New Results on the Impedance of Resistive Metal Walls of Finite Thickness

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Abstract

The resistive wall impedance of cylindrical vacuum chambers was first calculated over 40 years ago. The original results were valid for metal vacuum chamber walls which are thick compared to the skin depth at the frequencies of interest. Recently the subject has again become important for beam stability in the LHC where in particular the transverse impedance of the large number of graphite collimators to be installed could severely limit its performance, if the “thick wall” formulae were correct. The frequencies of the slow betatron waves in such a large machine are very low and thus the transverse impedance, originally found to be proportional to the inverse square root of frequency, could lead to instabilities.

However, when the skin depth exceeds the wall thickness, the transverse resistive wall impedance is strongly reduced and a number of papers have recently been published to estimate this reduction. However, all of these had restrictions of validity. Here we give a consistent derivation of the general expression for the transverse impedance of walls made of arbitrary materials which are valid at all frequencies. The results are compared with previous ones in the graphs for the LHC collimators - where they agree quite well, but also for the SPS-MKE kickers which are required for injection into the LHC and where the agreement is less perfect.
New results on the Impedance of Resistive Metal Walls of Finite Thickness

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August 2, 2005

1 Introduction

The longitudinal and transverse impedances of vacuum chambers with resistive walls were already calculated over 40 years ago in the seminal papers by Laslett, Neil and Sessler[1]. For (infinitely) thick vacuum chambers of length \( L \), with circular cross section of radius \( b \), the transverse resistive wall impedance can be written\(^1\)

\[
Z_\perp(\omega) = (1 + j)Z_0 \frac{\mu_r \delta L}{2 \pi b^3}
\]

where \( Z_0 = \sqrt{\mu_0 / \varepsilon_0} \) is the free-space impedance and \( \delta = \sqrt{2/\omega \mu \sigma_c} \) the skin depth for a wall material with conductivity \( \sigma_c \) and permeability \( \mu = \mu_0 \mu_r \). Alternatively it can be written \( \delta = \delta_0 \sqrt{\omega_0 / \omega} \) when it is expressed with the frequency independent factor \( \delta_0 = \sqrt{2/\omega_0 \mu \sigma_c} \), which is the skin depth at the revolution frequency \( \omega_0 = \beta_b c / R \) (where \( 2 \pi R \) is the machine circumference and \( \beta_b c \) the beam velocity).

The factor \( 1/b^3 \) in Eq.(1.1) causes a strong increase of the transverse impedance when the chamber radius \( b \) (or equivalently the half-height of a flat vacuum chamber or collimator jaws) becomes very small. The skin depth, proportional to the inverse square root of frequency \( \omega = 2\pi f \) and

\(^1\)Here the (circular) frequency \( \omega \) is assumed to be positive. For negative frequencies, the impedance can be found from the symmetry relation \( Z_\perp(-\omega) = -Z_\perp^*(\omega) \).
conductivity $\sigma_c$, will cause a further increase of the impedance at low frequencies, and in particular for bad conductors such as graphite used for the LHC collimators.

Renewed interest in this old subject is due to the important role of the resistive wall effect for collective beam stability in large circular proton accelerators or colliders such as the LHC. In particular, the numerous collimators in this machine are made of highly resistive graphite to withstand the high temperatures generated by the impact of high-energy protons, and will be moved into positions very close to the beam to protect the surrounding superconducting magnets from stray protons. These devices alone could create such high impedances as to severely limit the beam current and hence the performance of the collider.

However, Eq.(1.1) is no longer valid when the frequency is so low that the corresponding skin depth becomes larger than the wall thickness. It was pointed out long ago by Sacherer[2] that in this case the transverse impedance becomes proportional to $1/\omega$ and thus increases even more rapidly toward lower frequencies, “provided the impedance of any alternate current path outside the vacuum chamber is sufficiently high that all current flows through the walls”. Fortunately, there is nearly always such an alternate current path available due to structures outside the vacuum chamber which are made of conducting or magnetic materials.

Due to the importance of this effect for the LHC, a number of papers have been published recently on this subject[3, 4, 5, 6]. It was found that the real part of the transverse resistive wall impedance at high frequencies is proportional to $1/\sqrt{\omega}$, with decreasing frequency the behavior changes to $1/\omega$, goes through a maximum, and finally becomes proportional to $\omega$ as the frequency tends to zero. The real part of the transverse resistive wall impedance is thereby strongly reduced at low frequencies. At the same time, the imaginary part tends to a constant value which depends on radius, thickness and permeability of the wall, but also on the nature of the structure outside the wall proper. This behavior, similar to that of a resistance with a shunt inductance, has led to the currently popular name “inductive bypass”.

Unfortunately, the results do not always agree in detail in the frequency region of interest. Some of these approaches appear much simpler as only half of the required variables are used to compute the electromagnetic fields and hence the transverse impedance. One paper[3] gives a closed expression which shows correctly the transition from thin to thick wall impedance, but the derivation is no longer valid when the vacuum chamber wall is not so
highly conducting as to shield the electrostatic field. In another paper[4], the longitudinal impedance is calculated first, using circuit theory, but its conversion to the transverse one is not generally valid at all frequencies.

In this report, we show the exact calculation of the electromagnetic fields excited by an (infinitesimally ) thin, annular particle beam of finite radius with dipole modulation in a surrounding coaxial vacuum chamber of finite thickness. We consider 3 types of structures outside the vacuum chamber: a) a perfect conductor, most appropriate for machines with a thin metal wall “liner” inside a highly conducting vacuum chamber, such as planned for the LHC;

b) a perfect magnet, applicable to regions with iron pole pieces close to the vacuum chamber;

c) a chamber wall extending to infinity, a good approximation when the wall thickness is large compared to the skin depth.

For the case of a very thin metal wall (section 5), vacuum (or air) is assumed outside the chamber wall, with the same 3 types of boundary conditions at its outer edge.

2 Field calculations

From the Maxwell equations[7] one can obtain the (vectorial) electric wave equations for the electric and magnetic field strengths $\vec{E}$ and $\vec{H}$. Analytically it is simpler to work in the frequency domain, where all field strengths are taken to be proportional to $\exp(j\omega t)$, and the resulting equations are then called *Helmholtz equations* (see Appendix A). Combining the conduction current density $\vec{J}_c = \sigma_c \vec{E}$ with the displacement current density $j\omega \varepsilon \vec{E}$ by introducing the “complex permittivity” $\varepsilon_c = \varepsilon_r + \sigma_c/(j\omega)$ they can be written:

\[
\begin{align*}
[\Delta + \varepsilon_c j\omega^2] \vec{E} &= \frac{1}{\varepsilon} \text{grad} \rho + j\omega \mu \rho \vec{v}, \\
[\Delta + \varepsilon_c j\omega^2] \vec{H} &= -\rho \text{curl} \vec{v}.
\end{align*}
\] (2.1)

In a source free region, the (perturbed) charge density $\rho = 0$, and the RHSs of both equations are zero. Here $\Delta = \nabla^2$ is the *Laplacian operator*, $\varepsilon' = \varepsilon_c/\varepsilon_0 = \varepsilon_r - j\sigma_c/(\omega \varepsilon_0)$ the relative complex permittivity for a material with dielectric constant $\varepsilon = \varepsilon_r \varepsilon_0$ and conductivity $\sigma_c$. Also the permeability
\( \mu = \mu_0 \mu' \) may in general be complex when magnetic losses are non-zero. The relative complex permeability is written \( \mu' = \mu/\mu_0 = \mu_r (1 + j \tan \theta_M) \), where \( \tan \theta_M \) is the so-called loss tangent. In order not to perturb beam orbits, vacuum chamber materials are usually non-magnetic and hence \( \mu' = 1 \) in most cases.

