Partition Functions of Pure Spinors

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Abstract

We compute partition functions describing multiplicities and charges of massless and first massive string states of pure-spinor superstrings in $3, 4, 6, 10$ dimensions. At the massless level we find a spin-one gauge multiplet of minimal supersymmetry in $d$ dimensions. At the first massive string level we find a massive spin-two multiplet. The result is confirmed by a direct analysis of the BRST cohomology at ghost number one. The central charges of the pure spinor systems are derived in a manifestly $SO(d)$ covariant way confirming that the resulting string theories are critical. A critical string model with $\mathcal{N} = (2, 0)$ supersymmetry in $d = 2$ is also described.

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1 Introduction

During the last years, one of the fundamental piece of work in string theory is the construction of a quantizable model of superstring in 10 dimensions with manifest super-Poincaré invariance [1]. The progress in understanding gauge/gravity correspondences [2] makes clear than an understanding of string backgrounds in presence of RR fluxes are crucial to go beyond the supergravity level. The super-Poincaré invariant formulation [1] of ten dimensional superstring treats NSNS (Nevue-Schwarz) and RR(Ramond) fields on the same footing and it is the natural candidate to address this question (see [3] for a review).

The covariant formulation [1] is based on a set of worldsheet bosonic fields $\lambda^\alpha$ (and its complex conjugate $w_\alpha$) transforming as an $SO(10)$ spinor and satisfying a pure spinor constraint $\lambda \gamma^m \lambda = 0$. The pure spinor system is tensored with free fermions $(\theta^\alpha, p_\alpha)$ and bosons $x^m$ in such a way that the total conformal charge is zero and the Lorentz generators
associated to the spinorial variables (GS variables and ghost fields) have the same double poles of the Lorentz generator for worldsheet fermions of RNS. Additionally, the physical spectrum is constructed on a free-field action (for flat space) using a single non-hermitian BRST charge which is nilpotent when the pure spinor constraint is satisfied. It has been shown that the string spectrum identified with the BRST cohomology coincides with the light-cone spectrum of the GS strings [4, 5, 6]. This spectrum is found by solving the pure spinor constraint in an $SO(8)$ covariant way. Several checks have been performed for tree level and one-loop amplitudes [7, 8, 9].

It is important to recall that even if the pure-spinor formalism is manifestly super-Poincaré invariant, covariant computations are rather cumbersome due to the ghost constraints which have to be implemented at any stage. In papers [10, 11, 12] the authors reproduce the massless spectrum of open and closed superstrings using some additional ghost fields and imposing a grading constraint on the functional space to retrieve the correct constraints. Several studies [13, 14, 15, 16, 17] followed the original papers extending the analysis in different directions.

Recently, in [18], the authors exploited localization techniques to compute the zero mode part of the partition function for pure spinors in $d = 10$. The results were written in a $SO(10)$-covariant form and linked directly to the super-Yang-Mills multiplet dynamics describing the massless modes of the open ten-dimensional superstring. Indeed, it has been shown that not only the physical states (8 states of the vector multiplet and 8 states of the gluino) are represented, but also the complete set of ghosts and the antifields implementing the equations of motion. What about massive string modes? Massive string states enter in a somehow trivial way in most of the pure spinor string amplitudes studied so far. Even the simplest open string amplitude describing an open string in presence of a constant magnetic field has not been yet reproduced inside this formalism (see [19] for a study of the massless part). The main missing ingredient is a covariant description of the spectrum of the theory. Aim of this paper is to considering the first massive string state, as a first step towards a $SO(10)$ covariant string partition function. This is a crucial ingredient for the study of superstring spectra in non-trivial backgrounds like $AdS_5 \times S^5$. In particular, it would be nice to explain, by a direct computation in the pure spinor formalism, the spectrum and mass formula for string states on $AdS_5 \times S^5$ found via KK analysis in [20]-[22] (see [23] for generalizations to Dp-brane geometries).

In this paper we consider string theories based on pure spinors in dimensions $d = 4, 6, 10^3$. We refer the reader to [24, 25] for details in the definitions of the pure spinor strings in $d \neq 10$. We will compute the massless and first massive string state partition

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3We advise the reader that our definitions of pure spinors in $d = 4, 6$ differ from those used in [18] in the spinor representation chosen for $\lambda^a$. This explains why partition functions and central charges differ from those in that reference.

4Related work on the N=2 string formulation appeared in [26, 27]. Related work on the massless
functions. As in [18], we count states in a $SO(d)$ covariant way without solving the pure spinor constraint. From the massless partition function we extract the central charge of the pure spinor system and show that the resulting superstrings are critical in any dimension. In addition we will show that the massless partition functions of the pure spinor open string have precisely the degrees of freedom to describe $\mathcal{N} = 1$ SYM in $d = 4, 6, 10$. Closed superstrings based on these pure spinors realize $\mathcal{N} = 2$ supergravities in $d = 4, 6, 10$ with RR fields and branes and are suitable for studies of holography. Superstrings on $AdS_5 \times S^1$ has been recently proposed [29] as holographic duals of $\mathcal{N} = 1$ gauge theories in $d = 4$. The existence of critical pure-spinor superstrings realizing minimal Yang-Mill theories in any dimension open a new handle to covariant quantization of these AdS strings.

We will follow the strategy in [18]. Rather than compute the cohomology of the BRST operator defining the physical spectrum, we compute the $(-)^F$ weighted string partition function, $F$ being the worldsheet fermionic number keeping track also of $SO(d)$ charges. Since BRST operator is odd under $(-)^F$, paired states do not contribute to this index and therefore the string partition function counts only states in the cohomology. The string partition function results provide us then with shortcuts to the often lengthy cohomology analysis.

The paper is organized as follows: In Section 2 we introduce the pure spinor models in $d = 4, 6, 10$ and compute their string partition functions. In Section 3 we compute the cohomology at ghost number one. We find perfect agreement with the results coming from the string partition function. In section 4 we introduce a pure spinor system with $\mathcal{N} = (2, 0)$ supersymmetry in $d = 2$ and one with minimal supersymmetry in $d = 3$. In each case we derive the massless and first massive string spectrum both from counting of pure spinor states and cohomology analysis. In Section 5 we summarize our results and comment on future directions. Finally in appendix A we collect some details on the construction of massive multiplets in $d = 4, 6, 10$, and in appendix B we give details of anomalous Ward identities.

## 2 Pure spinors

Superstrings based on pure spinors are defined be the sigma model

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \left( \partial x^\mu \bar{\partial} x^\mu + p_A \bar{\partial} \theta^A + w_A \bar{\partial} \lambda^A \right)$$

(2.1)

Here $x^\mu$ describes the string coordinates in $\mathbb{R}^d$, $\theta^A$ are anticommuting variables and $p_A$ their conjugate momenta. The field $\lambda^A$ satisfy the pure spinor constraint

$$\lambda^\gamma{}^\mu \lambda = 0$$

(2.2)

cohomology of lower dimensional models appeared also in the interesting paper by A. Movshev [28].
The pure spinor constraint (2.2) induces a gauge invariance on $w_A$ (the conjugate momentum of $\lambda^A$):

$$\delta w_A = \Lambda_\mu (\gamma^\mu \lambda)_A$$  \hspace{1cm} (2.3)

Indices $\mu, A$(up) and $A$(down) run over the vector $\mathbf{V}_d$ and spinor representations $S_d, \bar{S}_d$ respectively of the Lorentz group $SO(d)$. Spinors are chosen to be Dirac in $d = 4$, symplectic Majorana Weyl in $d = 6$ and Majorana-Weyl in $d = 10$. The choice of the spinor representations $S_d$ is such that the vector representation $\mathbf{V}_d$ always appear in the product $S_d \times \bar{S}_d$ of two $\lambda$'s. The dimensions $S_d$ and Lorentz content in the various dimensions are resumed in table 1. The resulting string theories are critical in any dimension. This can be seen by notice that the naive central charge of the pure spinor system $c_{\lambda,w} = 2S_d - d$ cancels against that of the free $x^m, \theta$ system $c_{x,p,\theta} = -2S_d + d$. This naive counting of degrees of freedom will be confirmed below by a $SO(d)$ covariant derivation of the central charges from the pure spinor partition function.

We will organize the states according to the $U(1)$ charge $\Delta$:

$$\Delta = n_\lambda + n_\theta + 2n_x + 3n_w + 3n_p$$  \hspace{1cm} (2.4)

which is clearly a symmetry of (2.1) if we assign to $\alpha'$ charge $\Delta = 4$. The spectrum of string states and $\Delta$-charges can be read from the character valued partition function

$$Z(q|t) = \text{tr}_\mathcal{H} (-)^F q^{L_0} t^\Delta$$  \hspace{1cm} (2.5)

with $L_0$ the string level and $F$ the fermion number. The trace runs over the space of polynomials of the string modes $\lambda, \theta, p, w, x$ (and their worldsheet derivatives) satisfying the pure spinor constraint (2.2) and invariant under (2.3). The Fock space can then be written as a polynomials in the string modes:

$$\mathcal{H} = \{ F(\lambda_n, \theta_n, x_n, p_{n>0}, w_{n>0}) | 0 \} \mid \sum_m \lambda_{n-m} \gamma^\mu \lambda_m = 0 ; \ \delta F = 0 \}$$  \hspace{1cm} (2.6)

with $m, n = 0, 1, \ldots$. For simplicity we will always omit the contribution of bosonic zero modes $x_0$, i.e. we focus on the zero momentum spectrum.
The results will be written as polynomials in Lorentz representations made out of products of vector \( \mathbf{V}_d \) and spinor representations \( \mathbf{S}_d, \tilde{\mathbf{S}}_d \). In particular the contribution of a free worldsheet field \( \Phi_n \) in a given representation \( \mathcal{R} \) with scaling \( t^a \) will be written as:

\[
\frac{1}{(1 - t^a q^n)^\mathcal{R}} = 1 + t^a q^n \mathcal{R} + t^{2a} q^{2n} (\mathcal{R} \times \mathcal{R})_{\text{sym}} + \ldots \\
(1 - t^a q^n)^\mathcal{R} = 1 - t^a q^n \mathcal{R} + t^{2a} q^{2n} (\mathcal{R} \times \mathcal{R})_{\text{antisym}} + \ldots
\]

(2.7)

for a bosonic/fermionic worldsheet mode \( \Phi_n \) respectively. The blind partition function follows from replacing the representation \( \mathcal{R} \) by its dimension.

