group and homotopy theory we know that if $U$ belongs to the index $Q_1$ and $U'$ to $Q_2$ then the function $U'U$ belongs to $Q_1 + Q_2$. We can conclude by saying that if $U'(x)$ is to be a valid gauge transformation, its boundary value on $S^\infty_3$ must correspond to $Q_2 = 0$. Thus $Q_1 + Q_2 = Q_1$ and the homotopy index will not be affected by gauge transformations, consistent with the gauge-invariant expression Eq. (2.13).
Chapter 3

The Yang-Mills Instantons

3.1 Bogomolny Bound and some Instanton Solutions

At the classical level, instantons are not very different from static solutions of Minkowskian equations (solitons). This is obviously because static solutions involve only the spatial coordinates, i.e. the Euclidean subspace of Minkowskian space-time. In this chapter we will look, among finite-action configurations in any given homotopy sector $Q$ for some configurations that actually solve the field equation. The exact $Q=+1$ (or -1) solution we obtain will be called the instanton (or anti-instanton). We will also get some exact solutions with $Q = N$ (or -N), with $|N| > 1$. These are the multi-instanton (or anti-instanton) solutions. We begin with the trivial identity:

\[- \int d^4x \text{Tr}[(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2] \geq 0 \quad (3.1)\]

Using $\text{Tr}[F_{\mu\nu}F_{\mu\nu}] = \text{Tr}[\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}]$, this gives:

\[- \int d^4x \text{Tr}[F_{\mu\nu}F_{\mu\nu} + \tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} \pm 2\tilde{F}_{\mu\nu}F_{\mu\nu}] \geq 0 \quad \text{this gives} \quad (3.2)\]

\[- \int d^4x \text{Tr}[F_{\mu\nu}F_{\mu\nu}] \geq \mp \int d^4x \text{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}] \quad (3.3)\]

Using the expression of the Euclidean action $S$ Eq. (2.7) and the expression of the Pontryagin homotopy index $Q$ Eq. (2.13), we can convert this inequality in the Bogomolny bound:

\[S \geq \left( \frac{8\pi^2}{g^2} \right)|Q| \quad (3.4)\]

We obtained the Yang-Mills equation Eq. (2.8) by extremising the action $S$ through the variational principle $\delta S[A_{\mu}] = 0$, in fact Eq. (2.8) is just an explicit form of this principle. Since such small variations will keep the fields within the same sector, we see that after the variation we can still keep
in the same homotopy sector. In this way we can find solutions of Eq. (2.8) falling in a homotopy sector and these fields will extremise the action in this homotopy sector. From Eq. (3.2) we see that the absolute minimum value of $S$ i.e.:

$$S = \frac{8\pi^2}{g^2}|Q|$$

(3.5)

is attained in any given sector $Q$ when:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu
u\rho\sigma}F_{\rho\sigma} = \pm F_{\mu\nu}$$

(3.6)

Thus, selfdual and anti-selfdual configurations extremise $S$, and hence solve Eq. (2.8). Of course, there can be other extrema apart from the absolute minima of $S$. In other words there can be other solutions of Eq. (2.8) different from the selfdual or the anti-selfdual solution. We will content ourselves with the selfdual and anti-selfdual solutions, in which case we need to solve only Eq. (3.6) rather than the (possibly) more general Eq. (2.8). That fields satisfying Eq. (3.6) also satisfy Eq. (2.8) can be seen in a straightforward way:

$$D_\mu \tilde{F}_{\mu\nu} = \partial_\mu \tilde{F}_{\mu\nu} + [A_\mu, \tilde{F}_{\mu\nu}] = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \{ \partial_\mu (\partial_\rho A_\sigma + A_\rho A_\sigma) + [A_\mu, (\partial_\rho A_\sigma + A_\rho A_\sigma)] \} = 0$$

This is satisfied by any $F_{\mu\nu}$ of the form Eq. (2.4), obviously when $\tilde{F}_{\mu\nu} = \pm F_{\mu\nu}$ the last equation reduces to the field equation Eq. (2.8). The alternate proof using Eq. (3.6) brings out the added fact that such solutions in fact give the absolute minimum of the action $S$ in any given Q-sector. Since $F = \pm \tilde{F}$ corresponds to an extremum $S$, any selfdual or anti-selfdual $F$ is a solution. An **anti-instanton** solution is a anti-selfdual $F$ having $Q = -1$. An **instanton** solution is a selfdual $F$ having $Q = 1$. The corresponding actions are $S_I = -8\pi^2/g^2$ and $S_I = +8\pi^2/g^2$. Let us make the following ansatz for the gauge field:

$$A_\mu(x) = i\Sigma_{\mu\nu}\partial_\nu(\ln \phi(x))$$

(3.7)

where $\phi(x)$ is a scalar function to be determined and:

$$\Sigma_{\mu\nu} = \frac{1}{2} \begin{pmatrix} 0 & +\sigma_3 & -\sigma_2 & -\sigma_1 \\ -\sigma_3 & 0 & +\sigma_1 & -\sigma_2 \\ +\sigma_2 & -\sigma_1 & 0 & -\sigma_3 \\ +\sigma_1 & +\sigma_2 & +\sigma_3 & 0 \end{pmatrix}$$

(3.8)

Or:

$$\tilde{\Sigma}_{\mu\nu} = \tilde{\eta}^{i\nu\sigma}\sigma^i/2; \quad i=1,2,3 \quad \text{with:}$$

$$\tilde{\eta}^{i\mu\nu} = -\tilde{\eta}^{i\nu\mu} = \begin{cases} \varepsilon^{i\mu\nu} \quad \text{for } \mu, \nu = 1, 2, 3 \\ -\delta^{i\mu} \quad \text{for } \nu = 4 \end{cases}$$

(3.9)
We can also define:

\[ \Sigma_{\mu\nu} = \eta^{i\nu} \sigma^{i}/2; \quad i=1,2,3 \quad \text{with:} \]

\[ \eta^{i\mu} = \left\{ \begin{array}{ll}
\varepsilon^{i\mu} & \text{for } \mu, \nu = 1, 2, 3 \\
\delta^{i\mu} & \text{for } \nu = 4
\end{array} \right. \]  

\[ \Sigma_{\mu\nu} = \frac{1}{2} \begin{pmatrix}
0 & +\sigma_3 & -\sigma_2 & \sigma_1 \\
-\sigma_3 & 0 & +\sigma_1 & \sigma_2 \\
+\sigma_2 & -\sigma_1 & 0 & \sigma_3 \\
-\sigma_1 & -\sigma_2 & -\sigma_3 & 0
\end{pmatrix} \]  

(3.13)

and:

\[ \tilde{\Sigma}_{\mu\nu} = \Sigma_{\mu\nu} \]  

(3.14)

\[ \bar{\Sigma}_{\mu\nu} \] fulfill the properties:

\[ [\bar{\Sigma}_{\nu\sigma}, \bar{\Sigma}_{\nu\rho}] = i[\delta_{\mu\nu} \Sigma_{\sigma\rho} + \delta_{\rho\sigma} \Sigma_{\mu\nu} - \delta_{\mu\sigma} \Sigma_{\nu\rho} - \delta_{\nu\sigma} \Sigma_{\mu\rho}] \]  

(3.15)

\[ \varepsilon_{\mu\nu\alpha\beta} \bar{\Sigma}_{\beta\sigma} = \delta_{\mu\sigma} \bar{\Sigma}_{\nu\alpha} + \delta_{\nu\sigma} \bar{\Sigma}_{\alpha\mu} + \delta_{\alpha\sigma} \bar{\Sigma}_{\mu\nu} \]  

(3.16)

\[ \tilde{\Sigma}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \bar{\Sigma}^{\alpha\beta} = -\Sigma_{\mu\nu} \]  

(3.17)