In a circular-cylindrical coordinate system, appropriate for the model of a cylindrical or annular beam in a concentric vacuum chamber, the transverse components of the vector Helmholtz equation are coupled by the field strengths and are thus difficult to solve. Fortunately, the longitudinal components of the same vector equation for \( E_z \) and \( H_z \) reduce to separate scalar Helmholtz equations and can thus be solved more easily.

The homogeneous (scalar) Helmholtz equation can be solved by a product of functions in the 3 variables \( r, \theta, \) and \( z \). As shown in Appendix A, one obtains harmonic oscillator equations in \( \theta \) and \( z \), whose solutions are \( \exp(\pm jm\theta) \) and \( \exp(\pm jkz) \). In order for the solutions to be single valued, \( m \) must be an integer and is called the azimuthal mode number. Here we investigate pure dipole oscillations with \( m = 1 \). Choosing the \((-jkz)\) solution for the axial motion, it combines with the time dependence to give the factor \( \exp(j(\omega t - kz)) \), i.e. the fields are propagating waves with wave number \( k = \omega/v \) and phase velocity \( v = \omega/k = \beta c \) - which may in general differ from the beam velocity \( v_b = \beta c \).

The radial part of the homogeneous Helmholtz equation is then given by the differential equation for modified Bessel functions of \( m \)-th order and argument \( vr \), where the radial propagation constant is given by the square root of

\[
\nu^2 = k^2 - \omega^2 \mu \varepsilon_c = k^2(1 - \beta^2 \varepsilon'/\mu').
\] (2.2)

The sign of the square root for \( \nu \) has to be chosen such that the solutions decay for \( r \to \infty \).

For dipole oscillations, excited by a horizontal cosine modulation propagating along the particle beam, one can write the solutions as

\[
H_z = [C_1 I_1 (nu) + C_2 K_1 (nu)] \sin \theta
\]

\[
E_z = [D_1 I_1 (nu) + D_2 K_1 (nu)] \cos \theta,
\] (2.3)

Sine and cosine are interchanged for a purely vertical excitation.

As shown in appendix B, the Maxwell equations may be used again to obtain also the transverse field components from the longitudinal ones and their derivatives.
3 Source terms

The original model was a continuous, charged particle beam of constant density, and its (infinitesimal) transverse oscillations were replaced by surface charges and currents[1]. This description was also used in a previous study by the author[8], where furthermore the difference between beam and wave velocity was taken into account. However, this model is not very practical for bunched beams, and also becomes difficult to apply to continuous beams with a realistic radial density distribution without a hard edge.

Here we use the now more common model [9, 10] of an infinitesimally thin ring of charge with an azimuthal density modulation, which we extend by changing the velocity from that of the beam $v_b = \beta_c$ to that of the wave traveling on it $v = \omega/k$ which corresponds to betatron oscillations. In particular, for a circular accelerator or storage ring, the betatron wavelength must be an integer fraction of the circumference $\lambda = 2\pi R/n$, where $n$ is called the axial mode number. For the wave number $k = 2\pi/\lambda$ this leads to the simple relation $k = n/R$.

On the other hand, the betatron frequencies at a given position are given by $\omega = (n \pm Q)\omega_0$, where $Q$ is the betatron tune in the transverse direction under investigation, and $\omega_0 = \beta_c/R$ is the revolution frequency. The plus sign in $(n + Q)$ refers to fast waves (propagating along the beam in the direction of its velocity), while the minus sign (or negative $n$) holds for slow waves which propagate in the opposite direction of the beam. Only the slow waves can become unstable\(^2\), i.e. periodic energy exchange without continuous growth[11]. Thus the wave velocity of the slow wave becomes

$$v = \beta c = \frac{\omega}{k} = \beta_c \left[ 1 - \frac{Q}{n} \right], \quad (3.1)$$

which may be much smaller than the beam velocity, in particular when the mode number $n$ is equal to (or near) the integer part of the tune $[Q]$. For a fractional tune $q \leq 1/2$, the lowest wave velocity is given $\beta_c q/[Q]$ ($q$ should be replaced by $(1-q)$ when $q \geq 1/2$). This velocity can be quite small for large machines with high betatron tunes - e.g. it is only about 0.5% of the

\(^2\)Slow waves on a particle beam are said to carry negative energy, i.e. their amplitudes will grow exponentially when they couple to the positive energy waves propagating on the surrounding structure (the required energy comes from the longitudinal beam motion). Coupling of two positive energy waves only leads to 'beating'.

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beam velocity in the LHC with $q = 0.3$ and $|Q| = 60$. In such extreme cases, some of the usual relativistic approximations are no longer valid.

For a transversely oscillating beam, the perturbed charge and current densities are given by a series of terms proportional to $\cos m\theta$. The term $m = 1$ refers to dipole oscillations which is investigated here. Higher order terms with $m \geq 2$ (e.g. $m = 2$ quadrupole oscillations), are generally more stable than dipole oscillations and are of interest only when the dominant dipole oscillation is stabilized, e.g. by feedback.

