RETRODICTIVE QUANTUM STATE ENGINEERING

By

Kenneth Lyell Pregnell

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or in part for a degree in any university.

Kenneth Lyell Pregnell

“Only those who attempt the absurd will achieve the impossible” - F. Quimby
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In this chronological list of my refereed publications, those which are published in a conference proceedings are marked with an asterisk.


Abstract

This thesis is concerned with retrodiction and measurement in quantum optics. The latter of these two concepts is studied in particular form with a general optical multiport device, consisting of an arbitrary array of beam-splitters and phase-shifters. I show how such an apparatus generalizes the original projection synthesis technique, introduced as an in principle technique to measure the canonical phase distribution. Just as for the original projection synthesis, it is found that such a generalised device can synthesize any general projection onto a state in a finite dimensional Hilbert space. One of the important findings of this thesis is that, unlike the original projection synthesis technique, the general apparatus described here only requires a classical, that is a coherent, reference field at the input of the device. Such an apparatus lends itself much more readily to practical implementation and would find applications in measurement and predictive state engineering.

If we relax the above condition to allow for just a single non-classical reference field, we show that the apparatus is capable of producing a single-shot measure of canonical phase. That is, the apparatus can project onto any one of an arbitrarily large subset of phase eigenstates, with a probability proportional to the overlap of the phase state and the input field. Unlike the original projection synthesis proposal, this proposal requires a binomial reference state as opposed to a reciprocal binomial state. We find that such a reference state can be obtained, to an excellent approximation, from a suitably squeezed state.

The analysis of these measurement apparatuses is performed in the less usual, but completely rigorous, retrodictive formalism of quantum mechanics.
List of Symbols

The following is a listing of the most frequently used symbols in this thesis.

\( \hat{a} (\hat{a}^\dagger) \) an annihilation (creation) operator for an optical mode

\( n, |n\rangle \) a photon number, eigenstate

\( P(\cdots) \) probability density of \( \cdots \)

\( \text{Pr}(\cdots) \) probability of \( \cdots \)

\( \text{Pr}(i|j) \) Bayesian probability: probability of \( i \) given \( j \)

\( \mathbf{R}(N + 1) \) a unitary rotation matrix of \( N + 1 \) dimensions

\( r \) transmission coefficient of a beam-splitter

\( \hat{S} (\hat{S}^\dagger) \) the forward (backward) time unitary evolution operator of a linear optical device

\( t \) reflection coefficient of a beam-splitter

\( \text{Tr}[\cdots] \) trace of \( \cdots \)

\( \hat{U} (\hat{U}^\dagger) \) a forward (backward) time unitary evolution operator

\( \mathbf{U}(N + 1) \) a unitary transformation matrix of \( N + 1 \) dimensions

\( U_{nm} \) matrix elements of \( \mathbf{U}(N + 1) \)

\( |\alpha\rangle \) a coherent state

\( |\theta\rangle \) a phase state

\( |\theta_n\rangle \) a truncated phase state

\( \delta \theta = 2\pi/(N + 1) \), a discrete phase increment

\( |\tilde{\psi}\rangle \) an unnormalised state

\( \hat{\Gamma}_j \) a measurement device operator

\( \eta \) an efficiency

\( \hat{\Lambda}_i \) a preparation device operator
\( \hat{\mathcal{Z}}_i \) a (preparation) POM element

\( \hat{\Pi}_j \) a (measurement) POM element

\( \hat{\rho} \) a predictive density matrix

\( \rho_{n,m} = \langle n | \hat{\rho} | m \rangle \), density matrix elements of \( \hat{\rho} \) in the photon number basis

\( \hat{\rho}^{\text{ret}} \) a retrodictive density matrix

\( \Omega(N+1) \) a discrete Fourier transformation matrix of \( N + 1 \) dimensions

\( \Omega_{nm} = (N + 1)^{-1/2} \omega^{nm} \), the matrix elements of \( \Omega(N+1) \)

\( \omega = \exp[i2\pi/(N + 1)] \), the \( (N + 1)^{\text{th}} \) root of unity

\( \binom{n}{m} \) a binomial coefficient

\( \hat{1} \) unit operator of the infinite dimensional Hilbert space

\( \hat{1}_N \) unit operator of an \( N + 1 \) dimensional Hilbert space

\( \langle \cdots \rangle \) expectation value of \( \cdots \)

\( ^\wedge \) indicates a Hilbert space operator
List of Abbreviations

BS  Beam-splitter
DFT  Discrete Fourier transformation
MDO Measurement device operator. (For an unbiased measuring device this becomes the element of a POM.)
PDO Preparation device operator. (This includes information about the prepared state and its preparation probability. It can be normalised by dividing by its trace to become a density operator.)
POM Probability operator measure
PS  Phase-shifter
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Chapter 1

Introduction

Retrodictive state engineering concerns the generation of states that evolve backwards in time. At first glance this may appear to be in strong violation of our notion of causality. In the usual, that is predictive, formalism of quantum mechanics a state is produced by a preparation event and then evolves forwards in time. Retrodictive states, on the other hand, evolve backwards in time from the measurement event. As we shall see in this thesis, however, causality is not manifest in the time direction of evolution of the states. Rather it is encapsulated, and guaranteed by, an asymmetry in the normalisation of the operators associated with the preparation and measurement events.

It is the purpose of this chapter to introduce two intimately related concepts in the theory of quantum mechanics: retrodiction and the quantum theory of measurement. Retrodiction, in contrast to prediction, is a technique whereby information about some previous event is gained from knowledge of some related event occurring at a later time. If only partial knowledge is known about the later event, then retrodicting probabilities must suffice. Consider, for example, the event to be the outcome of some horse race, say the Melbourne cup. Typically a punter will try to predict the outcome of the race before it has happened. If he is an experienced punter, then he may favour a horse more if it has been in good form leading up to the race. That is to say he may increase the chances of the horse to win the race, based upon the results of the horse in previous races. In complete symmetry, the punter could retrodict the outcome of the race after it has happened. Take the unlikely situation where the punter was not privileged to know the outcome of the Melbourne cup, but does know how a particular horse performed in races after the Cup. If the horse is in
good form after the race, he may decide to favour the chance that the horse had won the Melbourne cup. The punter would then increase the chance he assigns to the horse having won the race, based upon the results of the horse in subsequent races.

In this situation, retrodiction is generally a redundant exercise since the results of the race can usually be found, after the race, in the public domain, for example in the newspaper or on the internet. For this reason bookmakers generally do not pay favourable odds to punters after the race has been run. However, such a technique does find useful applications when the past results cannot easily be found in the public domain. The rapidly growing field of secure communications is one such example where enormous efforts are placed to avoid information entering into the public domain. Unlike the punter in the above example, retrodiction is then a valuable tool for an eavesdropper attempting to acquire some useful information about a message, sent some time in the past, that he was not privileged to. Not only may information not enter into the public domain through the concerted efforts of a select few, but also it is possible that some part of the intended information may just become lost. Again, this is a real problem when information is sent down a noisy channel resulting in a distorted message being received by the receiver. In such cases the receiver can retrodict the intended message from the distorted one [63].

Despite the internal symmetry between the techniques of prediction and retrodiction there is, however, an undeniable asymmetry in the frequency in which we use prediction over retrodiction. This is because there is an inherent asymmetry in nature in that we remember the past and not the future. So although the techniques are time symmetric, because we can never be completely certain of the future, but at times can be completely sure of the past, prediction is more often than not the most used technique.

1.1 Retrodiction in quantum mechanics

1.1.1 Arrow of time

The perennial question as to why do we remember the events of the past as opposed to the events of the future is one of the great unresolved questions of theoretical physics. The observed distinction between past and future allows a direction, or an “arrow”, of time to be defined. Such an arrow is evidence of a fundamental asymmetry inherent in nature, which to date, is void of an orthodox physical explanation. In fact, there are at least two
other widely-believed versions of the arrow of time \[26\] aside from the psychological arrow just mentioned. The first of these is the cosmological arrow of time. This is the direction of time in which the universe is expanding rather than contracting. Such a result is the conclusion of the general theory of gravity. The second is the thermodynamical arrow and is encapsulated by the second law of thermodynamics which states that the disorder or entropy of an open system will not decrease over time. To illustrate, consider the frequently observed process of combustion. If we were to take a match and set it to a piece of paper we would not be alarmed if the piece of paper caught fire and after a short period of time turned to ash and smoke. However, if instead we were to observe the air above a pile of ash suddenly begin to go smokey and, for no apparent reason, begin to take fire we would think something strange was going on. Furthermore, if at the end of the process when the flames where finally extinguished there remained an unburnt piece of paper in place of the ash, we would be truly astonished. Both situations described are, obviously, the time reversal of each other and are therefore equivalent physical processes, however, there is an undeniable asymmetry in the frequency at which we observe one process over the other. The direction in which processes of this kind can proceed defines an arrow of time. A rather interesting argument was put forward by Hawking \[26\] to suggest that all three arrows necessarily point in the same direction.

In an attempt to provide a physical explanation for the thermodynamical arrow of time, it was suggested by Bohm \[18\], and von Neumann \[81\], that the “reduction of the wavefunction”, implicit in the orthodox interpretation of quantum measurement theory, introduces into the foundations of quantum physics a time asymmetric element, which in turn leads to the thermodynamical arrow of time. If this reduction were a real physical process, we would have a fundamental basis for the arrow of time.

It is the orthodox view that there are two distinct types of evolution present in the theory of quantum mechanics. There is the deterministic or reversible evolution, postulated by the Schrödinger or Heisenberg equations of motion, which is time symmetric. Such an evolution is represented in the theory by a unitary operator and, generally speaking, describes the evolution of a quantum system either not interacting, or interacting with only “small” systems. Then there is the non-deterministic and irreversible evolution resulting from the interaction of the system with a measuring device. Such an interaction is responsible for the “reduction of the wavefunction” and is represented in the theory by a non-unitary operator.
The question naturally arising from such a bifurcation is then: at what point does the evolution of a system cease to be deterministic and begin to be non-deterministic? That is, when do we consider a system to be “measured”? Such a question lies at the very heart of what is referred to as the “measurement problem”.

The idea of Bohm and von Neumann [18, 81] that such an irreversible process necessarily introduces into the theory a time asymmetric process, suggests that state reduction associated with the act of measurement is a physical process responsible, at least in part, for the thermodynamical arrow of time in the quantum domain. In response to these suggestions it was later shown by Aharonov, Bergmann and Lebowitz [3], following the work of Watanabe [84], that, although the reduction of the wavefunction is an irreversible process, it is not time asymmetric. They proceeded to demonstrate this by taking the standard probability expression of quantum mechanics, which implicitly deals with ensembles that have been “preselected” on the basis of some initial observation, and deriving a probability expression that favours neither the past nor the future. From such a time symmetric probability expression, they demonstrated how the conventional predictive, as well as a “retrodictive”, probability expressions can be obtained. It was their findings that an additional postulate was necessary to regain from the time symmetric expressions the conventional predictive probability expression. This led them to conclude;

“If, as we believe, the validity of this postulate and the falsity of its time reverse result from the macroscopic irreversibility of our universe as a whole, then the basic laws of quantum physics, including those referring to measurements, are as completely symmetric as those of classical physics.”

It was shown, “as a by-product” of their analysis, that a quantum state could be assigned based on the outcome of a future measurement event with as much justification as one, ordinarily, assigns the state base on the outcome of an initial preparation event. Since then the formalism of retrodiction in quantum mechanics has been further developed [14, 15, 49, 48] in addition to finding practical applications [47, 28, 29]. As we will see in Chapter 2, in the retrodictive formalism it is the preparation apparatus, as opposed to the measurement apparatus, that is responsible for the non-unitary evolution resulting in the reduction of the wavefunction. With the predictive and retrodictive formalisms being equally valid, it is apparent that the reduction of the wavefunction is a time symmetric process, while still
1.2 Structure of this thesis

Just as a preparation device is said to be responsible for the production of a predictive quantum state, the foundational work of [3] tells us that, with the same justification, a measurement device is responsible for the production of a retrodictive quantum state. It was the findings of [3] that selecting ensembles of systems favouring either preselection or postselection leads to the predictive or retrodictive formalism respectively. Traditionally, the conventional formalism of quantum mechanics has been the predictive formalism, presumably for the same reason as why a bookmaker will not pay favourable odds after a horse race. The retrodictive formalism is then usually derived from the conventional formalism using Bayes’ theorem [3, 14]. I begin this thesis from a more general approach. Following the work of Pegg, Barnett and Jeffers [49], I discuss in Chapter 2 a formalism of quantum mechanics which, similar to that of Aharonov et al. [3], is time symmetric. Such a formalism begins by postulating a probability measure in which both the preparation and measurement procedures are represented by the same class of operators. Along similar lines to the original paper [49], I demonstrate how the predictive and retrodictive formalism can be derived from this.

With both the preparation and measurement processes represented by the same class of operators, it is then possible to formally consider a hybrid process involving combinations of both preparation and measurement devices. I show that such a combined device can always be reduced to either a preparation device or a measurement device. This allows one to redefine the boundaries between preparation and measurement for a joint system under study.

Utilising this, I consider in this thesis linear optical multiport devices with equally many input ports as output ports. By considering everything except one of the input ports as a measuring device, I am able to effectively synthesize novel measuring devices. By changing the preparation device at the input port we can change the form of the synthesized measuring device. Such a technique was first introduced into quantum optics by Pegg, Barnett and Phillips [50, 54] as a way to measure the phase distribution of light in the quantum regime. Although this technique was novel, it was largely an in principle demonstration as the
reference states necessary to synthesise the measurement requires, in general, a non-classical reference state.

It is the findings of Chapter 3 that the non-classical reference state in the original projection synthesis can be replaced by an easily prepared coherent reference state, provided we extend the beam-splitter to a linear optical multiport with as many input ports as the maximum number of photons in the retrodictive state we wish to engineer. Such a finding demonstrates how any observable can be measured using only linear optics, coherent reference states and photo-detection. We find, in keeping with an emphasis on realistic measurement, that only photodetectors that can discriminate between zero, one and many photons are required.

Such a finding is important to the field of quantum optics because it demonstrates that any quantum optical observable can be measured using only three basic elements; linear optics, coherent states and single photon detection. As all these elements are experimentally available, this results offers an experimentally achievable way to measure any quantum optical observable.

We find, quite remarkably, that although any general retrodictive state can be produced by this method, it is difficult to physically implement the time reversed situation (that is swap the measured output with the prepared input and vice versa) to generate any general predictive state. The problem arises because of the difficulty associated with simultaneously producing multiple single photon states. By conditioning involving simultaneous photodetector outcomes it is not so difficult to produce the retrodictive equivalent of this. So although the range of predictive states which can be generated in the laboratory is relatively small, we find that the range of retrodictive states is, remarkably, quite large. With this in mind we investigate in Chapter 4 two experimental proposals which are proving difficult to implement due to a need for a non-classical predictive reference state. We consider redesigning the proposed experiments in such a way to replace the predictive reference state with a more readily prepared retrodictive state in order to achieve the same result. The experiments on the whole are much simpler, involving only linear optics, a coherent reference state and photo-detection.

Although the general experiment introduced in Chapter 3 can measure the probability distribution of any quantum optical observable, it does not provide a “single-shot” measurement of that observable. A single-shot measurement is defined as a measurement which can
project onto any eigenstate in the set of eigenstates representing the observable of interest with the correct probability. That is to say that the measurement device can produce the complete set of retrodictive states defined by the observable. Such an example is a perfect photon number measurement where the measurement device could project onto any one of the infinite photon number eigenstates \(|n\rangle\). Although such a measurement is not currently available, it is assumed that such a measurement could be performed in principle if we could eliminate all physical sources of imperfection inherent in the devices existing today. Conversely, an observable for which there does not exist even an imperfect single-shot measurement device is the observable canonically conjugate to photon number. Such an observable, commonly referred to as canonical phase, has had an interesting history in the field of quantum optics\(^1\). Despite these challenges, there nevertheless does exist a well-defined physical observable without, until now, a technique capable of providing a single-shot measurement. In Chapter 5 I present the first proposal capable of producing a single-shot measure of canonical phase. The technique is similar to the general measuring device introduced in Chapter 3, with one of the coherent reference states replaced with a binomial reference state. Such a tradeoff is the price paid to obtain a single-shot measuring device.

It is perhaps a little ironic then that the apparatus used to measure “measured phase”, which was introduced by Noh, Fougères and Mandel \([37, 39, 38, 40, 23]\) as a way to circumvent the difficulties associated with measuring canonical phase, can in fact be used to measure the canonical phase distribution for the range of fields considered by the same authors. I show in Chapter 5 that the only alteration to the apparatus needed is to either suitably squeeze one of the coherent reference states at the input of the optical multiport device or, even more simply, set the amplitudes of the three coherent reference states to a predetermined value. I conclude this thesis with some final remarks on the work presented within this thesis.

\(^1\)For a review see, for example, \([34]\) and for a chronological listing of most of the relevant papers on phase see the detailed bibliography of \([15]\).
Chapter 2

Quantum theory of preparation and measurement

Traditionally, the theory of quantum mechanics is regarded as a predictive theory. Take for example the situation were Alice prepares some system in a particular state which she then sends off to Bob who performs a measurement on it. We typically assign a quantum state $\hat{\rho}_i$ to the system Alice prepared based on the outcome $i$ of her preparation device and ask the question: with what probability will Bob observe the outcome $j$ at his measurement device given that Alice prepared the system in the state $\hat{\rho}_i$? It would also be perfectly reasonable, albeit less common, to ask this question in reverse. That is, given that Bob has observed the outcome $j$ at his measurement device, with what probability did Alice prepare the system in the state $\hat{\rho}_i$? Provided the answer to the first of these two questions is known then it is possible to infer an answer to the latter using Bayes’ theorem [19]. Alternatively, using the less usual but completely rigorous formalism of retrodiction [84, 3, 53, 1, 2, 1] it is possible to answer questions of this kind more directly. The theory assigns a retrodictive quantum state $\hat{\rho}_{\text{ret}}^j$ to the system based on the outcome $j$ of Bob’s measurement device. As opposed to the predictive state $\hat{\rho}_i$ that Alice assigns to the system which evolves forward in time, interestingly, the retrodictive state is said to evolve backwards in time, away from the measurement event.

To introduce both the predictive and retrodictive formalisms of quantum mechanics, it is perhaps simplest to begin with a theory which treats preparation and measurement on an equal footing. Following the work of Pegg et. al. [49] such a theory is introduced
in Sections 2.1 and 2.2. To show that the theory is consistent with all the prediction of quantum mechanics, the authors derive the conventional predictive formalism of quantum mechanics from their symmetric theory which is presented here in Section 2.3. From within this framework it is then a straightforward matter, as was originally done in [49], to extend this derivation to obtain the retrodiction formalism of quantum mechanics. This too is presented in Section 2.3. From there the work becomes original as I address in Section 2.5 the seemingly paradoxical issue associated with the very existence of a retrodictive state. We resolve this issue and explain why a predictive state can be used by Alice to send information to Bob some time into the future, but why Bob cannot use a retrodictive state to send useful information about the future backwards in time to Alice. This however does not render the concept of a retrodictive state as useless. Indeed, in Section 2.6 it is shown how, with the aid of post selection, retrodictive states can be conditionally generated. This is an important result to this thesis as the remaining chapters are focused on using post selection to engineer specific tailored retrodictive states to provide arbitrary measurements. We find that not only are retrodictive states practically useful, they also provide a deeper conceptual understanding of quantum mechanics where in Section 2.5 we briefly examine the measurement problem and highlight the ambiguity in defining when the state collapse ‘really’ occurs.

2.1 Preparation and measurement devices

Following Pegg et al. [49], consider the situation where Alice prepares a quantum system which is then automatically sent to Bob who performs a measurement upon it. Alice is capable of preparing the system in any one of a number of, not necessarily orthogonal, states labelled by $i = 1, 2, \ldots$. When the desired system is prepared she sends the label $i$ associated with her successful preparation event to a computer for recording. The combined process of preparation and recording of a particular event by Alice is described mathematically by a preparation device operator (PDO) $\hat{\Lambda}_i$. The PDO acts on the state space of the system and represents the successfully prepared state, including any biasing that might arise from the preparation device not being able to produce certain states or from Alice choosing not to record states of a particular kind. The set of all such operators $\{\hat{\Lambda}_i\}$ is sufficient to describe mathematically the preparation procedure dictated by Alice.
The measuring device has a readout mechanism which indicates the outcome of a measurement event labelled \( j = 1, 2, \ldots \). To each possible measurement event \( j \) a measurement device operator (MDO) \( \hat{\Gamma}_j \) is associated. This operator also acts in the state space of the system and, among other things, represents the measured state of the system. At Bob’s discretion, the label \( j \) of the actual outcome is sent to the computer for recording. If the computer receives a label from both Alice and Bob then it records the combined event \((i,j)\). If however the computer does not receive both a value of \( i \) and \( j \) then the single event recorded by either Alice or Bob is discarded by the computer automatically. Any biasing on behalf of Bob or of the measuring apparatus to record an event is also incorporated in the MDO. As an example, the MDO for a perfect von Neumann type measurement [81] which is faithfully recorded by Bob would be the projector formed by the eigenstate corresponding to the detection event \( j \). The combined process of detection and recording by Bob is called the measurement procedure and is completely described mathematically by the set of all MDOs \( \{\hat{\Gamma}_j\} \). For now no restrictions are imposed on the operators \( \hat{\Lambda}_i \) and \( \hat{\Gamma}_j \) other than they represent their associated preparation and measurement procedures. In the following section a single restriction upon these operators is postulated and from this postulate standard measurement theory [27] is derived.

By repeating the experiment many times, with both Alice and Bob independently performing their own preparation and measurement procedures, a list of combined events \((i,j)\) is compiled on the computer from which particular occurrence frequencies can be found.

### 2.2 Fundamental postulate

A sample space of mutually exclusive outcomes can be constructed from the combined recorded events \((i,j)\), such that each point in the space represents a particular event. To each identical event we assign the same point in the sample space. To this space a probability measure can be introduced that assigns probabilities between zero and one to each point so that the sum of the probabilities over the whole space is unity. This measure is such that the probability is proportional to the number of recorded events \((i,j)\) associated with each point, that is, the probability is proportional to the occurrence frequency of the recorded event \((i,j)\). The fundamental postulate of [49] is that the probability associated with each
point in this sample space is given by
\[
\Pr^{\Lambda\Gamma}(i,j) = \frac{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}_j]}{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}]},
\] (2.1)
where the trace is over the state space of the system and
\[
\hat{\Lambda} = \sum_i \hat{\Lambda}_i,
\] (2.2)
\[
\hat{\Gamma} = \sum_j \hat{\Gamma}_j.
\] (2.3)
That is to say that if a recorded event is chosen at random from the computer list, then
Equation (2.1) is the probability that this event is \((i,j)\).

The postulate of (2.1) imposes an implicit restriction upon the operators \(\hat{\Lambda}_i\) and \(\hat{\Gamma}_j\), in that these operators must be non-negative or non-positive definite to ensure that no probabilities \(\Pr^{\Lambda\Gamma}(i,j)\) are negative. Without loss of generality, we assume they are non-negative definite. The denominator in (2.1) is for normalisation purposes, that is to make certain that all probabilities assigned to the sample space add to unity. As a result, each set of operators \(\{\hat{\Lambda}_i\}\) and \(\{\hat{\Gamma}_j\}\) need only be specified up to an overall arbitrary constant, as this constant always cancels in the expression for the probabilities (2.1). This freedom will become useful later in defining a more convenient set of operators \(\{\hat{\Lambda}_i\}\) and \(\{\hat{\Gamma}_j\}\) such that the operators \(\hat{\Lambda}\) and \(\hat{\Gamma}\) have unit trace.

From (2.1), the following probabilities can be derived,
\[
\Pr^{\Lambda\Gamma}(i) = \sum_j \Pr^{\Lambda\Gamma}(i,j) = \frac{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}]}{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}]},
\] (2.4)
\[
\Pr^{\Lambda\Gamma}(j) = \frac{\text{Tr}[\hat{\Lambda} \hat{\Gamma}_j]}{\text{Tr}[\hat{\Lambda} \hat{\Gamma}]},
\] (2.5)
\[
\Pr^{\Lambda\Gamma}(j|i) = \frac{\Pr^{\Lambda\Gamma}(i,j)}{\Pr^{\Lambda\Gamma}(i)} = \frac{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}_j]}{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}]},
\] (2.6)
\[
\Pr^{\Lambda\Gamma}(i|j) = \frac{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}_j]}{\text{Tr}[\hat{\Lambda}_i \hat{\Gamma}_j]},
\] (2.7)
and can be interpreted as such: If one event is chosen at random from the list of combined recorded events on the computer, then (2.4) is the probability that this combined event includes the preparation and recording of the event \(i\). Similarly, \(\Pr^{\Lambda\Gamma}(j)\) is the probability that a randomly selected combined event will involve the detection and recording of measurement event \(j\). The expression (2.6) is the probability that if the preparation event \(i\)
was observed and recorded by Alice, then the corresponding measurement event \( j \) will be observed and recorded by Bob. The reverse situation may also be considered. That is, if Bob observed and recorded the detection event \( j \), then the probability that this detection event coincided with the preparation event \( i \) of Alice is given by (2.7).