Thus we obtain for the tensor field:

\[ F_{\mu\nu} = i\bar{\Sigma}_{\nu\sigma}(\partial_\mu \partial_\sigma \ln \phi - \partial_\mu (\ln \phi) \partial_\sigma (\ln \phi)) - i\bar{\Sigma}_{\mu\sigma}(\partial_\rho \partial_\sigma \ln \phi - \partial_\rho (\ln \phi) \partial_\sigma (\ln \phi)) - i\bar{\Sigma}_{\mu\nu}(\partial_\sigma \ln \phi)^2 \]

and for the dual tensor field:

\[ \tilde{F}_{\mu\nu} = i\varepsilon_{\mu\nu\alpha\beta}\tilde{\Sigma}_{\beta\sigma}(\partial_\alpha \partial_\sigma \ln \phi - \partial_\alpha (\ln \phi) \partial_\sigma (\ln \phi)) - \frac{1}{2}\tilde{\Sigma}_{\alpha\beta}(\partial_\sigma \ln \phi)^2 \]

\[ = i\bar{\Sigma}_{\nu\alpha}(\partial_\mu \ln \phi - \partial_\alpha (\ln \phi) \partial_\mu (\ln \phi)) - (\mu \leftrightarrow \nu) + i\Sigma_{\mu\nu} \partial_\sigma \partial_\sigma (\ln \phi) \]

Self-duality yields:

\[ \partial_\mu (\partial_\sigma \ln \phi) - (\partial_\mu \ln \phi)(\partial_\sigma \ln \phi) = \partial_\sigma (\partial_\mu \ln \phi) - (\partial_\sigma \ln \phi)(\partial_\mu \ln \phi) \]  

(3.18)

and also:

\[ \partial_\sigma \partial_\sigma (\ln \phi) + (\partial_\sigma \ln \phi)^2 = 0 \]  

(3.19)

the first is evident, and the second is:

\[ \Box \phi \phi = 0 \]  

(3.20)

When \( \phi \) is non-singular Eq. (3.20) reduces to \( \Box \phi = 0 \), with the only solution \( \phi = constant \) and \( A_\mu = 0 \). When we consider singular \( \phi \) we obtain more interesting solutions. For example, with \( \phi(x) = 1/|x|^2 \). At \( x \neq 0 \) we have:

\[ \Box \phi \frac{1}{\phi} \partial_\sigma \left( \frac{2x_{\sigma}}{|x|^4} \right) = 0 \]  

(3.21)
At \( x = 0 \), \( \Box (1/|x|^2) = -4\pi^2 \delta^4(x) \). Thus even at \( x = 0 \) \( \frac{\partial \phi}{\partial x} = -4\pi^2 |x|^2 \delta^4(x) = 0 \). Therefore \( \phi(x) = 1/|x|^2 \) solves Eq. (3.20) and so does the more general form:

\[
\phi(x) = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|x - a_i|^2}, \quad \mu = 1, 2, 3, 4
\]  

(3.22)

where \( a_i \) and \( \lambda_i \) are any real constants, which are respectively the center and the size of the instanton. The existence of solutions of arbitrary sizes is a necessary consequence of the scale invariance of the classical field theory.

Let us begin with \( N = 0 \), which corresponds to \( \phi(x) = 1 \) and \( A_\mu = 0 \). This is the trivial solution and belongs to the \( Q = 0 \) sector.

### 3.2 The Instanton Solution

We consider \( N = 1 \) in Eq. (B.7). With \( y_\mu = (x_\mu - a_1) \) and \( y^2 \equiv y_\mu y_\mu \), we have \( \phi(x) = 1 + \lambda_1^2/|y|^2 \) and:

\[
A_\mu(x) = -2i \lambda_1^2 \Sigma_{\mu\nu} \frac{y_\nu}{y^2(y^2 + \lambda_1^2)}
\]  

(3.23)

This solution is singular at \( y = 0 \). The singularity can be removed by a correspondingly singular gauge transformation. Since the self-duality condition and the equation of motion are gauge covariant, the resulting function will continue to satisfy them. The required gauge function is:

\[
U_1(y) = (y_4 + iy_3 \sigma_3)/|y|
\]  

(3.24)

this is because of appendix B Eq. (B.7) and from the definitions Eq. (3.9), Eq. (3.11)

\[
U_1(y) \frac{\partial}{\partial y_\mu} [U_1(y)]^{-1} = -2i \Sigma_{\mu\nu} (y_\nu/y^2), \quad \text{in appendix B:} \quad \Sigma_{\mu\nu} = -\frac{1}{2} \sigma_{\mu\nu} \quad \tilde{\Sigma}_{\mu\nu} = -\frac{1}{2} \tilde{\sigma}_{\mu\nu}
\]

and:

\[
U_1(y)^{-1} \frac{\partial}{\partial y_\mu} [U_1(y)] = -2i \Sigma_{\mu\nu} (y_\nu/y^2).
\]  

(3.25)

Thus we can write Eq. (3.23) as:

\[
A_\mu(x) = U_1(y)^{-1} \frac{\partial}{\partial y_\mu} [U_1(y)] \frac{\lambda_1^2}{y^2 + \lambda_1^2}
\]  

(3.26)

By gauge transforming with \( U_1(y) \):

\[
A_\mu(x) \rightarrow A'_\mu(x)
\]

\[
= [U_1(y)] [A_\mu + \partial_\mu] [U_1(y)]^{-1}
\]

\[
= (\partial_\mu U_1) U_1^{-1} \left( \frac{\lambda_1^2}{y^2 + \lambda_1^2} - 1 \right)
\]

\[
= - (\partial_\mu U_1) U_1^{-1} \frac{y^2}{y^2 + \lambda_1^2} = U_1 (\partial_\mu U_1^{-1}) \frac{y^2}{y^2 + \lambda_1^2}
\]
and finally:

$$A'_\mu (x) = -2i \Sigma_{\mu \nu} \frac{y_\nu}{y^2 + \lambda_1^2} = -2i \Sigma_{\mu \nu} \frac{(x - a_1)_\nu}{|(x - a_1)|^2 + \lambda_1^2} \quad (3.27)$$

This solution has no singularities at any \(x\) for any given \(\lambda_1 \neq 0\). The corresponding tensor field, using Eq. (2.4):

$$F'_{\mu \nu} = 4i \Sigma_{\mu \nu} \frac{\lambda_1^2}{||x - a_1||^2 + \lambda_1^2} \propto \frac{1}{x^4} \quad (3.28)$$

This solution can be compared with Eq. (B.12).

**Proposition 3.2.1.** The Pontryagin index of the instanton solution is \(Q = 1\).

**Proof.**

$$U_i(x) = \frac{(x_4 + ix_j \sigma_j)}{r} = \sum_\mu \hat{x}_\mu s_\mu \quad (3.29)$$

On comparing it with the general representation Eq. (2.12), this gauge function correspond to \(a_\mu = \hat{x}_\mu\). That is, every point on \(S^3\) is mapped on to the corresponding point, at the same polar angles, on \(S^3_{\text{int}}\). Thus the homotopy index \(Q\) must be equal to unity

$$Q = -\frac{1}{24\pi^2} \int_{S^3} dS \varepsilon^{ijk} Tr(\lambda^i \lambda^j \lambda^k) \quad (3.30)$$

where \(\lambda^k = U \partial^k U^{-1}\) To calculate \(Q\), we note that the surface integral extends over values of \(x\) on \(S^3\). Going from one value of \(x\) to another on \(S^3\) is a rotation of \(S^3\), and such a rotation can always be undone by a continuous change of \(U\), which does not affect \(Q\). Under a continuous change of \(U\), the integrand of Eq. (3.30) changes by an 4-divergence \(\partial_\mu X^\mu\) therefore, we can write the integrand as its value at any fixed \(x\), plus \(\partial_\mu X^\mu\), and the latter gives no contribution upon integration. Therefore, to calculate \(Q\) it suffices to evaluate the integrand at any fixed \(x\), and multiply the result by \(2\pi^2\), the volume of \(S^3\). It is most convenient to evaluate the integrand at \((x_4 = 1, x = 0)\). In the neighborhood of this point, we can set up 3-dimensional Cartesian axes \(x_1, x_2, x_3\) lying in \(S^3\). Then we have, at this point,

$$U = 1, \quad \partial^k U^{-1} = \partial^k \left( \frac{x_4 + ix \cdot \sigma}{r} \right) = \frac{i\sigma^k}{r} - \frac{x^k}{r^3} (x_4 + ix \cdot \sigma) = i\sigma^k$$

therefore \(\lambda^k = i\sigma^k\)

and finally

$$Q = (2\pi^2) \left( -\frac{i}{24\pi^2} \varepsilon^{ijk} Tr(\sigma^i \sigma^j \sigma^k) \right) = 1$$