Using the abbreviations $u = kr/\gamma$, $x = kb/\gamma$, and $s = ka/\gamma$, the source terms due to an infinitesimal annular ring of radius $a$ can be written (see Eqs.(6.32/35) in ref.[10])

\[
\begin{align*}
E_z^{(s)} & = jC \cos \theta F_1(u), \\
G_z^{(s)} & = jC \sin \theta \alpha_{TE} I_1(u), \\
E_\theta^{(s)} & = \gamma C \sin \theta \left[ \frac{F_1(u)}{u} + \beta \alpha_{TE} I'(u) \right], \\
G_\theta^{(s)} & = -\beta \gamma C \cos \theta \left[ F_1'(u) + \frac{\alpha_{TE} I_1(u)}{\beta u} \right],
\end{align*}
\]
\[ \tag{3.2} \]

where a prime designates the derivative w.r.t the argument, and

\[
F_1(u) = K_1(u) - \alpha_{TM} I_1(u). \tag{3.3}
\]

The factor $C$ is proportional to the dipole moment $P$:

\[
C = \frac{\omega P}{\pi \varepsilon_0 \gamma^2 a} I_1(s)e^{-jkz} \tag{3.4}
\]

Since all field components are proportional to this factor, we will normalize $jC K_1(x)$ to unity and introduce its actual value only when calculating the transverse impedance which can be written (see Appendix C) for a chamber of length $L^3$:

\[
Z_{\perp}(\omega) = -\frac{j L C}{k a P} F_1(s)e^{jkz} = \frac{j Z_0 L}{\pi a^2 \beta \gamma^2} F_1(s) I_1(s). \tag{3.5}
\]

The information about the fields is hidden in the coefficient $\alpha_{TM}$ of the function $F_1(s)$. This expression for the impedance still contains the contribution

\[ ^{3}\text{Here it was assumed that beam and wave velocities are equal; for unequal velocities, the calculation will be shown in a subsequent report.} \]
of the direct space charge for a beam of radius $a$ in a perfectly conducting chamber at radius $b$:

$$Z_\perp^{SC}(\omega) = -j \frac{LZ_0}{2\pi \beta \gamma^2} \left[ \frac{1}{a^2} - \frac{1}{b^2} \right].$$

(3.6)

Expanding the modified Bessel functions ($I_1(z) \approx z/2$) for the small arguments $s = ka/\gamma$ and $x = kb/\gamma$ and subtracting the direct space charge contribution then yields for the contribution of the wall resistivity

$$Z_\perp^{RW}(\omega) = -j \frac{LZ_0}{2\pi \beta \gamma^2 b^2} \left[ 1 - \alpha_{TM} \frac{k^2 b^2}{2 \gamma^2} \right].$$

(3.7)

The rest of this report is thus essentially devoted to the calculation of the coefficient $\alpha_{TM}$.

4 Field matching

For the circular cylindrical geometry investigated here, we divide the space into annular regions with constant material parameters $\varepsilon, \mu, \sigma$. The longitudinal components of the field strengths $E_z$ and $H_z$ can then be obtained by solving the homogeneous Helmholtz equations. Only in the region directly surrounding the beam the source terms have to be taken into account. The corresponding transverse field components can then be obtained from the Maxwell equations as shown in Appendix B.

At the interfaces of two regions ($r = \text{constant}$) all field strength components have to be matched, i.e., in the absence of surface charges and currents the tangential field strengths $E_z, E_\theta, H_z, H_\theta$ have to be continuous. Then the radial components of the displacement $D_r$ and of the induction $B_r$ are also continuous, i.e., matching of the radial components is redundant.

We will write the field components in the $p$-th region with a corresponding superscript and the radial propagation constant with a subscript. In particular, in a vacuum region with $\varepsilon' = \mu' = 1$ it is purely real

$$\nu_{\text{vac}} = k \sqrt{1 - \beta^2} = \frac{k}{\gamma},$$

(4.1)

where the relativistic factor $\gamma = 1/\sqrt{1 - \beta^2}$ refers to the wave energy and should not be confused with the beam energy factor $\gamma_b = 1/\sqrt{1 - \beta_b^2}$. 

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For a relativistic beam ($\beta \approx 1$) in a dielectric material (e.g., ceramic or glass), the radial propagation constant becomes purely imaginary when $\varepsilon_r \geq 1/\beta^2 \approx 1$:

$$\nu_{\text{die}l} = k \sqrt{1 - \varepsilon_r \beta^2} \approx jk \sqrt{\varepsilon_r - 1}. \quad (4.2)$$

Finally, in a good conductor (metal) at not extremely high frequencies where $\sigma/\omega\varepsilon_0 \gg \varepsilon_r$, one may neglect the small dielectric constant $\varepsilon_r \approx 1$ to get

$$\nu_{\text{met}} = \beta k \sqrt{\frac{\sigma}{\omega\varepsilon_0}} = \frac{1 + j}{\delta}, \quad (4.3)$$

where $\delta = \sqrt{2/\omega\mu\sigma}$ is the skin depth.

Since no free surface charges or currents can exist on material boundaries, matching the tangential electric and magnetic field strengths at radius $b_p$ between adjacent regions $(p)$ and $(p+1)$ gives simply the 4 equations

$$E_{z,\theta}^{(p+1)} = E_{z,\theta}^{(p)}, \quad H_{z,\theta}^{(p+1)} = H_{z,\theta}^{(p)}. \quad (4.4)$$

We will write the radial dependence of the field components in the $p$-th uniform region ($a_p \leq r \leq b_p$) as functions of $u_p = \nu_p r$, dividing the modified Bessel functions by their values at the surfaces $x_p = \nu_p b_p$. This normalization avoids extremely large or small values of the field components, which would occur as the modified Bessel functions are proportional to exponentials. Since all (modified) Bessel functions appearing in this report are of first order, we replace in the following the index by the argument to simplify the writing. Also leaving off the subscripts $p$ on $u = \nu r$ and $x = \nu b$ for clarity, and writing $\tilde{G} = Z_0 \tilde{H}$ as before, the axial fields components can be written

$$E_z^{(p)}(u) = E_p \left[ \frac{K_u}{K_x} - \alpha_p \frac{I_u}{T_x} \right],$$
$$G_z^{(p)}(u) = G_p \left[ \frac{K_u}{K_x} - \eta_p \frac{I_u}{T_x} \right]. \quad (4.5)$$

The azimuthal field components are obtained from the longitudinal components and their derivatives as shown in Appendix B ($u = \nu_p r$):

$$E_\theta^{(p)}(u) = -\frac{jk}{\nu_p} \left[ \frac{E_z^{(p)}(u)}{u} + \beta \mu_p' \frac{dG_z^{(p)}(u)}{du} \right],$$
$$G_\theta^{(p)}(u) = \frac{jk}{\nu_p} \left[ \frac{G_z^{(p)}(u)}{u} + \beta \epsilon_p' \frac{dE_z^{(p)}(u)}{du} \right]. \quad (4.6)$$
In the vacuum region inside the annular beam \((r \leq a)\), the field components are given by Bessel functions of the first kind \(I_1(u)\) since those of the second kind diverge for vanishing argument. However, the fields in this region are not required for matching. In the region outside the beam, but still inside the vacuum chamber proper, \((\text{region I: } a \leq r \leq b)\), \(\varepsilon' = \mu' = 1\), hence \(\nu = k\sqrt{1-\beta^2} = k/\gamma\), we use the expressions for the source fields derived in the last section to obtain the values of the coefficients \(E_1 = -G_1 = jCK_x(=1)\) which were normalized to unity:

\[
E_z^{(1)}(u) = \frac{K_u}{K_x} - \alpha_1 \frac{I_u}{I_x}, \quad G_z^{(1)}(u) = \eta_1 \frac{I_u}{I_x},
\]

\[
E_\phi^{(1)}(u) = -j\gamma \left[ \frac{1}{u} \left( \frac{K_u}{K_x} - \alpha_1 \frac{I_u}{I_x} \right) + \beta \eta_1 \frac{I_u'}{I_x} \right],
\]

\[
G_\phi^{(1)}(u) = j\gamma \left[ \eta_1 \frac{I_u}{I_x} + \beta \left( \frac{K_u'}{K_x} - \alpha_1 \frac{I_u'}{I_x} \right) \right],
\]

(4.7)

where we have introduced the modified field coefficients \(\alpha_1 = \alpha_{TM}I_x/K_x\) and \(\eta_1 = \alpha_{TE}I_x/K_x\).