Expression (2.6) may be used for prediction. If the PDO for Alice’s preparation event is known, say \( \hat{\Lambda}_i \), then (2.6) can be used to calculate the probability that Bob, some time in the future, will detect and record the event \( j \) represented by the MDO \( \hat{\Gamma}_j \). To do this, the operator \( \hat{\Gamma} \) must be known. This requires having some information about the measurement procedure of Bob. Expression (2.7), in contrast, can be used for retrodiction. When the detection event \( j \) is observed and recorded by Bob, a corresponding MDO \( \hat{\Gamma}_j \) can be assigned. With this knowledge, Bob can use expression (2.7) to calculate the probability that Alice, some time in the past, prepared and recorded the state represented by the PDO \( \hat{\Lambda}_i \). To do this he also needs to have some information about the preparation procedure of Alice, in that the operator \( \hat{\Lambda} \) needs to be known. For this reason, (2.7) can be viewed as the time reversal of Eqn (2.6) and vice versa.

### 2.3 Density matrices, POMs and unbiased devices

The postulate of Eqn (2.1) treats preparation and measurement on an equal footing. This is evident in that the class of operators representing each procedure are subject to the same constraints. In standard measurement theory there is, however, an asymmetry in the class of operators representing the preparation and measurement procedures. Density matrices, with unit trace, are used to describe the system prepared by Alice while a Probability Operator Measure (POM) \( \hat{\Pi} \), with elements \( \hat{\Pi}_j \) which sum to unity, is used to describe the measurement procedure of Bob. In what follows we relate the respective operators in these two formalisms and find an equivalence in the predictive ability of both formalisms. That is we derive the conventional predictive formalism of quantum mechanics from the postulate of Eqn (2.1). We find however, that the postulate of Eqn (2.1) allows for a wider class of measurements procedures than standard measurement theory allows.

Interestingly, quantum mechanics allows for the class of operators representing each procedure to be exchanged. In the standard theory where the class of operators is asymmetric, an unorthodox formalism arises. Such a formalism, known as the retrodictive formalism,
assigns a density matrix to the state of the system measured by Bob, while a POM can be used in some circumstances to describe the preparation procedure of Alice. In Section 2.3.2 we derive the retrodictive formalism from the postulate of Eqn (2.1) and, in doing so, find an expression relating the corresponding operators in each formalism.

### 2.3.1 Predictive formalism of quantum mechanics

Following the formalism of Ref. [49] it is understood that a PDO represents, among other things, the state of the system prepared by Alice. A normalised version of this operator can be introduced as

\[ \hat{\rho}_i = \frac{\hat{\Lambda}_i}{\text{Tr}[\hat{\Lambda}_i]} \]  

(2.8)

Since this operator is positive definite and has unit trace it serves as valid density matrix representing the state of the system prepared by Alice. Consider now the situation were Bob’s measurement procedure is such that \( \hat{\Gamma} \), the sum of the MDOs, is proportional to the unit operator acting on the state space of the system,

\[ \hat{\Gamma} = \gamma \mathbb{1} \]  

(2.9)

Such a measurement procedure is referred to as an unbiased measurement procedure [49]. For an unbiased measurement procedure it is useful to introduce the set of operators

\[ \hat{\Pi}_j = \gamma^{-1} \hat{\Gamma}_j \]  

(2.10)

Summing the above equation over all measurement outcomes \( j \) and comparing to Eqn (2.9) shows that the elements \( \hat{\Pi}_j \) sum to the identity. As these operators are non-negative definite it is appropriate to regard the set of such operators \( \{\hat{\Pi}_j\} \) as a POM. We note that the constant of proportionality \( \gamma \) always cancels in the expression for the probabilities (2.4)–(2.7) so it is appropriate to regard the POM elements as MDOs. Therefore it is understood that each POM element corresponds to the measuring and recording of a particular outcome by Bob.

By way of Eqns (2.8), (2.9) and (2.10), it is straightforward then to recast (2.6) as,

\[ \text{Pr}^{\Lambda_1}(j|i) = \text{Tr} \left[ \hat{\rho}_i \hat{\Pi}_j \right] \]  

(2.11)

thereby recovering the standard probability postulate of quantum mechanics [27]. In doing this however, an asymmetry in the normalisation properties of the two operators \( \hat{\rho}_i \) and
\( \hat{\Pi}_j \), representing the preparation and detection events respectively, is introduced. This asymmetry is not of a fundamental origin, but rather comes from the added restriction that the measurement process satisfy Eqn (2.9). Under this restriction, the symmetric postulate of (2.1) reduces to the asymmetric form given above. Interestingly, we show in Section 2.5 that the measurement procedure for all faithfully recording measurement devices must satisfy Eqn (2.9) in order to preserve causality. In this case the above equation is sufficient to predict the probability of all future measurement events \( j \), given that the system was initially prepared in the state \( \hat{\rho}_i \). In general, however, not all measurement procedures are required to faithfully record all outcomes. For example, in the operational phase measurements of Noh et al. certain photo-detector readings are not recorded because they do not lead to meaningful values being measured. This is a specific example where the MDOs do not sum to unity so the probabilities obtained from (2.11) must be appropriately renormalised to be consistent with the the statistics of the experiment. This is precisely what the more general measurement postulate leading to (2.6) does. Substituting Eqn (2.8) for \( \hat{\Lambda}_i \) in (2.6) we can express this in terms of density matrices as

\[
Pr_{\Lambda \Gamma}(j|i) = \frac{\text{Tr}[\hat{\rho}_i \hat{\Gamma}_j]}{\text{Tr}[\hat{\rho}_i \hat{\Gamma}]}.
\]

(2.12)

which is a more general expression than Eqn (2.11) as it accounts for experiments where measurement results may be discarded.

### 2.3.2 Retrodictive formalism of quantum mechanics

The decision to express the PDO as a density matrix was arbitrary. For an unbiased measurement device it gives a simple expression for the predictive formula \( Pr(j|i) \) of standard measurement theory. Alternatively, the MDO could be expressed in terms of a density matrix as

\[
\hat{\rho}^{\text{ret}}_j = \hat{\Gamma}_j / \text{Tr}[\hat{\Gamma}_j]
\]

(2.13)

which reduces the retrodictive formula \( Pr_{\Lambda \Gamma}(i|j) \) of Eqn (2.7) to

\[
Pr_{\Lambda \Gamma}(i|j) = \frac{\text{Tr}[\hat{\Lambda}_i \hat{\rho}^{\text{ret}}_j]}{\text{Tr}[\hat{\Lambda}_i \hat{\rho}^{\text{ret}}]}
\]

(2.14)

We say in general that the operation of the preparation device is unbiased if the PDOs are proportional to \( \hat{\Xi}_i \) where

\[
\sum_i \hat{\Xi}_i = \hat{1},
\]

(2.15)
that is, if the operators $\hat{\Xi}_i$ form the elements of a preparation device POM. For such a preparation procedure the retrodictive formula $\Pr^{\Lambda\Gamma}(i|j)$ of Eqn (2.14) simplifies to

$$\Pr^{\Lambda\Gamma}(i|j) = \text{Tr} \left[ \hat{\Xi}_i \hat{\rho}^{\text{ret}}_j \right],$$

(2.16)

which is identical to the standard predictive formula of Eqn (2.11) with the roles of preparation and measurement exchanged.

A specific example of a preparation device with an unbiased operation is where Alice prepares a spin-half particle in either the up state or the down state with equal probability. For such a preparation device, $\hat{\Lambda}$, the sum of the PDOs is proportional to the identity operator as the PDOs $\hat{\Lambda}^\uparrow$ and $\hat{\Lambda}^\downarrow$ associated with the up and down states are proportional to the projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ respectively. From Eqn (2.14) the retrodictive probability that Alice prepared the system in the up state given that Bob measured the system in the state $\hat{\rho}^{\text{ret}}_j = \hat{\Gamma}_j / \text{Tr}[\hat{\Gamma}_j]$ can be calculated to be

$$\Pr^{\Lambda\Gamma}(i = \uparrow | j) = \text{Tr} \left[ |\uparrow\rangle\langle\uparrow| \hat{\rho}^{\text{ret}}_j \right].$$

(2.17)

This is consistent with $\hat{\Xi}_\uparrow = |\uparrow\rangle\langle\uparrow|$ in Eqn (2.16).

In general however, many preparation procedures are biased in their operation in which case Eqn (2.16) would not apply. For example, the preparation device for an optical field is limited in that it cannot produce energies above some value. As such the sum of the PDOs is not the identity operator acting in the entire state space of the system. In the case of the spin-half particle, it may be such that Alice only produces states that are up or in and equal superposition of the states up and down. For such situations one must use the more general retrodictive formula of Eqn (2.14).

### 2.4 Time evolution

In the conventional approach, when a system evolves between the time of preparation $t_p$ and the time of measurement $t_m$, the density operator describing the state of the system $\hat{\rho}_i$ undergoes a unitary change. The final state of the system at the time of the measurement $\hat{\rho}_i(t_m) = \hat{U} \hat{\rho}_i \hat{U}^\dagger$, where $\hat{U}$ is a unitary operator, is then substituted for $\hat{\rho}_i$ in the appropriate probability formulae. Akin to this, the operator representing the state of the system in this approach, $\hat{\Lambda}_i$, is considered to undergo a unitary change and is replaced by $\hat{\Lambda}_i(t_m) = \hat{U} \hat{\Lambda}_i \hat{U}^\dagger$. Noting that $\text{Tr}[\hat{U} \hat{\Lambda}_i \hat{U}^\dagger] = \text{Tr}[\hat{\Lambda}_i]$, it is apparent from (2.8) that this approach is consistent
with the conventional approach and yields the standard predictive formula \( \text{(2.11)} \) with \( \hat{\rho}_i \) replaced by \( \hat{\rho}_i(t_m) \).

The time reversal of this is to consider the retrodictive picture, in which the general retrodictive probability formula \( \text{(2.14)} \) would read

\[
\Pr^\Lambda \Gamma(i|j) = \frac{\text{Tr}[\hat{\Lambda} \hat{\rho}_i \hat{\rho}^\text{ret}_j]}{\text{Tr}[\hat{\Lambda} \hat{\rho}^\text{ret}_j]} \tag{2.18}
\]

From the cyclic property of the trace this can be written as

\[
\Pr^\Lambda \Gamma(i|j) = \frac{\text{Tr}[\hat{\Lambda} \hat{\rho}^\text{ret}_j(t_p)]}{\text{Tr}[\Lambda \hat{\rho}^\text{ret}_j(t_p)]} \tag{2.19}
\]

where \( \rho^\text{ret}_j(t_p) = \hat{U}^\dagger \rho^\text{ret}_j \hat{U} \) is the retrodictive density operator evolved backwards in time to the time of the preparation. From equation \( \text{(2.19)} \) it could be suggested that the state collapse occurs at the time of the preparation, \( t_p \). This is not in conflict with the conventional predictive formalism as can be seen by replacing \( \hat{\rho}_i(t_m) \) by \( \hat{U} \hat{\rho}_i \hat{U}^\dagger \) in the predictive probability formula \( \text{(2.11)} \) and using the cyclic property of the trace to give the probability as \( \text{Tr}[\hat{\rho}_i \hat{U}^\dagger \hat{\Pi}_j \hat{U}] \). In this case the set of operators \( \{ \hat{U}^\dagger \hat{\Pi}_j \hat{U} \} \) constitute a valid POM and can therefore be interpreted as a measuring device that makes the measurement on the prepared state immediately after the the preparation time \( t_p \). This highlights the arbitrariness as to when the state collapse ‘really’ occurs, and it is seen to be a direct result of the ambiguity in defining the physical boundary of the measuring device.

### 2.5 Unidirectional flow of information

In their formalism of the quantum theory of preparation and measurement, Pegg et al. [49] defined an unbiased measuring device as one for which the sum of the MDOs are proportional to the unit operator. In this section we show that the physical requirement of causality requires that all faithfully recording measurement devices must be unbiased. Thus the arrow of time does not arise from quantum mechanics, it is inserted by means of the conventional postulate that sets of MDOs are represented by POMs but sets of PDOs are, in general, not. From this we are able to derive an expression representing the choices Alice makes in preparing a quantum system. The expression relates the PDO to the probability that she chooses to prepare the system in the state \( \hat{\rho}_i \). We call such a probability the \textit{a priori} probability as we explicitly show that Alice has sole choice in the states prepared by
the preparation device. As a consequence of causality we find, in contrast, that Bob cannot be assigned such an *a priori* probability as we show that he cannot choose the outcomes of his measurement device.

### 2.5.1 Unbiased measuring devices

With the arrangement we have described and the associated derived probabilities, we ask the question: can Bob employ retrodictive states to communicate with Alice at an earlier time by means of a series of a large number of experiments? Let us assume, for example, that Alice and Bob are a light day apart and the series of experiments takes an hour for Alice to prepare and equally as long for Bob to measure.

We take it that Bob has control over the choice of the measurement device and whether or not he records a measurement event that the readout on this device shows has happened. Utilizing this, Bob could then send a message by using his recording control to vary the probability that the preparation event \( i \) is recorded on the computer list, the expression for which is given by Eqn (2.4). However, Alice could only determine the probability for \( i \) and receive the message *after* Bob has made his contributions to the list. Essentially this process is making use of the list as a classical communication channel.

We could eliminate this classical means of communication as follows. If we ensure that Alice and Bob must record every preparation and measurement event shown by the readouts on their devices, then Eqn (2.4) becomes equal to the probability that events \( i \) and \( j \) are shown on the readouts of the preparation and measurement devices. We can thus eliminate the computer list entirely and apply the probability formulae to refer to the readout events themselves. It follows that Eqn (2.4) is then the probability that event \( i \) will be shown on Alice’s readout, which she can access before Bob receives his readout signal. Bob’s recording control is now useless and cannot help Bob play a part in determining Alice’s readout probability given by Eqn (2.4). Can Bob exert his free choice of *measuring device* to alter \( \hat{\Gamma} \) at will and thus send a message to Alice by changing \( \Pr A^\Gamma(i) \) in (2.4)? A controllable physical operation on the device can result in a unitary transformation of the MDOs that will alter \( \hat{\Gamma} \) to \( \hat{\Gamma}' = \hat{U}\hat{\Gamma}\hat{U}^\dagger \). If \( \hat{\Gamma}' \neq \hat{\Gamma} \) then Bob has a means of communicating with Alice at an earlier time via the retrodictive state by altering the probability of Eqn (2.4). In order to prevent this, and thus preserve causality, we need to ensure that \( \hat{\Gamma}' = \hat{\Gamma} \). The only way this can be done for all such transformations is for \( \hat{\Gamma} \) to be proportional to the unit operator as
this is the only operator that commutes with all possible unitary transformation operators. We note that Bob could also try to alter $\hat{\Gamma}$ by deciding not to make a measurement at all. To ensure that this is impossible, the single ‘non-measurement’ MDO must also be proportional to the unit operator.

We see, therefore, that causality, in the form of a unidirectional flow of information, demands that the sum of the MDOs for any measuring device for which the measurements are faithfully recorded are proportional to the elements of a probability operator measure. Causality thus ensures that any faithfully recording measuring device provides an unbiased measurement. On the other hand, there is no such restriction on the PDOs, because information is allowed to be transferred via the predictive state into the future. Faithfully recorded preparation devices therefore can be biased. Thus the original time symmetric equations of Pegg et al., when pertaining to controllable information flow, reduce to a unidirectional form that encapsulates the essence of the arrow of time. In this form, the predictive and retrodictive density operators in the absence of information about the actual preparation or measurement events are, respectively,

$$\hat{\rho} = \frac{\hat{\Lambda}}{\text{Tr}[\hat{\Lambda}]} \quad (2.20)$$

$$\hat{\rho}^{\text{ret}} = \frac{\hat{1}}{\text{Tr}[\hat{1}]} \quad (2.21)$$

where Eqn (2.20) comes from Eqns (2.5) and (2.8). The asymmetry in the normalisation of the non-negative operators in the usual postulate

$$\text{Pr}(j|i) = \text{Tr}[\hat{\rho}_i \hat{\Pi}_j] \quad (2.22)$$

for faithfully recording measurements can now be seen to result from causality.

The ‘no-information’ retrodictive density operator (2.21) is the state that is always sent back in time by a non-measurement. In this context it is totally controllable or deterministic. We wish to stress that this is the only retrodictive state that we can send back in time deterministically.

In this thesis, we shall show how to engineer a variety of retrodictive states that are not proportional to the unit operator by non-deterministic means. It is this lack of determinism that prevents such states from being used to send messages into the past. This can be seen explicitly from Eqn (2.27) and also (2.14) where general retrodictive states are generated conditioned, however, on the outcome $j$ of Bob’s measurement. Causality is preserved
in this case because Bob cannot reliably guarantee the outcome $j$ of the measurement, thus Alice must use Eqn (2.4) to try and infer a message from Bob which we constrained previously to be consistent with causality.

### 2.5.2 Independent *a priori* probability

We postulated in the preceding section that all faithfully recording measuring devices must, in order to preserve causality, be such that the sum of all the MDOs be proportional to the identity operator acting on the state space of the system. With such a condition it is then impossible for Bob to communicate with Alice through the outcomes of her preparation device, the probability of which is given by Eqn (2.4). This can be seen from (2.4) with $\hat{\Gamma} = \gamma \hat{1}$

$$\text{Pr}^\Lambda_1(i) = \frac{\text{Tr}[\hat{\Lambda}_i]}{\text{Tr}[\hat{\Lambda}]}$$

(2.23)

as this probability is $\hat{\Gamma}_j$-independent. Since this probability depends only on the PDOs describing Alice’s preparation procedure, we attribute choice to Alice and say that she can choose the probability $i$ of an outcome. We can therefore refer to this probability as Alice’s *a priori* probability $\text{Pr}^\Lambda(i)$,

$$\text{Pr}^\Lambda(i) = \frac{\text{Tr}[\hat{\Lambda}_i]}{\text{Tr}[\hat{\Lambda}]}$$

(2.24)

This *a priori* probability relates to Alice’s choice in the states that she decides to prepare, and is independent of Bob and his subsequent measurement procedure. In contrast, the probability that Bob observes the outcome $j$ at his measuring device, Eqn (2.5), does depend on Alice and her preparation procedure. This is a direct consequence resulting from the fact that Alice’s preparation procedure can be biased in its operation. That is $\hat{\Lambda}$ need not be proportional to the identity operator acting on the space of the system. As such Bob’s measurement outcomes are *not* independent of Alice and her preparation procedure and so Bob cannot be associated with an independent *a priori* probability $\text{Pr}^\Gamma(j)$.

We find that the condition of causality represented by Eqns (2.20) and (2.21) allow Alice to be assigned an independent *a priori* probability representing the choices she makes, but forbids such an assignment for Bob. Such an asymmetry, in that our choices can only influence the events of the future, again encapsulates the essence of the arrow of time in the form of unidirectional flow of information.

With Alice’s choice represented by the *a priori* probability of Eqn (2.24) we find, from
Eqn (2.8), that we can write \( \hat{\Lambda}_i \) as proportional to \( \Pr^{\Lambda}(i) \hat{\rho}_i \). The constant of proportionality, \( \text{Tr}[\hat{\Lambda}] \), always cancels in the expressions for the various probabilities so there is no loss in generality in setting it to unity. Then we have

\[
\hat{\Lambda}_i = \Pr^{\Lambda}(i) \hat{\rho}_i.
\]

(2.25)

It is now apparent how the PDO \( \hat{\Lambda}_i \), as well as the representing the state of the system, also contains information about the biasing in its preparation. The biasing factor is just the \textit{a priori} preparation probability. Because we have taken \( \hat{\Lambda} \) to have unit trace, it too is a density operator given by

\[
\hat{\Lambda} = \hat{\rho} = \sum_i \Pr^{\Lambda}(i) \hat{\rho}_i,
\]

(2.26)

which is the sum of all possible states that Alice prepares weighted by their \textit{a priori} probabilities of being prepared. Void of any knowledge of the preparation outcome \( i \), this is the best possible description of the state that Alice has prepared, when the only known information is the \textit{a priori} probabilities in which she prepares and records the states.

### 2.6 Conditional state generation

In principle it should be possible for Alice to generate any predictive state she so desires. Similarly, it should be possible, at least in principle, for Bob to measure in any basis he desires provided, of course, his measurement procedure is consistent with Eqn (2.9). Unfortunately, in practice, this is rarely the case as the range of precise preparation and measurement apparatuses are often limited. It is therefore always of considerable interest to find novel ways to extend the range of states, both predictive and retrodictive, that can be generated.

A technique which has long been used to generate a wide range of predictive states is what is referred to here as conditional preparation. It is a simple technique that begins with a entangled state at the initial time and, at some time later, involves a measurement on part of the system. The description of the remaining subsystem is then correlated to the outcome of the measurement event. In this section we show that the remaining subsystem can indeed be described mathematically by a predictive density operator conditioned on the outcome of the measurement event. By considering the measurement as part of the preparation procedure we are able to derive, from the fundamental postulate of Eqn (2.1),
a PDO associated with this event. From this we then derive an expression for the \textit{a priori} probability in which this conditional state is produced.

What is then interesting is to consider the time reversal of this situation. The technique, which could be referred to as conditional measurement, has a joint measurement on both subsystems, similar in form to a Bell-type measurement at the final time. In the retrodictive formalism a retrodictive state is assigned to the outcome of this joint measurement. This state evolves backwards in time until one of the subsystems is prepared. As a result of the correlations inherent in the retrodictive state, the remaining subsystem will be correlated (at earlier times!) to the outcome of the preparation event. By regarding this preparation procedure as part of the total measurement procedure we show that the remaining subsystem at earlier times can be described by a retrodictive state conditioned on the outcome of the preparation event. The interesting feature of a conditional measurement is that we have \textit{control} over the working of the preparation device. Utilizing this control allows a wider range of retrodictive states to be engineered than those produced by simple measurement devices alone. The importance of this technique is fundamental to the remainder of this thesis as all following sections are devoted to the generation of retrodictive states in this manner.

### 2.6.1 Predictive entangled state

Consider the situation illustrated in Figure 2.1. Depicted is a quantum system with subsystems $a$ and $b$ which are initially prepared in an entangled state described by the PDO $\hat{\Lambda}_i^{ab}(t_0)$ at some time $t_0$ associated with a preparation event $i$. After preparation the systems do not interact but evolve unitarily until a time $t_1$ where a measurement is performed on subsystem $a$. The evolved PDO of the system just prior to this measurement can be written as $\hat{\Lambda}_i^{ab}(t_1)$. The measurement event is associated with an MDO $\hat{\Gamma}_j^a$, corresponding to the measurement outcome $j$. At a later time $t_2$, subsystem $b$ is measured and the outcome $k$ is recorded. The later measurement event is described by the MDO $\hat{\Gamma}_k^b$, corresponding to the measurement outcome $k$.