Another way to prove that is to see that the action for this field reduces to:

$$S = -\frac{1}{2g^2} \int d^4x Tr[F'_{\mu \nu} F'_{\mu \nu}] = \frac{48\lambda_1^2}{g^2} \int \frac{d^4y}{(y^4 + \lambda_1^2)^2} = \frac{8\pi^2}{g^2} \quad (3.31)$$
and then knowing that for the selfdual solution $\tilde{F}_{\mu\nu} = F_{\mu\nu}$ and comparing with Eq. (3.5) which we write again:

$$S = \frac{8\pi^2}{g^2} |Q|$$

(3.32)

we see that $Q=1$.

### 3.3 N-Instanton Solutions

We obtain these solutions by inserting Eq. (B.7) in Eq. (3.7):

$$A_\mu(x) = i\Sigma_{\mu\nu}\partial_\nu[\ln(1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|y_i|^2})]$$

(3.33)

$$= -2i\Sigma_{\mu\nu}(\sum_{i=1}^{N} \frac{\lambda_i^2 y_{i\nu}}{|y_i|^2})/(1 + \sum_{j} \frac{\lambda_j^2}{|y_j|^2})$$

(3.34)

Where $(y_i)_\mu \equiv (x - a_i)_\mu$, using Eq. (3.25) we can write it as:

$$A_\mu(x) = \sum_{i=1}^{N} U^{-1}_1(y_i)\partial_\mu U_1(y_i)f_i(x)$$

(3.35)

where:

$$f_i(x) = \frac{(x - a_i)^2}{(x - a_i)^2 + \sum_{j} \frac{\lambda_j^2}{(x - a_j)^2}}$$

(3.36)

and $U_i(y_i)$ is the same kind of $Q = 1$ gauge function given by Eq. (3.24). This field $A_\mu(x)$ is a solution of the Yang-Mills equation since it has been derived from the selfduality condition. But it is also singular, it has $N$ singularities, at the points $x = a_i$, $i = 1 \cdots N$

$$f_j(x) \to \delta_{ij} \quad \text{as} \quad x \to a_i$$

(3.37)

$$A_\mu \to U^{-1}_1(x - a_i)\partial_\mu U_1(x - a_i) \quad \text{as} \quad x \to a_i$$

(3.38)

Thus, near any one of its singularities the N-instanton solution behaves like a pure gauge, just as did the one-instanton solution near its center. Therefore, the N-instanton solution can be thought of as a collection of $N$ individual single instantons. Near the core of the $i$th instanton ($x \simeq a_i$), the effect of the other instantons is negligible, and the N-instanton solution is dominated by that single instanton. From Eq. (3.37), we see that the singularities can be removed if we find a gauge function that reduces to $U_1(x - a_i)$ when $x$ approaches any of the $a_i$. We take:

$$U_N(x) = (Z_4 + iZ \cdot \sigma)/\sqrt{Z^2}$$

(3.39)

with:

$$Z_\mu(X) = \sum_{i=1}^{N} \beta_i \frac{y_{i\mu}}{y_i^2}$$

(3.40)
and $y_i = x - a_i$ where $\beta_i$ are constants. As $y_i \to 0$ for some $i$, $Z_\mu \to \beta_i y_i \mu / y_i^2$, $Z^2 \to \beta_i^2 / y_i^2$ and:

$$U_N(x) \to \beta_i(y_i + i \sigma \cdot y_i / y_i^2) y_i / \beta_i = U_1(y_i)$$

Gauge transformation of $A_\mu(x)$ by $U_N(X)$ will remove all the singularities at $x = a_i$:

$$A'_\mu(x) = U_N(x)(A_\mu + \partial_\mu)U^{-1}_N$$

$$A'_\mu(x) \xrightarrow{x \to a_i} U_1(y_i)[U_1(y_i)^{-1}\partial_\mu U_1(y_i)]U_1^{-1}(y_i) + U_1(y_i)\partial_\mu U_1^{-1}(y_i) + \text{possible finite terms}$$

$$= \partial_\mu[U_1(y_i)U_1^{-1}(y_i)] + \text{finite terms}$$

The first term vanishes, and the singularity at $x = a_i$ is removed. This holds for each $a_i$, $i = 1 \cdots N$. However, there is the danger that new singularities appear because of the gauge function $U_N(x)$ Eq. (3.39), that has another singularities at $Z_\mu = 0$. We still have the arbitrary real constants $\beta_i$ in the definition of $Z_\mu$, to play with.

**Proposition 3.3.1.** The Pontryagin index of the N-instanton solution is $Q = N$.

**Proof.** From the Eq. (2.13)

$$Q = -\frac{1}{16\pi^2} \int d^4x Tr[\tilde{F}'_{\mu\nu}F'_{\mu\nu}]$$

where $F'_{\mu\nu}$ is the tensor field corresponding to $A'_\mu$. Since the integrand is non-singular, we can exclude from the integration region N small spheres with centers at $x = a_i$ and radii $\varepsilon$ each, where $\varepsilon \to 0$. For any finite small $\varepsilon$, the integration is over $V_\varepsilon$, which stands for all space minus these N spheres. In $V_\varepsilon$ even the original solution $A_\mu(x)$ is non-singular and furthermore:

$$Tr[\tilde{F}_{\mu\nu}F_{\mu\nu}] = Tr[\tilde{F}'_{\mu\nu}F'_{\mu\nu}]$$

where $F_{\mu\nu}$ corresponds to $A_\mu$, since $A_\mu$ and $A'_\mu$ are related by gauge transformations. Hence:

$$Q = \lim_{\varepsilon \to 0} -\frac{1}{16\pi^2} \int d^4x Tr[\tilde{F}'_{\mu\nu}F'_{\mu\nu}]$$

(3.41)

The singularities of $A_\mu$ are excluded from $V_\varepsilon$ so we can use Gauss’s theorem:

$$Q = \lim_{\varepsilon \to 0} \left( \oint_{S_{\infty}} j_\mu d\sigma_\mu - \oint_{S_1} j_\mu d\sigma_\mu - \oint_{S_2} j_\mu d\sigma_\mu - \cdots - \oint_{S_N} j_\mu d\sigma_\mu \right)$$

(3.43)

where $j_\mu$ is the current. As $|x| \to \infty$, $A_\mu(x)$ behaves as $1/|x^3|$ and hence $\oint_{S_{\infty}} j_\mu d\sigma_\mu \to 0$. On the tiny spheres $(x \to a_i)$ $A_\mu$ becomes a pure gauge, with gauge function $U_1^{-1}(x - a_i)$. When $A_\mu$ is a pure gauge, we can use the result of a former proposition to obtain $\oint j_\mu d\sigma_\mu$ with gauge function $U_1$, in our case the gauge function is $U_1^{-1}$ and the value of the integral in each tiny sphere is -1, and we have:

$$Q = \lim_{\varepsilon \to 0} |(0 - (-1) - (-1) \cdots - (-1))| = N$$

(3.44)
Since these N-instanton solutions are selfdual, their action is \( S = 8\pi^2 Q/g^2 = 8\pi^2 N/g^2 \). In a Minkowskian space a static solution of N-solitary waves separated by finite distances, would be expected generally to have a different energy from N such isolated solitary waves. The difference in energy would be attributed to the interaction between these solitary waves. For our Euclidean system, the action functional S, which is somewhat analogous, has the remarkable property:

\[
S(N \text{ instatons}) = NS(\text{one instaton})
\]  

(3.45)

Anti-selfdual and N-anti-instanton solutions are obtained in the same way, by replacing \( \tilde{\Sigma}_{\mu\nu} \) by \( \Sigma_{\mu\nu} \) in the ansatz Eq. (3.7).