For a large energy factor \(\gamma\) or a small chamber radius \(b\) at not too high frequencies \(x = kb/\gamma \ll 1\), the Bessel functions are well approximated[13] by \(I_x \approx x/2, K_x = 1/x\). Then one gets at the surface of the vacuum chamber \((r = b, u = x)\):

\[
E_z^{(1)}(x) = \bar{\alpha}_1,
\]

\[
G_z^{(1)}(x) = \eta_1,
\]

\[
E_\phi^{(1)}(x) = -\frac{j\gamma}{x} \left[ \bar{\alpha}_1 + \beta \eta_1 \right],
\]

\[
G_\phi^{(1)}(x) = \frac{j\gamma}{x} \left[ \beta \bar{\alpha}_1 + \eta_1 - 2\beta \right],
\]

(4.8)

where we have introduced the complementary field coefficient

\[
\bar{\alpha}_1 = 1 - \alpha_1,
\]

(4.9)

in terms of which the resistive wall impedance is given simply by

\[
Z_\perp = -j \frac{LZ_0}{2\pi \beta \gamma^2 b^2} \bar{\alpha}_1,
\]

(4.10)
as shown in Appendix C. The parameter $\bar{\alpha}_1$ can be found by matching the 4 tangential field components at each interface between adjacent regions. But even for a simple wall matching at the inner and outer surface would then require the solution of 8 equations with the same number of unknowns. This formidable task can hardly be done analytically unless some simplifications are made.

## 5 Thin metal wall

The analysis can be considerably simplified when the wall thickness $t = d - b$ is small compared to skin depth $\delta$ as well as to the chamber radius $b < d$. In this case the electric field strength remains nearly constant across the wall, while the magnetic field strength changes by the current induced in the wall in perpendicular direction. The current is given by the product of the conductivity $\sigma_e$, the wall thickness $t$ with the electric field strength. Matching the tangential field strengths on the outside and inside of the vacuum chamber wall thus can be written

\[
E_z(d) = E_z(b), \quad E_\theta(d) = E_\theta(b), \\
H_z(d) = H_z(b) - \sigma_e t E_\theta(b), \quad H_\theta(d) = H_\theta(b) + \sigma_e t E_z(b).
\]  

Defining the dimensionless quantity $\zeta = Z_0 \sigma_e t$, and using again the abbreviation $\vec{G} = Z_0 \vec{H}$, we may write the last two conditions as

\[
G_z(d) = G_z(b) - \zeta E_\theta(b), \quad G_\theta(d) = G_\theta(b) + \zeta E_z(b).
\]  

The advantage of this approximation is not only the fact that the number of matching equations is halved, but also that the field solutions inside the conducting medium are not required.

Case a: the outer (vacuum) region extends to a perfect conductor (PC) at $r = d$, where $u = \nu d = y$; and thus the tangential electric field components must vanish there: $E_z(d) = E_\theta(g) = 0$. The latter condition leads to $G'_z(d) = 0$, hence we get relations between the unknown field coefficients which can be written

\[
\alpha^2_{PC} = \alpha_2 = \frac{K_y I_x}{K_x I_y} \approx \kappa^2, \quad \text{and} \quad \eta^2_{PC} = \eta_2 = \frac{K'_y I_x}{K_x I'_y} \approx -\kappa^2
\]  

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with $\kappa = h/g$. The approximations hold when the arguments of the Bessel functions are small compared to unity, e.g. in vacuum for large $\gamma$.

For an outer region extending to a perfect magnet (case b: PM), the tangential magnetic field components should tend to zero $G_z(d) = G_\theta(d)$ or $E_z'(d) = 0$ and thus $\alpha_2^{PM} = \eta_2$ while $\eta_2^{PM} = \alpha_2$. If the radius of the outer region tends toward infinity $(d \to \infty$, case c: INF), only Bessel functions of the second kind $K_1$ are permitted as $I_1(x)$ diverges and $\alpha_2^{INF} = \eta_2^{INF} = 0$.

All 3 cases can be combined, and in the small argument approximation one gets

$$\alpha_2 = -\eta_2 = \begin{cases} \kappa^2 & \text{case (a): PC} \\ -\kappa^2 & \text{case (b): PM} \\ 0 & \text{case (c): INF} \end{cases}$$

(5.4)

If the outermost region does not consists of air, its small permittivity ($\varepsilon' \approx 1.006$) will lead to weak Cerenkov radiation above an energy factor $\gamma \geq 14$, when the wave speed exceeds light velocity in the medium, and the radial propagation constant $\nu = k\sqrt{1 - \beta^2\varepsilon'\mu'}$ becomes imaginary.

In small argument approximation, the fields at the inner edge of the second region become

$$E_z^{(2)}(x) = (1 - \alpha_2) E_2,$$
$$G_z^{(2)}(x) = (1 + \alpha_2) G_2,$$
$$E_\theta^{(2)}(x) = -(1 - \alpha_2) \frac{j k}{\nu_2 x} (E_2 - \beta \mu'_2 G_2),$$
$$G_\theta^{(2)}(x) = (1 + \alpha_2) \frac{j k}{\nu_2 x} (G_2 - \beta \varepsilon'_2 E_2).$$

(5.5)

Assuming negligible wall thickness, matching with the fields at the outer edge of region I (Eq.(4.8)) yields 4 linear equations for the 4 unknown coefficients $\bar{\alpha}_1$, $\eta_1$, $E_2$, and $G_2$:

$$ (1 - \alpha_2) E_2 = \bar{\alpha}_1 \hspace{1cm} (a),$$
$$ \frac{k}{\nu_2} (1 - \alpha_2) (E_2 - \beta \mu'_2 G_2) = \gamma (\bar{\alpha}_1 + \beta \eta_1) \hspace{1cm} (b),$$
$$ \frac{k}{\nu_2} (1 + \alpha_2) G_2 = \eta_1 + \frac{j \gamma}{x} \zeta (\bar{\alpha}_1 + \beta \eta_1) \hspace{1cm} (c),$$
$$ \frac{k}{\nu_2} (1 + \alpha_2) (G_2 - \beta \varepsilon'_2 E_2) = \gamma (\beta \bar{\alpha}_1 + \eta_1 - 2\beta) - j x \zeta \bar{\alpha}_1 \hspace{1cm} (d).$$