As discussed in Section 2.6, we can obtain the same measured probabilities by redefining the second measurement to take place at an earlier time $t_1$ provided we replace the MDOs $\hat{\Gamma}_k^b$ with $\hat{\Gamma}_k^b = \hat{U}_b(t_2, t_1) \hat{\Gamma}_k^b \hat{U}_b(t_2, t_1)$ where $\hat{U}_b(t_2, t_1)$ is the unitary time displacement operator acting in the state space of system $b$. Formally these MDOs represent a different measuring
Figure 2.1: On the left is a bi-part system undergoing independent measurement of each sub-system at separate times. The square boxes represent preparation events and the domes represent measurement events. On the right is a single system undergoing measurement. By considering the measurement at time $t_1$ to be part of the preparation device, we show that these two systems are physically indistinguishable.

device, one in which the measurement takes place at an earlier time $t_1$. The probability of the joint event $(i, j, k)$ occurring is then, from Eqn (2.1)

$$\Pr(i, j, k) = \frac{\text{Tr}_{ab}[\hat{\Lambda}_{ab}^i \hat{\Gamma}_a^j \otimes \hat{\Gamma}_b^k]}{\text{Tr}_{ab}[\hat{\Lambda}_{ab}^i \hat{\Gamma}_a^j \otimes \hat{\Gamma}_b^k]},$$

(2.27)

where $\hat{\Lambda}_{ab}^i = \sum_i \hat{\Lambda}_{ab}^i$, $\hat{\Gamma}_a^i = \sum_j \hat{\Gamma}_a^j$, and $\hat{\Gamma}_b^k = \sum_k \hat{\Gamma}_b^k$ and the trace is over the state space of both subsystems. The probability that the later measurement event is $k$ if the earlier one is $j$ is given by

$$\Pr(k | i, j) = \frac{\Pr(i, j, k)}{\Pr(i, j)} = \frac{\Pr(i, j, k)}{\sum_k \Pr(i, j, k)}.$$

(2.28)

Substituting from Eqn (2.27) gives

$$\Pr(k | i, j) = \frac{\text{Tr}_b[\hat{\Omega}_{ij}^b \hat{\Gamma}_b^k]}{\text{Tr}_b[\hat{\Omega}_{ij}^b \hat{\Gamma}_b^k]},$$

(2.29)

where

$$\hat{\Omega}_{ij}^b \equiv \frac{\text{Tr}_a[\hat{\Lambda}_{ab}^i \hat{\Gamma}_a^j]}{\text{Tr}_{ab}[\hat{\Lambda}_{ab}^i \hat{\Gamma}_a^j]}.$$

(2.30)

Using the definition of $\hat{\Gamma}_b^k$ and the cyclic property of the trace we can rewrite this as

$$\Pr(k | i, j) = \frac{\text{Tr}_b[\hat{U}_b(t_2, t_1) \hat{\Omega}_{ij}^b \hat{U}_b^\dagger(t_2, t_1) \hat{\Gamma}_b^k]}{\text{Tr}_b[\hat{U}_b(t_2, t_1) \hat{\Omega}_{ij}^b \hat{U}_b^\dagger(t_2, t_1) \hat{\Gamma}_b^k]}.$$

(2.31)

It is clear from Eqn (2.29) that $\hat{\Omega}_{ij}^b$ is non-negative, so an alternative interpretation can be offered to the above configuration. The probability given by Eqn (2.31) is precisely what
2.6. CONDITIONAL STATE GENERATION

we would obtain by decomposing the entire dynamics into a single preparation event at time $t_1$ associated with the PDO $\hat{\Omega}_ij^b$ and a measurement event associated with the MDO $\hat{\Gamma}_k^b$ at time $t_2$. The preparation produces a state of the quantum subsystem $b$ that is conditioned on the preparation outcome $(i,j)$. This state evolves unitarily until a later time when a measurement is performed on this subsystem. If we wish to describe this state by a density operator instead of a PDO, we can define

$$\hat{\rho}_ij^b = \frac{\hat{\Omega}_ij^b}{\text{Tr}_b[\hat{\Omega}_ij^b]}$$

at time $t_1$.

The a priori probability for this state to be prepared is just the probability for the joint event $(i, j)$ in the absence of any measurement information about the subsystem $b$. Note that we are now considering the measurement event $j$ in the original interpretation as part of the joint preparation event. To find this a priori probability we put $\hat{\Gamma}_k^b = \hat{1}_b$ where $\hat{1}_b$ is the unit operator in the state space of subsystem $b$, which is the 'non-measurement' MDO \[49\]. From the formula corresponding to Eqn (2.1), we find the a priori probability for the state $\hat{\rho}_ij^b$ to be prepared is

$$\frac{\text{Tr}_{ab}[\hat{\Lambda}^ab\hat{\Gamma}_j^a \otimes \hat{1}_b]}{\text{Tr}_{ab}[\hat{\Lambda}^{ab}\hat{\Gamma}_j^a \otimes \hat{1}_b]}$$

This is precisely the result obtained by finding $\text{Tr}_b[\hat{\Omega}_ij^b]$, showing that $\hat{\Omega}_ij^b$ is just the state $\hat{\rho}_ij^b$ multiplied by the a priori probability that this state is produced. We are thus justified in representing the state of the remaining subsystem, after the measurement at $t_1$ is performed, by a predictive density operator prepared by a combined preparation device comprising the original preparation device and the first measurement device. Using this density operator as the state of the system immediately after $t_1$, we can calculate correctly the probabilities for any subsequent measurement.

The result derived in this subsection is one which most physicists accustomed to the predictive formalism would use intuitively. The purpose of deriving it formally is to establish a method for deriving the corresponding time-reversed result in the next subsection, which is not so intuitive.

2.6.2 Retrodictive entangled state

The retrodictive equivalent to the situation in the previous subsection is illustrated in Figure 2.2. We consider a quantum system consisting of two subsystems $a$ and $b$ which
undergo a joint measurement procedure at time $t_2$ which is characterised by the MDO $\hat{\Gamma}_{k}^{ab}(t_2)$ associated with the measurement outcome $k$. Before the measurement the systems do not interact but evolve unitarily from separate preparation procedures, the latest of which occurs at a time $t_1$. As discussed in Section 2.3 we can formally define the measurement procedure to occur at the earlier time $t_1$ and associate with it a different MDO $\hat{\Gamma}_{k}^{ab}(t_1) \equiv \hat{\Gamma}_{k}^{ab}$.

To each preparation event we assign a PDO corresponding to the outcome of the preparation device. For the preparation of subsystem $a$, occurring at time $t_1$, the PDO $\hat{\Lambda}^{a}_j$ is assigned based on the outcome $j$ of the preparation device, whereas the preparation of subsystem $b$, occurring at the earliest time of $t_0$, is associated with the PDO $\hat{\Lambda}^{b}_i$. Allowing for unitary evolution of the system between preparation events, we can write the PDO of subsystem $b$ at the later time $t_1$ as $\hat{\Lambda}^{b}_i = \hat{U}_b(t_0,t_1)\hat{\Lambda}^{b}_i \hat{U}_b^{\dagger}(t_0,t_1)$, where $\hat{U}_b(t_0,t_1)$ is the unitary time displacement operator of subsystem $b$ between times $t_0$ and $t_1$. The probability of the joint event $(i,j,k)$ occurring is then, from Eqn (2.31)

$$\Pr(i,j,k) = \frac{\text{Tr}_{ab}[\hat{\Lambda}^{a}_j \otimes \hat{\Lambda}^{b}_i \hat{\Gamma}_{k}^{ab}]}{\text{Tr}_{ab}[\hat{\Lambda}^{a} \otimes \hat{\Lambda}^{b} \hat{\Gamma}^{ab}]}$$

(2.34)

where $\hat{\Lambda}^{a} = \sum_{j} \hat{\Lambda}^{a}_{j}$, $\hat{\Lambda}^{b} = \sum_{i} \hat{\Lambda}^{b}_{i}$ and $\hat{\Gamma}^{ab} = \sum_{k} \hat{\Gamma}_{k}^{ab}$ and the trace is over the state space of both subsystems. Following similar steps to those presented in the preceding subsection, we arrive at an expression for the retrodictive probability $\Pr(i|j,k)$ that the earliest preparation event was $i$, given the combined outcome $(j,k)$ of the later preparation event and the final

Figure 2.2: On the left is a system, initially prepared at two different times by independent preparation procedures, undergoing a joint measurement. On the right is a single system undergoing measurement. By considering the preparation at time $t_1$ to be part of the measurement device, we show that these two systems are physically indistinguishable.
measurement event was observed, to be
\[
\Pr(i|j,k) = \frac{\text{Tr}_b[\hat{\Lambda}_b^b \hat{U}_b^b(t_0, t_1) \hat{\Phi}_{jk}^b \hat{U}_b^b(t_0, t_1)]}{\text{Tr}_b[\hat{\Lambda}_b^b \hat{U}_b^b(t_0, t_1) \hat{\Phi}_{jk}^b \hat{U}_b^b(t_0, t_1)]},
\]
(2.35)

where
\[
\hat{\Phi}_{jk}^b = \frac{\text{Tr}_a[\hat{\Lambda}_a^a \hat{\Gamma}_{ab}^a]}{\text{Tr}_{ab}[\hat{\Lambda}_a^a \hat{\Gamma}_{ab}^a]}
\]
(2.36)
is an operator acting solely in the state space of subsystem b. From (2.36) it is apparent that \(\hat{\Phi}_{jk}^b\) is a non-negative operator and, as such, this allows an alternative interpretation to be offered to the configuration. The probability of Eqn (2.35) is identical to that which would be obtained if the dynamics were decomposed into a single preparation event at time \(t_0\) associated with the PDO \(\hat{\Lambda}_b^b\) and a measurement event associated with the MDO \(\hat{\Phi}_{jk}^b\) at time \(t_1\). The measurement event is conditioned on the combined outcome \((j, k)\). As such, we interpret the second preparation event occurring at \(t_1\) to be part of the measurement event. If we wish to associate a retrodictive density operator to the measurement event instead of the MDO, we can define
\[
\hat{\rho}_{jk}^{\text{ret}} = \frac{\hat{\Phi}_{jk}^b}{\text{Tr}_b[\hat{\Phi}_{jk}^b]}
\]
(2.37)
where the mode label \(b\) is dropped for notational convenience. Such a definition is consistent with all the probability formulae and thus provides a valid alternative interpretation to the arrangement proposed above. The interesting feature of this retrodictive state is that it is generated conditioned on the preparation outcome \(j\) which we do have control over. This gives us some control over the retrodictive states that can be prepared but not total control as causality must still be preserved. In this situation causality is upheld because the generation of the retrodictive state is also conditioned on the measurement outcome \(k\) which we do not have control over. Indeed, we see by summing the MDO \(\hat{\Phi}_{jk}^b\) in Eqn (2.36) over \(j, k\) that the total MDO \(\hat{\Phi}^b\) is proportional to the identity operator acting on subsystem \(b\), showing that this interpretation is consistent with causality. The control that comes about from the preparation event \(j\) can generally be used to select the type of retrodictive states that are being created, rather than the specific state itself. It is this form of control that is exploited throughout this thesis to engineer retrodictive states which cannot be generated by simple measurement devices alone.
Chapter 3

Retrodictive state engineering

It has long been known that any physical observable can be represented mathematically by a set of POM elements. However, not until the projection synthesis technique introduced by Barnett and Pegg [12, 50], and later generalised by Phillips, Barnett and Pegg [54], was it illustrated how any such POM element could be synthesised experimentally. In terms of the retrodictive formalism, the technique can be regarded as starting with the selection of a standard POM element conditioned, typically, on a photodetection reading. By way of a beam-splitter and, generally, a non-classical reference state, the standard POM element is transformed non-unitarily backwards in time to a POM element representing the observable of interest. Viewed in this way projection synthesis is seen essentially as a form of retrodictive state engineering. Despite the theoretical implications of this technique, it does suffer from the practical drawback that it requires a non-classical reference state to generate the non-unitary transformation.

In this chapter, we extend the projection synthesis technique of Barnett and Pegg to include general linear optics of multiports, and find that a coherent states is all that is needed to synthesise any general optical POM element from a specific photodetection reading. This work provides a general technique for engineering any retrodictive optical state with a finite number of photon number-state components using currently available technology. It expands upon the work published by D. T. Pegg and myself [61].

I begin this analysis with a brief mathematical summary of a useful representation for a state expressible in a finite dimensional Hilbert space.
3.1 Factorizing states in finite dimensional Hilbert spaces

In general, any pure state can be expressed as a linear combination of basis states \( |\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle \), where here \(|n\rangle\) is an eigenstate of the photon number operator and satisfies the orthonormality condition \( \langle n|m\rangle = \delta_{n,m} \). By expressing each photon number state in terms of the vacuum state and the creation operator \( \hat{a}^\dagger \) as \( |n\rangle = (n!)^{-1/2} (\hat{a}^\dagger)^n |0\rangle \), it is possible to represent any state as an infinite summation of integer powers of the creation operator acting on the vacuum state. However, to any desired degree of accuracy, we can truncate this summation at some finite integer, \( N \), to give

\[
|\psi\rangle = \sum_{n=0}^{N} \frac{\psi_n}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle.
\] (3.1)

Alternatively, any state which exists in a finite dimensional Hilbert space can be created from the vacuum by repeated application of the operator \( (\hat{a}^\dagger - \beta_i^*) \) to give

\[
|\psi\rangle = \kappa \left[ \prod_{i=1}^{N} (\hat{a}^\dagger - \beta_i^*) \right] |0\rangle
\] (3.2)

where \( \kappa \) is a normalisation constant and \( \beta_i \) is a c-number. To relate the two expressions of \(|\psi\rangle\) given above, we act on both sides of (3.1) and (3.2) with a coherent state \( \langle \gamma | \) to give

\[
\kappa \prod_{i=1}^{N} (\gamma^* - \beta_i^*) = \sum_{n=0}^{N} \frac{\psi_n}{\sqrt{n!}} (\gamma^*)^n.
\] (3.3)

This expression effectively factorizes the polynomial of degree \( N \) on the right hand side and so, by setting \( \gamma = \beta_i \), the \( N \) values of \( \beta_i \) are the \( N \) complex roots of the equation in \( \gamma^* \) [20]

\[
\sum_{n=0}^{N} \frac{\psi_n}{\sqrt{n!}} (\gamma^*)^n = 0
\] (3.4)

The normalisation constant \( \kappa \) can be found after substituting the \( N \) values of \( \beta_i \) back into (3.2). For the remainder of this thesis I will refer to this equation as the characteristic polynomial. It is an important expression which relates the coefficients \( \psi_n \) in one representation to the coefficient \( \beta_i \) in another. As a matter of interest, it is worth mentioning that the complex coefficients \( \beta_i \) correspond to the \( N \) zeros in the Q-function representation of an \( N+1 \) dimensional state and are a necessary and sufficient set of parameters to describe any general quantum state in an \( N+1 \) dimensional Hilbert space.
3.2 Experimental realization of any discrete unitary transformation

A beam-splitter unitarily transforms two input modes of a travelling optical field to two output modes. This transformation is an example of the well known $U(2)$ group [90]. The natural extension to this is a multiport device consisting of $N + 1$ input modes and $N + 1$ output modes where a $(N + 1)$-dimensional unitary matrix $U(N + 1)$ describes the linear transformation of the mode operators. Recently, Reck et al. [62] have shown how any such transformation is equivalent to specific arrays of beam-splitters and phase-shifters. Such a proof opens the doors to more general experiments, including the recently proposed linear optics quantum computation scheme of Knill, Laflamme and Milburn [32]. Here we outline the proof introduced in [62] with the intent to use such a device to generalise the projection synthesis technique of Pegg and Barnett.

Consider a unitary ‘rotation’ matrix $R^\dagger(N + 1)$ that transforms a (normalised) arbitrary row vector in a $(N + 1)$-dimensional vector space into a unit vector in the same space,

$$
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}^T = e^{i\delta_N} \begin{pmatrix}
ie^{i\phi_0} \sin \theta_0 \\
ie^{i\phi_1} \sin \theta_1 \cos \theta_0 \\
ie^{i\phi_2} \sin \theta_2 \cos \theta_1 \cos \theta_0 \\
\vdots \\
ie^{i\phi_N} \sin \theta_N \cos \theta_{N-1} \ldots \cos \theta_0 \\
\cos \theta_N \cos \theta_{N-1} \ldots \cos \theta_0
\end{pmatrix}^T \cdot R^\dagger(N + 1) \tag{3.5}
$$

where the arbitrary vector is represented in generalised spherical coordinates and the phase factor $e^{i\delta_N}$ is included for generality. By writing the $(N + 1)^{th}$ row of a unitary matrix $U(N + 1)$ in terms of the generalised spherical coordinates, the action of the rotation matrix $R^\dagger(N + 1)$ can be seen to partially diagonalise the unitary $U(N + 1)$ into a reducible matrix

$$
U(N + 1) \cdot R^\dagger(N + 1) = \begin{pmatrix}
U(N) & 0 \\
0 & e^{i\delta_N}
\end{pmatrix}. \tag{3.6}
$$

Repeated application of rotation matrices of successively lower dimensions can then be used
to completely diagonalises the unitary $U(N+1)$

$$U(N+1) \cdot R^\dagger(N+1) \cdot R^\dagger(N) \ldots R^\dagger(2) = D$$  \hspace{1cm} (3.7)

where $D$ is a diagonal matrix with elements $e^{i\delta_n}$ along the leading diagonal. Inverting Eqn \eqref{eq:3.7} gives a factorised expression for $U(N+1)$ in terms of rotational matrices and a diagonal matrix as

$$U(N+1) = D \cdot R(2) \ldots R(N) \cdot R(N+1).$$  \hspace{1cm} (3.8)

The achievement of Reck \textit{et al.} was to realise that the necessary rotational matrices $R(m), m = 2, 3, \ldots, N+1$, and the diagonal matrix $D$ can be constructed from selected arrays of beam-splitters and phase-shifters. The unitary transformation can be physically implemented then by successively applying each array of beam-splitters and phase-shifters according to Eqn \eqref{eq:3.8}. To achieve this physical realisation we first need to consider the effect a lossless beam-splitter has on the input and output modes of a travelling optical field.

A lossless beam-splitter can be modelled by a linear, unitary transformation of the mode operators $(\hat{a}_0^\dagger, \hat{a}_1^\dagger)^T$ and $(\hat{b}_0^\dagger, \hat{b}_1^\dagger)^T$ for the appropriate input and output modes respectively. For the remainder of this thesis I take this transformation to be

$$\begin{pmatrix} \hat{b}_0^\dagger \\ \hat{b}_1^\dagger \end{pmatrix} = \begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{a}_0^\dagger \\ \hat{a}_1^\dagger \end{pmatrix},$$  \hspace{1cm} (3.9)

where $t = \cos \theta$ is the transmittance and $r = i \sin \theta$ is the reflectance. It should be mentioned that this transformation is not unique, indeed Reck \textit{et al.} select a different one in \cite{Reck1994}, however so long as the transformation matrix is unitary the net results will be equivalent. The above transformation is chosen for two reasons: firstly, in the limit where the beam-splitter becomes totally transmissive the transformation should approach the identity operator, and secondly, any phase change occurring upon reflection should be identical on both sides of the beam-splitter. To ensure unitary evolution, a $\pi/2$ phase change is then necessary upon reflection from either side of the beam-splitter. By inserting a phase-shifter at the input to mode 0, we can construct a general unitary transformation in $U(2)$ that will become the basic building block for this scheme. We represent such an element in Figure \ref{fig:beam-splitter} and denote the transformation matrix for such an element as

$$T = \begin{pmatrix} e^{i\phi} \cos \theta & i \sin \theta \\ ie^{i\phi} \sin \theta & \cos \theta \end{pmatrix}.$$  \hspace{1cm} (3.10)
Consider the case of a multiport device where there are \( N + 1 \) input modes and \( N + 1 \) output modes. A general beam-splitter described by the transformation in Eqn (3.10), combining modes \( p \) and \( q \), can be represented by a matrix \( T_{pq} \) which is an \( (N + 1) \)-dimensional identity matrix with the elements \( I_{pp} \), \( I_{pq} \), \( I_{qp} \) and \( I_{qq} \) replaced by the corresponding elements of \( T \). This matrix will act on the appropriate 2-dimensional subspace, leaving a \((N - 1)\)-dimensional subspace unchanged. Then by induction, it is straightforward to show that a rotation matrix introduced in Eqn (3.5) can be constructed from successive transformation matrices \( T_{p,q} \) as

\[
R^\dagger(N + 1) = \prod_{n=1}^{N} T^\dagger_{N,N-n,1}. 
\tag{3.11}
\]

To illustrate, take for example the rotational matrix \( R(4) = T_{3,0} \cdot T_{3,1} \cdot T_{3,2} \). Physically this is constructed from a linear array of three beam-splitters, each with one mode in common which we label as mode 3. Such an arrangement is illustrated in Figure 3.2. Then, by progressively applying linear arrays of beam-splitters, we can construct the the unitary matrix of interest. To implement the diagonal matrix \( D \) a phase shift at the exit of each the the linear arrays is necessary. The experimental setup of a general \( 4 \times 4 \) unitary transformation is illustrated in Figure 3.3. Extending this arrangement to \((N + 1)\)-dimensions is straightforward. It should be noted that only \( \binom{N+1}{2} = \frac{N(N+1)}{2} \) beam-splitter devices are required to construct a general \((N + 1)\)-dimensional unitary transformation.

Finally, a general algorithm introduced by Reck et al. [62] has been presented in this section. Such an algorithm allows any finite dimensional unitary transformation to be constructed from a finite number of beam-splitters and phase-shifters.

### 3.2.1 Generalised 50/50 beam-splitter

A transformation that will recur throughout this thesis is the generalised 50/50 beam-splitter. Such a device takes a single photon at the input and distributes it evenly across all modes such that a measurement will find, with equal probability, the photon in any one of the \( N + 1 \) output modes. Conversely, if a single photon is detected in any one of the outputs, then it is equally likely to have come from any one of the input modes. Mathematically, I describe such a transformation by the unitary matrix \( \Omega(N + 1) \), with elements

\[
\Omega_{n,m} = \frac{\omega^{nm}}{\sqrt{N + 1}}, \tag{3.12}
\]
3.2. DISCRETE UNITARY TRANSFORMATIONS

Figure 3.1: Basic element of any general unitary transformation of mode operators. On the left is a beam-splitter with transmittancy $\cos \theta$ and phase-shifter generating a phase shift of $\phi$ in input mode 0. On the right is a shorthand notation for the elements on the left. Note that the blackened side contains the phase-shifter at the input.

Figure 3.2: A linear array of optical elements. Such an array is found to produce a general rotation $R(4)$ necessary to partially diagonalise a unitary matrix $U(4)$.

Figure 3.3: A general linear optical transformation of four modes. The first row of elements from the bottom performs the rotation $R(4)$, while the second row performs the rotation $R(3)$ and so on. The four phase-shifters, as well as offsetting any phase shift introduced by the mirror, provide the diagonal transformation $D$. 
where $\omega = \exp[i2\pi/(N + 1)]$ is the $(N + 1)^{th}$ root of unity. The input and output mode operators related by such a transformation then form a discrete Fourier transform (DFT) pair. As such, for the remainder of this thesis I will refer to the transformation described by Eqn (3.12) as a discrete Fourier transform (DFT) in $(N + 1)$-dimensions. As a working example to illustrate the iterative procedure of Reck et al. described in the preceding section, I will derive the DFT in 4-dimensions,

$$\begin{align*}
\Omega(4) &= \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -1 & 1 & -1 \\
1 & -i & -1 & i
\end{pmatrix},
\end{align*}$$ (3.13)

which will be used later in this thesis.

Following the procedure outlined above, the elements in the last row of $\Omega(4)$ are equated to the corresponding elements of the generalised unit vector in Eqn (3.5), giving the set of four equations

$$\begin{align*}
\frac{1}{2} &= \exp[i(\phi_0 + \pi/2 + \delta_3)] \sin \theta_0 \\
-i \frac{1}{2} &= \exp[i(\phi_1 + \pi/2 + \delta_3)] \sin \theta_1 \cos \theta_0 \\
-i \frac{1}{2} &= \exp[i(\phi_2 + \pi/2 + \delta_3)] \sin \theta_2 \cos \theta_1 \cos \theta_0 \\
\frac{1}{2} &= \exp[i\delta_3] \cos \theta_2 \cos \theta_1 \cos \theta_0
\end{align*}$$ (3.14)

that can be solved simply to give

$$\begin{align*}
\phi_0 &= \pi & \sin \theta_0 &= \frac{1}{\sqrt{4}} \\
\phi_1 &= \pi/2 & \sin \theta_1 &= \frac{1}{\sqrt{3}} \\
\phi_2 &= 0 & \sin \theta_2 &= \frac{1}{\sqrt{2}}
\end{align*}$$ (3.15)

with $\delta_3 = \pi/2$. The rotation matrix $R(4) = T_{3,0} \cdot T_{3,1} \cdot T_{3,2}$ can then be constructed from Eqn (3.11) using the values in Eqn (3.15), where the reflection coefficient $\sin \theta_i$ and phases $\phi_i$ are terms belonging to the matrix $T_{3,i}$. Indeed, the rotation matrix serves to partially diagonalise $\Omega(4)$

$$\begin{align*}
\Omega(4) &\cdot R^\dagger(4) = \\
&= \begin{pmatrix}
-1/\sqrt{3} & e^{-i\eta} \sqrt{5/12} & e^{-i\pi/4}/2 & 0 \\
-1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} & 0 \\
-1/\sqrt{3} & e^{i\eta} \sqrt{5/12} & e^{i\pi/4}/2 & 0 \\
0 & 0 & 0 & i
\end{pmatrix}
\end{align*}$$ (3.16)
3.2. DISCRETE UNITARY TRANSFORMATIONS

<table>
<thead>
<tr>
<th>$t_{ij}, \phi_{i,j}$</th>
<th>$j$</th>
<th>$\delta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>1/2, 0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$1/\sqrt{3}, \pi/2$</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$1/\sqrt{2}, \pi/2$</td>
</tr>
</tbody>
</table>

Table 3.1: Reflection coefficients $t_{ij} = \sin \theta_{i,j}$ and phases $\phi_{i,j}$ and $\delta_j$ associated with the transformation matrices $T_{ij}$ and $D$. The collection of optical elements described by this table is sufficient to realise the transformation $\Omega(4)$, where the parameter $\eta$ is defined as $\exp(i\eta) = (3i - 1)/\sqrt{10}$, leaving a matrix in a 3-dimensional sub-space, where $e^{i\eta} = (3i - 1)/\sqrt{10}$. Repeating this procedure a further two times is sufficient to find $R(3)$ and $R(2)$ thereby completely diagonalising $\Omega(4)$. Once this is completed, the DFT may be expressed as a product of transformation matrices according to Eqns (3.8) and (3.11) as

$$\Omega(4) = D \cdot T_{1,0} \cdot T_{2,0} \cdot T_{2,1} \cdot T_{3,0} \cdot T_{3,1} \cdot T_{3,2} \quad (3.17)$$

where the reflection coefficients $r_{ij} = \sin \theta_{i,j}$ and phases $\phi_{i,j}$ and $\delta_j$ associated with the transformation matrices $T_{ij}$ and $D$ are listed in Table 3.1.