### 3.4 Symmetries and the Number of Instantons Parameters

It has to be noticed that in a 4-dimensional space our coupling constant g has dimension zero (constant), what does not occur for dimensions other than 4. This produces many problems in connection with infrarot-divergence above all for large instantons.

The Yang-Mills Instanton possesses symmetry under the full 4-dimensional conformal group, which has 15 generators corresponding to:

- Scale or dilatation transformations
- The four translations
- The six O(4) rotations
- The four special conformal transformations: \( x_\mu \rightarrow \left( \frac{x_\mu - \alpha_\mu x^2}{1 - 2\alpha_\mu x_\mu + x^2} \right) \) all this in addition to gauge transformations.

We would like to know the number of physically distinct solutions we can obtain from a particular N-instanton solution by exploiting the full symmetries of the system. For the case of the one-instanton solution. We have already seen five parameters, \( \lambda \) and \( a_\mu \), corresponding to dilatation and translation symmetry. The effect of the remaining generators of the conformal group (O(4) rotations and special conformal transformations) can be undone by gauge transformations and translations (see Jackiw and Rebbi [35], [36], [37], [38]). Local gauge transformations are not the same kind of symmetries as spatial rotations. Their action produces not a physically distinct system, but the same system in a different (gauge) convention. One fixes the gauge by some gauge condition and then derives physical consequences. For the global gauge transformations (i.e. \( x \)-independent) however, it has been proved (see Coleman [15]) that these are symmetries like rotations since they lead to conservation laws. In our case the global gauge transformations are elements of SU(2) therefore global symmetry gives three more degrees of freedom, adding these 3 parameters to the 5 corresponding to dilatation and translation symmetry we obtain a total of eight free parameters for the single instanton. The solution
expressed by Eq. (3.27) reveals only five of these eight free parameters for the single instanton. In order to exploit the full symmetry of the system we have to substitute the specific gauge function $U_1(x)$ in Eq. (3.24) by a global element in the group space. We would expect $8N$ parameters in an $N$-instanton solution. We have already seen that the $N$-instanton solution looks like a one-instanton solution near each of the $a_i$. Equivalently, when the sizes $\lambda_i$ of the individual instantons are small compared with their separations, the $N$-instanton solution will look like a collection of $N$ (almost) non-overlapping single instantons. This count of $8N$ degrees of freedom could be reduced to $8N - 3$ if we decide on physical grounds to ignore global transformations as symmetries. In Eq. (3.33) we have only $5N$ parameters because of the limitations of the ansatz Eq. (3.7) (for a rigorous derivation see Atiyah and Ward [1]).
Chapter 4

Some aspects in QCD

4.1 The Vacuum in a Quantum Non-Abelian Gauge Theory

In this section we will continue with the non-abelian Yang-Mills theory, we will work with SU(2), but our analysis will also hold for self-coupled gauge fields of any higher SU(N) group in (3+1) dimensions, with minor modifications and this is because of the Bott’s Theorem, that states that any continuous mapping of \( S^\infty_3 \) into any general simple Lie group \( G \) can be continuously deformed into a mapping of \( S^\infty_3 \) into an SU(2) subgroup of \( G \). QCD is essentially an SU(3) theory, with fermions added.

Let us recall the equations of our system in Minkowskian space:

\[
S = \frac{2}{g^2} \int d^4 x \text{Tr}[F_{\mu\nu} F^{\mu\nu}] = \frac{1}{g^2} \int d^4 x \text{Tr}[(\partial^0 A + \nabla A^0 - [A^0, A]) \cdot E + \frac{1}{2}(E^2 + B^2)] = 4.1
\]

\[
= \frac{2}{g^2} \int d^4 x \text{Tr}[\partial^0 A \cdot E + \frac{1}{2}(E^2 + B^2) - A^0(\nabla \cdot E + [A, E])] = 4.2
\]

we see that the canonical variable are \( A \) and \( 2/g^2 E \) here we have introduced the analogs of the electric and magnetic fields: \( E^i_a = F^i_{a0} \) and \( B^j_i = -\frac{1}{2} \varepsilon_{ijk} F^k_{aj} \) \( i, j, k = 1, 2, 3 \)

The first step is to identify the minimum-energy configurations (the classical vacua) of this system, so that a suitable vacuum state may be constructed around them. The set of classical vacua of this system are the pure gauges satisfying \( F_{\mu\nu} = 0 \), or equivalently according to Eq. (2.9):

\[
A_\mu(x_\mu) = e^{-\alpha(x_\mu)} \partial_\mu e^{\alpha(x_\mu)}
\]

where \( \alpha(x_\mu) \) is any traceless anti-hermitian \((2 \times 2)\) matrix function, \( \alpha \) is time-dependent, we can remove it by partially fixing the gauge so that \( A_0(x_\mu) = 0 \) and then we have \( \alpha(x_i) \) where \( i = 1, 2, 3 \).

Further, we can restrict ourselves to those \( \alpha \) which satisfy \( e^{\alpha} = 1 \) at all points \( |x| = \infty \). We will
show later that quantum tunnelling as calculated by the appropriate Euclidean functional integral will connect the $\alpha = 0$ configuration with only those classical vacua which satisfy $[e^{\alpha(x)}]_{x \to \infty} = 1$. Therefore three-dimensional space is compacted into the surface of a hypersphere $S^3_\infty$. Each such function gives a mapping of this $S^3_\infty$ into the group space of SU(2). Such mappings can be classified into homotopy sectors $\pi_3(SU(2)) = \mathbb{Z}$ and we can adapt Eq. (2.20) to this situation (with $N$ instead of $Q$):

$$ N = \frac{1}{24\pi^2} \int d^3 x \varepsilon_{ijk} Tr[(e^{\alpha} \nabla_i e^{-\alpha})(e^{\alpha} \nabla_j e^{-\alpha})(e^{\alpha} \nabla_k e^{-\alpha})] $$

Consequently the set of classical vacua:

$$ A_i(x) = e^{-\alpha(x)} \nabla_i e^{\alpha(x)} $$

(4.3)

can be divided into a discrete infinity of sectors, each sector associated with a given homotopy class $N$ of the corresponding gauge group element $e^{-\alpha(x)}$. For convenience, we have taken $e^\alpha = 1$ as $x \to \infty$, an other constant than 1 is also possible. The extent of gauge equivalence under time-independent transformations (which are the only ones left in the $A_0 = 0$ gauge) is fixed by Gauss’s law. As generalised to Yang-Mills theory this law states that:

$$ D_i E^i = \partial_i E^i + [A_i, E^i] = 0 $$

(4.4)

Here $E^i \equiv F^{0i}$ is the Yang-Mills "electric" field in 2x2 matrix form, this equation has no time derivatives and is therefore a constraint. It has to be imposed as a condition on physical states which are required to satisfy $D_i E^i(\Psi)_{phys} = 0$. Under infinitesimal transformations $e^{-\delta \Lambda(x)} \simeq 1 - \delta \Lambda(x)$ we have $A_i \to A_i + [A_i, \delta \Lambda] + \nabla_i (\delta \Lambda) = A_i + D_i (\delta \Lambda)$, where $D_i$ is the covariant derivative. We consider for a moment only those gauge functions $\Lambda(x)$ which satisfy $\Lambda(x) \to 0$ as $x \to \infty$, the operator generating the gauge transformations $e^{-\Lambda(x)}$ is:

$$ U = \exp(\frac{2i}{g^2} \int d^3 x Tr((D_i \Lambda)^E_i)) = \exp(-\frac{2i}{g^2} \int d^3 x Tr(\Lambda D_i E^i)) $$

(4.5)

By virtue of Gauss’s law Eq. (4.4):

$$ U_\Lambda |\Psi\rangle_{phys} = |\Psi\rangle_{phys} $$

(4.6)

Thus Gauss’s law forces gauge equivalence only under gauge transformations, which satisfy $\Lambda(x \to \infty) = 0$. By contrast, consider the transformation $g_1(x)$ which takes the $N=n$ sector into the $N=n+1$ sector. The corresponding gauge function $\Lambda_1(x)$ does not satisfy $\Lambda_1(x \to \infty) = 0$. Therefore, states in different homotopy sectors are not forced to be gauge equivalent by Gauss’s law.