(5.6)
Eq.(5.6 b) can be simplified with Eq.(5.6 a) to \((1 - \alpha_2)G_2 = -\eta_1\), thus one can replace \(E_2\) and \(G_2\) in Eqs.(5.6 c and d) to get two linear equations for \(\bar{a}_1\) and \(\eta_2\). If the outer region (layer 2) is also vacuum, then \(\alpha_2 = \eta_2 = 0\), and \(\mu'_2 = \varepsilon'_2 = 1\). Solving the equations yields:

\[
\bar{a}_1 = -\frac{\beta \gamma}{\zeta} \frac{\beta \gamma \zeta - 2j x}{1 + x^2 + j x \beta \gamma (\zeta / 2 + 2 / \zeta)}.
\]  

(5.7)

For \(\gamma \rightarrow \infty\), the leading term is proportional to \(\gamma\), and the transverse impedance - which is divided by \(\gamma\) - will remain finite for large energy factors.

6 Finite wall thickness

In the wall region, \(b \leq r \leq d\), the longitudinal field components can be written in terms of modified Bessel functions of both first and second kind \(I_1\) and \(K_1\). We abbreviate \(\nu_2 r = u_2\) and \(\nu_2 b = x_2\) and write for short \(I_1(u_2) = I_w, I_1(x_2) = I_x\) and \(K_1(u_2) = K_w, K_1(x_2) = K_x\), Furthermore we leave off the azimuthal factors by defining \(E_z = E_{z0} \cos \theta, G_\theta = G_{\theta0} \cos \theta\), while \(G_z = G_{z0} \sin \theta\), and \(E_\theta = E_{\theta0} \sin \theta\):

\[
E_{z0}^{(2)}(r) = E_2 \left[ \frac{K_w}{K_x} - \frac{\alpha_2}{I_x} I_u \right],
\]

\[
G_{z0}^{(2)}(r) = G_2 \left[ \frac{K_w}{K_x} - \frac{\eta_2}{I_x} I_u \right],
\]

(6.1)

where \(\alpha_2 = -F_2/E_2\) and \(\eta_2 = -H_2/G_2\). In general, the complex radial propagation constant \(\nu = \nu_2\) is a function of frequency, given by the root (with a positive real part)

\[
\nu = k \sqrt{1 - \beta^2 \varepsilon' \mu'},
\]

(6.2)

where \(\varepsilon' = \varepsilon_r + \sigma_c/(j \omega \varepsilon_0)\) is the *complex permittivity* and \(\mu' = \mu_r (1 + j \tan \theta_M)\) the *complex permeability* for a wall material with *electrical conductivity* \(\sigma_c\), relative permeability \(\mu_r\) and magnetic loss tangent \(\tan \theta_M\).

For a good conductor such as most metals, \(\varepsilon_r \approx 1\) and \(\sigma_c \gg \omega \varepsilon_0\) at not too high frequencies, the radial propagation constant is well approximated by \(\nu = (1 + j) / \delta\), where \(\delta = \sqrt{2 / (\omega \mu \sigma_c)}\) is the *skin depth*. In a source free region,
the azimuthal field components are given by the relations (see Appendix II)

\[
E_{\phi 0}(r) = -\frac{jk}{\nu^2} \left[ \frac{E_z}{r} + \beta \mu'_z \frac{dG_z}{dr} \right],
\]

\[
G_{\phi 0}(r) = \frac{jk}{\nu^2} \left[ \frac{G_z}{r} + \beta \varepsilon r \frac{dE_z}{dr} \right]. 
\]  

(6.3)

At the inner wall \((r = b, u_2 = x_2)\) one gets from Eqs. (6.1) simply \(E_{\phi 0}^{(2)}(b) = E_2(1 - \alpha_2)\), \(G_{\phi 0}^{(2)}(b) = G_2(1 - \eta_2)\). For a metal chamber, for which \(\nu_2 \approx (1 + j)/\delta\), the transverse field components Eqs. (6.3) become with \(jk/\nu^2 = k\delta^2/2\)

\[
E_{\phi 0}^{(2)}(b) = -\frac{k\delta^2}{2b} \left[ E_2(1 - \alpha_2) + \beta \mu'_2 x_2 G_2 (Q_2 - \eta_2 P_2) \right]
\]

\[
G_{\phi 0}^{(2)}(b) = \frac{k\delta^2}{2b} \left[ G_2(1 - \eta_2) + \beta \varepsilon_2 x_2 E_2 (Q_2 - \alpha_2 P_2) \right] 
\]  

(6.4)

where \(P_2 = I'_x / I_x\) and \(Q_2 = K'_x / K_x\).

The coefficients \(\alpha_2\) and \(\eta_2\) are determined by the boundary conditions at the outer chamber wall \(r = d\) where \(u_2 = y_2\). For the 3 cases of the wall extending to a very good conductor (case a), to a high-permeability magnet (case b) or a very thick wall (case c), we get the same expressions for \(\alpha_2\) and \(\eta_2\) Eqs. (5.3) derived for the thin wall in the last section. We rewrite the expressions for the azimuthal field components with the abbreviations

\[
p = k^2 \delta^2, \quad q = kb, \quad r = \mu'_2 \beta k \delta,
\]  

(6.5)

one gets \(k\delta^2/2b = p/q, \quad jk\beta \mu'_2 / \nu = (1 + j)r/2, \quad \)and \(jk\beta \varepsilon'_2 / \nu = (1 - j)/r, \) and the azimuthal field components become

\[
E_{\phi 0}^{(2)}(b) = -\frac{p}{q} E_2(1 - \alpha_2) - (1 + j) G_2 (Q_2 - \eta_2 P_2) \frac{r}{2},
\]

\[
G_{\phi 0}^{(2)}(b) = \frac{p}{q} G_2(1 - \eta_2) + (1 - j) E_2 (Q_2 - \alpha_2 P_2) \frac{1}{r}. 
\]  

(6.6)

The matching conditions at the inner chamber wall, \(r = b\), for the longitudinal field components with Eqs. (4.8) then yield simply \(E_2(1 - \alpha_2) = \tilde{\alpha}_1\) and \(G_2(1 - \eta_2) = \eta_1\). Substitution of these expressions into Eqs. (6.4) gives

\[
E_{\theta}^{(2)}(b) = -\frac{p}{q} \tilde{\alpha}_1 - (1 + j) Q_\eta \frac{r}{2q} \eta_1,
\]

\[
G_{\theta 0}^{(2)}(b) = \frac{p}{q} \eta_1 + (1 - j) Q_\alpha \frac{1}{rq} \tilde{\alpha}_1, 
\]  

(6.7)