It should be noted that the factorisation of Reck et al. is not unique. In fact, we showed in [59] how the 8-port interferometer used by Noh, Fougeres and Mandel [37, 38, 40, 23] to measure operational phase does, with some additional phase-shifters, reconstruct the DFT in 4-dimensions. Not only are the values of the transmission coefficients different for each beam-splitter, but the 8-port interferometer produces the desired transformation using only four beam-splitters, a reduction of two over the general technique.

For a general DFT in $(N+1)$-dimension, Törmä and Jex [74] have shown how plate beam-splitters\(^1\) can be used to decrease the number of optical elements necessary in the construction. In reference [74], it was shown how a general DFT can be constructed using plate beam-splitters, where the number of optical elements scales as $N+1$, where as the general technique requires $(N+1)^2$, a square root reduction.

\(^1\)A plate beam-splitter is an optical element similar to a beam-splitter, however, the transmittivity of the element varies along the length of the device. For a review see, for example, [74].
Figure 3.4: A multiport device capable of producing any retrodictive state that can be written as a superposition of $N + 1$ photon number states. The desired state is produced at the input mode 0 of the multiport when each photo-detector at the output registers a single photocount, except the one in mode 0, which detects the vacuum. The reference field $|\alpha_i\rangle$ in input mode $i$ is in a coherent state.

3.3 Retrodictive state engineering

Consider the apparatus illustrated in Figure 3.4; it is a multiport device consisting of $N + 1$ input ports and $N + 1$ output ports. In all output ports a photodetector capable of distinguishing between zero, one, and more than one photon is positioned. In $N$ of the input ports, a coherent state $|\alpha_m\rangle_m$ with $m = 1, 2, \ldots, N$ is present, where the label $m$ numbers the corresponding input port. For any photodetector reading at the detectors, there will be some retrodictive state that propagates backwards in time out of the free input port, which we label as port 0. For a given photocount pattern, the specific form of the retrodictive state will depend on the construction of the device and the amplitudes of the $N$ coherent input states. To controllably engineer different retrodictive states, both the construction of the device could be altered in addition to changing the amplitudes of the $N$ coherent reference states. However, to maintain an emphasis on a physically implementable measuring device, we will not consider altering the multiport device to engineer different retrodictive states, rather we shall consider the more practical alternative of altering the amplitudes of the $N$ coherent reference states. This leaves us with $N$ controllable parameters, so at most we could hope to engineer an arbitrary retrodictive state of $N + 1$ dimensions as in Eqn (3.2), where the additional free parameter is constrained by the normalisation condition.
3.3. RETRODICTIVE STATE ENGINEERING

By energy conservation, to obtain a retrodictive state in port 0 containing at most \( N \) photons we need to register a photocount sequence tallying \( N \) photocounts. As we wish to use photodetectors that can only discriminate between zero, one and many photocounts, resembling that of practical photodetectors, the only possible detection sequence available is when all \( N \) photodetector registers a single photocount and one detector, which we take without loss of generality to be \( D_0 \), detects no photocounts. For now we consider the photodetectors to have unit quantum efficiency and consider in Appendix A the implications of nonunit efficiency. For this case, the POM element for this detection event is simply the tensor product of the single mode POM elements,

\[
\hat{\Pi}(0) = |0\rangle_0\langle 0| \prod_{m=1}^{N} |1\rangle_m\langle 1|,
\]

where the index on the POM element labels the mode where no photocounts where detected. We can write this as a projector \( \hat{\Pi}(0) = |\Psi\rangle\langle \Psi| \), where

\[
|\Psi\rangle = |0\rangle_0 \prod_{m=1}^{N} |1\rangle_m
\]

is a multimode state. Following the formalism introduced in Chapter 2 we define a multi-mode retrodictive state at the output ports of the device as the MDO \(|\Psi\rangle\langle \Psi|\). We note that this state requires no renormalisation as its trace is already unity. The evolution of this state backwards through the multiport device can be represented by a unitary operator, \( \hat{S}^\dagger \), acting in the total Hilbert space of the \( N + 1 \) optical modes. We restrict the action of the device to linear transformations of the mode operators

\[
\hat{S}^\dagger \hat{a}_m^\dagger \hat{S} = \sum_{j=0}^{N} U_{mn}^* \hat{a}_n^\dagger
\]

where \( U_{mn} \) is an element of a unitary matrix. Such transformations were discussed in the preceding section, where it was shown how any finite dimensional unitary transformation of mode operators could be constructed from beam-splitters and phase-shifters.

At the entry of the device the retrodictive state \(|\Psi\rangle\langle \Psi|\) has evolved backwards in time

\[
\hat{\rho}^{\text{ret}} = \hat{S}^\dagger |\Psi\rangle\langle \Psi| \hat{S}.
\]

Also, the PDO describing the preparation event \( \hat{\Lambda} = \hat{\Lambda}_0 \otimes \hat{\Lambda}_{1,2,...,N} \) can be introduced, where

\[
\hat{\Lambda}_{1,2,...,N} = \prod_{m=0}^{N} |\alpha_m\rangle_m\langle \alpha_m|
\]
represents the $N$ coherent reference states and $\hat{\Lambda}_0$ is the PDO for mode 0. As discussed in Section 2.6.2 by redefining the boundaries between the preparation and measurement apparatuses we can interpret all of Figure 3.4 except input mode 0 as a measuring device. Generalising the derivation of Section 2.6.2 to $N + 1$ systems, we project the $N + 1$ mode retrodictive state $\hat{\rho}^{\text{ret}}$ onto the $N$ mode PDO $\hat{\Lambda}_{1,2,\ldots,N}$ to produce a single mode MDO for this enlarged measuring apparatus as,

$$\hat{\Gamma}_0(0) = \text{Tr} \left[ \hat{\rho}^{\text{ret}} \hat{\Lambda}_{1,2,\ldots,N} \right] = |\tilde{\psi}\rangle_0 \langle \tilde{\psi}|. \quad (3.23)$$

where the trace is taken over all modes except mode 0. It follows then that the normalised retrodictive state $\hat{\rho}^{\text{ret}}_0$, at the entry of the device in input port 0, generated by the measurement event associated with this MDO is $\hat{\Gamma}_0(0)/\text{Tr}[\hat{\Gamma}_0(0)]$. This propagates backwards in time from the measurement event. Because both $\hat{\rho}^{\text{ret}}$ and $\hat{\Lambda}_{1,2,\ldots,N}$ are not mixed, the single mode retrodictive state is a projection operator formed from the unnormalised state

$$|\tilde{\psi}\rangle_0 = \left( \prod_{m=1}^{N} \langle \alpha_m | \right) \hat{S}^\dagger |\Psi\rangle, \quad (3.24)$$

which is the (unnormalised) retrodictive state at the input port 0 of the device, propagating backwards in time away from the measurement event. Equation (3.23) can be viewed as the time reversal of the predictive case where some subsystem of the total system is measured, resulting in an (unnormalised) predictive state propagating forward in time representing the state of a smaller system.

To evaluate Eqn (3.24), we write the multimode retrodictive state in terms of the creation operators acting on the vacuum state

$$|\Psi\rangle = \left( \prod_{m=1}^{N} \hat{a}_m^\dagger \right) |0\rangle \quad (3.25)$$

where $|0\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \ldots |0\rangle_N$ is the multimode vacuum. The evolution of this state backwards in time through the optical elements results in a general multimode entangled state at the input of the device, given by

$$\hat{S}^\dagger |\Psi\rangle = \left( \prod_{j=1}^{N} \left( \hat{S}^\dagger \hat{a}_j^\dagger \hat{S} \right) \right) |\hat{S}^\dagger |0\rangle = \left[ \prod_{i=1}^{N} \left( \sum_{j=0}^{N} U_{ij}^* \hat{a}_j^\dagger \right) \right] |0\rangle \quad (3.26)$$

where, by energy conservation, the vacuum state transforms to the vacuum state and
3.3. RETRODICTIVE STATE ENGINEERING

Eqn (3.20) has been used to transform the mode operators. Projecting onto the coherent reference states gives the single mode retrodictive state as

\[ |\tilde{\psi}_0\rangle = \left( \prod_{j=1}^{N} \langle \alpha_j | 0 \rangle_j \right) \left[ \prod_{i=1}^{N} \left( U_{i0}^* \hat{a}_0^\dagger + \sum_{j=1}^{N} U_{ij}^* \alpha_j^* \right) \right] |0\rangle_0. \]  

(3.27)

For simplicity, we define the parameter \( \beta_i \), for \( i = 0, 1, \ldots, N \) as

\[ U_{i0} \beta_i = -\sum_{j=1}^{N} U_{ij} \alpha_j \]  

(3.28)

and

\[ \bar{\kappa} = \exp \left( -\frac{1}{2} \sum_{j=1}^{N} |\alpha_j|^2 \right) \prod_{i=1}^{N} U_{i0}^* \]  

(3.29)

which simplifies Eqn (3.27) to a form resembling that introduced in Section 3.1

\[ |\tilde{\psi}_0\rangle = \bar{\kappa} \left[ \prod_{i=1}^{N} \left( \hat{a}_0^\dagger - \beta_i^* \right) \right] |0\rangle_0, \]  

(3.30)

which can represent any state in a \( N+1 \) dimensional Hilbert space provided arbitrary control over the parameters \( \beta_i \) is available. It should be noted that \( \bar{\kappa} \), as opposed to \( \kappa \), is not a normalisation constant and is constrained by the elements of the unitary transformation matrix and the amplitudes of the coherent reference states as defined by Eqn (3.29).

To engineer a specific retrodictive state, we need to determine the amplitudes of the \( N \) coherent reference states \( \alpha_m \). To do this we need to invert Eqn (3.28). A simple way of doing this is to make use of the unitary nature of the matrix elements \( U_{ij} \) by incorporating two additional variables, \( \alpha_0 \) and \( \beta_0 \). To make sure this does not alter the set of \( N+1 \) equations we set the value of \( \alpha_0 \) to be zero. The set of equations can now be written as

\[ U_{i0} \beta_i = -\sum_{j=0}^{N} U_{ij} \alpha_j \quad \text{for } i = 0, 1, \ldots, N \]  

(3.31)

which can be inverted simply to give

\[ \alpha_j = -\sum_{i=0}^{N} U_{ij}^* U_{i0} \beta_i \quad \text{for } j = 0, 1, \ldots, N. \]  

(3.32)

The constraint on \( \alpha_0 \) imposes a restriction on the set of values taken by \( \beta_i \),

\[ \sum_{i=0}^{N} |U_{i0}|^2 / \beta_i = 0. \]  

(3.33)
As the \( N \) values of \( \beta_i, i = 1, 2, \ldots, N \), are predetermined by the retrodictive state that we wish to engineer, this equation serves to define the value of the additional variable \( \beta_0 \). We note, from Eqn \((3.32)\), that there is still enough flexibility in the \( N \) coherent amplitudes \( \alpha_j \) to generate any set of \( N \) values of \( \beta_i \).

So we find with control of \( N \) coherent reference states we can, upon detection of a specific photocount pattern, engineer any retrodictive state that can be sufficiently well approximated in an \( N + 1 \) dimensional Hilbert space. The unitary evolution of the device required is represented by a linear transformation of the mode operators and can be implemented physically by beam-splitters and phase-shifters alone. Surprisingly, almost any lossless multimode device can be used since the values of the coherent amplitudes can be adjusted accordingly.

### 3.3.1 State optimisation

While the above multiport device is capable of producing a large set of retrodictive states, sometimes only a particular state or states derived from it by, for example, a phase shift, is all that may be required. In such cases it is worth optimising the probability with which the apparatus can generate the desired state. Since the apparatus can be arranged so to produce any retrodictive state whenever the photocount sequence \((0, 1, \ldots, 1)\) is observed at the detectors \( D_0, D_1, \) etc, this problem is equivalent to maximizing the probability with which such a sequence is observed at the readouts of the detectors. This is just the probability of a measurement event for which the corresponding MDO is given by Eqn \((3.23)\).

This probability for any input state will be proportional to \(|\bar{\kappa}|^2\) as can be seen from \((3.30)\). So the problem of maximising the probability in which the apparatus can produce a single retrodictive state associated with the detection sequence \((0, 1, \ldots, 1)\), reduces to optimising the variable \(|\bar{\kappa}|^2\), where \(\bar{\kappa}\) is given by \((3.29)\).

It is rather a surprising result that almost any lossless multimode device can be used to construct a general retrodictive state. In fact, the only necessary condition that must be satisfied by the unitary evolution of the device is a photon entering the device from input port 0 must have a nonzero probability of exiting the device at each of the \( N + 1 \) output ports. This condition can be stated as \(|U_{i0}| \neq 0\) for all \( i \), and can be derived from Eqn \((3.29)\) where if \(|U_{i0}|\) was zero, for any \( i = 1, 2, \ldots, N \), the constant \(\bar{\kappa}\) would be zero. The probability for detecting the state \(|\psi\rangle\) would then be zero, meaning that this state can never
be detected with this apparatus. Also if $|U_{00}|$ were to be zero we would, from Eqn (3.33), lose the freedom in choosing one of the variables $\beta_i$ for $i = 1, 2, \ldots, N$ which would imply restrictions on the allowed retrodictive states that can be engineered.

In determining the value of $\bar{\kappa}$, it is necessary to evaluate the sum of the squares of the field strengths $\alpha_j$. From Eqn (3.32) this can be simply expressed as a linear combination of the square of the characteristic roots,

$$
\sum_{j=0}^{N} |\alpha_j|^2 = \sum_{i=0}^{N} |U_{i0}|^2 |\beta_i|^2,
$$

so we can explicitly write $|\bar{\kappa}|^2$ as

$$
|\bar{\kappa}|^2 = \exp \left( - \sum_{i=0}^{N} |U_{i0}|^2 |\beta_i|^2 \right) \cdot \prod_{i=1}^{N} |U_{i0}|^2.
$$

As the values of $\beta_i$ are set by the specific state we wish to engineer, it is interesting to note that $|\bar{\kappa}|^2$ depends only on the elements in the first column of the transformation matrix which we do have freedom in choosing, provided

$$
\sum_{i=0}^{N} |U_{i0}|^2 = 1 \quad \text{and} \quad |U_{i0}| \neq 0
$$

where the last condition summarises the argument given earlier. Optimising over these variables then implies

$$
\frac{\partial |\bar{\kappa}|^2}{\partial x_i} = 0, \quad i = 0, 1, \ldots, N
$$

where $x_i = |U_{i0}|^2$. Taking Eqn (3.36) as a constraint, this problem is best handled using Lagrange multipliers. The end result will be a set of $N + 1$ values for the transmission coefficients which will optimise the probability in which the apparatus can detect the state $|\psi\rangle$.

### 3.4 Retrodictive phase state with linear optics

We found that the probability of producing the retrodictive state associated with the detection sequence $(0, 1, \ldots, 1)$ is proportional to the constant $|\bar{\kappa}|^2$. We see from Eqn (2.5) that it also depends upon the input state to the apparatus. It is useful then to define a measure independent of the initial state that represents the efficiency of production of a retrodictive state. To construct such a measure, we first consider a von Neumann type measurement.
corresponding to the projection $|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is the normalised version of $|\tilde{\psi}\rangle$ given by (3.2). We take as our standard a von Neumann measuring device whose POM includes this projector. We define our measure $P_\psi$ then as the ratio of the probability of observing the photodetection sequence $(0,1,\ldots,1)$ to the probability of observing the outcome associated with the state $|\psi\rangle$ with the von Neumann measuring device $\Pr(\psi)$, with the same input state. From (2.5), (3.2) and (3.30), we see that this relative probability is equal to $|\bar{\kappa}/\kappa|^2$, where $\kappa$ is a normalisation constant defined in (3.2).

As mentioned earlier one of the applications of retrodictive state engineering is for projection synthesis. In this case the retrodictive state production efficiency $P_\psi$ has the following interpretation. When the generated retrodictive state is used as an MDO for a measurement of a particular observable, $P_\psi$ is the ratio of the probability of a successful detection event to the corresponding probability for the case where the POM element is an ideal von Neumann projector. Thus the relative probability $P_\psi$ is also a measure of the efficiency of the projection synthesis.

With this defined, we illustrate the effect the chosen transformation can have on the efficiency of the multiport device when used for projection synthesis. For this illustration, we calculate the efficiency $P_\psi$ of generating the retrodictive state $3^{-1/2}(|0\rangle + |1\rangle + |2\rangle)$ for three different multiport configurations. The first multiport considered is the simplest experimental configuration, consisting of three input ports and three output port formed by a linear array of two beam-splitters with a single mode in common. The t:r ratio of each beam-splitter was somewhat arbitrarily chosen as 1:1. The tradeoff between ease of construction and efficiency is evident in this design, giving the relative probability $P_\psi$ as a small 0.8%. This can be compared to the optimal configuration which is derived using Lagrange multipliers and can generate the retrodictive state with an efficiency $P_\psi$ of 14.9%. Such a configuration is compared, out of interest, to the DFT in 3-dimensions which is found to have an efficiency of 13.3%, almost as good as for the optimal configuration [61].

The procedure to generate the required retrodictive state is to detect one photon at $D_1$ and $D_2$, and no photons at $D_0$. Solving the characteristic polynomial for $\tilde{\psi}_0 = \tilde{\psi}_1 = \tilde{\psi}_2 = 1$ gives

$$\beta_1 = \frac{-1}{\sqrt{2}} + i\sqrt{\frac{1}{2} - 1/2}, \quad \beta_2 = \beta_1^*$$  (3.38)

The coherent state amplitudes $\alpha_1$ and $\alpha_2$ are then calculated from Eqn (3.32) once the unitary transformation $R$ is specified. The following three subsections outline the procedure.
3.4. RETRODICTIVE PHASE STATE WITH LINEAR OPTICS

Figure 3.5: The simplest optical arrangement to produce the retrodictive state proportional to \( |0\rangle + |1\rangle + |2\rangle \) at the input to mode 0. The reference fields \( |\alpha_1\rangle_1 \) and \( |\alpha_2\rangle_2 \) are in coherent states while photo-detectors \( D_0, D_1 \) and \( D_2 \) are in each output mode of the two 50/50 beam-splitters. When the photo count sequence (0,1,1) is detected at detectors \( D_0, D_1 \) and \( D_2 \) respectively, the desired retrodictive is produced.

of such a calculation for the three transformations mentioned above.

3.4.1 Simple configuration

Consider the two 50/50 beam-splitters, \( BS_1 \) and \( BS_2 \), illustrated in Figure 3.5. The beam-splitters are arranged such that one of the output modes of \( BS_1 \), labelled \( a_0 \), is the input mode to \( BS_2 \). Accordingly, the second input mode of \( BS_1 \) is labelled \( a_1 \) while the second input mode of \( BS_2 \) is \( a_2 \). To each of the three outputs a photocounter is positioned. Each photocounter is labelled \( D_0, D_1, D_2 \) corresponding to the appropriate mode which it is positioned.

As introduced in Section 3.2 above, the transformation matrix for \( BS_1 \), a symmetric 50/50 beam-splitter coupling modes \( a_0 \) and \( a_1 \), is written as

\[
T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & i & 0 \\
i & 1 & 0 \\
0 & 0 & \sqrt{2}
\end{pmatrix},
\] (3.39)

with \( \cos \theta = \sin \theta = 1/\sqrt{2} \) in Eqn (3.39). Similarly a transformation matrix \( T_2 \) for \( BS_2 \) can be assigned. Since the beam-splitters act in succession, the joint transformation the two beam-splitters have upon the three mode incident field is found by the matrix product of
T₂ and T₁ which is simply calculated to be

\[
U = T₂T₁ = \frac{1}{2} \begin{pmatrix}
1 & i \sqrt{2} \\
-i \sqrt{2} & \sqrt{2} & 0 \\
i & -1 & \sqrt{2}
\end{pmatrix}
\] (3.40)

With these values of |Uᵢ₀|², the value of |κ|² is calculated for this optical configuration from Eqn (3.35) to be 1.36 × 10⁻³, where the value of β₁ and β₂ is given above. This gives the relative probability Pₚ of producing the retrodictive phase state as 0.8%. The amplitudes of the coherent fields required at the inputs of this apparatus can be calculated from Eqn (3.32) to be \(α₁ = \frac{1}{\sqrt{2}}\) and \(α₂ = -i\sqrt{2} - 1\). This agrees with the results of Clausen et al. [20] who studied this particular example from a different viewpoint.

### 3.4.2 Optimal configuration

In general there will exist some 6-port configuration of optical elements that will optimise the probability in which the retrodictive truncated phase state will be produced. It will consist of at most six beam-splitters and an equal number of phase-shifters. To determine the best transformation describing such a configuration it is necessary to optimise the parameter |κ|² in Eqn (3.35) over the transformation matrix elements |Uᵢ₀|², i = 0, 1, 2.

For notational convenience we denoted these variables as \(xᵢ\) as introduced in Section 3.3.1. Writing β₁ = |β|eᵢθ, Eqn (3.35) becomes

\[
|κ|² = x₁x₂ \exp\left[-|β|²(x₁ + x₂) - \frac{|β|²}{x₀}(x₁² + x₂² + 2\cos 2θx₁x₂)\right]
\]

\[
= x₁x₂ \exp(Θ)
\] (3.41)

where

\[
Θ = -\sqrt{2}\left(x₁ + x₂ + \frac{x₁² + x₂² + 2\cos 2θx₁x₂}{x₀}\right)
\] (3.42)

with |β|² = \(\sqrt{2}\) and \(\cos 2θ = \frac{1}{\sqrt{2}} - 1\). The function to optimise, \(f(x₀, x₁, x₂) = |κ|²\), is a function of three variables which are subject to the normalisation constraint \(ϕ(x₀, x₁, x₂) = x₀ + x₁ + x₂ = 1\). A solution to such a problem is easily obtained by method of Lagrange multiplier [17]. The method involves finding the minimum of the function

\[
F(x₀, x₁, x₂) = f(x₀, x₁, x₂) + λϕ(x₀, x₁, x₂),
\] (3.43)
over the unknown variables \( x_i \), with \( \lambda \) referred to as a Lagrange multiplier. The minimum is obtained when \( \partial F / \partial x_i = 0 \) giving the following set of three equations,

\[
\begin{align*}
\frac{\partial F}{\partial x_0} &= f \frac{\partial \Theta}{\partial x_0} + \lambda = 0 \\
\frac{\partial F}{\partial x_1} &= f \frac{1}{x_1} + f \frac{\partial \Theta}{\partial x_1} + \lambda = 0 \\
\frac{\partial F}{\partial x_2} &= f \frac{1}{x_2} + f \frac{\partial \Theta}{\partial x_2} + \lambda = 0
\end{align*}
\]  

where

\[
\begin{align*}
\frac{\partial \Theta}{\partial x_0} &= \frac{\sqrt{2}}{x_0^2} (x_1^2 + x_2^2 + 2 \cos 2\theta x_1 x_2) \\
\frac{\partial \Theta}{\partial x_1} &= -\frac{\sqrt{2}}{x_0} (x_0 + 2x_1 + 2 \cos 2\theta x_2) \\
\frac{\partial \Theta}{\partial x_2} &= -\frac{\sqrt{2}}{x_0} (x_0 + 2x_2 + 2 \cos 2\theta x_1).
\end{align*}
\]

Subtracting Eqn (3.45) from Eqn (3.46) after substituting Eqns (3.48) and (3.49) gives the expression

\[
(x_1 - x_2) \cdot \left[ \frac{1}{x_1 x_2} + \frac{2(2\sqrt{2} - 1)}{x_0} \right] \cdot f = 0. 
\]  

(3.50)

It is not desirable for \( f = 0 \), as this would imply \( |\tilde{\kappa}|^2 = 0 \), giving the probability for a successful detection as zero. The variables \( x_i \) are defined to be non-negative, making it impossible for the term in the square brackets to be zero. The minimum is then achieved only when \( x_1 = x_2 \). By again subtracting, this time Eqn (3.44) from Eqn (3.45), the following third order polynomial in \( x_1 \) can be be derived

\[
(4\sqrt{2} - 2)x_1^3 - (4\sqrt{2} + 2)x_1^2 + (4 + \sqrt{2})x_1 - 1 = 0,
\]

(3.51)

where the normalisation constraint has been used to remove the variable \( x_0 \). Numerical methods provided a single real root for this expression in the domain \( 0 \leq x_1 \leq 0.5 \) as \( x_1 = 0.28205 \). The first column of the transformation matrix which optimises the probability in which a truncated phase state can be detected is then

\[
\begin{align*}
|U_{00}|^2 &= 0.4359_1 \\
|U_{10}|^2 &= 0.2820_5 \\
|U_{20}|^2 &= 0.2820_5.
\end{align*}
\]  

(3.52)
which gives the optimal efficiency $P_\psi$ of producing the retrodictive truncated phase state using any multiport configuration as 14.92%. It is interesting to note that the other six matrix elements along with the three unspecified phases of $U_{i0}$ are not necessary in optimising $P_\psi$. Since these elements are unspecified the physical realisation of the optimal optical multiport is not unique. As such these elements will generally be set by the experimenter when realising the particular optical configuration most suited to them provided Eqn (3.52) holds. Once realised, then the amplitudes of the necessary coherent reference states can be derived from Eqn (3.52).