Before entering the details of the computation let us sketch our general strategy to count pure spinor states. Written in string modes, the pure spinor constraint reads

\[
\lambda_0 \gamma^m \lambda_0 = 0 \\
\lambda_0 \gamma^m \lambda_1 = 0
\]

(2.8)

and so on. This condition restricts the number of representations that appear in the tensor product of two pure spinors. Explicitly:

\[
\lambda_0 \times \lambda_0 : \quad (\mathbf{S}_d \times \mathbf{S}_d)_{\text{sym}} - \mathbf{V}_d \\
\lambda_0 \times \lambda_1 : \quad (\mathbf{S}_d \times \mathbf{S}_d) - \mathbf{V}_d
\]

(2.9)

and so on. In other words, polynomials in \( \lambda \) are given by symmetric product of the spinor representation \( \mathbf{S}_d \) where the vector representation \( \mathbf{V}_d \) in each bispinor product is deleted.

Analogously gauge invariance requires that \( F \) depends on \( w \) only via the gauge invariant combinations \( J^B_A = \mathcal{P}_{BD}^{AC} w_C \lambda^D \) with \( \mathcal{P}_{BD}^{AC} \) a projector on the gauge invariant components:

\[
d = 4 : \quad [1,0] + [0,1] + 2[0,0] \\
d = 6 : \quad [011]_0 + [000]_1 + [000]_0 \\
d = 10 : \quad [01000] + [00000]
\]

(2.10)

They realize the Lorentz \( J_{mn} \) and ghost currents \( \Delta \) in \( d = 4, 6, 10 \). In addition the extra singlet in \( d = 4 \) represents the axial current \( J_5 \) while in \( d = 6 \) the triplet realizes the \( Sp(1) \) \( \mathcal{R} \)-symmetry current \( J_{(ij)} \).

The contribution of \( \lambda, w \) to the partition function (2.5) can then be computed by counting polynomials \( \lambda_0^n, \lambda_1 \lambda_0^n, w_1 \lambda_0^n \) with bispinor products restricted according to (2.9,2.10).
Finally the total partition function follows by multiplying the pure spinor result with the contribution coming from the free \( \theta, p, x \)-system:

\[
Z_{\theta,p,x}(q|t) = (1 - t)^{S_d} \prod_{n=1}^{\infty} \frac{(1 - q^n t^1)^{S_d}}{(1 - q^n t^2)^{V_d}} (1 - q^n t^3)^{\bar{S}_d}
\]

As we mentioned before here and below we omit the contribution \((1 - t^2)^{-V_d}\) coming from the bosonic zero modes \(x_0^m\), i.e. we consider spacetime constant fields. The total partition function will be then written as:

\[
Z(q|t) = Z_{\lambda,w}(q|t) Z_{\theta,p,x}(q|t) = Z_0(t) + q Z_1(t) + \ldots
\]

The finite polynomials \(Z_\ell(t)\) encode the informations about multiplicities and charges of string states. Aim of this work is to evaluate \(Z_{0,1}(t)\) for the massless and first massive string state in \(d = 4, 6, 10\).

Some comments about the relation between the string partition function and the Q-cohomology of the corresponding string theory are in order. First the string partition sums over all ghost number states while the cohomology analysis here will be often restricted to ghost number one. The agreement between the twos does not imply that higher spin cohomology is empty (although this is mostly the case) but only that higher ghost number states, if exist, come in field/antifield pairs. Second the string partition function counts states off-shell at a fixed momentum while states in the cohomology of \(Q\) are often on-shell\(^5\). This implies in particular that the cohomology of \(Q\) at zero momenta is empty every time massive states come on-shell since \((\partial^2 + m^2)\phi = 0\) implies \(\phi = 0\) if \(p = 0\). This will be confirmed by the computation of the string partition function at the first massive level in \(d = 6, 10\) where states come always in worldsheet bosonic/fermionic pairs and the resulting partition function cancels. We stress the fact that this cancelation is not related to supersymmetry since we are counting states keeping track of their \(SO(d)\) representations, it is a cancelation between fields and antifields. In the pure spinor formalism it is hard to separate the two contributions without spoil \(SO(d)\) covariance. As we will see physical states in \(d = 6, 10\) at the first massive level can be isolated by keeping the contributions from the modes \(\lambda_1, \theta_1\) separated from those coming from their momenta \(p_1, x_1, w_1\). The resulting spectrum organizes into a massive on-shell spin-two multiplet of the minimal supersymmetry in \(d = 6, 10\). In \(d = 4\) the massive multiplet comes off-shell and the string and cohomology results therefore agree. This is also the case for massless states in \(d = 4, 6, 10\).

In addition we will show how that the central charges of the Virasoro algebra coming from the partition function of pure spinors match the naive counting above in \(d = 4, 6, 10\).

\(^5\)J.F.M. thanks R. Russo for discussions on this point
To this purpose one rewrites the massless partition function of pure spinors $Z_0$ in terms of a free system of infinitely many fields [18]:

$$Z_0(t) = \prod_{n=1}^{\infty} (1 - t^n)^{-N_n}$$

(2.13)

with some $N_n$. In terms of this free description the central charge of the Virasoro current can be read from the logarithmic divergent term in the small $x$ expansion of $\log Z_0(t = e^x)$:

$$- \log Z_0(e^x) = \log(x) \sum_n N_n + \ldots = \frac{1}{2} c_{\text{vir}} \log(x) + \ldots$$

(2.14)

In $d$-dimensions the theory is critical if the central charge of the $(\lambda, \theta, p, w)$-system computed in this way cancel that of the free bosons, i.e. if $c_{\text{vir}} = -d$. Notice that in order to do this computation we do not need to determine $N_n$, but simply expand the left hand side of this expression and find the coefficient of $\log x$ as $\frac{1}{2} c_{\text{vir}}$. The results will be shown in agreement with the naive counting of degrees of freedom of pure spinors in $d = 4, 6, 10$.

### 2.1 $d = 4$

At the massless level the pure spinor constraint in $D = 4$ can be written as:

$$\lambda_0^{\alpha} \bar{\lambda}_0^{\dot{\alpha}} = 0$$

(2.15)

with $\lambda_0^4 = \{\lambda_0^0, \bar{\lambda}_0^0\}$ and $\alpha, \dot{\alpha} = 1, 2$. States in $d = 4$ will be labelled by their representations under the $SO(4) \sim SU(2) \times SU(2)$ Lorentz group and the $U(1)$ charge. The $U(1)$ charge will be traced by the powers of $t$, while Lorentz quantum numbers will be labelled by the $[j_1, j_2]$ spins. We introduce the short hand notation:

$$\lambda_0^{\alpha} : s \ t \equiv [\frac{1}{2}, 0] \ t \ \ \ \bar{\lambda}_0^{\dot{\alpha}} : c \ t \equiv [\frac{1}{2}, 0] \ t$$

(2.16)

Polynomials satisfying (2.15) are built of symmetric combinations of either $\lambda_0^\alpha$ or $\bar{\lambda}_0^{\dot{\alpha}}$:

$$\lambda_0^\alpha : ([\frac{1}{2}, 0] + [0, \frac{3}{2}]) t^n$$

(2.17)

In particular all products containing both $s$ and $c$ representations have been suppressed in (2.17). Here and below we will often suppressed Lorentz indices when write polynomials $\lambda_0^\alpha$ of pure spinors. Complete symmetrization between the $\lambda^4$ will be always understood.

The massless partition function follows then by summing up (2.17) over $n$ and multiplying by the free $\theta_0$ -contribution $(1 - t)^{s+c}$.

$$Z_0(t) = \text{tr}_{\mathcal{H}_0} (-)^F q^{L_0} t^J = (1 - t)^{s+c} \left[ \frac{1}{(1 - t)^s} + \frac{1}{(1 - t)^c} - 1 \right]$$

$$= 1 - s \ c \ t^2 + (s + c) t^3 - t^4$$

$$= 1 - 4 \ t^2 + 4t^3 - t^4$$

(2.18)
The two series in the bracket come from symmetric polynomials of $\lambda_0^a = s$ and $\bar{\lambda}_0^d = c$ respectively while the minus one subtracts the overcounted identity. The last line display the blind dimensions. It is important to notice that the contribution of $\theta_0$ cancels exactly the denominators (2.18) leaving a finite polynomial $Z_0(t)$. This will be always the case in any dimension.