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where we defined for short
\[ Q_\alpha = q \frac{Q_2 - \alpha_2 P_2}{1 - \alpha_2}, \quad Q_\eta = q \frac{Q_2 - \eta_2 P_2}{1 - \eta_2}. \] (6.8)

After multiplication by \( jq \), the matching conditions for the azimuthal field components yield:
\[
\begin{align*}
\bar{a}_1 \left[ \gamma^2 + j \beta \right] + \eta_1 \left[ \beta \gamma^2 - (1 - j) \frac{r}{2} Q_\eta \right] &= 0, \\
\bar{a}_1 \left[ \beta \gamma^2 + (1 + j) \frac{1}{r} Q_\alpha \right] + \eta_1 \left[ \gamma^2 + j \beta \right] &= 2 j \beta \gamma^2. \quad (6.9)
\end{align*}
\]

These are 2 linear equations for the 2 unknowns \( \bar{a}_1 \) and \( \eta_1 \). With the identity \( \gamma^4 - \beta^2 \gamma^4 = \gamma^2 \) one finds the determinant
\[
det = \gamma^2 \left( 1 + 2 j p - \beta \left[ \frac{1 + j}{r} Q_\alpha - (1 - j) \frac{r}{2} Q_\eta \right] \right) + Q_\alpha Q_\eta - p^2 \quad (6.10)
\]
and hence \( \bar{a}_1 = -2 j \beta \gamma^2 \left( \beta \gamma^2 - (1 - j) Q_\eta r / 2 \right) / \det \). The transverse resistive wall impedance is given by Eq. (4.10) and becomes
\[
Z_\perp (\omega) = j \frac{L Z_0}{\pi b^2} \frac{\beta - (1 - j) Q_\eta \frac{r}{2 \gamma^2}}{1 + 2 j p - \beta \left[ \frac{1 + j}{r} Q_\alpha - (1 - j) \frac{r}{2} Q_\eta \right] + \frac{Q_\alpha Q_\eta - p^2}{\gamma^2}}. \quad (6.11)
\]

For large \( \gamma \) (\( \beta \to 1 \)) and low frequencies \( \omega \ll \sigma_c / (\mu' \varepsilon_0) \) also \( r \ll 1 \) one can neglect several terms to get the simpler expression
\[
Z_\perp (\omega) = j \frac{L Z_0}{\pi b^2} \frac{1}{1 + 2 j p - (1 + j) \frac{1}{r} Q_\alpha + (1 - j) \frac{r}{2} Q_\eta}. \quad (6.12)
\]

For \( \omega \to 0 \) also \( p, q, \) and \( r \to 0 \), while \( (1 + j) Q_\alpha / r \to -1 / \mu'_2 \). Thus the transverse impedance tends to a finite value
\[
Z_\perp (0) = j \frac{L Z_0}{\pi b^2} \frac{1}{1 + 1 / \mu'_2}. \quad (6.13)
\]

For the usual case \( \mu' = 1 \) \( Z_\perp \) tends thus to \( j Z_0 L / 2 \pi b^2 \).
At very low frequencies $x < y \ll 1$ (corresponding to a very large skin depth $\delta \gg d > b$), expansion of the Bessel functions yields approximately $P_2 = -Q_2 = 1/x_2 = (1 - j)\delta/2b$ and one gets

$$Q_\alpha \approx (1 - j) \frac{k\delta}{2} \frac{1 + \alpha_2}{1 - \alpha_2},$$

(6.14)

where the coefficients $\alpha_2 = -\eta_2$ are given by Eqs. (5.4) $\alpha_2 = \kappa^2 = (b/d)^2$ for case a (PC), $-\kappa$ for case b (PC), and 0 for case c (INF). The coefficient $r = \mu_0^2 \beta k\delta$ is proportional to the skin depth, and thus $Q_\alpha/r$ tends to zero. On the other hand, $Q_\eta r/2$ and $p$ are proportional to $\delta^2 \propto 1/\omega$, and thus $Q_\eta r/2$ becomes independent of frequency. The transverse impedance is then approximated by

$$Z_\perp(\omega) = \frac{jLZ_0}{\pi b^2} \frac{1}{1 + (1 - j)Q_\eta \frac{r}{2}}$$

(6.15)

Thus at low frequencies, the real part of the impedance is proportional to $\omega$, with increasing frequency it goes through a maximum, then decreases as $1/\omega$ (the “Sacherer region”). Finally, when the frequency becomes high enough that the skin depth is small compared to the chamber radii and the wall thickness, both real and imaginary parts decrease as $\omega^{-\frac{1}{2}}$ in agreement with the standard “thick wall” formula.

7 Walls consisting of several layers

When the inside of a wall is made of poorly conducting material is often covered by a thin layer of highly conducting material to reduce the resistive wall effect. For non-conducting wall materials such a coverage is also useful to avoid static charging.

For such geometries, matching of 4 tangential field components at each interface leads to a set of 4 $n$ linear equations containing (modified) Bessel functions of various complex arguments. Only in exceptional cases these can be replaced by small or large argument approximations and analytic solution can be attempted. In general, it is necessary to solve these equations by computer, and then it is often found that these equations are “ill-conditioned”, i.e. their determinant is close to zero and one cannot get sufficient accuracy with standard computer precision. The original code LAWAT[8] (short for LAyered WAAll Transverse) was first written over 30 years ago, later converted
to MATHEMATICA, improved and used for evaluating the transverse resistive wall impedance for a number of large storage rings[14]. Since it was based on direct numerical evaluation the results are not always accurate.

Therefore it has been found preferable to use symbolic computer programs such as MATHEMATICA[12] to obtain the solutions algebraically before evaluating them numerically. The algebraic expressions are much too lengthy to be useful, but their numeric evaluation is no longer of limited accuracy. The computer code written with this technique has been called LAWAT2000, similar to the original code LAWAT.