### 3.4.3 Discrete Fourier Transform

It is interesting to note that there is an intimate relation between the truncated phase state that we are trying to generate (in retrodiction) and the photon number eigenstates that we are generating it from. In a 3-dimensional Hilbert space these sets of states form what is known as a discrete Fourier transform pair. That is they satisfy the condition

$$\langle n | \theta_m \rangle = \frac{\exp(i2\pi nm/3)}{\sqrt{3}}, \quad \text{with} \quad n, m = 0, 1, 2,$$

(3.53)

where $|n\rangle$ is a photon number eigenstate and $|\theta_m\rangle$ is a phase state in three dimensions. Both sets of states form an orthonormal basis in the three dimensional Hilbert space. With this in mind, it then is natural to consider how effective a discrete Fourier transform multiport device is at transforming the three mode photon number state $|0\rangle_0 |1\rangle_1 |1\rangle_2$ into a single mode truncated phase state.

The optical multiport of interest was introduced in Section 3.2.1 as a generalisation of a 50/50 beam-splitter. The matrix elements of the transformation matrix are given by Eqn (3.12) which are proportional to powers of $\omega = \exp(i2\pi/3)$. With the matrix elements defined, the probability in which a truncated phase state can be detected using this optical configuration is calculated to be 13.33% provided the amplitude of the two coherent reference states required at the inputs of the multiport are $\alpha_1 = -1.2591$ and $\alpha_2 = -0.1551$.

From the symmetry involved in such a transformation and the intimate relation existing between photon number states and truncated phase states it is not surprising that this optical configuration is near optimal. Perhaps what is even more surprising is that the DFT is not the optimal configuration as one might be tempted to think. A closer look at the physical process involved suggests why this is not the case.
The optical multiport, in conjunction with the non-unitary projection onto the coherent reference states, takes a 3-mode photon number state and transforms it into a single mode retrodictive phase state. The process is conditioned on observing the particular photocount pattern \((0, 1, 1)\) at the three photo-detectors. Such a phot-count pattern inadvertently introduces an asymmetry into the variable \(|\bar{\kappa}|^2\) which we aim to optimise, with respect to the transformation matrix elements \(|U_{i0}|^2\). Accordingly, the optimal optical arrangement needs to contain a bias in the transformation matrix elements to account for the inherent asymmetry introduced by the detection event. It is then clear why the unbiased DFT is not the optimal optical configuration. However, in the example considered in this section the variable \(|\bar{\kappa}|^2\) is symmetric with respect to the transformation matrix elements \(|U_{10}|^2\) and \(|U_{20}|^2\) leading to an optimal configuration where \(|U_{10}|^2 = |U_{20}|^2\) as derived in the preceding subsection. As the DFT satisfies this condition it is now clear why such a transformation is near optimal, but not exactly optimal.

3.5 Generalised measuring apparatus

In what has preceded, only the retrodictive states corresponding to particular photocount patterns were studied. As all detection events give rise to some retrodictive state there is a large number of retrodictive states not considered. In what follows, we generalise the scheme presented above to include all detection events and identify the class of retrodictive states that can be generated by a given multiport configuration.

Consider the optical multiport introduced in Section 3.3. Injected to all but input mode zero was a coherent reference state with a controllable amplitude and positioned at each of the \(N + 1\) output modes is a photocounting device. To study the potential generation of novel retrodictive states we now consider the idealised situation where the photocounting device can discriminate between 0, 1, 2, \ldots, \(M\) photons, where \(M\) is some arbitrary number. Take the situation where by the \(i\)-th photodetector, \(D_i\), registers \(n_i\) photons, with \(n_i \leq M\). Based on the outcome of this measurement event we assign the multimode retrodictive state

\[
|\Psi\rangle = \prod_{i=0}^{N} |n_i\rangle_i = \prod_{i=0}^{N} \left(\hat{a}_i^\dagger\right)^{n_i} \frac{n_i!}{\sqrt{n_i!}} |0\rangle,
\]

(3.54)

at the output of the multiport device. The reasoning of such an assignment follows that of the particular case introduced in Section 3.3. With the same unitary operator \(\hat{S}^i\) responsible
CHAPTER 3. RETRODICTIVE STATE ENGINEERING

for the evolution of this state backwards in time it becomes a straightforward calculation, akin to that presented earlier, to derive the single mode retrodictive state $|\tilde{\psi}\rangle_0$ as

$$|\tilde{\psi}\rangle_0 = \bar{\kappa} \left[ \prod_{i=0}^{N} \left( \hat{a}_0^\dagger - \beta_i^* \right)^{n_i} \right] |0\rangle$$  \hspace{1cm} (3.55)

resulting in a more general expression for $|\bar{\kappa}|^2$ as

$$|\bar{\kappa}|^2 = \exp\left( -\sum_{i=0}^{N} |U_{i0}|^2 |\beta_i|^2 \right) \cdot \prod_{i=0}^{N} \frac{|U_{i0}|^{2n_i}}{n_i!}. \hspace{1cm} (3.56)$$

The variable $\beta_i$ remains unchanged to that previously defined in Eqn (3.31) and is also subject to the same constraint given by Eqn (3.56) which is restated here for convenience as

$$\sum_{i=0}^{N} |U_{i0}|^2 \beta_i = 0. \hspace{1cm} (3.57)$$

By allowing for more general detection events we generalise the retrodictive states defined by Eqn (3.55) to include some states of higher dimensions. This happens in two ways. Firstly, by considering detection events other than zero photons at detector $D_0$ we introduce terms that are powers of $(\hat{a}_0^\dagger - \beta_0^*)$ into the retrodictive state. Since the parameter $\beta_0$ is constrained by Eqn (3.57) above this does not provide an additional degree of freedom necessary to produce arbitrary states in $N+2$ dimensions. However, by selecting particular transformation coefficients there is some freedom in choosing particular values of $\beta_0$ which will allow for the generation of some states in higher dimensions. Secondly, by allowing for multi-photon detection events at all other detectors we introduce powers in terms like $(\hat{a}_0^\dagger - \beta_i^*)$ which raise the dimensions of the retrodictive state space but do not introduce additional free parameters necessary to provide general states of higher dimensions. It is quite remarkable that the set of all possible retrodictive states that can be constructed by this multiport device are characterised by the set of parameters $\beta_i$ which are chosen subject to the constraint of Eqn (3.57). Even more remarkable is the ease at which these parameters are physically realised through the amplitudes of the coherent reference states and the chosen multiport device.

To illustrate how the apparatus can generate retrodictive states with higher number state components it will be shown how the state proportional to $|0\rangle - |N+1\rangle$ can be generated from an $N+1$ multiport device. Such a state is a specific example of a general quantum state in $N+2$ dimensions. This state is interesting not just for its extreme non-classical nature but also because it has applications in parameter estimation techniques. It
was shown in [87] that such a state, could it be produced, is necessary to estimate with minimum uncertainty the quantity $\theta = \omega t$ of a harmonic oscillator, where $\omega$ is the angular frequency of the oscillator and $t$ is the time over which the system oscillates.

Solving the characteristic polynomial for $\tilde{\psi}_0 = -\tilde{\psi}_{N+1} = 1$ gives the $N + 1$ solutions

$$\beta_n = \frac{2(N+1)}{(N+1)!} e^{in\tilde{\theta}}, \quad (3.58)$$

with $\tilde{\theta} = 2\pi/(N + 1)$ and $n = 0, 1, \ldots, N$. The solution allows the state to be written in factored form as

$$|0\rangle - |N + 1\rangle = \kappa \prod_{n=0}^{N} \left( \hat{a}^\dagger - |\beta| e^{in\tilde{\theta}} \right) |0\rangle \quad (3.59)$$

with $|\beta| = \frac{2(N+1)}{(N+1)!}$. By comparison to the general expression for a retrodictive state, Eqn (3.55), the desired retrodictive state is generated from this apparatus when each of the $N + 1$ photo-detectors detect a single photon.

With $\beta_i$ defined, Eqn (3.57) acts as a constraint on the matrix elements of the optical multiport device. To satisfy such a constraint it is necessary for the transformation to have matrix elements of equal magnitude, with the DFT being one such example. Taking the DFT to be the optical transformation, the coherent reference states required at the input of the multiport are $\alpha_n = -\delta_{n,1}|\beta|$. This equates physically to a simple configuration that has a vacuum state at the input of modes $2, 3, \ldots, N$ and a coherent state with an amplitude of $-|\beta|$ at input mode one. The efficiency with which this apparatus can generate such a retrodictive state is then

$$P_\psi = 2 \exp(-|\beta|^2) \frac{(N + 1)!}{(N + 1)^{N+1}}, \quad (3.60)$$

which unfortunately decreases exponentially with the number of dimensions $N + 1$.

In summary, what has been presented in this chapter is a general apparatus that can generate any retrodictive state which can be represented in a finite dimensional Hilbert space. Such an apparatus can be used to generalise the projection synthesis technique of Barnett and Pegg to include optical multiport devices. The most notable advantage of such a generalisation is to replace the non-classical reference state required in the original projection synthesis technique with simply prepared coherent reference states. Such a substitution allows for the generation of extremely non-classical retrodictive states from very classical coherent reference states. In addition, this technique only requires photo-detectors that can discriminate between none, one and many photons in a single mode which is more
like realistic detectors, where as the original technique requires some detection mechanism that can detect $N$ photons in a single mode. Overall, the technique presented here is far more practical than the original projection synthesis technique whilst still being able to generate all states in a Hilbert space with a dimension equal to the number of inputs, and even some states in a higher dimensional space than this.

Finally we note that the required $N$ coherent input states can be generated from a single coherent state field by means of a suitable linear array of beam-splitters with phase-shifters in their outputs and vacuum states in all remaining input ports. This array can be incorporated into the general multiport device giving a larger multiport that can generate the required retrodictive states with just one non-vacuum coherent state input. This provides a method of projection synthesis using just one coherent reference state replacing the exotic reference state of the original projection synthesis device 12.
Chapter 4

Simplifying experiments with exotic retrodictive probe states

Different measurement schemes aim to measure different properties of a quantum state of light. While some measurement schemes, such as homodyne \[65, 89\] and heterodyne \[82, 88\] detection, are physically realisable there are many which are not. Such proposals \[52, 68\], while still theoretically interesting, generally require some form of state preparation which is not achievable with present technology. However, while the list of predictive states which can be prepared in the laboratory is relatively small, the list of retrodictive states is significantly larger as we have seen in the previous chapter. The asymmetry in generating more exotic retrodictive states than the predictive counterpart is inherent in the ease at which a photon number state can be measured as opposed to prepared. With this in mind we ask the question: is it possible to modify these proposed experiments such that the necessary state preparation requires an exotic retrodictive quantum state rather than a predictive state, thereby making the scheme more realisable with current technology?

In this chapter we take two such proposals which are proving difficult to implement physically and redesign them to utilise the non-classical properties of retrodictive quantum states. The first such proposal, introduced by Pegg, Phillips and Barnett \[52\], was designed to measure the the phase variance of light. The scheme was novel but would be difficult to implement as it requires a two component probe field of the form \[c_0|0\rangle + c_1|1\rangle\]. Even using the quantum scissors device of \[51\] to generate such a state by truncating a coherent state is by no means trivial \[6\]. The other proposal is that of Steuernagel and Vaccaro \[68\] to
measure directly the density matrix element \( \rho_{N,N+\lambda} \) of light. Again, the proposal requires a two-component probe field of the form \(|0\rangle + |\lambda\rangle\) to perform the experiment. In general, no such predictive state is experimentally available.

In this chapter we propose a single experiment which is capable of measuring both the density matrix elements \( \rho_{N,N+\lambda} \) in addition to the phase moments \( \langle \cos(\lambda \theta) \rangle \) and \( \langle \sin(\lambda \theta) \rangle \) of light \(55, 57\). The experiment is simple, consisting of only two beam-splitters and one phase-shifter. Remarkably, the only predictive states required for this proposal are a vacuum state and a reference state which is easily derived from a coherent state. The experiment has been designed to utilise the non-classical features of retrodictive states thus removing the emphasis of the two component predictive probe field necessary in the previous proposals. Because of the inability of detectors to reliably discriminate between large photon numbers in short time intervals, this the scheme is practical only for relatively weak fields.

4.1 Phase variance

While there are some differences in various theoretical quantum descriptions of the phase of light, a common feature is that there should be some uncertainty relation between photon number and phase. Thus the quantum nature of phase should be manifest as an uncertainty, that is as a non-zero variance in the phase probability distribution. This uncertainty should be most pronounced for states of light with very small photon number variances as must pertain, for example, to states that do not differ very much from the vacuum. By contrast, strong coherent states of light, which approximate classical states, should have sharply defined values of phase. For this reason experimental investigations into the quantum nature of the phase of light \(25, 24, 37, 39, 38, 40\) have paid particular attention to finding the width of the phase distribution of states of light with low mean photon number. As the variance of the phase angle \(\theta\) itself depends critically on the \(2\pi\) window assigned to its range of values, such experiments are usually directed at measuring the phase cosine and sine variances \((\Delta \cos \theta)^2\) and \((\Delta \sin \theta)^2\). For small phase variances one might expect from expanding the classical series that \((\Delta \cos \theta)^2 + (\Delta \sin \theta)^2 \approx (\Delta \theta)^2\). Simple balanced homodyne techniques, sometimes referred to as phase measurements, can be used to obtain a distribution of a suitably defined operational, or measured, sine and cosine of phase.
For states with a small enough phase variance, this distribution can give a very good approximation to the canonical phase distribution \([78]\) where the canonical phase is defined as the complement of the photon number operator\(^1\) and can be described mathematically by the formalism in \([43, 10, 44]\). For weak fields in the quantum regime, however, which by necessity have broader phase distributions, significant divergences occur between the canonical and operational phase distributions. This is also true for the operational phase defined and measured by Noh et al. \([37, 39, 38, 40]\) whose distribution width for coherent states has a maximum divergence from that of the canonical distribution for mean photon numbers around unity. More recently other techniques have also been suggested that focus on measuring directly the phase properties of weak fields. These include projection synthesis \([12, 50]\) for measuring the canonical phase distribution and a two-component probe technique \([52]\) for measuring the canonical phase cosine or sine variance. As previously mentioned these techniques rely on engineering specifically tailored (predictive) probe states that, although possible in principle, will be very difficult in practice and have so far not been produced. Thus on one hand there are techniques that use easily prepared states but which do not measure the canonical phase variances and on the other there are techniques that measure canonical phase variances but rely on exotic quantum states.

In this section we examine the possibility of measuring the canonical phase cosine and sine variances of optical fields, with a particular interest in weak fields, by using input states which are easily produced in the laboratory, that is, coherent states. We find that, even though the two-component predictive probe states needed for the technique of \([52]\) are effectively not available at present, it is not difficult to use a retrodictive two-component probe state for our purposes.

### 4.1.1 Mean of the phase sine

Different quantum descriptions of phase will yield different values of \(\langle \cos(\lambda \theta) \rangle\) and \(\langle \sin(\lambda \theta) \rangle\). For canonical phase,

\[
\langle \cos(\lambda \theta) \rangle = \frac{1}{2} \sum_{p=0}^{\infty} (\rho_{p,p+\lambda} + \rho_{p+\lambda,p})
\]  

\[(4.1)\]

---

\(^1\)Equivalently, canonical phase can be defined as the quantity possessing a probability distribution that is invariant under a photon number shift. For further definitions in regards to defining canonical phase see, for example, \([34]\). We return to discussing various concepts of phase in section \([5.1.1]\).
and

\[ \langle \sin(\lambda \theta) \rangle = \frac{i}{2} \sum_{p=0}^{\infty} (\rho_{p,p+\lambda} - \rho_{p+\lambda,p}) \] (4.2)

where \( \rho_{n,m} \) are elements of the optical density matrix. One method of obtaining Eqns (4.1) and (4.2) is the limiting procedure of [13, 10, 14]. Here

\[ \langle \cos(\lambda \theta) \rangle = \lim_{s \to \infty} \langle \cos(\lambda \hat{\phi}_\theta) \rangle \] (4.3)

where \( \hat{\phi}_\theta \) is the Hermitian phase operator acting on a \((s+1)\)-dimensional Hilbert space. For a mixed state \( \hat{\rho} \),

\[ \langle \cos(\lambda \hat{\phi}_\theta) \rangle = \text{Tr} \left[ \hat{\rho}_s \cos(\lambda \hat{\phi}_\theta) \right] \] (4.4)

where \( \hat{\rho}_s \) is the truncation of \( \hat{\rho} \) onto the \((s+1)\)-dimensional subspace. Within this formalism, where the operators \( \cos \hat{\phi}_\theta \) and \( \sin \hat{\phi}_\theta \) commute, it is possible to derive the relations

\[ \langle \cos^2 \theta \rangle = \frac{1}{2} \left[ 1 + \langle \cos(2\theta) \rangle \right] \] (4.5)

which will be used later in calculating the variance in the phase sine and cosine. Equations (4.1) and (4.2) are the same results that would have been obtained by using

\[ \langle \cos(\lambda \theta) \rangle = \text{Tr} \left[ \hat{\rho} \hat{C}_\lambda \right] \] (4.6)

\[ \langle \sin(\lambda \theta) \rangle = \text{Tr} \left[ \hat{\rho} \hat{S}_\lambda \right] \] (4.7)

where the operators

\[ \hat{C}_\lambda = \frac{1}{2} \sum_{p=0}^{\infty} |p + \lambda \rangle \langle p | + |p \rangle \langle p + \lambda | \] (4.8)

\[ \hat{S}_\lambda = \frac{i}{2} \sum_{p=0}^{\infty} |p + \lambda \rangle \langle p | - |p \rangle \langle p + \lambda | \] (4.9)

act on the usual infinite-dimensional Hilbert space.

Our proposed measurement technique uses the beam-splitter arrangement shown in Figure 4.1. A controllable reference field in a coherent state \( |\alpha \rangle_2 = \sum_n a_n |n \rangle_2 \) is in the input mode 2 of a 50/50 symmetric beam-splitter \( BS2 \). The state of the system \( \hat{\rho}_0 \) to be measured is in the input mode 0 of beam-splitter \( BS1 \) and a vacuum state is in input mode 1 of \( BS1 \). The transmission and reflection coefficients of \( BS1 \) remain, for now, unspecified. Photon detectors \( D_2, D_0 \) and \( D_1 \) are in the output mode 2 and output mode 0 of \( BS2 \) and in the output mode 1 of \( BS1 \). We shall assume for now that these detectors
can count photons with perfect efficiency, no dark counts and negligible dead time. We address such an assumption in Appendix A where it is shown how to correct for some of these imperfections. It should be noted that this technique is similar in principle to the projection synthesis of [12, 50, 54] in which the unknown quantity is measured by observing specific relative frequencies of particular event in an experiment. As such, multiple copies of the state $\hat{\rho}_0$ will be necessary to obtain accurate sampling.

In the generalised measurement theory introduced in Chapter 2, a set of MDOs is assigned to describe the total measurement procedure which is a two-step process incorporating both measurement and recording of the desired outcome. With appropriate normalisation of the MDO corresponding to a particular measurement result, a retrodictive state can be assigned to the field immediately prior to the measurement event. This state evolves backwards in time until the field interacts with the preparation apparatus. The preparation apparatus is described in general by a set of PDOs with elements corresponding to possible outcomes of the preparation event. The probability of the joint preparation and measurement event occurring is given by the projection of the evolved PDO onto the associated MDO, that is, by the trace of the product of the MDO and the PDO. To avoid unnecessary complications, we assume the PDO we assign to the input fields of the device in Figure 4.1 describe the fields at their entry to the beam-splitters. The free evolution in the intermediate mode 0 between the beam-splitters only changes the phase of the field in this mode so, by choosing the distance between beam-splitters to be an integer number of wavelengths, we can ignore this evolution. In practice, even if this is not the case such a phase shift can
be compensated by adjusting the phase of the reference field $|\alpha\rangle_2$. Finally we can ignore the free evolution in all the output modes as these do not affect the photocount probabilities. We denote the total (forward time) unitary operator for the actions of beam-splitters $BS_1$ and $BS_2$ as $\hat{S} = \hat{S}_2 \otimes \hat{S}_1$ where $\hat{S}_2$ acts on states in modes 2 and 0, and $\hat{S}_1$ acts on states in modes 1 and 0.

Since the preparation procedure is always the same there is only one PDO, so we denote this by $\hat{\Lambda}$. The initial combined PDO for the three input field is the tensor product of the individual density matrices is then

$$\hat{\Lambda} = \hat{\rho}_0 \otimes |0\rangle_{11} \langle 0| \otimes |\alpha\rangle_{22} \langle \alpha|.$$  (4.10)

The MDO $\hat{\Pi}(n_0, n_1, n_2)$ for the detection of $n_0$, $n_1$ and $n_2$ photons in output modes 0, 1 and 2 respectively is, up to an arbitrary constant which is set to unity, equivalent to the POM element

$$\hat{\Pi}(n_0, n_1, n_2) = |n_0\rangle_{00} \langle n_0| \otimes |n_1\rangle_{11} \langle n_1| \otimes |n_2\rangle_{22} \langle n_2|.$$  (4.11)

as the detection event is unbiased. The probability for the detection of $n_0$, $n_1$ and $n_2$ photons at detectors $D_0$, $D_1$ and $D_2$ respectively is, from Eqn (2.6) with $\hat{\Gamma} = \hat{1}$,

$$Pr(n_0, n_1, n_2) = \text{Tr} \left[ \hat{\Lambda} \hat{S}_1^\dagger \hat{\Pi}(n_0, n_1, n_2) \hat{S}_2 \right],$$  (4.12)

where the trace is taken over all modes. Substituting from Eqns (4.10) and (4.11) this can be rewritten as

$$Pr(n_0, n_1, n_2) = \text{Tr}_0 \left[ \hat{\rho}_0 \hat{\Gamma}_0(n_0, n_1, n_2) \right].$$  (4.13)

Here $\hat{\Gamma}_0(n_0, n_1, n_2)$ is the MDO for the measurement event where the measuring device which is defined as everything in Figure 4.1 except the state to be measured, $\hat{\rho}_0$. This can be expressed as

$$\hat{\Gamma}_0(n_0, n_1, n_2) = n_0 \text{Tr}_{1,2} \left[ |0\rangle_{11} \langle 0| \otimes |\alpha\rangle_{22} \langle \alpha| \hat{S}_1^\dagger \hat{\Pi}(n_0, n_1, n_2) \hat{S}_2 \right],$$  (4.14)

where, from Eqn (4.11),

$$|\tilde{\psi}\rangle_0 = \text{Tr}_{1,2} \left[ |0\rangle_{11} \langle 0| \otimes |\alpha\rangle_{22} \langle \alpha| \hat{S}_1^\dagger \hat{\Pi}(n_0, n_1, n_2) \hat{S}_2 \right].$$  (4.15)

The state $|\tilde{\psi}\rangle_0$ is interpreted as the unnormalised retrodictive state of the field in input mode 0 associated with the detection of $n_0$, $n_1$ and $n_2$ photons. We can see from Eqn (4.15) that
the retrodictive fields \(|n_0\rangle_0\) and \(|n_2\rangle_2\) associated with the measurements in the output mode 0 and output mode 2 evolve backwards in time and are entangled by means of beam-splitter BS2. This entangled state is projected onto \(|\alpha\rangle_2\) to yield an unnormalised retrodictive probe state

\[
|q\rangle_0 = 2\langle\alpha|\hat{S}^\dagger_2|n_0\rangle_0|n_2\rangle_2
\] (4.16)
in the intermediate mode 0, that is, between the two beam-splitters. As we shall see later this is a retrodictive two-component state which performs a similar function to the predictive two-component probe of [52]. The state \(|q\rangle_0\) is entangled by beam-splitter BS1 with the retrodictive state from the measurement outcome of the detector D1. This state in turn is projected onto the vacuum in input mode 1 to give the unnormalised retrodictive state \(|\tilde{\psi}\rangle_0\) for projection onto the state to be measured. We remark here that if the state \(\hat{\rho}_0\) is a coherent state then in the predictive picture there is no entanglement at all because all input states are coherent. The entanglement mentioned above occurs only in retrodiction.