We therefore have a distinct topological vacuum $|N\rangle$ in each sector. These are not the true vacua since they will tunnel into one another. We can anticipate this, at least in a semiclassical expansion,
since the Yang-Mills system permits finite-action instanton solutions. The correct vacua will be linear combinations of the topological vacua:

\[ |\theta\rangle = \sum_{N=-\infty}^{\infty} e^{iN\theta} |N\rangle \] (4.7)

Consider the unitary operator \( U(g_1) \) which executes the gauge transformation \( g_1 \) and gives \( U(g_1)|N\rangle = |N + 1\rangle \). Since \([U(g_1), H] = 0\) and \( U(g_1) \) is unitary, the eigenstates of \( H \) must satisfy \( U(g_1)|\theta\rangle = e^{-i\theta}|\theta\rangle \), which is fulfilled by Eq. (4.7). Finally, if \( B \) is any gauge-invariant operator, \([B, U(g_1)] = 0\). Hence

\[ 0 = \langle \theta | [B, U(g_1)] | \theta' \rangle = (e^{-i\theta'} - e^{-i\theta}) \langle \theta | B | \theta' \rangle \]

Therefore, \( \langle \theta | B | \theta' \rangle = 0 \) if \( \theta \neq \theta' \). Hence each \( |\theta\rangle \) is the vacuum of a separate sector of states, unconnected by any gauge-invariant operators. Let us turn to the Euclidean functional integral which gives the tunnelling amplitude. The fields \( A_\mu \) to be integrated over must approach vacuum (pure gauge) configurations at the boundary of Euclidean space-time. This boundary is a surface \( S^3_\infty \). Since a pure gauge field has the form \( A_\mu = e^{-\alpha} \partial_\mu e^\alpha \), where \( e^{-\alpha} \) is a element of \( SU(2) \), each given boundary condition is a mapping of \( S^3_\infty \rightarrow SU(2) \). Such mappings are again classifiable into homotopy sectors with an associated Pontryagin Index Eq. (2.13):

\[ Q = -\frac{1}{16\pi^2} \int d^4x \text{Tr}[\tilde{F}_{\mu\nu} F_{\mu\nu}] = \frac{1}{24\pi^2} \oint_{S^3_\infty} d\sigma^\mu \varepsilon^{\mu\nu\alpha\beta} \text{Tr}[A_\nu A_\alpha A_\beta] \]

The Euclidean functional integral in the sector \( Q \) will be:

\[ \lim_{\tau \to \infty} [G(\tau)]_Q = \lim_{\tau \to \infty} \text{Tr}(e^{-\frac{H\tau}{\hbar}}) = \int \{ D[A_\mu] \} Q \exp[-S_E + S_{gf}] \] (4.8)

where \( S_{gf} \) are the gauge-fixing terms [17].
Next, consider the $A_0 = 0$ gauge, and distort the boundary of space-time into a large closed cylinder (Fig.6). The two flat surfaces of the cylinder stand for all of three-dimensional space, at $\tau = \pm \infty$ respectively, while the curved surface stands for the boundary of space $x \to \infty$ at all $\tau$. In the $A_0$ gauge, the curved surface will make no contribution to the surface integral for $Q$. Hence $Q = N_+ - N_-$, where $N_\pm = \frac{1}{{2\pi^2}} \int dx^4 \varepsilon_{ijk} T [A_i A_j A_k]_{\tau = \pm \infty}$; furthermore, one can use the freedom of time-independent gauge transformations to set $\alpha(x) = 0$ $(A_i(x) = 0)$ at $\tau = -\infty$. Then $N_- = 0$ and $Q = N_+$; the functional integral can be interpreted, in this gauge, as the Euclidean transition amplitude connecting the $N=0$ sector to the $N=Q$ sector. Finally, consider the curved surface of the cylinder where, in the $A_0 = 0$ gauge, $0 = A_0(x \to \infty, \tau) = [e^{-\alpha(x,\tau)}(\partial/\partial \tau)e^{\alpha(x,\tau)}]_{x \to \infty}$. Hence $[e^{\alpha(x,\tau)}]_{x \to \infty}$ is independent of $\tau$. Since in the initial configuration at $\tau = -\infty$ we have chosen $e^{\alpha(x,-\infty)} = 1$, therefore in the final configuration at $\tau = \infty$, again $e^{\alpha(x,-\infty)} = 1$. Thus transition takes place only to those classical vacua where $[e^{\alpha(x,\tau)}]_{x \to \infty} = 1$. Our homotopy analysis of the classical vacua, which we based on this restriction, is hence justified. Finally the Euclidean functional integral would be $(\tau \to \infty)$:

$$\langle \theta | e^{-\frac{\theta}{4} | \theta \rangle = \sum_{N,Q} e^{-iQ\theta} \langle N + Q | e^{-\frac{\theta}{4}Q} | N \rangle = 2\pi \delta(0) \sum_Q e^{-iQ\theta} \int \mathcal{D}[A_\mu] e^{-S_{Euc}} =$$

$$= 2\pi \delta(0) \int \mathcal{D}[A_\mu] e^{-4\theta \int \mathcal{D}[F_{\mu\nu} \tilde{F}_{\mu\nu}] \exp(-S_{Euc}) + \frac{i\theta}{16\pi^2} \int \mathcal{D}[D_{\nu}] e^{-S_{Euc}}$$

We have an extra term in the Minkowskian Lagrangian $\Delta \mathcal{L}_\theta = \frac{\theta}{16\pi^2} \int Tr(F_{\mu\nu} \tilde{F}_{\mu\nu})$, this term is a total divergence and will not affect the classical Yang-Mills equations. The problem is that with this additional term the system violates P and T symmetry. The exception is the $\theta = 0$ case when the extra term vanishes.