Acknowledgments

I want to thank Dr. Elias Metral (CERN) and Dr.Bill Ng (FNAL) for careful checking of all equations and suggesting many improvements and corrections, as well as Dr Francesco Ruggiero for continuous encouragement to complete this work. I also want to thank Dr. Metral for providing the plots of the transverse resistive wall impedance of LHC collimators and SPS MKE kickers.
Appendix A: The Helmholtz Equations

In the frequency domain, all field quantities are taken to be proportional to \( \exp(j\omega t) \), and thus all time derivatives \( \partial/\partial t \) are replaced by \( j\omega \). Then the Helmholtz equations for the Fourier transforms of the electric and magnetic field strengths \( \tilde{E} \) and \( \tilde{H} \), resp. the electric displacement \( \tilde{D} \), and the magnetic induction \( \tilde{B} \), excited by a charge density \( \rho \) and a current density \( \tilde{J} \) can be written

\[
\begin{align*}
curl \tilde{H} &= \tilde{J} + j\omega \tilde{B}, \\
curl \tilde{E} &= -j\omega \tilde{B}, \\
div \tilde{B} &= 0, \\
div \tilde{D} &= \rho.
\end{align*}
\] (A.1)

For a material with permittivity \( \varepsilon \), permeability \( \mu \), and conductivity \( \sigma_c \), the electric displacement is given by \( \tilde{D} = \varepsilon \tilde{E} \), the magnetic induction by \( \tilde{B} = \mu \tilde{H} \). In general, the current density \( \tilde{J} \) is the sum of the conduction current density \( \tilde{J}_c = \sigma_c \tilde{E} \) and the convection current density \( \rho \tilde{v} \). One can combine the conduction and displacement current terms in the \( curl \tilde{H} \) equation by defining the complex permittivity \( \varepsilon_c = \varepsilon' \varepsilon_0 \) with

\[
\varepsilon' = \varepsilon_r + \frac{\sigma_c}{j\omega \varepsilon_0},
\] (A.2)

where \( \varepsilon_r \) is the (relative) dielectric constant, usually negligible compared to the second term. Then the Maxwell equations, expressed with only \( \tilde{H} \), \( \tilde{E} \), and the (perturbed) charge density \( \rho \) can be written

\[
\begin{align*}
curl \tilde{H} &= \rho \tilde{v} + j\omega \varepsilon_c \tilde{E}, \\
curl \tilde{E} &= -j\omega \mu \tilde{H}, \\
div \tilde{H} &= 0, \\
div \tilde{E} &= \rho/\varepsilon.
\end{align*}
\] (A.3)

Taking the \( curl \) of the 2 Maxwell curl equations, using the vector identity \( curl \ curl = \grad \ div - \Delta \), where \( \Delta = \nabla^2 \) is the Laplacian operator, and substituting the two Maxwell \( div \) equations one obtains

\[
[\Delta + \omega^2 \mu \varepsilon_c] \tilde{H} = -\rho \ curl \tilde{v},
\]
\[ [\Delta + \omega^2 \mu \varepsilon_c] \vec{E} = \frac{1}{\varepsilon} \nabla \rho + j\omega \mu \rho \vec{v}. \quad (A.4) \]

In (circular) cylindrical coordinates the Laplacian operator is

\[ \Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \quad (A.5) \]

In this coordinate system, the equations for the transverse components of the electric and magnetic field strengths are coupled and thus difficult to solve, but those for the longitudinal components \( \vec{H} \) and \( \vec{E} \) remain separate scalar Helmholtz equations

\[
\begin{align*}
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon_c \right] H_z &= -\frac{\rho}{r} \left( \frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \\
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon_c \right] E_z &= \frac{1}{\varepsilon} \frac{\partial \rho}{\partial z} + j\omega \mu v_z \rho.
\end{align*}
\quad (A.6)\]

The homogeneous equations can be solved by separation of variables. For this we write \( H_z \) (or \( E_z \)) = \( R(r) \Theta(\theta) Z(z) \) to get

\[ \frac{1}{r \frac{d}{dr} \left( r \frac{d}{dr} \right)} + \frac{1}{r^2 \frac{d}{d\theta} \Theta} + \omega^2 \mu \varepsilon_c = -\frac{d^2 Z}{Z dz^2}. \quad (A.7) \]

The LHS is a function of \( r \) and \( \theta \), while the RHS is a function of \( z \) only, hence both sides must equal a constant which we call \( k^2 \). This yields the harmonic oscillator equation \( d^2 Z/dz^2 + k^2 Z = 0 \) with the solutions \( Z(z) = \exp(\pm jkz) \). Bringing the \( \theta \) dependent terms to the RHS, then calling the new separation constant \( m^2 \), one gets again a harmonic oscillator equation \( d^2 \Theta/d\theta^2 + m^2 \Theta = 0 \) with the solutions \( \Theta(\theta) = \exp(\pm jm\theta) \) or \( \cos m \theta \) and \( \sin m \theta \).

Since the azimuth \( \theta \) is a cyclic variable, \( m \) must be an integer for the solutions to be single valued and is called the azimuthal mode number. For pure dipole oscillations \( m = \pm 1 \), and the solution can also be written in terms of \( \cos \theta \) and \( \sin \theta \).

Reinserting the time dependence, the axial motion is seen to be a wave proportional to \( \exp(j(\omega t - k z)) \), with phase velocity \( v = \omega/k = \beta c \) and the wave number \( k = \omega/v \).
The radial dependence of both longitudinal field components $H_z$ and $E_z$ is then given by the equation
\[
\left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) - \frac{m^2}{r^2} - \nu^2 \right] R(r) = 0, \tag{A.8}
\]
where the radial propagation constant is given by
\[
\nu^2 = k^2 - \omega^2 \mu \varepsilon_c = k^2 (1 - \beta^2 \varepsilon' \mu'). \tag{A.9}
\]
This can be recognized as the differential equation for (modified) Bessel functions of order $m$ and argument $u = kr \sqrt{1 - \beta^2 \varepsilon' \mu'}$, where the sign of the square root has to be chosen such that the solutions decay for $r \to \infty$. For pure dipole oscillations, excited by a horizontal cosine modulation propagating along the particle beam, one can write the solutions as
\[
H_z = [C_1 I_u + C_2 K_u] \sin \theta
\]
\[
E_z = [D_1 I_u + D_2 K_u] \cos \theta. \tag{A.10}
\]
Sine and cosine are interchanged for a purely vertical excitation. It will turn out that we need only the solutions of the homogeneous Helmholtz equations since all region considered are source free except the one containing the beam where the source terms will be determined separately.

**Appendix B: Transverse Field Components**

In circular cylindrical coordinates, the radial and azimuthal components of the Maxwell curl equations in a source-free region ($\rho = 0$) can be written
\[
\frac{1}{r} \frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = j \omega \varepsilon_c E_r, \\
\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = j \omega \mu E_\theta, \tag{B.1}
\]
and
\[
\frac{1}{r} \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -j \omega \mu H_r, \\
\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j \omega \mu H_\theta. \tag{B.2}
\]
where we have again combined conduction and displacement current density terms by using the complex permittivity \( \varepsilon_c = \varepsilon_0 (\varepsilon_r - j\sigma_e / \omega) \). For propagating waves \( \propto \exp \left( j(\omega t - k z) \right) \) the derivatives become \( \partial / \partial t = j \omega \) and \( \partial / \partial z = -j k \).