In addition to giving new insight, working in terms of the retrodictive probe state has practical calculational advantages. In our case, the fields that evolve backwards originate from single photon number states associated with the measurement outcomes in contrast to the fields that evolve forwards which contain multi-photon superpositions states.

The simplest possible retrodictive probe is associated with the detection event \(n_0 = n_2 = 0\). In this case we find that the retrodictive probe state \(|q\rangle_0\) is just the vacuum and so, with the detector D1 detecting \(n_1 = N\) photons, \(|\tilde{\psi}\rangle_0\) is just proportional to \(|N\rangle_0\). Thus only the diagonal matrix elements of \(\hat{\rho}_0\) are obtainable from the measured probabilities.

The next simplest retrodictive probes are associated with the measurement result \(n_2 = 0, n_0 = 1\) and \(n_2 = 1, n_0 = 0\). For a symmetric beam-splitter the output mode operators are related to the input mode operators by the unitary transformation of Eqn (3.9). For a 50/50 beam-splitter \(\theta = \pi/4\). Using this value for BS2 we easily find, by writing \(|1\rangle_0 = \hat{b}^\dagger_0|0\rangle_0\), the unnormalised retrodictive probe state for \(n_2 = 0, n_0 = 1\) to be

\[
\sqrt{2}|q\rangle_0 = a^*_0|1\rangle_0 + ia^*_1|0\rangle_0
\] (4.17)

When detector D1 registers \(n_1 = N\) photons, this two-component retrodictive probe evolves backwards through BS1 and becomes entangled. In retrodiction, the detection of \(N\) photon in the output mode 1 of BS1 acts like a photon source to the retrodictive probe state. The result of the input mode being in a vacuum state is to raise the photon occupation number
in the state $|q\rangle_0$ by $N$. This can be seen explicitly from Eqn (4.15), where it is shown in the Appendix B that the operator $\hat{S}_1^+|N\rangle_1$ is proportional to the raising operator to the $N$th power. Using this expression it is trivial to derive the retrodictive probe state at the input to mode 0 as

$$\sqrt{2}|\tilde{\psi}\rangle_0 = (ir)^N \left(ia_1^*|N\rangle_0 + ta_0^*\sqrt{N+1}|N+1\rangle_0\right)$$

where $t = \cos \gamma$ and $r = \sin \gamma$ are the transmission and reflection coefficients of $BS_1$ respectively.

This in turn allows the MDO $\hat{\Gamma}_0(1, N, 0)$ to be calculated from Eqn (4.14) and hence, from Eqn (4.13), the measurable probability

$$\Pr(1, N, 0) = \frac{1}{2}r^{2N} \left[|a_1|^2\rho_{N,N} + t^2|a_0|^2(N+1)\rho_{N+1,N+1} + \left(i\rho_{N+1,N}a_0a_1^*\sqrt{N+1}\rho_{N+1,N} + c.c\right)\right]$$

From this we see that the measured probability depends upon the first off-diagonal matrix elements $\rho_{N+1,N}$, and its complex conjugate, of the input field $\hat{\rho}_0$. Unfortunately, is also depends upon the diagonal elements $\rho_{n,n}$. To remove this dependence and obtain a measurable value for $\langle \sin \theta \rangle$ in Eqn (4.1) we also measure the probability in which $n_2 = 1$, $n_0 = 0$ and $n_1 = N$ photon are detected. We find this occurs with a probability given by

$$\Pr(0, N, 1) = \frac{1}{2}r^{2N} \left[|a_1|^2\rho_{N,N} + t^2|a_0|^2(N+1)\rho_{N+1,N+1} - \left(i\rho_{N+1,N}a_0a_1^*\sqrt{N+1}\rho_{N+1,N} + c.c\right)\right].$$

For the first experiment we choose the phase of coherent reference state $|\alpha\rangle_2$ so that $a_n$ are real and positive. From Eqns (4.19), (4.20) and (4.22) an expression for the mean of the sine in terms of the measurable probabilities can then be written as

$$\langle \sin \theta \rangle = \sum_N \frac{\Pr_0(0, N, 1) - \Pr_0(1, N, 0)}{2tr^2N|a_0a_1|\sqrt{N+1}}$$

where the subscript on the probability refers to the first experiment.

### 4.1.2 Variance of the phase cosine and sine

The next simplest possible retrodictive probe originates from the measurement event $n_0 = n_2 = 1$. Again for the 50/50 beam-splitter $BS_1$ we have $\theta = \pi/4$. Writing $|1\rangle_0 = \hat{b}_0^+|0\rangle_0$ and $|1\rangle_2 = \hat{b}_2^+|0\rangle_2$ we obtain from Eqn (3.9), the two-component retrodictive probe state

$$-i\sqrt{2}|q\rangle_0 = a_0^*|2\rangle_0 + a_2^*|0\rangle_0$$
leading to
\[
- i \sqrt{2} \hat{\psi}_0 = \langle ir \rangle^N \left( a_2^* |N\rangle_0 + a_0^* t^2 \sqrt{(N+1)(N+2)/2} |N+2\rangle_0 \right). \quad (4.23)
\]

From Eqns (4.13) and (4.14), this gives the probability of detecting a single photon at detectors $D_0$ and $D_2$ and $N$ photons at $D_1$ as
\[
Pr(1, N, 1) = \frac{1}{2} r^2 N \left[ |a_2|^2 \rho_{N,N} + t^4 (N+2)(N+1)/2 |a_0|^2 \rho_{N+2,N+2}
+ \left( a_0 a_2^* t^2 \sqrt{(N+2)(N+1)/2} \rho_{N+2,N} + c.c \right) \right]. \quad (4.24)
\]

To allow the experiment with this probe to be conducted simultaneously with the experiment to find $\langle \sin \theta \rangle$, we take $a_0$ and $a_2$ to be real and positive and find $Pr_0(1, N, 1)$ by simply replacing $a_0 a_2^*$ in Eqn (4.24) by $|a_0 a_2|$. After measuring the probabilities $Pr_0(1, N, 0)$, $Pr_0(0, N, 1)$ and $Pr_0(1, N, 1)$ the experiment is repeated with a phase shift of $\pi/2$ in the reference state $|\alpha\rangle_2$, which has the effect of changing $a_n$ to $a_n \exp(i n \pi/2)$. Thus now $a_0 = |a_0|$, $a_1 = i |a_1|$, and $a_2 = - |a_2|$ in Eqns (4.19), (4.20) and (4.24), yielding $Pr_1(1, N, 0)$, $Pr_1(0, N, 1)$ and $Pr_1(1, N, 1)$. From Eqn (4.1) we can then obtain the mean phase cosine from the measured results as
\[
\langle \cos \theta \rangle = \sum_N \frac{Pr_1(1, N, 0) - Pr_1(0, N, 1)}{2 t^2 r^2 N |a_0 a_1| \sqrt{N+1}}. \quad (4.25)
\]

We also find from (4.1) that
\[
\langle \cos(2\theta) \rangle = \sum_N \frac{Pr_0(1, N, 1) - Pr_1(1, N, 1)}{2 t^2 r^2 N |a_0 a_2| \sqrt{(N+1)(N+2)/2}}. \quad (4.26)
\]

After these values are obtained from the measured probabilities, the mean square phase cosine can be found from Eqn (4.5) and finally the phase cosine variance calculated as $\langle \cos^2 \theta \rangle - \langle \cos \theta \rangle^2$. Further, we can also write from the phase formalism of [43, 10, 44]
\[
\langle \sin^2 \theta \rangle = \frac{1}{2} \left[ 1 - \langle \cos(2\theta) \rangle \right], \quad (4.27)
\]
which allows us also to find the phase sine variance from the measured probabilities.

We have already assigned a value to the phase of the reference state $|\alpha\rangle_2$ but still have freedom to choose its mean photon number $|\alpha|^2$. To avoid having to renormalise very small probabilities, it is worth maximising the denominator, and hence the numerator, in Eqns (4.24) and (4.26). Thus we should choose a reference state to maximise $|a_0 a_1|$ and for Eqn (4.26) we should maximise $|a_0 a_2|$. The former and latter are maximised for
mean photon numbers of 0.5 and 1 respectively. The experiment could in principle be run for both these values but in practice it would be simpler just to use a compromise value between 0.5 and 1. We have yet to choose the reflection to transmission ratio of the beamsplitter $BS_2$. Again it is useful to choose a ratio which maximises the denominators of the terms in Eqn (4.21). The optimum value of the reflection coefficient $\sin \gamma$ for each term is $(1 + 2N)^{-1/2}$. For Eqn (4.26) the optimum value of $\sin \gamma$ for each term is $(1 + N)^{-1/2}$. If necessary the experiment could be repeated for different values of $N$ but, given we are mainly interested in weak fields, the spread in values of $N$ should not be huge. Thus a compromise value of around $\langle n \rangle$ should be adequate for determining both Eqns (4.21) and (4.26), where $\langle n \rangle$ is the mean photon number of the field to be measured. Thus for fields with a mean photon number around unity a 50/50 beam-splitter would be quite suitable. For stronger fields an increase in the transmission of $BS_2$ would be desirable.

The above measurable quantities can also be obtained by means of a two-component probe field technique suggested in [52]. There are very important differences however. In [52], where states are assigned to the probe fields according to the usual predictive quantum formalism, the required probe states are, as acknowledged in that paper, very difficult to prepare. The preparation method suggested for [52] was optical truncation using quantum scissors [51, 13], so the measurement would require three beam-splitters in all, with separate experiments being run with each different probe. More seriously, the preparation of the probe in a one-photon and vacuum superposition in [52] requires the injection of a single photon state into one input port and the probe in a two-photon and vacuum superposition requires the simultaneous injection of a single photon into two input ports. In contrast to the technique suggested in [52], the method proposed here has real practical advantages. Only two beam-splitters are used and, apart from the state to be measured, the only states injected into other input ports are vacuum and coherent states. These are not only considerably easier to prepare, their coherence lengths can allow longer gating times, reducing the effect of dead times. The retrodictive one-photon states needed to construct the retrodictive probe originate from photon detection events and are thus more readily available than their predictive counterparts which originate from preparation events. Further, because the retrodictive probe states are produced by the detection events, all three probe states, including the retrodictive vacuum state used for measuring the diagonal density matrix elements, are produced in the one experiment. There is no need to run
It is interesting to compare this approach with the retrodictive analysis of the quantum scissors device \[51\]. In the retrodictive picture, the state to be truncated is in one input mode of a beam-splitter with detectors in the two output modes. When one of these detects one photon and the other detects zero photons, the retrodictive state in the other input mode is a superposition of the vacuum and one photon states, so the actual cutting out of the higher photon state components is done at this beam-splitter. The other beam-splitter of the quantum scissors creates a predictive entangled state. Projection of the retrodictive state onto this state effectively converts the retrodictive state into a predictive state with the coefficients of the vacuum and one photon components interchanged. The beam-splitter \(BS2\) in Fig. 4.1 can be regarded as the part of the quantum scissors that creates the retrodictive two-component state. As we wish to use this retrodictive probe directly, there is no need to employ another beam-splitter to convert it to a predictive probe. This also dispenses with the necessity to produce and inject single-photon fields. Finally, the insight provided by the retrodictive formalism of quantum mechanics has enabled us to propose a relatively simple experiment to measure some of the canonical phase properties of light. There is now less need to define separate operational phase properties based on easily performed experiments.

### 4.2 Higher order phase moments

For any given distribution the variance only provides a single measure of the spread of that distribution. For states of light such as \(c_0|0\rangle + c_1|1\rangle + c_2|2\rangle\) which contain only diagonal, off-diagonal and next off-diagonal elements in the density matrix, when expressed in the photon number basis, we see from Eqns (4.1) and (4.2) that the first two moments are the only non-zero measures which can be acquired about the distribution. In general, however, knowledge of higher order moments is necessary if a more complete understanding is required of the distribution. Indeed, the set of all such moments provides enough information to reconstruct the complete probability distribution \[57\], which, for a continuous distribution, is generally an infinite set. Although such a reconstruction could be done in principle provided all such moments could be measured, practically, it would be a very tedious way to acquire the probability distribution. Instead, it would be more practical if these moments were used as
a measure to compare directly between different states of light, particularly if there were a simple procedure to measure these moments in practice. Of course if it were found from the photon statistics during the experiment that the field is truncated or sufficiently weak then only a small number of moments would be needed to construct the phase probability.

In this section we show how the higher order moments \( \langle \cos(\lambda \theta) \rangle \) and \( \langle \sin(\lambda \theta) \rangle \) can be measured using the same apparatus as introduced in the previous section. The only modification necessary to the experiment is to replace the coherent reference state in input mode 2 with a mixed reference state \( \hat{\rho}_2(\lambda) \). For now we do not specify the form of \( \hat{\rho}_2(\lambda) \) but we do remark that such a state can be derived easily from the original coherent reference state with the preparation technique dependent upon the particular moment which is being measured. We write the modified PDO for the new experiment as,

\[
\hat{\Lambda} = \hat{\rho}_0 \otimes |0\rangle_1 \langle 0| \otimes \hat{\rho}_2(\lambda).
\]

(4.28)

Since the remainder of the experiment is unchanged, the probability in which we observe the detection event \((n_0, n_1, n_2)\) at detectors \(D_0, D_1\) and \(D_2\) respectively is given by Eqn (4.13), with the obvious replacement of \(\hat{\Lambda}\) above. After substituting Eqns (4.11) and (4.28) for the MDO and PDO respectively, we can derive the same simplified expression for the probability as given in Eqn (4.14), where now the MDO, \(\hat{\Gamma}_0\), which, when normalised represents the retrodictive state propagating backwards in time out of input mode 0, is mixed

\[
\hat{\Gamma}_0(n_0, n_1, n_2) = 1 \langle 0| \hat{S}_1^n |n_1\rangle_1 \text{Tr}_2 [\hat{\rho}_2(\lambda)|z\rangle \langle z|]_1 \langle n_1| \hat{S}_1^n |0\rangle_1,
\]

(4.29)

where \(|z\rangle = \hat{S}_2^n |n_0\rangle|n_2\rangle_2\) is defined for notational convenience. It can be seen from Eqn (4.29) that the mixing arises because the coherent reference state \(|\alpha\rangle_2\) has been replaced by a mixed state \(\hat{\rho}_2(\lambda)\). Interestingly, it turns out that the mixing is a necessary feature of this measurement procedure as opposed to most measurement techniques which go to great lengths to preserve the purity in their systems. With \(|z\rangle\) defined above as the two-mode retrodictive state at the input of \(BS_2\), we can interpret the single mode operator \(\text{Tr}_2 [\hat{\rho}_2(\lambda)|z\rangle \langle z|]\) which is the projection of \(|z\rangle\) onto the mixed reference state in input mode 2 as the unnormalised retrodictive state \(\hat{\rho}_q^{\text{ret}}\) in the intermediate mode 0. That is the state which propagates backwards in time away from the input mode 0 of \(BS_2\) and is incident on output mode 0 of \(BS_1\). The subscript \(q\) is attached as a reminder that this state is a mixed generalisation of the retrodictive probe state \(|q\rangle_0\) in Eqn (4.16). With the remainder
of the experiment unchanged, the retrodictive state $\hat{\rho}_q^{\text{ret}}$ undergoes the same non-unitary evolution at $BS_1$ as did the state $|q\rangle_0$ when the reference state was pure.

Previously in measuring $\langle \cos \theta \rangle$, the retrodictive probe state $|q\rangle_{00}$ was a two-component probe state engineered to contain only diagonal and first off-diagonal term in the density matrix, while when measuring the second cosine moment $\langle \cos(2\theta) \rangle$, only the diagonal and second off-diagonal terms in $|q\rangle_{00}$ were non-zero. In keeping with this trend we are going to require a retrodictive probe state $\hat{\rho}_q^{\text{ret}}$ which contains only diagonal and $\lambda$ off-diagonal matrix elements if we are to generalise this technique to measure $\langle \cos(\lambda \theta) \rangle$. We find that there are two necessary steps required to generate such a retrodictive state. First, a total of $\lambda$ photons need be detected across both detectors $D_0$ and $D_2$ such that $n_0 + n_2 = \lambda$. The specific sequence in which this happens is, for now, not important. What is important is that a total of $\lambda$ photons be incident on the output of $BS_2$ and evolve backwards through the beam-splitter. From energy conservation the retrodictive state $|z\rangle$ at the entry of $BS_2$ will be, in general, some linear combination of all two mode photon number states that sum to $\lambda$,

$$|z\rangle = S_2^\dagger|n_0\rangle_0|\lambda - n_0\rangle_2 = \sum_{m=0}^{\lambda} z_m |n\rangle_0 |\lambda - m\rangle_2.$$  \hspace{1cm} (4.30)

In addition to this we require the probe field $\hat{\rho}_2(\lambda)$ to be of the form

$$\hat{\rho}_2(\lambda) = \sum_n \varrho_{n,n} |n\rangle_{22} \langle n| + \left( \sum_n \sum_{m \geq \lambda} \varrho_{n,n+m} |n\rangle_{22} \langle n+m| + \text{H.c.} \right)$$  \hspace{1cm} (4.31)

which is a state with the first $(\lambda - 1)$ off-diagonal matrix elements as zero. A specific example of such a state would be $|0\rangle + |\lambda\rangle$ which is an extremely non-classical state and therefore very difficult to produce. Alternatively, we could produce such a state as an equal mixture of coherent states $|\alpha_j\rangle$ with $\alpha_j = |\alpha| \exp(i2\pi j/\lambda)$ as

$$\hat{\varrho}(\lambda) = \frac{1}{\lambda} \sum_{j=0}^{\lambda-1} |\alpha_j\rangle \langle \alpha_j| = \sum_{n,m} \delta_{\lambda,n-m} \varrho_{n,m} |n\rangle \langle m|,$$  \hspace{1cm} (4.32)

where $\delta_{\lambda,n-m} = \frac{1}{\lambda} \sum_{j=0}^{\lambda-1} \exp[i(n-m)2\pi j/\lambda]$ is a periodic Kronecker delta function and $\varrho_{n,m} = \langle n|\alpha_0\rangle \langle \alpha_0| m\rangle$. The last line in Eqn (4.32) can be derived by expressing $|\alpha_j\rangle$ in the number state basis and summing over $j$. The periodic Kronecker delta function is zero unless $(n - m)$ is an integer multiple of $\lambda$, in which case it is one. So, by mixing coherent
states of selected values of $\alpha_j$ all matrix elements except the leading diagonal, the $\lambda$\textsuperscript{th} off-diagonal, the $(2\lambda)$\textsuperscript{th} off-diagonal and so on in Eqn (4.32) average to zero. This is precisely the form required of the reference state. To achieve this state in practice, one would initially run the experiment with a coherent state of zero phase, $|\alpha_0\rangle$. To obtain reliable statistics the experiment needs to be run many times. Each time the experiment is run the phase of the coherent reference state is adjusted by $2\pi/\lambda$. To ensure even sampling, the experiment would need to be run an equal number of times for each coherent reference state $|\alpha_j\rangle$. To ensure the reference state is then mixed, the statistics of the experiments are compiled without discriminating between the different values of $\alpha_j$ of the coherent reference states.

It should be mentioned that the state $\hat{\rho}_2(\lambda)$ in Eqn (4.32) need not be derived from a pure coherent state. In general, this mixing procedure will remove selected off-diagonal elements in any general state. So if the initial coherent state is mixed to begin with, as may often be the case in experiments, then this procedure will still produce a valid reference state.

With these two requirements satisfied we can evaluate the projection of reference state $\hat{\rho}_2(\lambda)$ onto the two-mode retrodictive state $|z\rangle$ as

$$\hat{\Gamma}_q = \text{Tr}_2 [\hat{\rho}_2(\lambda)|z\rangle\langle z|] = \sum_{n=0}^{\lambda} |z_n|^2 \hat{\rho}_{\lambda-n,\lambda-n}|n\rangle_{00}\langle n| + (z_0 z_\lambda^* \hat{\rho}_{0\lambda}|0\rangle_{00}\langle \lambda| + H.c.) \quad (4.33)$$

which is the unnormalised expression for the mixed retrodictive probe state $\hat{\rho}_q^\text{ret} = \hat{\Gamma}_q / \text{Tr}[\hat{\Gamma}_q]$. From this expression we see that the necessary off-diagonal element $|0\rangle_{00}\langle \lambda|$ and the conjugate are apparent in the retrodictive probe state, while all other off-diagonal elements are not. To evaluate the final retrodictive state at the input mode 0 of BS\textsubscript{1} we follow the evolution of the intermediate retrodictive probe state backwards through BS\textsubscript{1}. The evolution, which is conditioned on detector $D_1$ detecting $n_1 = N$ photons while zero photon are in the input, can be viewed as a non-unitary transformation of the intermediate retrodictive probe state. The result, identical to the that presented in the preceding section, is to raise the photon occupation number of the intermediate retrodictive probe state by $N$ photons. This can be explicitly derived from Eqn (4.29), using the expression for $1\langle 0|\hat{S}_1^+|n_1\rangle_1$ derived in Appendix B to be

$$\hat{\Gamma}_0 = \sum_{n=0}^{\lambda} \sigma_{n,m}|N+n\rangle_{00}\langle N+n| + (\sigma_{0,\lambda}|N\rangle_{00}\langle N + \lambda| + H.c.) \quad (4.34)$$

where the constant

$$\sigma_{n,m} = r^{2N} t^{n+m} z_n^* s_m \hat{\rho}_{\lambda-n,\lambda-n} \left( \frac{N+n}{n} \right)^{1/2} \left( \frac{N+m}{m} \right)^{1/2} \quad (4.35)$$
4.2. HIGHER ORDER PHASE MOMENTS

is introduced for notational convenience and $t = \cos \gamma$ and $r = \sin \gamma$ are the transmission and reflection coefficients of $BS_1$. The probability of detecting $n_0$, $N$ and $(\lambda - n_0)$ photons at detectors $D_0$, $D_1$ and $D_2$ is then given by the overlap of the total MDO, $\hat{\Gamma}_0$, and the state to be measured,

$$\Pr_0(n_0, N, \lambda - n_0) = \sum_{n=0}^{\lambda} \sigma_{n,n} \rho_{N+n,N+n} + (\sigma_{0,\lambda} \rho_{N+\lambda,N} + c.c.),$$  (4.36)

where $\rho_{n,m} = \langle n | \hat{\rho}_0 | m \rangle_0$ is the matrix coefficient of the state to be measured. A label is attached to the probability to indicate that this is the first experiment. To extract the off-diagonal terms from the probability we need to remove the diagonal terms in Eqn (4.36). This is done by repeating the experiment with a $\pi/\lambda$ phase shift applied to the input mode 2 of $BS_2$. The effect is to alter the phase of the mixed reference state $\hat{\rho}_2(\lambda)$ such that the element $\rho_{n,m}$ transforms to $\exp[i(n-m)\pi/\lambda] \rho_{n,m}$, thereby changing $\sigma_{0,\lambda}$ to $-\sigma_{0,\lambda}$. The probability of such an outcome with this phase shift is then

$$\Pr_1(n_0, N, \lambda - n_0) = \sum_{n=0}^{\lambda} \sigma_{n,n} \rho_{N+n,N+n} - (\sigma_{0,\lambda} \rho_{N+\lambda,N} + c.c.)$$  (4.37)

where the subscript denotes this as the second experiment. If we set the phase of the mixed reference field to offset any phase shift induced by $BS_1$ such that $z_0 z_0^* \rho_{0,\lambda}$ is real and positive, then by comparison to Eqn (4.1), we find that there is sufficient information to obtain a measured value for the $\lambda^{th}$ cosine moment from the statistics of these two experiments as

$$\langle \cos(\lambda \theta) \rangle = \sum_{N} \frac{\Pr_0(n_0, N, \lambda - n_0) - \Pr_1(n_0, N, \lambda - n_0)}{4 \sigma_{0,\lambda}}.$$  (4.38)

To obtain a measured value for the $\lambda^{th}$ sine moment we need to repeat the procedure a further two times, once with a phase shift of $\pi/(2\lambda)$ applied to the reference field in mode 2 and then again with a phase shift of $3\pi/(2\lambda)$. Writing the probabilities for the outcomes of such experiments as $\Pr_{1/2}(n_0, N, \lambda - n_0)$ and $\Pr_{3/2}(n_0, N, \lambda - n_0)$ respectively, we can obtain an expression for $\langle \sin(\lambda \theta) \rangle$ in Eqn (4.2) in terms of measured probabilities as

$$\langle \sin(\lambda \theta) \rangle = \sum_{N} \frac{\Pr_{1/2}(n_0, N, \lambda - n_0) - \Pr_{3/2}(n_0, N, \lambda - n_0)}{4 \sigma_{0,\lambda}}.$$  (4.39)

It can be seen from the two equations above that the observed probabilities need to be rescaled before a measured value of the sine and cosine moments can be obtained. If the
scaling factor is large then the measured probabilities will be small in which case a large number of experiments will need to be performed before reliable statistics can be obtained from the data. To avoid this the denominator in Eqn (4.38) and (4.39), \( \sigma_0, \lambda \), should be optimised over all free parameters thereby minimising the number of experiments needed for a given level of accuracy. This can be done in two ways. To begin with we consider the optimum detection sequence at detectors \( D_0 \) and \( D_2 \).