### 4.2 Massless Fermions in QCD

When massless spin-1/2 fields are coupled to the Yang-Mills fields, they lead to further modifications in the behaviour of the vacuum state. The system of massless fermions coupled to non-abelian gauge fields has some relevance to hadron physics. Quantum chromodynamics (QCD) is a candidate to describe hadron physics and is a non-abelian gauge theory in (3+1) dimensions. The gauge group ('colour' group ) is SU(3), and therefore calls for eight gauge fields $A^a_\mu$ $a = 1, 2, \ldots 8$. These can be represented collectively by an anti-hermitian traceless $3 \times 3$ matrix, analogous to the Yang-Mills system. Let $\lambda_a/2$ be the hermitian traceless generators of SU(3) in the fundamental $3 \times 3$ representation. They satisfy $[\lambda_a/2, \lambda_b/2] = if^{abc}\lambda_c/2$ where $f^{abc}$ are the structure constants of the group SU(3), and $Tr(\lambda^a \lambda^b) = 2\delta^{ab}$ finally $A_\mu(x, t) = \frac{g}{2i} \sum \lambda^a A^a_\mu(x, t)$, where $g$ is the coupling constant we have also the Eq. (2.4): $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. In QCD, these gauge fields
couple to several species of spin-1/2 quarks. Each species is represented by the Dirac field \( \Psi^f_\alpha \), where \( f \) is a flavour index, \( \alpha \) is a colour index and the spin index has been suppressed. For a given flavour (up, down, strange, charm, top, bottom) \( \Psi^f_\alpha \) (\( \alpha = 1, 2, 3 \)), transforms according to the fundamental representation of the colour SU(3) group. The quarks have masses \( m_f \), these in general may depend on flavour but not on colour, which is an exact gauge symmetry of the model. Our gauge-invariant Lagrangian density for the model is:

\[
\mathcal{L}(x, t) = \sum_{f, \alpha, \beta} \bar{\Psi}_\alpha^f (i \gamma^\mu D_\mu - m_f)_{\alpha\beta} \Psi^\beta_f + \frac{1}{2g^2} Tr[F_{\mu\nu}^\rho F^{\mu\nu}] \tag{4.9}
\]

where \( (D_\mu)_{\alpha\beta} = (\partial_\mu + A_\mu)_{\alpha\beta} \). We can do the decomposition: \( \Psi = \Psi_L + \Psi_R \) where:

\[
\Psi_L, R = \frac{1}{\sqrt{2}} \gamma_5 \Psi
\]

Indirect evidence on quark masses suggests that the masses of the up and down quarks are much smaller than the masses of quarks of other flavours. When \( m_f = 0 \) the model Eq. (4.9) enjoys further symmetries beyond SU(3):

\[
\mathcal{L}(x, t) = \sum_f \bar{\Psi}_L^f (i \gamma^\mu D_\mu) \Psi_L + \sum_f \bar{\Psi}_R^f (i \gamma^\mu D_\mu) \Psi_R + \frac{1}{2g^2} Tr[F_{\mu\nu}^\rho F^{\mu\nu}]
\]

\[
= \sum_f \bar{\Psi}_L' (i \gamma^\mu D_\mu) \Psi_L' + \sum_f \bar{\Psi}_R' (i \gamma^\mu D_\mu) \Psi_R' + \frac{1}{2g^2} Tr[F_{\mu\nu}^\rho F^{\mu\nu}].
\]

Where:

\[
\Psi_L' = R_j^i \Psi_L^j
\]

and \( R \in SU(\nu) \), therefore \( SU(\nu)_L \times SU(\nu)_R \) is a symmetry of \( \mathcal{L} \) when \( m = 0 \).

Let the Dirac operator be \( i \not{D} = i D_\mu \gamma^\mu = i(\partial_\mu + A_\mu)\gamma^\mu \), we can determine the spectrum of this operator \( i \not{D} \psi_\lambda = \lambda \psi_\lambda \) we have \( \not{D}^+ = -\not{D} \), therefore all the eigenvalues of \( \not{D} \) are \( \{i\lambda_k, -i\lambda_k\} \). Quarks modify the weight in the euclidean partition function by the fermionic determinant:

\[
\prod_f det[i \not{D}(A_\mu) - m_f] \tag{4.10}
\]

Using the fact that non-zero eigenvalues come in pairs, the determinant for each flavour can be written as:

\[
det[i \not{D} - m] = m^\nu \prod_{\lambda \neq 0} (i\lambda - m) = m^\nu \prod_{\lambda \geq 0} (i\lambda - m)(-i\lambda - m) = m^\nu \prod_{\lambda \geq 0} (\lambda^2 + m^2) \tag{4.11}
\]

where \( \nu \) is the number of zero modes. Note that the integration over fermions gives a determinant in the numerator. This means that (as \( m \to 0 \)) the fermion zero mode makes the tunneling amplitude
vanish and individual instantons cannot exist. In the QCD vacuum, however, chiral symmetry is spontaneously broken and the quark condensate $\langle \bar{q}q \rangle$ is non-zero. The quark condensate is the amplitude for a quark to flip its chirality, so we expect that the instanton density is not controlled by the current masses, but by the quark condensate, which does not vanish as $m \to 0$. The experimental value for the quark condensate is $\langle \bar{q}q \rangle \approx -(230\text{MeV})^3$. The perturbations theory gives no explanation to this value.

### 4.2.1 Banks-Casher Relation

The quark propagator $P(x,y) = -\langle x| (i\slashed{D})^{-1} |y \rangle$ is given by:

$$P(x,y) = -\sum_{\lambda} \frac{\psi_{\lambda}(x)\psi_{\lambda}^+(y)}{\lambda + im} \tag{4.12}$$

using the fact that the eigenfunctions are normalized:

$$\int d^4x \text{Tr}(P(x,x)) = -\sum_{\lambda} \frac{1}{\lambda + im}$$

$$\langle \bar{\Psi} \Psi \rangle = \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \text{Tr} \left( \frac{1}{\slashed{D} + m} \right) =$$

$$= \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \sum_{\lambda_i \geq 0} \left( \frac{1}{i\lambda_i + m} + \frac{1}{-i\lambda_i + m} \right) = \lim_{m \to 0} \int_0^\infty d\lambda \rho(\lambda) \frac{2m}{\lambda^2 + m^2}$$

$$= \frac{1}{2} \lim_{m \to 0} \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \frac{2m}{\lambda^2 + m^2} = \lim_{m \to 0} \frac{1}{2} \rho(im) \pi \rho(\lambda = 0) \tag{4.13}$$

in the last step we have performed a complex integration over the positive imaginary axis and we have finally:

$$\langle \bar{\Psi} \Psi \rangle = \pi \rho(\lambda = 0)$$

this is known as the Banks-Casher relation (1980). The interpretation of this relation is (‘t Hooft):

In instanton field, it exists an $\psi_0(x)$ such that $\slashed{D} \psi_0(x) = 0$, the equation for this zero-mode is:

$$\psi_0(x) = \frac{1}{\pi r^2} \frac{\rho}{\rho^2 + r^2} \frac{1}{x \cdot \gamma \phi}$$

in singular gauge, where $\phi^\alpha = \frac{1}{\sqrt{2}} \varepsilon^{\alpha\beta} \phi^\beta$; $\alpha, \beta = 1, 2$ (Dirac & colour) i.e. $\phi$ has only 2 non-zero components. This zero-mode is localized around instanton ($|\psi|^2 \sim r^6$) and is chiral i.e. $\frac{1+\gamma_5}{2} \psi_0(x) = 0$, $\frac{1-\gamma_5}{2} \psi_0(x) = \psi_0(x)$.

Each Instanton contributes with f zero-modes (one per flavour) to $\rho(\lambda = 0)$ Eq. (4.13). The result

$$\langle \bar{\Psi} \Psi \rangle = -\int_0^\infty d\lambda \rho(\lambda) \frac{2m}{\lambda^2 + m^2}$$
shows that the order in which we take the chiral and thermodynamic limits is crucial. In a finite system, the integral is well behaved in the infrared and the quark condensate vanishes in the chiral limit. This is consistent with the observation that there is no spontaneous symmetry breaking in a finite system. A finite spin system, for example, cannot be magnetized if there is no external field. If the energy barrier between states with different magnetization is finite and there is no external field that selects a preferred magnetization, the system will tunnel between these states and the average magnetization is zero. Only in the thermodynamic limit can the system develop a spontaneous magnetization. However, if we take the thermodynamic limit first, we can have a finite density of eigenvalues arbitrarily close to zero. In this case, the $\lambda$ integration is infrared divergent as $m \to 0$ and we get a finite quark condensate Eq. (4.13). Studying chiral symmetry breaking requires an understanding of quasi-zero modes, the spectrum of the Dirac operator near $\lambda = 0$. If there is only one instanton the spectrum consists of a single zero mode, plus a continuous spectrum of non-zero modes. In the chiral limit, fluctuations of the topological charge are suppressed, so one can think of the system as containing as many instantons as anti-instantons. In this way we can explain the non-vanishing value of the quark condensate by introducing an instanton density, further calculations give $\langle \bar{q}q \rangle \approx - (240\text{Mev})^3$, in concordance with the phenomenological value. There is no explanation of this value in the framework of perturbation theory.