Even and odd powers of $t$ in (2.18) correspond to spacetime bosonic and fermionic degrees of freedom respectively. The polynomial $P_0(t)$ describes the off degrees of freedom of a massless $\mathcal{N} = 1$ vector multiplet in $d = 4$ with content $(A_\mu - \Lambda; \psi_\alpha, \bar{\psi}_\dot{\alpha}; D)$ with $\Lambda$ parametrizing the gauge invariance. Notice that the powers of $t$ describe precisely twice the dimensions of these fields, i.e. 0 for the gauge parameter, 1 for the vector, $3/2$ for the gaugino and 2 for the auxiliary field $D$.

Now let us consider the first massive string level. We have an extra pure spinor constraint and gauge invariance:

$$\begin{align*}
\lambda_0^a \bar{\lambda}_1^\dot{a} + \lambda_1^a \bar{\lambda}_0^\dot{a} &= 0 \\
\delta w_{1\alpha} &= \Lambda_{1\alpha\dot{a}} \lambda_0^a, \quad \delta \bar{w}_{1\dot{\alpha}} = \Lambda_{1\alpha\dot{a}} \bar{\lambda}_0^d
\end{align*}$$

It is important to notice that when combined with (2.15), eq. (2.19) implies

$$\lambda_0^a \bar{\lambda}_1^\dot{a} = \lambda_1^a \bar{\lambda}_0^\dot{a} = 0$$

therefore operators satisfying the pure spinor constraint contain either $\lambda_{0,1}$ or $\lambda_{0,1}^\dot{a}$. Invariance under (2.19) implies that $w_{1\alpha}$ appear only in the combination $w_{1\alpha} \lambda^\beta$ and $w_{1\dot{\alpha}} \lambda^{\dot{\beta}}$. In appendix B, we derive eq. (2.20) by using covariant equations.

The $SU(2)^2 \times U(1)$ content of pure spinor states satisfying (2.19) is then given by:

$$\begin{align*}
\lambda_1 \lambda_0^a : & \quad ([\frac{1}{2}, 0] \times [\frac{3}{2}, 0] + [0, \frac{1}{2}] \times [0, \frac{3}{2}]) \ t^{n+1} \\
w_1 \lambda_0^a : & \quad ([\frac{1}{2}, 0] \times [\frac{3}{2}, 0] + [0, \frac{1}{2}] \times [0, \frac{3}{2}]) \ t^{n+3} \quad n \geq 1
\end{align*}$$

The total contribution coming with $\lambda_1, w_1$ modes follows then by summing up over $n$ and multiplying by the contribution $(1 - t)^{s+c}$ coming from $\theta_0$'s. One finds:

$$Z_{\lambda_1, w_1} = (1 - t)^{s+c} \left( \frac{s(t + t^3)}{(1 - t)^s} + \frac{c(t + t^3)}{(1 - t)^c} - (s + c)t^3 \right)$$

Finally one should add the contributions coming from $\theta_1, p_1, x_1$:

$$Z_{\theta_1, x_1, p_1} = [s \ c \ t^2 - (s + c) (t + t^3)] \ Z_0(t)$$

with $Z_0(t)$ given by (2.18). Summing (2.23) and (2.22) one finally finds:

$$Z_1(t) = s \ c \ (1 - t)^{s+c}$$
Even and odd powers of $t$ correspond to spacetime bosonic and fermionic degrees of freedom respectively. By expanding (2.24) one finds the bosonic field content $(g_{\mu \nu}, b_{\mu \nu}, 4A_{\mu}, \varphi)$. This multiplet is generated by acting with all supercharges on a vector field, i.e. it has $4 \times 2^4$ degrees of freedom, and contains as a highest helicity state a spin two particle.

Finally we can compute the central charge of the $d = 4$ system. Plugging the massless partition function for pure spinors $Z_{\lambda_0}$ in (2.14) one finds

$$- \log Z_0(t)(e^x) = -2 \log(x) + \ldots$$

leading to $c_{\text{vir}} = -4$, therefore the theory is critical in $d = 4$!

### 2.2 $d = 6$

The massless pure spinor constraint in $d = 6$ can be written as:

$$\lambda^{[A}_{0i} \lambda^{B]}_i = 0$$

(2.26)

with $A = 1, \ldots, 4$, $i = 1, 2$, $\mu = 1, \ldots, 6$. This implies that in the symmetric product of $n \lambda_0$’s only the representation

$$\lambda^n_0 : \lambda^{(A_1}_0 \ldots \lambda^{A_n)}_0 = [00n]_{\frac{n}{2}}t^n$$

(2.27)

will survive. Here $[n_1n_2n_3]_j$ denote the $SO(6)$ dynkin labels and $SU(2)$ spin. In this notation $S_d = [001]_\frac{d}{2}$. The massless partition function can then be written as

$$Z_0(t) = (1 - t)^{S_d} \sum_{n=0}^{\infty} [00n]_{\frac{n}{2}}t^n$$

$$= 1 - [100]_0 t^2 + [001]_{\frac{1}{2}} t^3 - [000]_1 t^4$$

$$= 1 - 6 t^2 + 8 t^3 - 3 t^4$$

(2.28)

The cohomology $Z_0(t)$ now describes the off degrees of freedom of a massless $\mathcal{N} = 1$ vector multiplet in $d = 6$, i.e. $(A_\mu - \Lambda; \psi^A_i; D_{ij})$ with $i, j = 1, 2$.

At the first string level one has an extra constraint and a gauge invariance:

$$\lambda^{[A}_{0i} \lambda^{B]}_i = 0$$

$$\delta w_{1Ai} = \Lambda_{1[AB]} \lambda^{B}_{0i}$$

(2.29)

Gauge invariant states satisfying the pure spinor constraints are built in terms of the
following monomials:
\[ \lambda_1 \lambda_0^n : \left( [00, n+1]_{\frac{n+1}{2}} + [10, n-1]_{\frac{n-1}{2}} \right) t^{n+1} \]
\[ \theta_1 \lambda_0^n : \left( -[001]_{\frac{1}{2}} \times [00n]_{\frac{n}{2}} \right) t^{n+1} \]
\[ w_1 \lambda_0^n : \left( [0, 1, n]_{\frac{n-1}{2}} + [0, 0, n-1]_{\frac{n-1}{2}} \right) t^{n+3} \quad n > 0 \]
\[ x_1 \lambda_0^n : [100] \times [00n]_{\frac{n}{2}} t^{n+2} \]
\[ p_1 \lambda_0^n : \left( -[001]_{\frac{1}{2}} \times [00n]_{\frac{n}{2}} \right) t^{n+3} \]

In addition one has an extra \((1-t)^8\) coming from powers of \(\theta^A_i\). Collecting the various contributions one finds \(q \, P_{1\,\text{phys}}(t) = Z_{\lambda_1, \theta_1}(t) = -Z_{w_1, p_1, x_1}(t)\) with:

\[
P_{1\,\text{phys}}(t) = -t^2 [1, 0, 0]_0 + t^3 [0, 1, 0]_{\frac{3}{2}} + t^4 ( [2, 0, 0]_0 - [0, 0, 0]_1 )
- t^5 [1, 1, 0]_{\frac{1}{2}} + t^6 ([0, 2, 0]_0 + [1, 0, 0]_1) - t^7 [0, 1, 0]_{\frac{3}{2}} + t^8 [0, 0, 0]_0
= -6 t^2 + 8 t^3 + 17 t^4 - 40 t^5 + 28 t^6 - 8 t^7 + t^8
\]

The polynomial \(P_{1\,\text{phys}}(t)\) describes the degrees of freedom of a massive spin two multiplet in \(d = 6\): \((g_{\mu\nu} - \Lambda_{\mu}; \psi_{\mu A}^i - \Lambda^i_A; C^-_{\mu\nu\rho}, C_{\mu}^{ij} - \Lambda^{ij}; \lambda^i_A; \varphi)\), see Appendix A for the construction of the massive supermultiplet.

For the central charges one finds:
\[
- \log Z_0(t)(e^x) = -3 \log(x) + \ldots
\]
leading to \(c_{\text{vir}} = -6\) and therefore the theory is critical in \(d = 6\)!

### 2.3 \(d = 10\)

The pure spinor condition in \(d = 10\) reads:
\[
\lambda_0 \gamma^m \lambda_0 = 0
\]

This condition restrict the symmetric products of pure spinors \(\lambda_0\) to the representation:
\[
\lambda_0^n : [0000n] t^n
\]

Summing up over \(n\) one finds
\[
Z_0(t) = (1-t)^{16} \sum_{n=0}^{\infty} [0000n] t^n
= 1 - 10_\nu \, t^2 + 16_\nu \, t^3 - 16_\nu \, t^5 + 10_\nu \, t^6 - t^8
\]
The cohomology $Z_0(t)$ now describes the on degrees of freedom of a massless $\mathcal{N} = 1$ vector multiplet in $d = 10$, i.e. $(A_\mu - \Lambda, \psi^\alpha)$ and their antifields.