The scaling factor \( \sigma_0, \lambda \) is proportional to \( z_0 z_\lambda^* \) which are the coefficients of the multimode retrodictive state \( |z\rangle \). Although an explicit form of \( z_m \) in Eqn (4.30) is difficult to evaluate for a general detection sequence, it is not so difficult to calculate the case when \( m = 0, \lambda \).

By writing \( |n_0\rangle_0 = (n_0!)^{-1/2} (\hat{b}_0^\dagger)^{n_0} |0\rangle \) in Eqn (4.30) and similarly for \( |\lambda - n_0\rangle_2 \) we can obtain from Eqn (3.9) the expression

\[
    z_0 z_\lambda^* = (-i^{tr})^{\lambda} (-1)^{n_0} \binom{\lambda}{n_0}.
\]

(4.40)

It is straightforward to see that this is optimised when \( t = r = 2^{1/2} \) and \( n_0 \), the number of photons detected at \( D_0 \), is \( \lambda/2 \) when \( \lambda \) is even and \( (n \pm 1)/2 \) when \( \lambda \) is odd. Unfortunately, even with the optimised variables the scaling factor still scales exponentially with \( \lambda \) as can be seen from Eqn (4.40) with \( (tr)^{\lambda} = 2^{-\lambda} \). One way of avoiding this is to include all detection events at detectors \( D_0 \) and \( D_2 \) that sum to \( \lambda \), not just the single optimum case mentioned above. This is a more efficient process since we are not discarding data that can potentially be used to measure the moments. To see just how much of an improvement can be obtained we rewrite the coefficient \( \sigma_{n,m} \) in Eqn (4.35) as

\[
    \sigma_{n,m} = g_{n,m} (2i)^{-\lambda} (-1)^{n_0} \binom{\lambda}{n_0}.
\]

(4.41)

Substituting this into the expression for the total MDO, Eqn (4.34), we find that summing \( \hat{\Gamma}_0 \) over \( n_0 \) with a weighting factor of \((-1)^{n_0}\) has the effect of replacing \( \sigma_{0,\lambda} \) with \( i^{-\lambda} g_{0,\lambda} \) since \( 2^{-\lambda} \sum_n \binom{\lambda}{n_0} = 1 \). Taking the trace of this expression with the state to be measured leads to a modified expression for the moments as

\[
    \langle \cos(\lambda \theta) \rangle = \sum_N \frac{\widetilde{P}_{r_0}(N, \lambda) - \widetilde{P}_{r_1}(N, \lambda)}{4 g_{0,\lambda}} (4.42)
\]

\[
    \langle \sin(\lambda \theta) \rangle = \sum_N \frac{\widetilde{P}_{r_{1/2}}(N, \lambda) - \widetilde{P}_{r_{3/2}}(N, \lambda)}{4 g_{0,\lambda}} (4.43)
\]

where we now have taken the phase of \( \hat{g}(\lambda) \) such that \( i^{-\lambda} g_{0,\lambda} \) is real and positive and defined
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\[
\widetilde{P}_{r_0}(N, \lambda) = \sum_{n_0} (-1)^{n_0} P_{r_0}(n_0, N, \lambda - n_0)
\]  

(4.44)

as the corresponding weighted sum of measured probabilities. So we find by including all measurement outcomes that sum to \(\lambda\) at the output of BS2 we gain a modest reduction in the number of experiments that need to be performed since the scaling factor increases to

\[
g_{0,\lambda} = r^{2N} (-it)^{\lambda} \frac{\theta_{0,\lambda}}{\sqrt{N}} \left(\frac{N + \lambda}{N}\right)^{1/2}.
\]  

(4.45)

Unfortunately there is still an exponential dependence on the measurement outcome \(N\) and \(\lambda\) in this scaling factor. Although this cannot be removed it can be minimised. To do this we replace \(t = \cos \gamma\) and \(r = \sin \gamma\) in Eqn (4.45) and find the minimum over \(\cos^2 \gamma\). After some simple algebra we find the optimum \(r : t\) ratio of BS1, \(\tan \gamma\), to be \(\sqrt{2N/\lambda}\). Since we are mainly interested in weak fields, the spread of values that \(N\) and \(\lambda\) take should not be large. As such a fixed value of \(\tan \gamma\) around \(\sqrt{2\langle n\rangle}\) should generally suffice when measuring lower order moments, where \(\langle n\rangle\) is the mean photon number of the state to be measured. For higher order moments, a decrease in this ratio could be desirable. To complete the optimisation protocol we could adjust the strength of the coherent reference state to maximise \(\theta_{0,\lambda}\) in the expression for \(g_{0,\lambda}\). This is perhaps the easiest of the three optimizations to implement physically. We find, by writing \(\theta_{0,\lambda}\) as \(\exp(-|\alpha|^2)\alpha^\lambda/\sqrt{\lambda}\) and differentiating the modulus with respect to the mean photon number \(|\alpha|^2\), the optimum strength of the coherent reference state to be \(|\alpha|^2 = \lambda/2\). This is independent of \(N\), the detection outcome at \(D_1\). This is advantageous as the coherent strength need only be adjusted each time a different moment is to be measured, not for each value of \(N\) within a measurement.

A point of interest is the non-classical retrodictive probe state \(\rho_{q^\text{ret}}\) in the intermediate mode 0. It was the motivation of this work to utilise easily prepared retrodictive probe states in lieu of the more difficult to prepare predictive counterparts to design more practical experiments. What has been proposed utilised a mixed retrodictive probe that was similar in function to the highly non-classical state \(|0\rangle + |\lambda\rangle\). What is interesting about this work is the way in which the non-classical retrodictive probe was generated from a mixture of classical coherent states. Remarkably, it is the mixing of the coherent state that provided the retrodictive probe with the necessary off-diagonal terms to measure the \(\lambda\)th sin and cosine
phase moments. This is in contrast to most experiments which strive to avoid classically induced uncertainty in their design.

4.3 Measuring the density matrix elements of light

It is now well established that the quantum state of light can be measured. The first experimental evidence of this [67, 66] followed the work of Vogel and Risken [79], where it was shown that the Wigner function could be reconstructed from a complete ensemble of measured quadrature amplitude distributions. The authors of [67, 66] measured the quadrature distributions using balanced homodyne techniques. In the case of inefficient homodyne detectors, a more general s-parameterized quasiprobability distribution is obtained resulting in a smoothed Wigner function. In either case, to obtain the quasiprobability phase space distribution from the measured data a rather complicated inverse transformation is required.

Novel techniques which avoid this transformation are aimed at measuring the quasiprobability distribution more directly. This can be achieved, for example, in unbalanced homodyne counting experiments [8, 83], where a weighted sum of photocount statistics are combined to obtain a single point in the phase space distribution. The entire distribution is then obtained by scanning the magnitude and phase of the local oscillator over the region of interest while repeating the photon counting at each point. Perhaps the most direct method of obtaining a quasiprobability distribution is to use heterodyne [65] or double homodyne [82] detection techniques where the \( Q \) function is measured. The \( Q \) function is related to the Wigner function through a convolution with a Gaussian distribution which effectively washes out many of the interesting quantum features. It is possible to recover these features by deconvoluting the \( Q \) function, however this requires multiplying by an exponentially increasing function thereby introducing a crucial dependence on sampling noise [80].

A different approach has been suggested by Steuernagel and Vaccaro [68], who have proposed an interesting nonrecursive scheme to measure not the quasiprobability distribution, but rather the density operator in the photon number basis. The scheme is relatively direct in that only a finite number of different measurements are required to determine each matrix element. The experimental arrangement of the proposal is illustrated in Figure [4.2] It
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Figure 4.2: Experimental proposal of Steuernagel and Vaccaro to measure the density matrix elements $\rho_{N,N+\lambda}$ of an optical field. At the output of the 50/50 beam-splitter are photodetectors $D_0$ and $D_2$. The field to be measured is $\hat{\rho}_0$ while $|0\rangle + |\lambda\rangle$ is a reference state.

consists of a single 50/50 beam-splitter with input modes 0 and 2. At each of the two output modes there is a photodetector label $D_0$ and $D_2$ respectively. The state to be measured is input mode 0 of the beam-splitter while the field in input mode 2 is a reference field. The most practical arrangement of this scheme requires the reference field to be in the state proportional to $|0\rangle_2 + e^{-i\lambda \theta} |\lambda\rangle_2$. With such a reference field Steuernagel and Vaccaro then considered the case when the the two photodetectors detected a total of $N + \lambda$ photons. For a detailed mathematical analysis of this proposal the reader is referred to the original paper [65], however, useful insight into the problem can be gained by considering the dynamics in the retrodictive formalism.

Following the arguments introduced in Chapter 3, we assign, conditioned on detector $D_0$ detecting $n_0$ photons and detector $D_2$ detecting $N + \lambda - n_0$ photons, the retrodictive state $|n_0\rangle_0 |N + \lambda - n_0\rangle_2$ just prior to the detection event. Denoting the unitary action of the beam-splitter by the operator $\hat{S}$ which acts in the joint Hilbert space of mode 0 and mode 2, we can follow the evolution of the state backwards in time and find the retrodictive state at the input of the beam-splitter as

$$|z\rangle = \hat{S}^\dagger |n_0\rangle_0 |N + \lambda - n_0\rangle_2 = \sum_{m=0}^{N+\lambda} z_m |m\rangle_0 |N + \lambda - m\rangle_2$$

(4.46)

which, from conservation of energy, is a general superposition of all two-mode photon number states that sum to $N + \lambda$. Projecting the predictive reference state in input mode 2, which is an equal superposition of the vacuum state and a $\lambda$-photon state, onto $|z\rangle$ we find the single mode unnormalised retrodictive state at the input mode 0 of the beam-splitter is
\[ \sqrt{2} |\tilde{\psi}\rangle_0 = z_N |N\rangle_0 + z_{N+\lambda} e^{i\lambda \theta} |N + \lambda\rangle_0. \]  

(4.47)

The joint probability for such a detection event is then given by the overlap of the unnormalised retrodictive state \(|\tilde{\psi}\rangle_0\) and the state to be measured, \(\hat{\rho}_0\), which from Eqn (4.47) above is

\[ \Pr_\theta(n_0, N + \lambda - n_0) = \frac{1}{2} \left[ |z_N|^2 \rho_{N,N} + |z_{N+\lambda}|^2 \rho_{N+\lambda,N+\lambda} + \left( z_N^* z_{N+\lambda} e^{i\lambda \theta} \rho_{N,N+\lambda} + c.c \right) \right], \]

(4.48)

where \(\rho_{n,m} = <n|\hat{\rho}_0|m>_0\). To determine the value of the off-diagonal element \(\rho_{N,N+\lambda}\) the entire experiment need to be repeated four times in total, each time with a different phase shift \(\theta\) applied to the reference field. By selecting the four phase shift settings as \(\theta = j\pi/\lambda\) where \(j = 0, 1/2, 1, 3/2\) a value of \(\rho_{N,N+\lambda}\) can be extracted from the measured probabilities as

\[ \rho_{N,N+\lambda} = \frac{\Pr_0(N+\lambda) - i \Pr_{1/2}(N+\lambda) - \Pr_1(N+\lambda) + i \Pr_{3/2}(N+\lambda)}{2z_N^* z_{N+\lambda}} \]  

(4.49)

where \(\Pr_j(N+\lambda)\) is shorthand for \(\Pr_\theta(n_0, N + \lambda - n_0)\). So what was originally proposed by Steuernagel and Vaccaro was an insightful way to relate the probability in which particular detection events occur in an experiment to specific off-diagonal elements in the density matrix description of an optical field. This might be viewed as a generalisation of a direct photon-counting experiment where the probabilities of the detection events give the diagonal elements of the density matrix. The problem however is that the reference state \(|0\rangle + |\lambda\rangle\) is extremely difficult to produce in practice. Even to use the quantum scissors device to truncate a coherent state for the case when \(\lambda = 1\) is by no means trivial. Remarkably, by introducing a second beam-splitter with the vacuum in one of the inputs, it is possible to replace the non-classical reference state with a mixture of coherent states and achieve the same results \([56, 60]\). The experiment from which the measured probabilities are observed is exactly that which was proposed in the preceding section. The difference between the previous sections work and this is in the way in which the probabilities are combined to ascertain something different about the state we are observing. To see how the off-diagonal matrix element \(\rho_{N,N+\lambda}\) can be measured using the double beam-splitter device introduced in the previous section, consider the probability of the detection event \((n_0, N, \lambda - n_0)\) at
4.3. MEASURING THE DENSITY MATRIX ELEMENTS OF LIGHT

detectors \( D_0, D_1 \) and \( D_2 \) respectively. From Eqn (4.35), we rewrite this here as

\[
\Pr_0(n_0, N, \lambda - n_0) = \sum_{n=0}^{\lambda} \sigma_{n,n} \rho_{n+n,N+n} + (\sigma_{0,\lambda} \rho_{N+\lambda,N} + \text{c.c.}).
\]  

(4.50)

If instead of summing over \( N \), we combine the four measured probabilities \( \Pr_j(n_0, N, \lambda - n_0) \), \( j = 0, 1/2, 1, 3/2 \), relating to the four different phase shift settings and normalise we can extract the matrix element \( \rho_{N,N+\lambda} \) akin to expression (4.49) we used to describe the approach of Steuernagel and Vaccaro as

\[
\rho_{N,N+\lambda} = \frac{\Pr_0(N, \lambda) - i\Pr_{1/2}(N, \lambda) - \Pr_1(N, \lambda) + i\Pr_{3/2}(N, \lambda)}{4\sigma_{0,\lambda}^*},
\]

(4.51)

where \( \Pr_j(N, \lambda) \) is shorthand for \( \Pr_j(n_0, N, \lambda - n_0) \). Since we are not modifying the double beam-splitter experiment, merely changing the way in which we combine the measured probabilities, all the optimisation protocols suggested in the previous section apply. As such we may write Eqn (4.51) in terms of the weighted sum of probabilities \( \tilde{\Pr}_j(N, \lambda) \) introduced in Eqn (4.44) as

\[
\rho_{N,N+\lambda} = \frac{\tilde{\Pr}_0(N, \lambda) - i\tilde{\Pr}_{1/2}(N, \lambda) - \tilde{\Pr}_1(N, \lambda) + i\tilde{\Pr}_{3/2}(N, \lambda)}{4\sigma_{0,\lambda}^*},
\]

(4.52)

which includes all detection events at detectors \( D_0 \) and \( D_2 \) that sum to \( \lambda \).

It is interesting to note that Steuernagel and Vaccaro proposal would still work if the reference state were replaced by a general truncated state \( \hat{\rho}^{tr} = \sum_{n,m=0}^{\lambda} \rho_{n,m} |n\rangle\langle m| \) in \( \lambda + 1 \) dimensions. Using the mixing technique presented in the previous section all but the diagonal and \( \lambda^{th} \) off-diagonal elements would need to be removed to make the reference state of the form \( \hat{\rho}^{mix}_{\text{mix}} = \sum_{n=0}^{\lambda} \rho_{n,n} |n\rangle\langle n| + \rho_{0,\lambda} |0\rangle\langle \lambda| + \rho_{\lambda,0} |\lambda\rangle\langle 0| \). As the contribution from the diagonal elements are removed when the probabilities are subtracted, this state is, for all purposes considered here, equivalent to the state \( |0\rangle + |\lambda\rangle \). Unfortunately, generating a predictive state in a finite number of dimensions, or equivalently, truncating a state in a finite number of dimensions is still a difficult process.

So we find, from Eqn (4.47), that the proposal of Steuernagel and Vaccaro takes the truncated predictive state \( |0\rangle + |\lambda\rangle \) and simultaneously turns it into a retrodictive state while raising the photon occupation number by \( N \). The double beam-splitter proposal presented here does these two operations separately. At the first beam-splitter \( BS_1 \) (in the retrodictive picture), the mixed retrodictive state is truncated and turned into a retrodictive state \( \hat{\rho}^{ret}_q \), the retrodictive equivalent of \( \hat{\rho}^{tr}_{\text{mix}} \). The second beam-splitter performs the second
of these operations which is to raise the photon occupation number of the retrodictive state by $N$. The result is a retrodictive probe equivalent to that in the Steuernagel and Vaccaro scheme which can be used to measure individual elements of the optical density matrix in the photon number basis. Both proposals detect a total of $N + \lambda$ photons. The advantage of the the double beam-splitter arrangement is that it naturally truncates the reference state in a $\lambda + 1$ dimensional sub-space when turning it into a retrodictive state. This, in conjunction with the mixing technique, allows an ordinary coherent state to be used as a reference state in lieu of the non-classical truncated state, of which $|0\rangle + |\lambda\rangle$ is a specific example. So again we find an asymmetry in the ease at which a retrodictive state can be produced in practice as opposed to the predictive counterpart.

In summary, by utilising the more readily prepared retrodictive quantum states we were able to take two experimental proposals that were proving difficult to implement physically and redesign them in a such a way to make them more implementable with current technology. Both proposals are non-recursive in that the measured quantity is extracted directly from observed probabilities of selected measurement events, in a similar fashion to the original projection synthesis technique of [12, 50] 54. Interestingly, one of these methods allows the density matrix elements of an unknown field to be obtained quite simply from the density matrix of a mixed local oscillator state, even when the unknown field is in a pure state.
Chapter 5

Quantum optical phase and its measurement

The quantum mechanical nature of the phase of light has been studied since the beginnings of quantum electrodynamic theory [21] and with renewed interest recently. The study of quantum phase is distinguished from the study of many other quantum observables by the difficulties inherent not only in finding a theoretical description but also in finding methods for measuring the phase observable so described [45]. Despite the method proposed in Chapter 3 and others like it [12, 54], to engineer any general retrodictive state expressible in a finite number of dimensions, a “single-shot” measure of quantum optical phase has been illusive. In this chapter I present the first proposed method capable, at least in principle, of providing a single shot measure of canonical phase. This work has been published in our paper [58]. The technique is simple, involving only beam-splitters, phase-shifters and photodetectors which can discriminate between zero, one and many photons. Following this I show that the eight-port interferometer used by Noh, Fougères and Mandel [37, 38, 23] to measure their operational phase distribution of light [37, 39, 38, 40, 11, 11] can, remarkably, also be used to measure the canonical phase distribution for weak optical fields [59]. A binomial reference state is required for this purpose which we show can be obtained, to an excellent degree of approximation, from a suitably squeezed state.
5.1 Single-shot measure of quantum optical phase

5.1.1 Canonical phase

Quantum-limited phase measurements of the optical field have important applications in precision measurements of small distances in interferometry and in the emerging field of quantum communication, where there is the possibility of encoding information in the phase of light pulses. Much work has been done in attempting to understand the quantum nature of phase. Continuing our discussion of phase in Section 4.1, we note that some approaches have been motivated by the aim of expressing phase as the complement of photon number \[|\psi\rangle\]. Examples of these approaches include the probability operator measure approach \[\text{Ref. 27, 64}\], a formalism in which the Hilbert space is doubled \[\text{Ref. 36}\], a limiting approach based on a finite Hilbert space \[\text{Ref. 10, 34}\] and a more general axiomatic approach \[\text{Ref. 34}\]. Although these approaches are quite distinct, they all lead to the same phase probability distribution for a field in state \[|\psi\rangle\] as a function of the phase angle \(\theta\) \[\text{Ref. 34}\]:

\[
P(\theta) = \left| \frac{1}{2\pi} \sum_{n=0}^{\infty} \langle \psi | n \rangle \exp(in\theta) \right|^2
\]

where \(|n\rangle\) is a photon number state and \(\theta\) can take on any value within a \(2\pi\) window which we arbitrarily take as \(0 \leq \theta \leq 2\pi\). Leonhardt et al. \[\text{Ref. 34}\] have called this common distribution the “canonical” phase distribution to indicate a quantity that is the canonical conjugate, or complement, to photon number. This distribution is shifted uniformly when a phase-shifter is applied to the field and is not changed by a photon number shift. In accord with our previous discussion of phase, we can continue to adopt this definition here and use the term canonical phase to denote the quantity whose distribution is given by (5.1). We can write the definition in Eqn (5.1) in a more compact notation as \(|\langle \psi | \theta \rangle|^2\), where we use the (improper) state vector

\[
|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(in\theta) |n\rangle.
\]

We note that this state is not orthogonal to a state of different phase \(|\theta'\rangle\), even in the sense of a Dirac delta function. Interestingly, many of the difficulties associated with finding an Hermitian phase operator in the infinite-dimensional Hilbert space can be attributed to this fact which is due to the semi-bounded property of the associated Hilbert space. Howbeit, we refer to the state defined by Eqn (5.2) as a phase state. In the case of a mixed state \(\hat{\rho}\),
we can generalise Eqn (5.1) to
\[ P(\theta) = \text{Tr}[\hat{\rho}|\theta\rangle\langle\theta|]. \]  
(5.3)

Using the definition of the phase state it can be shown that the set of all such operators $|\theta\rangle\langle\theta|$ form a resolution of the identity operator in the infinite dimensional Hilbert space,
\[ \int_{0}^{2\pi} |\theta\rangle\langle\theta| d\theta = \hat{1}. \]  
(5.4)

As it is obvious from Eqn (5.3) that $|\theta\rangle\langle\theta|$ is a non-negative operator, the set of all such operators constitutes a valid POM, with elements $\hat{\Pi}_\theta = |\theta\rangle\langle\theta|$. Since the set of POM elements $\hat{\Pi}$ is sufficient to derive the canonical phase distribution of Eqn (5.1), we see that the phase POM provides a way of representing the phase observable without defining a phase operator. By redefining phase in terms of a POM, we can associate the observable phase with outcomes of a measurement apparatus while maintaining consistency with other descriptions of canonical phase\(^1\).

Despite these theoretical advances, much less progress has been made on ways to measure canonical phase. Homodyne techniques can be used to measure phase-like properties of light but are not measurements of canonical phase. For very weak states of light an adaptive technique can improve the homodyne methods to provide a quite good approximate measurement of canonical phase\(^8\). These have been implemented experimentally recently\(^8\). Using the apparatus described in Chapter 2 or even the original projection synthesis technique described in\(^12\), it is possible in principle to measure the canonical phase distribution by a series of experiments on a reproducible state of light but there has been no known way of performing a single-shot measurement. Even leaving aside the practical issues, the concept that a particular fundamental quantum observable may not be measurable, even in principle, has interesting general conceptual ramifications for quantum mechanics. A different approach to the phase problem, which avoids difficulties in finding a way to measure canonical phase, is to define phase operationally in terms of observables that can be measured\(^\text{34}\). The best known of these operational phase approaches is that of Noh et al.\(^37\)\(^39\)\(^38\)\(^10\)\(^11\)\(^41\)\(^23\). Although the experiments to measure this operational phase produce excellent results, they were not designed to measure canonical phase as defined here and, as shown by the the measured phase distribution\(^37\)\(^38\)\(^10\)\(^23\), they

\(^1\)The consistency of quantum descriptions of phase has been studied in\(^77\). There it is shown how the POM description can be derived from the limiting approach of\(^43\)\(^14\)\(^10\).
do not measure canonical phase. In this chapter I show how, despite these past difficulties, it is indeed possible, at least in principle, to perform a single-shot measurement of canonical phase in the same sense that the experiments of Noh et al. are single-shot measurements of their operational phase.

A single-shot measurement of a quantum observable must not only yield one of the eigenvalues of the observable, but repeating the measurement many times on systems in identical states should result in a probability distribution appropriate to that state. If the spectrum of eigenvalues is discrete, the probabilities of the results can be easily obtained from the experimental statistics. Where the spectrum is continuous, the probability density is obtainable by dividing the eigenvalue range into a number of small bins and finding the number of results in each bin. As the number of experiments needed to obtain measurable probabilities increases as the reciprocal of the bin size, a practical experiment will require a non-zero bin size and will produce a histogram rather than a smooth curve.