For horizontal dipole modes \( H_z, H_r, E_\theta \propto \sin \theta \) (hence \( \partial / \partial \theta \rightarrow \cos \theta \)), while \( E_z, E_r, H_\theta \propto \cos \theta \) (hence \( \partial / \partial \theta \rightarrow -\sin \theta \)). Furthermore we abbreviate the magnetic field strength multiplied by the free space impedance

\[
\vec{G} = Z_0 \vec{H},
\]

which has the same dimensions as the electric field strength. With \( \omega / c = \beta k \) the above equations can be rewritten as

\[
\begin{align*}
\frac{G_z}{r} + jk G_\theta &= j \beta k \varepsilon' E_r \quad (a), \\
-jk G_r - \frac{dG_z}{dr} &= j \beta k \varepsilon' E_\theta \quad (b),
\end{align*}
\]

and

\[
\begin{align*}
\frac{E_z}{r} + jk E_\theta &= -j \beta k \mu' G_r \quad (a), \\
-jk E_r + j \beta k \mu' G_\theta &= -j \beta k \mu' G_\theta \quad (b),
\end{align*}
\]

By reordering we get 2 sets of 2 linear equations for the 2 unknowns: Eqs. (B.4 a) and (B.5 b) constitute 2 equations for \( E_r \) and \( G_\theta \):

\[
\begin{align*}
 j \beta k \varepsilon' E_r - jk \mu' G_\theta &= \frac{G_z}{r}, \\
-jk E_r + j \beta k \mu' G_\theta &= \frac{dE_z}{dr},
\end{align*}
\]

while Eqs. (B.4 b) and (B.5 a) yield 2 equations for \( E_\theta \) and \( G_r \):

\[
\begin{align*}
 j \beta k \varepsilon' E_\theta + jk G_r &= - \frac{dG_z}{dr}, \\
jk E_\theta + j \mu' \beta k G_r &= \frac{E_z}{r},
\end{align*}
\]

Both sets have the same determinant \( k^2 (1 - \beta^2 \varepsilon' \mu') = \nu^2 \). The azimuthal field components thus become

\[
G_\theta = \frac{jk}{\nu^2} \left[ \frac{G_z}{r} + \beta \varepsilon' \frac{dE_z}{dr} \right],
\]

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\[ E_\theta = \frac{j k}{\nu^2} \left[ \frac{E_z}{r} + \beta \mu' \frac{dG_z}{dr} \right], \]  

(B.8)

**Appendix C: The transverse Impedance**

The usual definition of the transverse impedance (in the horizontal plane) can be written

\[ Z_\perp(\omega) = \frac{j}{P} \int_{-\infty}^{\infty} dz \left[ E_x - v_z B_y \right] e^{j \omega t}, \]  

(C.1)

where \( P \) is the dipole moment of the charge distribution in the beam. Since all field components will turn out to be proportional to it, it actually drops out in the calculation.

The integrand can also be written in terms of the transverse derivative of the longitudinal electric field strength

\[ Z_\perp(\omega) = -\frac{1}{kP} \int_{-\infty}^{\infty} dz \frac{\partial E_z}{\partial x} e^{j \omega t}, \]  

(C.2)

which has the advantage that the magnetic field strength need not be known explicitly. Following the definitions used in ref.\([10]\) we can rewrite the transverse impedance as

\[ Z_\perp(\omega) = -\frac{1}{kP^2} \int dV E_z J^*_z, \]  

(C.3)

where the complex conjugate of the source current density of a dipole modulated ring beam of radius \( a \) is

\[ J^*_z = \frac{P}{\pi a^2} \delta (r - a) \cos \theta e^{j \omega t}. \]  

(C.4)

In Eq.(3.2), the longitudinal electric field strength was obtained in the form \( E_z = jC F_1(s) \cos \theta \).

Since \( s = ka/\gamma \ll 1 \), the function \( F_1(s) = K_1(s) - \alpha_{TM} I_1(s) \) is approximately \( 1/s - \alpha_{TM} s/2 \). The coefficient \( C \) was defined in Eq. (3.4) as

\[ C = \frac{\omega P}{\pi \varepsilon_0 \nu^2 \gamma^2 a} J_1(s) e^{-j \omega t}. \]  

(C.5)
Substituting these expressions we see that the longitudinal dependence drops out and the integral over $dz$ just yields the length $L$ of the resistive element. This becomes the machine circumference $2\pi R$ when the whole vacuum chamber is considered. The integral over the azimuthal coordinate is simply $\int_0^{2\pi} \cos^2 \theta = \pi$. The integral over the radial coordinate, due to the delta function, yields the electric field strength at the beam radius $r = a$:

$$Z_\perp(\omega) = -\frac{jLC}{k\alpha P} e^{j k z} F_1(s) \approx -\frac{jLZ_0}{2\pi \beta \gamma^2} \left[ \frac{1}{\alpha^2} - \alpha_{TM} \frac{k^2}{2\gamma^2} \right]. \quad (C.6)$$

By subtracting and adding $1/b^2$ in the bracket, we can recognize the usual space charge term, including a perfectly conducting wall at $r = b$:

$$Z_{\perp}^{SC} = -\frac{jLZ_0}{2\pi \beta \gamma^2} \left[ \frac{1}{\alpha^2} - \frac{1}{b^2} \right], \quad (C.7)$$

which disappears for $\gamma \to \infty$, and the contribution due to the wall resistance

$$Z_{\perp}^{RW} = -\frac{jLZ_0}{2\pi \beta \gamma^2} \left[ \frac{1}{b^2} - \alpha_{TM} \frac{k^2}{2\gamma^2} \right]. \quad (C.8)$$

Finally, introducing the modified coefficient $\alpha_1 = \alpha_{TM} I_x/K_x \approx \alpha_{TM} k^2 b^2/(2\gamma^2)$ we get with $\bar{\alpha}_1 = 1 - \alpha_1$ the short expression

$$Z_{\perp}^{RW} = -\frac{jLZ_0}{2\pi b^2} \frac{\bar{\alpha}_1}{\beta \gamma^2}. \quad (C.9)$$

which remains finite for $\gamma \to \infty$ since $\bar{\alpha}_1 \propto \gamma^2$. 

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In the first figure we show a log-log plot of the transverse impedance of an LHC collimator with 1 m length and a radius of 2 mm (the minimum half-gap is actually only 1.2 mm) as function of frequency, computed with Eq.(6.11)[15]. The results are compared to those obtained with the expressions from ref.[3] for the same geometry. The difference at this scale is so small that it cannot be seen. Also shown are results obtained with the classical “thick-wall” formula which are much too large at low frequencies.

The second figure shows the same but with a thin, highly conducting metal layer deposited on the graphite collimator jaws.

The last figure shows a log-linear plot of the transverse impedance of the SPS-MKE kickers which were installed in the SPS for its operation as LHC injector. There is a difference of nearly 30% with the results computed with the expressions in ref.[3]. This difference can be explained by the absence of electrostatic shielding for a poor conductor like Ferrite, which was a basic assumption of that theory.

References

Figure 1: Transverse impedance of Graphite LHC Collimators

Figure 2: Transverse impedance of metalized LHC Collimators
Figure 3: Transverse impedance of SPS-MKE kicker