As usual at level one one has new constraints and gauge invariances:

$$\lambda_0 \gamma^\mu \lambda_1 = 0$$
$$\delta w_{1\alpha} = \Lambda_1 \mu (\gamma^\mu \lambda)_{0\alpha}$$

(2.35)

This restricts the allowed monomials to the representations:

$$\lambda_1 \lambda_0^n : ([0000, n+1] + [0010, n-1]) \ t^{n+1}$$
$$\theta_1 \lambda_0^n : -[00001] \times [0000n] \ t^{n+1}$$
$$w_1 \lambda_0^n : ([0100, n-1] + [0000, n-1]) \ t^{n+3} \ n > 0$$
$$x_1 \lambda_0^n : [10000] \times [0000n] \ t^{n+2}$$
$$p_1 \lambda_0^n : -[00010] \times [0000n] \ t^{n+3}$$

(2.36)

times $\theta_\alpha$’s contributing an extra $(1 - t)^{16_s}$.

Collecting all the pieces and summing up over $n$ one finds

$$q P_1^{\text{phys}}(t) = Z_{\lambda_1, \theta_1}(t) = -Z_{w_1, p_1, x_1}(t)$$

with

$$P_1^{\text{phys}}(t) = -t^2 [10000] + t^3 [00010] + t^4 [20000] - t^5 ( [00001] + [10010] )$$
$$+ t^6 ( [01000] + [10000] ) - t^8 ( [00000] + [01000] ) + t^9 [00001]$$
$$= -10 t^2 + 16 t^3 + 54 t^4 - 160 t^5 + 130 t^6 - 46 t^8 + 16 t^9$$

(2.37)

This is precisely the content of a massive spin two multiplet in $D = 10$:

$$(g_{\mu \nu} - \Lambda_\mu ; \psi_{\mu \dot{\alpha}} - \Lambda_{\dot{\alpha}}; \lambda_\alpha; C_{\mu \nu \sigma} - \Lambda_{\mu \nu}, A_\mu - \Lambda; )$$

with $128_B - 128_F$ physical degrees of freedom.

For the central charges one finds

$$- \log Z_0(t)(e^x) = -5 \log(x) + \ldots$$

(2.38)

leading to $c_{\text{vir}} = -10$ and therefore the theory is critical in $d = 10$ !

### 3 Cohomology

In the present section we derive the general form of the field equations coming from $Q$-invariance at ghost number one. The solution for $d = 4$ is worked out in full details. The case $d = 10$ has been already discussed in [30].
3.1 General results

Physical states are in one-to-one correspondence with states in the cohomology of the BRST operator

\[ Q = \int dz \lambda^A d_A \quad d_A = p_A - \frac{1}{2} \partial x_m \gamma^m \theta - \frac{1}{8} \gamma^m \theta \gamma^m \partial \theta \]  

(3.1)

By \( \int \) we will always mean \( \frac{1}{2\pi \alpha'} \oint \). The operator generates a symmetry of the lagrangian and is nilpotent if the constraint (2.2) is satisfied. The set of rules that we are using are recalled for convenience. The fundamental fields have been already introduced in the previous sections. In terms of these fields, one defined the composite operators \( \Pi_m = \partial x^m + \frac{1}{2} \theta \gamma^m \partial \theta \), \( d_A, \partial \theta A \) (see [24, 25] for the conventions in \( d = 4, 6 \)) which satisfy the affine Kac-Moody algebra

\begin{align*}
&d_A(z) d_B(w) \to - \frac{\alpha'}{(z-w)} \gamma^m_{AB} \Pi_m(w), \quad d_A(z) \Pi_m(w) \to \frac{\alpha'}{(z-w)} \gamma^m_{m,AB} \partial \theta B(w), \\
&\Pi_m(z) \Pi_n(w) \to - \frac{\alpha'}{(z-w)^2} \delta^{B}_{m,n}, \quad d_A(z) \partial \theta B(w) \to \frac{\alpha'}{(z-w)^2} \delta_A^B, 
\end{align*}

(3.2)

The BRST transformations of the fields read

\begin{align*}
&Q x^m = \frac{1}{2} \lambda^A \gamma^m_{AB} \theta B, \quad Q \theta^A = \lambda^A, \quad Q \Pi_m = \lambda_A \gamma^m_{AB} \partial \theta B, \\
&Q d_A = - \Pi_m \gamma^m_{AB} \lambda^B, \quad Q \lambda^A = 0, \quad Q w_A = - d_A, 
\end{align*}

(3.3)

The BRST transformation rules are nilpotent up to gauge variations. Indeed applying twice the BRST charge on \( w_A \) we get \( Q^2 w_A = \Pi_m \gamma^m_{AB} \lambda^B = \delta_{m} w_A \) with \( \delta_A w_A = A^m \gamma^m_{AB} \lambda^B \). Therefore, the only vertex operators which are admissible are those which are gauge invariant under the gauge transformations for the \( w \)'s.

Physical states appear in the cohomology of the BRST operator \( Q \) at ghost number \( n_\lambda - n_w = 1 \). We denote string vertices by \( U_\ell^{(q)} \) with \( \ell \) labelling the string level and \( q \) the ghost charge. The vertices \( U_\ell^{(q)} \) are defined up to gauge transformations \( \delta U_\ell^{(q)} = Q U_\ell^{(q-1)} \).

At the massless level one finds:

\begin{align*}
U_0^{(1)} &= : \lambda^A A_A : \\
U_0^{(0)} &= : \Omega :
\end{align*}

(3.4)

with \( A_A, \Omega \) arbitrary superfields\(^6\). On a generic superfield \( A(x, \theta) \) the BRST operator \( Q \) acts as a supersymmetric derivative:

\[ QA = \lambda^A D_A A \]  

(3.5)

---

\(^{6}\)Normal order here and below follows the definition [31]: \( AB : (z) = \frac{1}{2\pi i} \oint \frac{dy}{y-z} A(y) B(z) \) and \( : ABC : (z) = \frac{1}{(2\pi i)^2} \lim_{w \to z} \left[ f_{Cw} + f_{Cz} \right] \frac{dy}{y-z} A(y) B(w) C(z) \).
Acting with $Q$ and imposing $QU^{(1)}_0 = 0, \delta U^{(1)}_0 = QU^{(0)}_0$ one finds
\[ D_{(AA)B} = \gamma^m_{AB} A_m, \quad \delta A_A = D_A \Omega, \quad \delta A_m = \partial_m \Omega, \] (3.6)

It is not hard to see that this gives the degrees of freedom of a vector multiplet in $d$ dimensions.

Let us consider now the first massive string level. For the ghost number 0,1 vertices one finds:
\[ U^{(0)}_1 = : \partial \theta^A \Omega_A : + : \Pi^m \Gamma_m : + : d_A A^A : + : J_A C^A \Phi_A : \] (3.7)
\[ U^{(1)}_1 = : \partial \lambda^A A_A : + : \lambda^A \partial \theta^B B_{AB} : + : \lambda^A d_B C^B_A : + : \lambda^A \Pi^m H_{Am} : + : J_A B^C F^A_{CB} : \] (3.8)

Again superfields $A_A, B_{AB}, ... \Omega, \Gamma, ...$ are functions of the (super)spacetime coordinates $(x^m, \theta^A)$. We denote by $J_{AB}$ the gauge invariant combinations
\[ J_{AB} = P_{BD}^{AC} : w_D \lambda^C : , \] (3.9)

with $P_{BD}^{AC}$ a projector into the gauge invariant components satisfying $\delta J_{AB} = 0$. For example in $d = 4$, $J_{AB} = J_B^A + J_{mn} \gamma^{mnB} + J_5^B \gamma_5 A$, where $J$ is the ghost current, $J_{mn}$ are the Lorentz generators in the pure spinor sector and $J_5$ generates chiral rotations of the pure spinors. In a similar way the Lorentz content of $J_{AB}$ in $d = 6, 10$ is given in (2.10).

In taking products of $J_{AB}$ and $\lambda^A$ one should take particular care with normal orderings. In particular, two operators that are independent at the classical level can mix under normal ordering. The first few of these relations take the form:
\[ \Xi_{BC}^A J_{AB}^B \lambda_C = 0 \Rightarrow \Xi_{BC}^A : J_{AB}^B \lambda_C : = -\alpha' \Xi_{BA}^A \partial \lambda^B , \] (3.10)
\[ K_{BCD}^A J_{AB}^B \lambda_C \lambda^D = 0 \Rightarrow K_{BCD}^A : J_{AB}^B \lambda^C \lambda^D : = -2 \alpha' K_{(BC)}^A \lambda^C \partial \lambda^B \]

The right hand side here refers to product of functions rather than operators. This implies that a relation that hold at the classical level $\alpha' \to 0$ due to pure spinor constraints can fail in the full quantum theory. We will refer to these relations as anomalous Ward identities. In particular the first of these equations implies that operators $\partial \lambda^A$ and $J_{AB} \lambda^C$ appearing in $U^{(1)}_1$ are not independent, i.e. the vertex operator $U^{(1)}_1$ is defined up to the algebraic gauge transformations
\[ \delta F_{BC}^A = \Xi_{BC}^A , \quad \delta A_A = -\alpha' \Xi_{AB} \] (3.10)

where $\Xi_{BC}^A$ is a gauge parameter satisfying (3.9). These gauge transformations can be used to restrict $F_{BC}^A$ to those components satisfying $F_{BC}^A J_{AB} \lambda^C \neq 0$ at the classical level. Here will always work in this gauge. In a similar way $K_{BCD}^A$ reflects a relation between operators appearing at ghost number 2 and will be important in our analysis below. The
Lorentz content of $\Xi^A_{BC}$ and $K^A_{BCD}$ depends on the dimensions and will be worked out below in some relevant cases.