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Appendix A

Semi-Classical Methods in Path Integrals

Let us consider a general action $S[x]$ which is not necessarily quadratic. The transition amplitude associated with this action, is given by:

$$\langle x_f, \frac{T}{2} | x_i, -\frac{T}{2} \rangle = N \int \mathcal{D}x e^{\frac{i}{\hbar} S[x]} \tag{A.1}$$

This integral is oscillatory. However, we know that, in such a case, we can rotate to the Euclidean space and the integrand become well behaved. We will continue to use the real time description keeping in mind the fact that in all our discussions, we are assuming that the actual calculations are always done in the Euclidean space and the results are rotated back to Minkowski space. The classical trajectory satisfies the equation:

$$\frac{\delta S_E[x]}{\delta x(t)} \bigg|_{x=x_{cl}} = 0 \tag{A.2}$$

Therefore, it provides an extremum of the exponent in the path integral. Furthermore, the action is a minimum for the classical trajectory. Thus, we can expand the action around the classical trajectory. Namely $x(t) = x_{cl}(t) + \eta(t)$ then we have:

$$S[x] = S[x_{cl} + \eta] \tag{A.3}$$
$$= S[x_{cl}] + \frac{1}{2} \int \int dt_1 dt_2 \eta(t_1) \frac{\delta^2 S[x_{cl}]}{\delta x_{cl}(t_1) \delta x_{cl}(t_2)} \eta(t_2) + O(\eta^3) \tag{A.4}$$

and the transition amplitude becomes:

$$\langle x_f, \frac{T}{2} | x_i, -\frac{T}{2} \rangle = N \int \mathcal{D}\eta \exp\left[\frac{i}{\hbar}(S[x_{cl}] + \frac{1}{2} \int \int dt_1 dt_2 \frac{\delta^2 S[x_{cl}]}{\delta x_{cl}(t_1) \delta x_{cl}(t_2)} \eta(t_2) + O(\eta^3))\right] \tag{A.5}$$
\begin{equation}
\frac{NK}{\sqrt{\det\left(\frac{1}{\hbar}\frac{\delta^2S[x_{cl}]}{\delta x_{cl}(t_1)\delta x_{cl}(t_2)}\right)}} e^{\frac{i}{\hbar}(S[x_{cl}] )} \tag{A.6}
\end{equation}

where we have used that:

\begin{equation}
\int \mathcal{D}\eta e^{\frac{i}{\hbar}\int_{t^i}^{t_f} dt \eta(t)O(t)\eta(t)} = K \sqrt{\det O(t)} \tag{A.7}
\end{equation}

When \(\det\left(\frac{1}{\hbar}\frac{\delta^2S[x_{cl}]}{\delta x_{cl}(t_1)\delta x_{cl}(t_2)}\right) = 0\), the path integral has to be evaluated more carefully using the method of collective coordinates.

For a general action:

\begin{equation}
S[x] = \int_{t^i}^{t_f} dt \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) \tag{A.8}
\end{equation}

we have:

\begin{equation}
\frac{\delta^2S[x_{cl}]}{\delta x_{cl}(t_1)\delta x_{cl}(t_2)} = -(m \frac{d^2}{dt^2} + V''(x))\delta(t_1 - t_2) \tag{A.9}
\end{equation}

Therefore, we see that:

\begin{equation}
\det\left(\frac{1}{\hbar}\frac{\delta^2S[x_{cl}]}{\delta x_{cl}(t_1)\delta x_{cl}(t_2)}\right) \propto \det\left(\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + V''(x))\right)\right) \tag{A.10}
\end{equation}

We are interested in evaluating determinants of the form \(\det\left(\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + W(x))\right)\right)\) in the space of functions where \(\eta(t_i) = 0 = \eta(t_f)\).

Let us write the eigenvalue problem for the following operator:

\begin{equation}
\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + W(x))\right)\psi_W^{(\lambda)} = \lambda\psi_W^{(\lambda)} \tag{A.11}
\end{equation}

with the initial value conditions \(\psi_W^{(\lambda)}(t_i) = 0\) and \(\psi_W^{(\lambda)}(t_i) = 1\), then:

\begin{equation}
\frac{\det\left(\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + W_1(x)) - \lambda\right)\right)}{\det\left(\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + W_2(x)) - \lambda\right)\right)} = \frac{\psi_W^{(\lambda)}(t_f)}{\psi_W^{(\lambda)}(t_f)} \tag{A.12}
\end{equation}

where \(W_1\) and \(W_2\) are two bounded potentials. We note that for \(\lambda\) coinciding with an eigenvalue of one of the operators, the left hand side will have a zero or a pole. But so will the right hand side for the same value of \(\lambda\) because in such a case \(\psi^{(\lambda)}\) would correspond to an energy eigenfunction satisfying the boundary conditions at the end points. Since both the left and the right hand side of the above equation are meromorphic functions of \(\lambda\) with identical zeroes and poles, they must be equal. It follows from this result that:

\begin{equation}
\frac{\det\left(\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + W_1(x))\right)\right)}{\psi_W^{(\lambda)}(t_f)} = \frac{\det\left(\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + W_2(x))\right)\right)}{\psi_W^{(\lambda)}(t_f)} = \text{constant} \tag{A.13}
\end{equation}

That is, this ratio is independent of the particular form of the potential \(W(x)\) and can be used to define the normalization constant in the path integral. For example we can define a quantity \(N\) by:

\begin{equation}
\frac{\det\left(\frac{1}{\hbar}\left((m \frac{d^2}{dt^2} + W(x))\right)\right)}{\psi_W^{(\lambda)}(t_f)} = \pi\hbar N^2 \tag{A.14}
\end{equation}
As a consequence of the above discussion $N$ is independent of $W$, and we have the desired formula for evaluating Gaussian functional integrals:

$$\frac{NK}{\sqrt{\det\left(\frac{1}{\hbar} \delta^2 S[x_{cl}] \delta x_{cl}(t_1) \delta x_{cl}(t_2)\right)}} e^{\frac{i}{\hbar} \left(S[x_{cl}]\right)} = \left[\pi \hbar \psi_W^{(0)}(t_f)\right]^\frac{1}{2} e^{\frac{i}{\hbar} \left(S[x_{cl}]\right)} \quad (A.15)$$

We note here that the classical equations following from our action have the form:

$$m \frac{d^2x_{cl}}{dt^2} + V'(x_{cl}) = 0 = \frac{d}{dt} \left(m \frac{d^2x_{cl}}{dt^2} + V'(x_{cl})\right)$$

or:

$$\left(m \frac{d^2}{dt^2} + V''(x_{cl})\right) \frac{dx_{cl}}{dt} = 0 \quad (A.16)$$

We can see that:

$$\psi_W^{(0)}(t) \propto \frac{dx_{cl}}{dt} \propto p(x_{cl}) \quad (A.17)$$

and finally:

$$\langle x_f, T \mid x_i, -T \rangle = \frac{N}{\sqrt{p(x_f)}} e^{\frac{i}{\hbar} \left(S[x_{cl}]\right)} \quad (A.18)$$
Appendix B

Another Way to Obtain the Yang-Mill Instanton

Let us define for $\mu, \nu = 1, 2, 3, 4$:

\[ \sigma_\mu = (\sigma, -i) \quad \sigma_\mu^+ = (\sigma, i), \]
\[ \sigma_{\mu\nu} = i(\sigma_\mu \sigma_\nu^+ - \delta_{\mu\nu}) \]