Acting with the BRST charge on the vertex operator $U_1^{(1)}$, one obtains the following equations of motion

$$\lambda^A \lambda^B \partial \theta^C \left(D_{(AB)}B_C - \gamma^s_{C(A}H_{B)s} \right) = 0 ,$$

(3.11)

$$\lambda^A \lambda^B \Pi_s \left(D_{(A}H_{B)} - \gamma^s_{C(A}C^C_{B)} \right) = 0 ,$$

(3.12)

$$\lambda^A \lambda^B \partial C \left(D_{(A}C_{B)} + F_{(AB)}^C \right) = 0 ,$$

(3.13)

$$\lambda^B J_A^B : \left(D_B F^A_{CB} - K^A_{BCD} \right) = 0 ,$$

(3.14)

$$\lambda^A \lambda^B \partial \theta^C \left(D_{(A}H_{B)} - \gamma^s_{C(A}H_{B)s} \right) = 0 .$$

(3.15)

The contributions of $K^A_{BCD}$ to (3.14) and (3.15) cancel against each other in $QU_1^{(1)}$ according to (3.9) and allow us to treat these two equations independently. It is important to notice that the introduction of this superfield ensures the gauge invariance under the symmetry (3.10) of the equations of motion (3.11)-(3.15). Taking for example eq. (3.14) and by performing the gauge transformation (3.10), one can see that the compensating gauge transformation is

$$\delta K^A_{BCD} = D_D \Xi^A_{CB}$$

(3.16)

In addition, the equations of motion are invariant under the gauge transformations

$$\delta A_A = \Omega_A + \alpha' \gamma^m_{AB} \partial_m A^B - \alpha' D_B \Phi^B_A ,$$

$$\delta B_{AB} = - D_A \Omega_B + \gamma^m_{AB} \Gamma_m ,$$

$$\delta H_{Am} = D_A \Gamma_m - \gamma_{m,AB} \Lambda_B^A ,$$

$$\delta C^A_B = - D_A \Lambda^B - \Phi^A_B ,$$

$$\delta F^A_{CB} = D_C \Phi^A_B .$$

(3.17)

These gauge transformations are needed in order to select the physical states and they are used to set some of the field to zero. Indeed the gauge parameters $\Omega_A, \Gamma_m, \Lambda^A, \Phi^B_A$ can be used to set

$$A_A = \gamma^{mAB} B_{AB} = \gamma^{mAB} H_{mB} = \mathcal{P}^{BD}_{AC} C^A_B = 0$$

(3.18)
respectively. The resulting fields should be plugged into the BRST equations (3.11-3.15). Formally the first four equations can be written as: \( H = dB, \ C = dH, \ F = dC, \ K = dF, \) projected on specific representations carried by the worldsheet operators multiplying them. These equations, as we will see, allow us to express all fields in terms of a single superfield \( B_{[mnp]} \). Below we will show this for \( d = 4 \). The case \( d = 10 \) has been worked out in [30]. There is an important difference between \( d = 4 \) and \( d = 6, 10 \). In \( d = 4 \) we will find that the massive fields are off-shell while in \( d = 6, 10 \) are on-shell. This is due to the fact that in \( d = 4 \) there is no enough supersymetries to build the Laplacian operator.

\[ \text{3.2} \quad d = 4 \]

We use the following notation for Dirac matrices: \( \gamma^{mnp} \) for antisymmetrized combinations of the Dirac matrices \( \gamma^m \). A symmetric bispinor \( \psi^{(AB)} \) can be decomposed as follows

\[ \psi^{(AB)} = \psi_m^{m} \gamma^{(AB)} + \psi^{mn} \gamma^{(AB)}; \]

an antisymmetric bispinors \( \psi^{[AB]} = \psi^{C[AB]} + \psi^{mnp} \gamma^{[AB]} \) and \( \psi^{mnpq} \gamma^{[AB]} \). The indices are raised and lowered with the antisymmetric tensor \( C^{[AB]} \).

The pure spinors are represented by a Dirac spinor \( \lambda^A \) \( (A = 1, \ldots , 4) \) and the pure spinor constraint is \( \lambda \gamma^m \lambda = 0 \). As a warming-up exercise, we compute the cohomology at massless level. The most general massless vertex operator is

\[ U^{(1)}_0 = \lambda^A A_A (x, \theta) \]  

(3.19)

and the gauge symmetry is given by the scalar superfield \( \Omega \). The BRST condition implies \( \lambda^A \lambda^B D_A A_B = 0 \) up to gauge transformations \( \delta A_A = D_A \Omega \). Therefore, the most general solution is given by

\[ U^{(1)}_0 = \lambda^A (D_A M + \gamma^{5}_{AB} D^B M_5) \]  

(3.20)

The first term can be removed by a gauge transformation, but the second term represents an element of the cohomology. Notice that we have to require the reality condition in order not to spoil this symmetry of the theory. This implies that the dofs are represented by a real scalar superfield \( M_5 \). This computation appeared also in [24].

The BRST condition on the physical states gives the equations of motion in (3.11)-(3.15). However, one has to remove the factors in front of the equations and to project the equation along the pure spinor directions. In the \( d = 4 \), a symmetric bispinor is decomposed as \( \lambda^A \lambda^B = \frac{1}{4} \gamma^{AB}_m \psi^m + \frac{1}{6} \gamma^{AB}_{mn} \psi^{mn} \) and \( \psi^m = 0 \) because of the pure spinor condition. This implies for example that the first equation (3.11) becomes

\[ \gamma^{AB}_{mn} (D_{(A} B_{B)C} - \gamma^{s}_{C(A} H_{B)s}) = 0 , \]  

(3.21)
and so on. It is undoubtedly convenient to use SU(2) × SU(2) (Weyl) indices to label SO(4) representations. In terms of these definitions the superfields decompose as follows

\[ A_A = \{A_\alpha, A_\dot{\alpha}\}, \]
\[ B_{AB} = \{B_{\alpha\beta}, B_{\dot{\alpha}\dot{\beta}}, B_{\alpha\dot{\beta}}, B_{\dot{\alpha}\beta}\}, \]
\[ H_{Am} = \{H_{\alpha\dot{\beta}\dot{\beta}}, H_{\dot{\alpha}\beta\dot{\beta}}\}, \]
\[ C^B_A = \{C^\alpha_\beta, C^\dot{\beta}_\dot{\alpha}, C^\beta_\alpha, C^{\dot{\beta}}_\dot{\alpha}\}, \]
\[ F^A_{BC} = \{F^{\alpha}_{(\beta\gamma)}, F^{\dot{\alpha}}_{(\dot{\beta}\dot{\gamma})}\}, \]

where we have used the \(\Xi^{\beta\dot{\beta}}_\alpha, \Xi^{\dot{\beta}\beta}_\dot{\alpha}\)-gauge symmetry in order to put \(F^A_{BC}\) in this form. The currents \(J^A_B\) become \(\{J_\alpha\beta, J_{\dot{\alpha}\dot{\beta}}\}\) and satisfy the anomalous Ward identities (3.9) which expressed in Weyl indices become

\[ :J_{\alpha\beta}\lambda^\beta := -\alpha'\partial\lambda_\alpha, \]
\[ :J_{\alpha\dot{\beta}}\dot{\lambda}^{\dot{\beta}} := 0, \]
\[ :J_{\alpha\dot{\beta}}\lambda^\beta\lambda^\gamma := -\alpha'\partial\lambda_\alpha\lambda^\gamma - \alpha'\delta^\gamma_\alpha\partial\lambda_\beta\lambda^\beta, \]
\[ :J_{\alpha\dot{\beta}}\dot{\lambda}^{\dot{\gamma}} := 0, \]
\[ :J_{\alpha\dot{\beta}}\lambda^\beta\lambda^\gamma := 0. \]

(3.23)

and the hermitian conjugates. For simplicity in what follows we omit complex conjugate equations that can be easily found by exchanging dotted and undotted indices. The Ward identities (B.2) follow from (3.9) by introducing \(K_{\beta\sigma}\) and its complex conjugate. Eqs (3.11-3.15) become

\[ D_{(\alpha B_\beta)\gamma} = 0, \]
\[ D_{(\alpha B_\beta)\dot{\gamma}} - H_{(\alpha\beta)\dot{\beta}} = 0, \]
\[ D_{(\alpha H_\beta)\dot{\beta}} - C_{\beta\dot{\beta}} = 0, \]
\[ D_{(\alpha C^\gamma_\beta)} + F^{\gamma}_{(\alpha\beta)} = 0, \]
\[ D_{(\alpha C^{\dot{\gamma}}_\dot{\beta})} = 0, \]
\[ D_{(\beta F^{\sigma}_{\alpha})\gamma} - K_{(\gamma(\alpha\delta^\sigma_\beta)} = 0, \]
\[ D_{\alpha A_\beta} + B_{\alpha\beta} + \alpha'\partial_{\beta\dot{\alpha}}C^{\dot{\beta}}_\alpha - 2\alpha' D_{(\tau_{F^\tau_{(\alpha\beta)}}} + \alpha' K_{\alpha\beta} = 0, \]

(3.28)
These equations are gauge invariant with respect to the gauge symmetry (3.17) that in Weyl indices become
\begin{align*}
\delta A_\alpha &= \Omega_\alpha - \alpha' D_\gamma \phi^\gamma_\alpha + \alpha' \partial_\alpha \bar{\lambda}^\beta, \\
\delta B_{a\beta} &= -D_a \Omega_\beta, \\
\delta B_{\dot{\beta}a} &= -\bar{D}_\dot{\beta} \Omega_\alpha + \Gamma_\alpha \dot{\beta}, \\
\delta H_{a\dot{\beta}\dot{\gamma}} &= D_a \Gamma_{\dot{\beta}\dot{\gamma}} - \epsilon_{a\dot{\beta}\dot{\gamma}} \bar{\lambda}^\beta, \\
\delta C_{\dot{\alpha}a} &= -\phi^\alpha_\beta - D_a \bar{\lambda}^\beta, \\
\delta C_{\dot{\alpha} \dot{\beta}} &= -\bar{D}_\dot{\alpha} \Lambda^\beta, \\
\delta F^{(\alpha\gamma)}_{\alpha\beta} &= D_{\beta} \phi^{(\alpha\gamma)}.
\end{align*}

The gauge symmetries $\Omega_\alpha, \Gamma_{\alpha\dot{\beta}}, \Lambda^\gamma, \Phi^\beta_\alpha$ can be used to set
\begin{align*}
A_\alpha &= B_{a\dot{\beta}} + B_{\dot{\beta}a} = H^{\alpha}_{a\dot{\beta}} = C^\beta_\alpha = F_{\alpha\gamma}^\alpha = 0 \tag{3.30}
\end{align*}
and similar for their complex conjugates.

Now it is easy to solve the equations. From (3.24-3.27), one finds
\begin{align*}
B_{a\beta} &= D_a T_\beta, \\
H_{a\dot{\beta}\dot{\gamma}} &= D_{(a} B_{\beta)\dot{\gamma}}, \\
C_{\dot{\beta}a} &= D_{(a} H_{\beta)\dot{\gamma}}^\alpha, \\
F_{\alpha\beta}^{(\alpha\gamma)} &= K_{a\beta} = 0 \tag{3.31}
\end{align*}
Then eq. (3.28) gives
\begin{align*}
T_\alpha &= -\frac{2\alpha'}{3} \partial_\alpha \bar{D}_a B^{\sigma \dot{\alpha}}, \tag{3.32}
\end{align*}
Therefore the full cohomology at ghost number one can be written in terms of an unconstrained superfield $B_{a\dot{\beta}} - B_{\dot{\beta}a}$. This is in complete agreement with the result of the string partition function (2.24). Notice that in obtaining this result it is crucial that in $d = 4$ $\lambda \partial \bar{\lambda} = \bar{\lambda} \partial \lambda = 0$ or in covariant form $\lambda^{mnp} \partial \lambda = 0$. This is not the case in $d = 6, 10$ and therefore the last equation (3.15) gives a non-trivial Laplacian eq. for $B_{mnp}$ in these dimensions. In all cases the multiplet starting with $B_{mnp}$ contains a spin two particle but in $d = 6, 10$ fields are on-shell and the multiplets are shorter. This was explicitly shown in [30] for the case of $d = 10$.

4 Low dimensional models

Here we consider pure spinor constructions in $d = 2, 3$ dimensions.
4.1 $\mathcal{N} = (2, 0)$ model

We consider the pure spinor system

$$\begin{align*}
\lambda \bar{\lambda} &= 0, \\
\delta w &= \Lambda \lambda, \\
\delta \bar{w} &= \Lambda \bar{\lambda},
\end{align*}$$

(4.1)

with $\Lambda$ is the gauge parameter. As before we add the anticommuting variables $(\theta, \bar{\theta})$ and their conjugates $(p, \bar{p})$. The pure spinor constraint allows us to set either $\lambda = 0$ or $\bar{\lambda} = 0$ and therefore the $\lambda, w, \theta, p$ system has naive central charge $c_{\lambda, w, \theta, p} = 2 - 4 = -2$. Therefore the system is critical in $d = 2$. We denote by $X, \bar{X}$ the bosonic degrees of freedom.

The covariant derivatives and supersymmetric line elements are defined by

$$d = p + \bar{\theta} \partial x, \quad \bar{d} = \bar{p}, \quad \Pi = \partial x, \quad \bar{\Pi} = \partial \bar{x} + \bar{\theta} \partial \theta,$$

(4.2)

The two supercharges realize an $\mathcal{N} = (2, 0)$ supersymmetry. We should stress that even if these definitions look asymmetrical, by similarity transformations they can be put in a symmetric form. The BRST charge can be written as

$$Q = \int (\lambda d + \bar{\lambda} \bar{d}),$$

(4.3)

and acts as follows

$$\begin{align*}
Q \theta^\alpha &= \lambda^\alpha, \quad Q \bar{x} = \lambda \bar{\theta}, \\
Q \lambda^\alpha &= 0, \quad Q w_\alpha = -d_\alpha, \\
Q \bar{d} &= -\lambda \bar{\Pi}, \quad Q \bar{\Pi} = \bar{\lambda} \partial \theta,
\end{align*}$$

(4.4)

Acting on a generic superfield $A(x^m, \theta)$ one finds

$$QA = \lambda D A + \bar{\lambda} \bar{D} A, \quad D = \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{x}} , \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}}$$

(4.5)

The worldsheet theory is now invariant under (apart from $\Delta$) the extra $U(1)$ symmetry defined by the charge assignments:

$$J' = n_{d_\alpha} + n_{w_\alpha} + n_{\Pi}$$

(4.6)

This new symmetry will be traced by the parameter $t'$. It is easy to compute the partition function of this model. The pure spinor constraint requires that only polynomials of either $(\lambda, w)$ or $(\bar{\lambda}, \bar{w})$ are allowed. One finds:

$$Z_{\lambda, w}(q, t) = \left(\frac{2}{1 - t} - 1\right) + q \left(\frac{2(t + t' t^3)}{(1 - t)} - 2 t' t^3\right) + \ldots$$

(4.7)
Multiplying by the free contributions of $\theta_{0,1}, x_1, d_1$:

$$Z_{\theta_{0,1}, x_1, p_1} = (1 - t)^2 \left[ 1 + q(t^2 + t' t^2 - 2t - 2t' t^3) + \ldots \right] \quad (4.8)$$

one finds

$$Z_0(t) = 1 - t^2$$
$$Z_1(t) = t' t^2 (1 - t)^2 - t^2 (1 - t)^2$$ \quad (4.9)

Interestingly enough, one finds a pair of multiplets at the first massive level with opposite statistics and different $J'$ charge. Below we will confirm this result by explicit analysis of the cohomology.

For the central charges one finds

$$-\log Z_0(t)(e^x) = -\log(x) + \ldots$$ \quad (4.10)

leading to $c_{\text{vir}} = -2$ therefore the theory is critical in $d = 2$!

Now let us consider the ghost number one cohomology. We start with the massless level. The vertex operator is $U^{(1)}_0 = \lambda \bar{A} + \bar{\lambda} A$ and the gauge symmetry is generated by $U^{(0)}_0 = \Omega$. The BRST symmetry implies that

$$DA = D\bar{A} = 0$$ \quad (4.11)

and its most general solution is $A_\alpha = D_\alpha M$ where $M$ is an arbitrary superfield. $M$ can be gauged away using $\Omega$ and therefore there is no cohomology at ghost number one. Notice that at zero momentum this is not true any longer and one finds that there is an element of the cohomology given by the monomial $\bar{\lambda} \theta$. The cohomology at zero momentum contains then the identity operator and a fermionic state at order $t^2$ in agreement with (4.9).

Let consider the first massive level. At ghost number zero one has the vertex:

$$U^{(0)}_1 =: \partial \theta^\alpha \Omega_\alpha : + \Pi^m \Gamma_m : + : d_\alpha \Lambda^\alpha : + : J^\alpha \Phi_\alpha :$$ \quad (4.12)

$QU^{(0)}_1 = 0$ implies

$$J^\alpha = \Omega^\alpha = \bar{\Gamma} = 0 \quad D\Gamma + \bar{\Lambda} = \bar{D}\Gamma - \bar{\Lambda} = 0$$

therefore the cohomology at ghost number zero is given in terms of a single unconstrained superfield $\Gamma$. The content of $\Gamma$ reproduces $t'$ term in (4.9). This is an important difference with the case $d = 4$ where the cohomology at ghost number zero was shown to be empty.

The remaining states appear at ghost number one. Acting with $Q$ on (4.12) one finds the gauge transformations:

$$\delta U^{(1)}_1 =: \partial \lambda^\alpha \Omega_\alpha : + : \bar{\lambda} \partial \theta \bar{\Gamma} : + : \bar{\lambda} \Pi \Lambda : - : \lambda \Pi \bar{\Lambda} : - : d_\alpha \lambda^\alpha \Phi_\alpha : + \ldots$$ \quad (4.13)
with dots denoting supersymmetric derivatives of the gauge superfields. The gauge transformations (4.13) can be used to set all components of $U_1^{(1)}$ appearing in (4.13) to zero. One finds the gauge fixed vertex operator

$$U_1^{(1)} =: \lambda \partial \bar{\theta} \tilde{B} : + : \lambda \bar{\Pi} \bar{H} : + \left( : \lambda \partial \theta B : + : \lambda \bar{B} C : + : J \lambda F : + \text{h.c.} \right)$$ (4.14)

Imposing $QU_1^{(1)} = 0$ a simple algebra leads to the following eqs.:

$$B = \bar{B} = C = \bar{C} = F = \bar{F} = H = 0$$
$$D \tilde{B} = D H = 0$$ (4.15)

At zero momenta the degrees of freedom coming from (4.15) are $\tilde{B} = -t^2(1 - t)$ and $H = t^3(1 - t)$. Altogether they reproduce the $t^0$ states in (4.9). It is important to stress that although the cohomology at ghost number 0,1 already matches the string partition function this does not imply that the higher ghost number cohomology is empty. In particular there is an infinite tower of states $\bar{\Pi} \lambda^n$ that clearly belong to the cohomology, but for $n > 1$ they come always in pairs field/antifield with opposite statistics.