It is straightforward to prove:

\[ \tilde{\sigma}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \sigma_{\alpha\beta} = \sigma_{\mu\nu} \]
\[ (x \cdot \sigma)(x \cdot \sigma^+) = r^2 \]

Let us make also the following definition:

\[ U = i \frac{x \cdot \sigma}{r} \quad U^{-1} = -i \frac{x \cdot \sigma^+}{r} \]

We can deduce the following expression for the vector field in the pure gauge Eq. (2.9):

\[ A_{\mu \text{ pure-gauge}} = U \partial^\mu U^{-1} = i \frac{\sigma_{\mu\nu} x_\nu}{r^2} \]

For a finite-energy solution with $Q = 1$, let us make the following ansatz:

\[ A_\mu = i \frac{\sigma_{\mu\nu} x_\nu}{r^2} f(r^2) \]

$f(0) = 0$ (regularity at $r = 0$), $f(\infty) = 1$ (finite energy, $Q = 1$). To make Eq. (B.7) this a solution, we only need to choose $f$ such that $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ (selfdual condition for the instantons).
Proposition B.0.1. From Eq. (B.7) we can derive the following expressions for the tensor field:

\[ F_{\mu \nu} = \frac{-2i f}{r^2} \left( \frac{(1-f)}{r^2} \sigma_{\mu \nu} + \frac{1}{r^2} \left[ f' - \frac{f(1-f)}{r^2} \right] \left[ (\sigma_{\mu \alpha} x_{\alpha \nu} - \sigma_{\nu \beta} x_{\alpha \mu}) \right] \right), \quad (B.8) \]

\[ \tilde{F}_{\mu \nu} = \frac{-2i f}{r^2} \sigma_{\mu \nu} + \frac{1}{r^2} \left[ f' - \frac{f(1-f)}{r^2} \right] \left[ \varepsilon_{\mu \alpha \beta} (\sigma_{\alpha \lambda} x_{\lambda \beta} - \sigma_{\beta \lambda} x_{\lambda \alpha}) \right], \quad (B.9) \]

Proof. We will proof only the first identity Eq. (B.8):

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + (A_\mu A_\nu - A_\nu A_\mu) = \\
i(\sigma_{\nu \mu}) f \left( \frac{r^2}{2} \right) - \frac{2x_{\mu} f' - 2x_{\mu} f}{r^2} - \frac{2x_{\mu} f}{r^2} f = \sigma_{\mu \alpha} \sigma_{\nu \beta} \sigma_{\alpha \delta} \sigma_{\beta \mu} + \sigma_{\nu \beta} \sigma_{\alpha \mu} \sigma_{\beta \delta} \sigma_{\mu \alpha} \\
+ \frac{i}{f^2} 1 f^2 \sigma_{\mu \alpha} \sigma_{\nu \beta} \sigma_{\alpha \delta} \sigma_{\beta \mu} = \frac{-2f^2}{r^4} \left[ (\sigma_{\mu \alpha} \sigma_{\nu \beta} - \sigma_{\nu \beta} \sigma_{\alpha \mu}) \right] \\
= \frac{i}{r^2} \left[ (\sigma_{\mu \alpha} - \sigma_{\nu \beta}) \frac{f}{r^2} + 2f \left( \sigma_{\mu \alpha} x_{\beta \nu} - \sigma_{\alpha \beta} x_{\mu \nu} \right) - 2f \left( \sigma_{\mu \alpha} x_{\beta \nu} - \sigma_{\nu \beta} x_{\alpha \mu} \right) \right]
\]

The latter parenthesis is:

\[
(\sigma_{\mu \alpha} \sigma_{\nu \beta} - \sigma_{\nu \beta} \sigma_{\mu \alpha}) = (\sigma_{\mu \alpha} (1) \sigma_{\nu \beta} - \sigma_{\nu \beta} (1) \sigma_{\mu \alpha}) = \\
i(\sigma_{\mu} \sigma_{\alpha}^+ - \sigma_{\nu} \sigma_{\beta}^+)(1) i(\sigma_{\beta} \sigma_{\nu}^+ - \sigma_{\nu} \sigma_{\beta}^+)(1) i(\sigma_{\alpha} \sigma_{\mu}^+ - \sigma_{\mu} \sigma_{\alpha}^+) = \\
= \sigma_{\mu} \sigma_{\alpha}^+ \sigma_{\beta} \sigma_{\nu}^+ - \sigma_{\mu} \sigma_{\nu}^+ \sigma_{\alpha} \sigma_{\beta}^+ + \sigma_{\nu} \sigma_{\beta}^+ \sigma_{\mu} \sigma_{\alpha}^+ - \sigma_{\nu} \sigma_{\alpha}^+ \sigma_{\beta} \sigma_{\mu}^+ = \sigma_{\beta} \sigma_{\nu}^+ \sigma_{\mu} \sigma_{\alpha}^+ - \sigma_{\nu} \sigma_{\alpha}^+ \sigma_{\beta} \sigma_{\mu}^+ + \sigma_{\mu} \sigma_{\alpha}^+ \sigma_{\beta} \sigma_{\nu}^+ - \sigma_{\mu} \sigma_{\nu}^+ \sigma_{\alpha} \sigma_{\beta}^+
\]

and finally using \((x \cdot \sigma)(x \cdot \sigma^+) = r^2\) we have:

\[
-\frac{f^2}{r^4} x_{\mu} x_{\beta} (\sigma_{\mu \alpha} \sigma_{\nu \beta} - \sigma_{\nu \beta} \sigma_{\mu \alpha}) = \\
-\frac{f^2}{r^4} [(\sigma_{\mu} \sigma_{\nu}^+ - \sigma_{\nu} \sigma_{\mu}^+) r^2 + x_{\mu} x_{\beta} (\sigma_{\nu} \sigma_{\beta}^+ - \beta \sigma_{\nu}^+)] + x_{\alpha} x_{\nu} (\sigma_{\alpha} \sigma_{\beta}^+ - \sigma_{\beta} \sigma_{\alpha}^+) =
\]

and now using \(\sigma_{\nu} \sigma_{\beta}^+ = \frac{\sigma_{\beta}^+}{r^2} + \delta_{\nu \beta}\), we obtain:

\[
-\frac{f^2}{r^4} x_{\mu} x_{\beta} (\sigma_{\mu \alpha} \sigma_{\nu \beta} - \sigma_{\nu \beta} \sigma_{\mu \alpha}) = = 2i \left[ \frac{\sigma_{\mu \nu}}{f^2} + x_{\mu} x_{\beta} \frac{\sigma_{\nu \beta}}{r^4} f^2 - x_{\alpha} x_{\nu} \frac{\sigma_{\mu \alpha}}{r^4} f^2 \right]
\]

and altogether:

\[
\partial_\mu A_\nu - \partial_\nu A_\mu + ig (A_\mu A_\nu - A_\nu A_\mu) = \\
= 2i [-\sigma_{\mu \nu} \frac{f}{r^2} + \sigma_{\mu \nu} \frac{f^2}{r^4} + (f' - f \frac{f}{r^2} + f^2) (\sigma_{\nu \beta} x_{\alpha \mu} - \sigma_{\mu \alpha} x_{\alpha \nu})]
\]
Thus $F_{\mu\nu}$ is self-dual if the second term vanishes:

$$f' - \frac{f(1 - f)}{r^2} = 0 \quad (B.10)$$

the general solution satisfying the required boundary conditions is:

$$f(r^2) = \frac{r^2}{\rho^2 + r^2} \quad (B.11)$$

where $\rho$ is an arbitrary scale parameter, the translational invariance permits to displace the center at an arbitrary point $r_0$, for which it is necessary to replace $r$ by $r - r_0$ and

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} = -2i \frac{\rho^2}{(r^2 + \rho^2)^2} \sigma_{\mu\nu} \quad (B.12)$$

which is the field tensor for an instanton with $Q = 1$. 

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