### 4.2 $d=3$

Next we consider a pure spinor $\lambda^\alpha$ in $d = 3$ satisfying the constraints:

$$\lambda^{(\alpha \lambda \beta)} = 0$$ (4.16)

transforming in the vector representation of $SO(3)$, i.e. the 3 of $SU(2)$. The gauge invariance reads

$$\delta w_\alpha = \Lambda_\alpha^\beta \lambda^\beta$$ (4.17)

In addition one adds a $(p_\alpha, \theta^\alpha, x^m)$ system. It is easy to list the set of invariant monomials satisfying (2.19):

$$1, \lambda_0^\alpha, \lambda_1^\alpha, \theta_1^\alpha, \lambda_0^{[\alpha \beta]} \lambda_0^{\beta}, x_1^m, x_1^m \lambda_0^\alpha, p_{1\alpha}, p_{1\alpha} \lambda_0^\beta, w_{1\alpha} \lambda_0^\beta$$ (4.18)

Collecting all contributions and multiplying by $(1 - t)^2$ one finds:

$$Z_0(t) = 1 - 3 t^2 + 2 t^3$$
$$Z_1(t) = 4 t^3 - (3 + 5) t^4 + 4 t^5$$ (4.19)

Like in the $d = 4$ case one finds a massless gauge multiplet in $d = 3$ and a massive long multiplet containing a spin two particle. For the central charges one finds

$$- \log Z_0(t)(e^x) = -2 \log(x) + \ldots$$ (4.20)
leading to $\frac{1}{2} c_{\text{vir}} = -4$ therefore the theory is non-critical in $d = 3!$

In this case it is rather easy to compute the complete spectrum at massless and first massive level. The crucial point here in comparison with higher dimensional examples is that the pure spinor constraint are very strong leading to a non-trivial cohomology only at ghost number one. Indeed we will identify all states in the partition function result (4.19) as ghost number 1 states.

We write the pure spinor constraint as $\lambda \gamma^m \lambda = 0$ and the gauge symmetry as $\delta w_\alpha = \Lambda_m (\gamma^m \lambda)$. The combinations $J_\alpha^\beta =: w_\alpha \lambda^\beta :$ are gauge invariant. A bispinor $\psi^{\alpha \beta}$ is decomposed into $\psi^{\alpha \beta} = \epsilon^{\alpha \beta} \psi + \gamma_m^{\alpha \beta} \psi^m$. The Dirac matrices $\gamma_m^{\alpha \beta}$ are symmetric and real, they satisfies the Fierz identities $\gamma_m^{\alpha \beta} \gamma^m_{\gamma \delta} = 0$. The notation, the action and the BRST symmetry are described in [33].

The most general vertex operator of ghost number 1 at the massless level is

$$U^{(1)}_0 = \lambda^\alpha A_\alpha (x, \theta)$$

(4.21)

and the gauge symmetries are represented by a scalar superfield $\Omega$. The gauge transformations are given by $\delta A_\alpha = D_\alpha \Omega$. Imposing the BRST invariance and using the pure spinor condition, we see that there is no constraint on the superfield $A_\alpha$. However, by using the gauge symmetry, we can easily see that

$$A_\alpha = a^{(\alpha \beta)}(x) \theta^\beta + u_\alpha(x) \theta^2, \quad \delta a_{\alpha \beta}(x) = \partial_{\alpha \beta} \omega(x).$$

(4.22)

These are the dofs for an off-shell super-Yang-Mills in 3d in agreement with (4.19).

For the massive spectrum, we consider the first massive level, by expanding the vertex operator into conformal spin 1 worldsheet operators.

We have that the most general vertex operator is given by

$$U^{(1)}_1 = \partial \lambda^\alpha A_\alpha + \lambda^\alpha \partial \theta^\beta B_{\alpha \beta} + \lambda^\alpha \Pi^\beta \gamma H_{\alpha (\beta \gamma)} + \lambda^\alpha d_\beta C_{\alpha}^\beta + : J_\alpha^\beta \lambda^\gamma : F_{\alpha \gamma}^\gamma$$

(4.23)

which is invariant under the gauge symmetry

$$\delta F_{\alpha \beta}^\gamma = \Xi_{\alpha \beta}^\gamma, \quad \delta A_\alpha = -\alpha' \Xi_{\alpha \beta}^\gamma.$$

(4.24)

Notice that this gauge symmetry removes completely the field $F$, but this has an effect on the superfield $A_\alpha$. The most general gauge transformation is generated by the vertex operator

$$U^{(0)}_1 = \partial \theta^\alpha \Omega_\alpha + \Pi^{(\alpha \beta)} \Gamma_{(\alpha \beta)} + d_\alpha \Lambda^\alpha + J_\alpha^\beta \phi^\beta_{\alpha}.$$  

(4.25)
and the gauge transformations are given by
\[
\begin{align*}
\delta A_\alpha &= \Omega_\alpha - \alpha' \partial_{\alpha} \Lambda^\beta - \alpha' D_{\beta} \phi^\beta_\alpha \\
\delta B_{\alpha\beta} &= -D_\alpha \Omega_\beta + \Gamma_{(\alpha\beta)} \\
\delta H_{\alpha(\beta\gamma)} &= D_{\alpha} \Gamma_{\beta\gamma} + \epsilon_{\alpha(\beta} \Lambda_{\gamma)} \\
\delta C^\alpha_\beta &= -D_{\beta} \Lambda^\alpha - \phi^\beta_\alpha \\
\delta F^\gamma_{\alpha\beta} &= D_{(\alpha} \phi^\gamma_{\beta)}.
\end{align*}
\]
(4.26)
The gauge symmetry of \( F \) is a subset of the gauge symmetry (4.24).

By imposing the BRST symmetry, we have the only condition (the other conditions are trivially satisfied)
\[
\epsilon^{\alpha\beta} \left( D_\beta A_\alpha + B_{\alpha\beta} + \alpha' \partial_{\gamma} (\alpha \gamma C^\beta_\gamma) - \alpha' D_{(\alpha} F^\gamma_{\beta)\gamma} \right) = 0,
\]
(4.27)
Now, we can use the gauge symmetry to removing several fields. 1) Use \( \Xi^\alpha_\beta \) to remove \( F^\gamma_{\beta\gamma} \), 2) use \( \phi^\beta_\alpha \) to remove \( C^\beta_\alpha \), 3) use \( \Lambda_\alpha \) to set \( \epsilon^{\alpha\beta} H_{\alpha(\beta\gamma)} = 0 \), 4) use \( \Gamma_{\alpha\beta} \) to kill the symmetric part of \( B_{\alpha\beta} \), 5) use \( \Omega_\alpha \) to set \( A_\alpha = 0 \). From eq. (4.27) we are left with \( \epsilon^{\alpha\beta} B_{\alpha\beta} = 0 \). THis implies that the only physical content of the theory stays in the superfield \( H_{\alpha(\beta\gamma)} \). So, the physical states are given by the vertex operator
\[
U_{1}^{(1)} = \lambda^\alpha \Pi^{\beta\gamma} H_{\alpha(\beta\gamma)}(x, \theta)
\]
(4.28)
for any \( H_{\alpha(\beta\gamma)} \). By expanding it into components, we have
\[
H_{\alpha(\beta\gamma)} = h_{\alpha(\beta\gamma)}(x) \theta^\delta + \hat{h}_{\alpha(\beta\gamma)}(x) \theta^2
\]
(4.29)
which contains 8 bosons and 8 fermions. Comparing the \( SO(3) \) content in (4.29) with that found in (4.19) we find a perfect agreement. The fields are not on-shell as to be expected by general considerations.

5 Conclusions and Summary

In this paper we consider pure-spinor critical strings in dimensions \( d = 4, 6, 10 \). We determine the spectrum of massless and first massive string states in any dimension. The results are written in terms of \( SO(d) \) covariant string partition functions tracing Lorentz representations and an extra \( U(1) \) charge \( \Delta \). The Fock space is defined by string modes satisfying the pure spinor constraint and the induced gauge invariance. This results into restrictions on the allowed representations entering in the product of pure spinors \( \lambda^A_n \) and/or their conjugated momentum \( w^A_n \). The outcome of this analysis is displayed in tables 2 and 3 for the massless and first massive level respectively.
In particular specifying to \( n = 2 \) in table 5 we see that the vector components \([\frac{1}{2}, \frac{1}{2}], [100]\) and \([0000]\) are missing in the symmetric products of two pure spinors \( \lambda_0^A \) as expected.

The massless partition function follows by collecting the representations in table (5) weighted by \( t^n \), summing up over \( n \) and multiplying them by the contribution of the free fields \( \theta, x, p \). One finds

\[
Z_{d=0}^d(t) = (1 - t)^{S_d} \sum_n \lambda_0^n t^n
\]

with \( SO(d) \) content

\[
Z_{d=4}^d(t) = 1 - 4 \nu t^2 + (2 s + 2 c) t^3 - t^4
\]
\[
Z_{d=6}^d(t) = 1 - 6 \nu t^2 + 24 s t^3 - 3 t^4
\]
\[
Z_{d=10}^d(t) = 1 - 10 \nu t^2 + 16 s t^3 - 16 c t^5 + 10 \nu t^6 - t^8
\]

In each dimension this is precisely the content of a gauge vector supermultiplet of minimal supersymmetry.

At the first massive states in \( d = 4 \) one finds

\[
P_{d=4}^d(t) = -4 \nu t^2 (1 - t)^2 s + 2 c
\]

Even and odd powers of \( t \) correspond to bosonic and fermionic degrees of freedom respectively. Expanding (5.3) one finds the degrees of freedom of a long multiplet of minimal supersymmetry with a spin two particle as the highest helicity state. Notice that this represents the content of a long multiplet with a spin two highest helicity state. In \( d = 6, 10 \) fields come always in pairs with anti-fields and the total partition function
vanishes. Physical states can be identified with those coming from $\lambda_1, \theta_1$ string modes. One finds $P_1(t) \equiv Z_{\lambda_1, \theta_1}(t) = -Z_{w_1, p_1, x_1}(t)$ with

$$P_{d=6}^1(t) = -6v t^2 + 24s t^3 + (20 - 3)t^4 - 220st^5 + (10 + 36)t^6 - 24ct^7 + t^8 \tag{5.4}$$
$$P_{d=10}^1(t) = -10v t^2 + 16s t^3 + 54t^4 - (16c + 144)t^5 + (10v + 120)t^6 - (1 + 45)t^8 + 16s t^9$$

This reproduces the content of a massive spin-two multiplet in $d = 6, 10$ dimensions. In particular, in $d = 10$ one finds the $128_B - 128_F$ degrees of freedom of the ten-dimensional open superstring. It would be nice to extend these results to higher string modes.

The pure spinor CFT’s introduced here open new scenarios for studies of holographic correspondences between gravity and minimal SYM gauge theories. Indeed critical closed strings can be constructed in a similar way by tensoring two copies of the CFT described here. In particular the spectra is given by the square of (5.2, 5.3, 5.4) and correspond to $N = 2$ supergravities in $d = 4, 6, 10$. The non-perturbative spectrum of these theories always comprehends brane where the boundary gauge theories described in the present work live. It would be nice to explore applications of the pure spinor descriptions along these lines.

The techniques developed here also apply to a large class of interesting conformal field theories defined via constraints. In particular in section (4) we show how a critical string describing a two-dimensional CFT with $\mathcal{N} = (2, 0)$ supersymmetry can be described in terms of two bosonic variables satisfying a constraint. It would be nice to apply these ideas to the study of elliptic genera of other constrained systems like strings moving on algebraic surfaces.

In [18], the group structure of the pure spinor space, which is a conical space (see [32]), was used to compute the zero-mode part of the partition function. One can wonder whether the same technique applies to massive states in terms of the associated Kac-Moody algebras.

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Appendices

A Massive Supermultiplets

In this appendix we organize states in the string spectrum in multiplets of minimal supersymmetry in $d$ dimensions. Supermultiplets can be constructed by acting with the raising supersymmetry charges on a highest weight states (see [21] for details):

$$[n_1, n_2, ...] = \sum_{\epsilon_i = 0, 1} (-t)^{2+\epsilon} \dim [n_1 + \epsilon_i q^i_1, n_1 + \epsilon_i q^i_2, ...]^*$$

(A.1)

with $\epsilon = \sum \epsilon_i$, $Q^i = [q^i_1, q^i_2, ...]$ the weights of the raising supercharges $i = 1, ..., S_d/2$ in the Dynkin basis. Finally $[n_1, n_2, ...]^* \equiv [n_1, n_2, ...] - [n_1 - 1, n_2, ...]t^{-2}$ labels the $SO(d)$ Dynkin labels of the h.w.s.. The term with minus sign subtract the unphysical components, e.g. a massive vector in $d$ dimensions is written as $[100..]^* = [100..] - [000..]$ and so on. Alternatively one can write the polynomial with definitely positive coefficients in terms of $SO(d-1)$ representations. Here we prefer to keep $SO(d)$ covariance.

In table 4 we list the supercharges weights for minimal supersymmetry in dimensions $d = 4, 6, 10$. Plugging the supercharges into (A.1) one finds that the content of vector multiplets $-t^2[\frac{1}{2}, \frac{1}{2}], -t^2[1, 0, 0, 0], -t^2[1, 0, 0, 0, 0, 0]$ match precisely the polynomials (5.2) describing the massless string spectrum. In a similar way the content of a spin two multiplets $t^4[1, 1], t^4[2, 0, 0, 0, 0]$ in $d = 6, 10$ match precisely the result $P^\text{phys}_1(t)$ given by (5.4) for the first massive string level in $d = 6, 10$.

<table>
<thead>
<tr>
<th>D</th>
<th>$Q^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$[-\frac{1}{2}, 0], [-\frac{1}{2}, 0]$</td>
</tr>
<tr>
<td>6</td>
<td>$[0 0 -1]<em>{\pm \frac{1}{2}}, [-1 0 1]</em>{\pm \frac{1}{2}}$</td>
</tr>
<tr>
<td>10</td>
<td>$[0 0 0 0 -1], [-1 0 1 0 -1], [-1 1 -1 1 0], [0 -1 0 1 0], [0 -1 1 -1 0], [0 0 -1 0 1], [-1 0 0 0 1]$</td>
</tr>
</tbody>
</table>

Table 4: Supercharges

B Ward Identities

Here we derive eq. (2.20) using anomalous Ward identities.

The pure spinors are represented by a Dirac spinor $\lambda^A (A = 1, \ldots, 4)$ satisfying $\lambda \gamma^m \lambda = 0$. Therefore, there are eight gauge invariant combinations

$$J =: w_A \lambda^A :, \quad J_{mn} =: w_A (\gamma_{mn})^A_B \lambda^B :, \quad J_{mnpq} =: w_A (\gamma_{mnpq})^A_B \lambda^B :$$

(B.1)
The last one is a pseudoscalar quantity since it can be rewritten in term of $\gamma^5$ as follows

$$J_{mnop} = \epsilon_{mnop} w_A (\gamma^5)_B^A \gamma^B$$

and we use the notation $w_A (\gamma^5)_B^A \gamma^B \equiv J^5$.

These gauge invariant operators satisfy the anomalous Ward identities

\begin{align*}
:J_{mn}^5 \lambda^B : \gamma_{BC}^m - \frac{1}{2} : J^B \lambda^B : \gamma_{n,BC}^m = & \alpha' \gamma_{n,BC} \partial \lambda^C, \\
: J_{mn}^5 \lambda^B : \gamma_{BC}^m - \frac{1}{2} : J^B \lambda^B : \gamma_{n,BC}^m = & \frac{3}{2} \lambda^A \partial \lambda^B \gamma_{n,BC} + \frac{\alpha'}{2} \lambda^D \partial \lambda^B ( \gamma_{mn}^5 )_D^A \gamma_{BC}^m, \\
\end{align*}

Now, in the $d=4$ case there is a new Ward identity relating the Lorentz generator $J_{mn}$ and the pseudoscalar $J^5$. Notice that we can rewrite the generator $J_{mn}$ as follows

$$J^5_{mn} \lambda^B : \gamma_{BC}^m - \frac{1}{2} : J^B \lambda^B : \gamma_{n,BC}^m = \alpha' (\gamma^5 \gamma_{n,BC} \partial \lambda)^A.$$ 

which are derived by using the Fierz identity and the pure spinor condition. Now, we can contract both sides of the second equation with $g^{n,AD}$ (and renaming $D \rightarrow A$) and we find

$$- : J_{mn}^5 \lambda^B : \gamma_{BC}^m - 2 : J^B \lambda^B : \gamma_{n,AD}^m = 4 \alpha' (\gamma^5 \partial \lambda)^A + 2 \alpha' (\gamma^5 \partial \lambda)^A + \frac{\alpha'}{2} (\gamma_{mn}^5 \partial \lambda)^B (\gamma_{mn}^5 \partial \lambda)^A.$$ 

and finally contracting both sides with $\gamma_{AB}^m$, we finally conclude that

$$\gamma_{AB}^m \partial \lambda = 0.$$ 

Notice that the commuting nature of $\lambda$’s implies that $\lambda g^5 \gamma^m \lambda = 0$ and, the pure spinor condition implies that $\lambda \gamma^m \partial \lambda = 0$, it turns out that due to pure spinor condition and the properties of Dirac gamma matrices in $d=4$, we have that the axial part of the bispinor $\lambda^A \partial \lambda^B$ vanishes. Thus, this proves eq. (2.20)

**References**


7The following conclusion can also be obtained using only the Ward identity (B.2), by first contracting with $\gamma_{n,AD}$ and then by $(\gamma^5 \gamma^m)_{AB}$. 

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