Although the experiments of Noh et al. were not designed to measure canonical phase, it is helpful to be guided by their approach. In addition to their results being measured and plotted as a histogram, some of the experimental data are discarded, specifically photon count outcomes that lead to an indeterminacy of the type zero divided by zero in their definitions of the cosine and sine of the phase \[37, 38, 40, 23\]. The particular experiment that yields such an outcome is ignored and its results are not included in the statistics. Such a measurement procedure is a specific example of a biased measurement procedure introduced in Chapter 2, the statistics of which are governed by the general expression of Eqn \[2424\].

We seek now to approximate the continuous distribution \(5.3\) by a histogram representing the probability distribution for a discrete observable \(\theta_m\) such that when the separation \(\delta\theta\) of consecutive values of \(\theta_m\) tends to zero the continuous distribution is regained. A way to do this is first to define a (proper) state vector

\[
|\theta_m\rangle = \frac{1}{(N + 1)^{1/2}} \sum_{n=0}^{N} \exp(in\theta_m)|n\rangle.
\]

There are \(N + 1\) orthogonal states \(|\theta_m\rangle\) corresponding to \(N + 1\) values \(\theta_m = m\delta\theta\) with \(\delta\theta = 2\pi/(N + 1)\) and \(m = 0, 1, \ldots, N\). This range for \(m\) ensures that \(\theta_m\) takes values between 0 and \(2\pi\). Then, if we can find a measurement technique that yields the result \(\theta_m\) with a probability of \(\text{Tr}[\hat{\rho}|\theta_m\rangle\langle\theta_m|]\), the resulting histogram will approximate a continuous
distribution with a probability density of $\delta \theta^{-1} \text{Tr}[\hat{\rho}|\theta_m\rangle\langle\theta_m|]$. It follows that, as we let $N$ tend to infinity, there will exist a value of $\theta_m$ as close as we like to any given value of $\theta$ with a probability density approaching $P(\theta)$ given by (5.3). If we keep $N$ finite so that we can perform an experiment with a finite number of outcomes, then the value of $N$ must to be sufficiently large to give the resolution $\delta \theta$ of phase angle required and also for $|\psi\rangle$ to be well approximated by $\sum_n \langle n|\psi\rangle|n\rangle$ where the sum is from $n = 0$ to $N$. The latter condition ensures that the terms with coefficients $\langle n|\psi\rangle$ for $n > N$ have little effect on the probability $\text{Tr}[\hat{\rho}|\theta_m\rangle\langle\theta_m|]$. As we shall be interested mainly in weak optical fields in the quantum regime with mean photon numbers of the order of unity, the phase resolution $\delta \theta$ desired will usually be the determining factor in the choice of $N$.

When $N$ is finite, the states $|\theta_m\rangle$ do not span the whole Hilbert space, so the projectors $|\theta_m\rangle\langle\theta_m|$ will not sum to the unit operator $\hat{1}$ in this space. Thus these projectors by themselves do not form the elements of a POM in this space. Conveniently however, they do sum to the identity operator in an $N + 1$ dimensional Hilbert space, $\hat{1}_N$.

$$\sum_{m=0}^{N} |\theta_m\rangle\langle\theta_m| = \hat{1}_N. \tag{5.6}$$

To complete the POM acting in the whole Hilbert space we need to include an element $\hat{1} - \hat{1}_N$, such that the sum of all elements is the identity in the whole space. If we are to discard the outcome associated with this element, that is, treat an experiment with this outcome as an unsuccessful attempt at a measurement in a similar way that Noh et al. [37] [38] [40] [23] treated experiments with indeterminate outcomes, then we should associate each of the non-discarded events with a MDO $\hat{\Gamma}_m$ proportional to $|\theta_m\rangle\langle\theta_m|$. From Eqn (2.12), with Eqn (5.6) above, the probability that the recorded outcome of a measurement is the phase angle $\theta_m$ is given by

$$\text{Pr}(\theta_m) = \frac{\text{Tr}[\hat{\rho}|\theta_m\rangle\langle\theta_m|]}{\text{Tr}[\hat{\rho}\hat{1}_N]}. \tag{5.7}$$

We now require a single-shot measuring device that will reproduce this probability in repeated experiments.

### 5.1.2 Experimental proposal

We demonstrated in Section 3.3 that a multiport device is capable of producing a single retrodictive truncated phase state of the form given by Eqn (5.5). The probability of
Figure 5.1: Multiport device for measuring phase. The input and output modes are labelled 0, 1, ..., N from the left. In input mode 0 is the field in state $|\psi\rangle_0$ to be measured and in input mode 1 is the reference field in state $|B\rangle_1$. Vacuum states form the other inputs. There is a photodetector in each output mode. If all the photodetectors register one count except the detector $D_m$ in output mode $m$, which registers no counts, then the detector array acts as a digital pointer mechanism indicating a measured phase angle of $\theta_m$.

Generating such a state is proportional to $\text{Tr}[\hat{\rho}|\theta_m\rangle\langle\theta_m|]$ and was conditioned on observing the photodetection sequence $(0, 1, 1, \ldots, 1)$ at the $N+1$ photodetectors $D_i$, $i = 0, 1, \ldots, N$. In all input modes, except the one containing the state to be measured, a coherent field was present. To construct a device capable of producing a single shot measure of phase in the sense outlined above, we need a device which is capable of generating all of the $N+1$ retrodictive states of the form in Eqn (5.5), with $m = 0, 1, \ldots, N$. We ask the question: can we modify this apparatus such that each of the $N+1$ permutations of the detection sequence $(0, 1, 1, \ldots, 1)$ generates one of the $N+1$ truncated phase states? If we can, then we have an apparatus capable of performing a single shot measurement of phase.

Retaining the linear optical multiport and the photodetectors at each of the output modes means that the only modification we can make to the apparatus is to allow for more general reference states than coherent states. We consider the simplest case of replacing only one of the coherent reference state $|\alpha\rangle_1$ in input mode 1 with a general reference state $|B\rangle_1 = \sum_{n} b_n |n \rangle_1$, and replace all others with a vacuum field. Such a multi-mode input state is written as

$$
\left( \prod_{i=2}^{N} |0\rangle_i \right) |B\rangle_1
$$

(5.8)

Following on from Section 5.5, we now consider the retrodictive state that is generated with this multi-mode reference state when each photodetector registers a single photon, except
5.1. SINGLE-SHOT MEASURE OF QUANTUM OPTICAL PHASE

one, which registers zero photons. There are $N + 1$ different ways in which this can happen resulting from the the $N + 1$ different detectors, we consider all of them. If we label the detector which does not detect any photons as the $m^{th}$ detector, then we can assign a POM element to the multi-photon detection event as

$$\hat{\Pi}(m) = |0\rangle_m \langle 0| \prod_{j \neq m}^N |1\rangle_j \langle 1|,$$

(5.9)

where the index on the product is taken over all mode labels, $j = 0$ to $N$, but does not include $j = m$. We can write this operator as a projector $\hat{\Pi}(m) = |\Psi_m\rangle \langle \Psi_m|$, where

$$|\Psi_m\rangle = |0\rangle_m \prod_{j \neq m}^N |1\rangle_j.$$

(5.10)

It follows then from Section 3.3 with the multi-mode reference state given by Eqn (5.8), that the single-mode retrodictive state at the input of mode 0 conditioned on detecting one photon in each output mode, except the $m^{th}$ mode, is

$$|\tilde{\psi}_m\rangle_0 = \kappa_1 |B\rangle \left( \prod_{i=2}^N \langle 0| \right) \hat{\mathbf{S}}^\dagger |\Psi_m\rangle.$$

(5.11)

The operator $\hat{\mathbf{S}}^\dagger$ again represents the unitary evolution of the multi-mode state, backwards in time, and is characterised by the mode transformation matrix elements $U_{ij}$ in Eqn (3.20). We require such a device for which the associated unitary matrix is

$$U_{ij}^* = \frac{\omega^{ij}}{\sqrt{N + 1}}$$

(5.12)

where $\omega = \exp[-i2\pi/(N + 1)]$ that is, a $(N + 1)^{th}$ root of unity. This is precisely the transformation introduced in Section 3.2.1 as the discrete Fourier transformation in $N + 1$ dimensions. Substituting Eqn (5.10) into the expression for the retrodictive state (5.11) and writing $|1\rangle_j$ as $\hat{a}_j^\dagger |0\rangle$ gives, after some algebra,

$$|\tilde{\psi}_m\rangle_0 = \kappa_1 |B\rangle \left( \prod_{i=2}^N \langle 0| \right) \left( \prod_{j \neq m}^N \hat{\mathbf{S}}^\dagger \hat{a}_j^\dagger \hat{\mathbf{S}} \right) |0\rangle.$$

(5.13)

where $|0\rangle$ is the multi-mode vacuum which is invariant under the linear transformation of $\hat{\mathbf{S}}^\dagger$. Substituting Eqn (5.20) for the mode transformation and using the matrix elements of (5.12) simplifies this expression to

$$|\tilde{\psi}_m\rangle_0 = \kappa_1 |B\rangle \left[ \prod_{j \neq m}^N (\hat{a}_0^\dagger + \omega^j \hat{a}_1^\dagger) \right] |0\rangle_0 |0\rangle_1.$$

(5.14)
where $\kappa_1 = (N + 1)^{-N/2}$.

To evaluate Eqn (5.14) we divide both sides of the identity

$$X^{N+1} + (-1)^N = (X+1)(X+\omega)(X+\omega^2)\ldots(X+\omega^N)$$

(5.15)

by $X + \omega^m$ to give, after some rearrangement and application of the relation $\omega^m (N+1) = 1$,

$$\prod_{j \neq m}^N (X + \omega^j) = (-1)^N \omega^{mN} \frac{1 - (-X\omega^{-m})^{N+1}}{1 - (-X\omega^{-m})}. \quad (5.16)$$

The last factor is the sum of a geometric progression. Expanding this and substituting $X = x/y$ gives eventually the identity

$$\prod_{j \neq m}^N (x + \omega^j y) = \sum_{n=0}^N x^n (-\omega^m y)^{N-n}. \quad (5.17)$$

We now expand $|B\rangle_1$ in terms of photon number states as $|B\rangle_1 = \sum_{n=0}^N b_n |n\rangle_1$ and put $x = a_0^\dagger$ and $y = a_1^\dagger$ in Eqn (5.17). With this we find that the retrodictive state of Eqn (5.14) becomes

$$|\tilde{\psi}_m\rangle_0 = \kappa_2 \sum_{n=0}^N (-1)^{N-n} \binom{N}{n}^{-1/2} \omega^{-mn} b_{N-n}^\dagger |n\rangle_0 \quad (5.18)$$

where $\kappa_2 = \kappa_1 \omega^{-m} (N!)^{1/2}$. We see then that, if we let $|B\rangle_1$ be the binomial state

$$|B\rangle_1 = 2^{-N/2} \sum_{n=0}^N (-1)^n \binom{N}{n}^{1/2} |n\rangle_1, \quad (5.19)$$

then Eqn (5.18) is proportional to $\sum_n \omega^{-mn} |n\rangle_0$, that is, to $|\theta_m\rangle_0$.

The unnormalised retrodictive state at the input of mode 0 of the device, $|\tilde{\psi}_m\rangle_0$, can be associated with a MDO $\hat{\Gamma}_0(m)$ for the measuring device consisting of everything in Figure 5.1 except the state to be measured. We found above that this MDO is proportional to the projector $|\theta_m\rangle_0\langle\theta_m|$. With this we see from Eqn (2.12) that the probability that zero photons are detected in output mode $m$ and one photon is detected in all the other output modes, given that only outcomes associated with the $(N+1)$ events of this type are recorded in the statistics, is consistent with Eqn (5.7), where we note that the proportionality constant $\kappa_2$ will cancel from this expression. Thus the measurement event that zero photons are detected in output mode $m$ and one photon is detected in all the other output modes can be taken as the event that the result of the measurement of the phase angle is $\theta_m$. Thus the photodetector with zero photocounts, when all other photodetectors have
registered one photocount, can be regarded as a digital pointer to the value of the measured phase angle.

We have shown, therefore, that it is indeed possible in principle to conduct a single-shot measurement of canonical phase to within any given non-zero error, however small. This error is of the order $2\pi/(N + 1)$ and will determine the value of $N$ chosen.

While the aim of this section is to establish how canonical phase can be measured in principle, it is worth briefly considering some practical issues. Although we have specified that the photodetectors need only be capable of distinguishing among zero, one and more than one photons, reflecting the realistic case, there are other imperfections such as inefficiency. These will give rise to errors in the phase measurement, just as they will cause errors in a single-shot photon number measurement. In practice, there is no point in choosing the phase resolution $\delta\theta$ much smaller than the expected error due to photodetector inefficiencies, thus there is nothing lost in practice in keeping $N$ finite. A requirement for the measuring procedure is the availability of a binomial state. Such states have been studied for some time \[69, 22\] but their generation has not yet been achieved. In practice, however, we are usually interested in measuring weak fields in the quantum regime with mean photon numbers around unity \[37, 40\] and even substantially less \[72\]. Only the first few coefficients of $|n\rangle_0$ in Eqn (5.18) will be important for such weak fields. Also, it is not difficult to show that the reference state need not be truncated at $n = N$, as indicated in Eqn (5.19), as coefficients $b_n$ with $n > N$ will not appear in Eqn (5.18). Thus we need only prepare a reference state with a small number of its photon number state coefficients proportional to the appropriate binomial coefficients. Additionally, of course, in a practical experiment we are forced to tolerate some inaccuracy due to photodetector errors, so it will not be necessary for the reference state coefficients to be exactly proportional to the corresponding binomial state coefficients. These factors give some latitude in the preparation of the reference state. The multiport depicted in Figure 5.1 can be constructed in a variety of ways. As mentioned in Section 5.2.1 Reck et al. provide an algorithm for constructing a triangular array to realise such a transformation. In addition, it was also discussed how the plate beam-splitter design of Törnä and Jex \[74\] will provide the same transformation with a quadratic reduction in the number of optical elements necessary.
5.2 Canonical phase distribution for weak optical fields

In this section I show that the eight-port interferometer used by Noh, Fougères, and Mandel [37, 38, 23] to measure their operational phase distribution of light can also be used to measure the canonical phase distribution of weak optical fields. Such a result was originally published in Ref. [59] by D. T. Pegg and myself. Here I show that both the multimode projection synthesis technique introduced in Chapter 3 and a method for measuring the canonical phase distribution based on the single-shot method presented above can be implemented with the eight-port interferometer. The obvious difference between these two methods is the reference states required at the input of the interferometer. In the case of the multi-mode projection synthesis technique, it is necessary to combine the state being measured with three coherent fields at the input of the interferometer, while for the single-shot technique the state being measured is combined with a single binomial state and two vacuum states. We find, remarkably, that the binomial reference state can be obtained to an excellent degree of approximation from a suitably squeezed state. Given that the operational phase measurements have already been conducted using the eight-port interferometer and single photon detectors provides encouraging support that there should be no insurmountable physical challenges preventing the measurement of the canonical phase distribution for weak optical fields.

5.2.1 Reconstructing and measuring the phase distribution

As phase is an intrinsically continuous observable, akin to position, it is necessary to represent the phase distribution by a continuous probability density as opposed to a discrete probability distribution. In a practical experiment, however, only discrete probability distributions can be observed. Accordingly, the continuous probability density can be approximated by observing the probability that the outcome falls in a particular range, called a bin. The probability divided by the associated bin size, can be plotted as a discrete probability density. It follows then that in the limit as the bin size tends to zero, the continuous probability density can be obtained from the discrete density. In practice to obtain a continuous probability experimentally for an unknown state would take an infinite time so, in general, a continuous probability density can only be measured to within some nonzero error. We show in this section by decomposing the canonical phase distribution of (5.1)
into the Fourier coefficients, that truncated states, for example, with at most $N$ photons cannot produce a phase distribution $P(\theta)$ with oscillations more rapid than $\exp(i\theta N)$. It follows then that for such a state $2N + 2$ points are the minimum needed to determine the phase distribution.

Making use of the Dirac delta function we can express the canonical phase distribution as

$$P(\theta) = \int P(\phi)\delta(\theta - \phi) d\phi. \tag{5.20}$$

After writing the delta function as an infinite summation of orthonormal functions $\exp(in\theta)$, we see that the phase distribution can be expressed as a weighted sum of the orthonormal functions

$$P(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \alpha_n \exp(in\theta), \tag{5.21}$$

where the weighting coefficients

$$\alpha_n = \int P(\phi)\exp(-in\phi) d\phi \tag{5.22}$$

are the $n^{th}$ order exponential phase moments. From (5.21), the infinite set of coefficients $\alpha_n$ then provides an equivalent representation to the continuous distribution $P(\theta)$ of (5.1). We refer to Eqn (5.21) as the Fourier decomposition of the canonical phase distribution, where the Fourier coefficients are just the weighting coefficients of Eqn (5.22). In general each Fourier coefficient can take on any independent value within a unit circle on the complex number plane, provided however $\alpha_n^* = \alpha_{-n}$. This condition can be derived in a straightforward manner by taking the complex conjugate of Eqn (5.22).

The Fourier coefficients are a set of values general enough to represent the probability distribution for any state $|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle$ contained within the infinite dimensional Hilbert state. In practice however, all states possess only a finite amount of energy. As such, most states can be sufficiently well represented by truncating $|\psi\rangle$ at some finite energy value $N$, however large,

$$|\psi\rangle = \sum_{n=0}^{N} \psi_n |n\rangle. \tag{5.23}$$

It is shown in Appendix C that the Fourier coefficients $\alpha_n$ associated with such states are zero for all $|n| > N$. In such case the probability distribution can be written as a finite summation of oscillating functions

$$P(\theta) = \frac{1}{2\pi} \sum_{n=-N}^{N} \alpha_n \exp(in\theta). \tag{5.24}$$
So for states represented by Eqn (5.23), this introduces a maximum limit of $2\pi/N$ to the period in which the probability distribution can oscillate. Thus we see, for states with a finite number of energy terms, that the continuous phase probability density can be represented by a finite set of $N+1$ unique complex numbers\(^2\). So, provided there is a way of obtaining the $N+1$ coefficients, we can reconstruct the continuous phase probability distribution.

One way of obtaining the nonzero coefficients is to sample the continuous distribution \((5.24)\) at evenly spaced angles $\gamma_m = \theta_m/2 = 2\pi m/(2N+2)$,

$$P(\gamma_m) = \frac{1}{2\pi} \sum_{n=-N}^{N} \alpha_n \exp(in\gamma_m),$$

(5.25)

where $m = 0, 1, \ldots, 2N+2$ to ensure the entire distribution is sampled completely. We can invert this expression by taking the discrete Fourier transform of these $2N+2$ values to give the Fourier coefficients in terms of the measured probability densities as

$$\alpha_n = \frac{\pi}{N+1} \sum_{m=0}^{2N+1} \exp(-in\gamma_m) \cdot P(\gamma_m).$$

(5.26)

Thus we need at least $2N+2$ points to reproduce the distribution. For a state that is not truncated, we will need an infinite number of such points. However if such a field is sufficiently weak for the phase distribution to be obtained to a good approximation by projection onto a phase state truncated after the $N$ photon component then a correspondingly smaller number of points is needed. We now consider how to obtain such points by experimental measurement. Such a measurement, as previously mentioned, would be associated with the projection of the state $|\psi\rangle$ onto the phase state of Eqn (5.2). Since the state $|\psi\rangle$ is truncated after $n = N$, there would be no observable difference if instead we projected onto the truncated phase state of Eqn (5.5), with $\gamma_m = \theta_m/2$. Since the outcome of this would be represented by a discrete observable, it can be associated with a probability $Pr(\gamma_m)$, as opposed to a density. Indeed, projecting $|\psi\rangle$ onto both phase states we find that

$$2\pi P(\gamma_m) = (N+1) \cdot Pr(\gamma_m)$$

(5.27)

showing the probability is proportional to the probability density at the point $\theta = \gamma_m$. So, after substituting Eqns (5.26) and (5.27) into (5.21), we arrive at an expression for the

\(^2\)In general there are $2N+1$ nonzero Fourier coefficients in Equation (5.24), however $N$ of them are just the complex conjugate, leaving $N+1$ unique coefficients.
5.2. CANONICAL PHASE DISTRIBUTION FOR WEAK OPTICAL FIELDS

Continuous canonical phase distribution for a truncated state $|\psi\rangle$ as

$$P(\theta) = \frac{1}{4\pi} \sum_{n=-N}^{N} \sum_{m=0}^{2N+1} \exp[in(\theta - \gamma_m)] \Pr(\gamma_m).$$ \hspace{1cm} (5.28)

We note that this is already normalised \cite{59}. This expression depends only on the $2N + 2$ measurable probabilities $\Pr(\gamma_m)$, as opposed to the infinite amount necessary to measure the phase distribution of a non-truncated state. So for states definitely containing at most $N$ photons, it is possible, at least in principle, to obtain the continuous canonical phase distribution perfectly.

In practice although $P(\theta)$ can be reconstructed from just $2N + 2$ experimental points, obtaining just this minimum number of points leaves no room for error and there must always be statistical errors when determining probabilities. Also there will in general be experimental errors. For this reason it may be preferable to make direct measurements of probabilities at a sufficiently large number of phase settings to identify points severely affected by errors. The possibility of directly plotting the histogram from the measurements gives this approach an advantage over the technique of Section 4.2 which we could use only for reconstructing the distribution using a similar procedure to the one outlined above.

5.2.2 Eight-port interferometer

In general, any method which can measure the minimum $2N + 2$ probabilities $\Pr(\gamma_m)$ described in the preceding section can produce the continuous canonical phase distribution associated with the truncated state $|\psi\rangle$ of Eqn (5.23). The original projection synthesis of \cite{12} was the first proposed technique capable, at least in principle, of measuring such probabilities. The technique, although theoretically simple, is very difficult to implement practically as it requires the use of an exotic reciprocal binomial state as a reference field, which, to date, has yet to be produced in the laboratory. An extension of the original projection synthesis proposal is the recently proposed multi-mode technique \cite{61}, presented here in Chapter 3. As discussed, the advantage of this technique is that it would need only coherent references states to obtain the probability $\Pr(\gamma_m)$ necessary to produce the canonical phase distribution. By replacing one of the coherent references states with a binomial state, we saw in the preceding section how $N + 1$ probabilities $\Pr(\theta_m)$ associated with the phase angles $\theta_m$ can be obtained from a single measurement apparatus. For such a measurement the input mode operators need to be related to the output mode operators
CHAPTER 5. QUANTUM OPTICAL PHASE AND ITS MEASUREMENT

Figure 5.2: Eight-port interferometer for measuring the canonical phase distribution of weak fields. The field in state $|\psi\rangle_0$ to be measured is in the input mode labelled $In_0$, while the reference field in state $|r_j\rangle_j$, $j = 1, 2, 3$ is in input mode $In_j$. A photodetector is in each output port. The dotted phase-shifters are for mathematical convenience only, and do not affect the results.

by a discrete Fourier transformation. In general, this is not necessary for the case when the reference states are coherent states as the amplitudes and phases can be adjusted to compensate. We saw in Section 3.4, however, that such a transformation of mode operators did produce near optimal results for the specific case when $N + 1 = 4$. So, with such a multiport, in conjunction with $N + 1$ photodetectors that can discriminate between zero, one, and many photons, it should be possible to obtain experimentally, for the first time, a direct measurement of the canonical phase distribution for a weak optical field. This is in contrast to reconstructing the distribution from its moments, as we have suggested earlier, or by reconstructing the complete state first and then calculating the distribution (see references given in [50]).

Given that operational phase was introduced to circumvent the problems associated with measuring canonical phase, it is perhaps a surprising coincidence that the experiment of Noh et al. [37, 40] and Torgerson and Mandel [72, 73] to measure operation phase does, in fact, use a linear optical multiport of the form mentioned above. To see that the linear multiport does in fact relate the input mode operators to the output mode operators by a discrete Fourier transformation consider the eight-port interferometer used by Noh et al. and Torgerson and Mandel in their experiments. Such an apparatus, illustrated in Figure 5.2, consists of four 50:50 symmetric beam-splitters at the corners of a square. The phase-shifter
5.2. CANONICAL PHASE DISTRIBUTION FOR WEAK OPTICAL FIELDS

labelled \(-i\) between the two beam-splitters on the right shifts the phase by \(\pi/2\). The field state \(|\psi\rangle_0\) to be measured is in input mode 0. The phase-shifter in input mode 1 allows the phase of the reference field state \(|r_1\rangle_1\) to be changed, similarly for the phase-shifter in input mode 2. In the experiment to measure operational phase, the reference states \(|r_2\rangle_2\) and \(|r_3\rangle_3\) in input mode 2 and 3 were in the vacuum states. The dotted phase-shifters before detectors D_1, D_2 and D_3, which are not present in the original interferometer, are merely inserted here for mathematical convenience. As the detectors detect photons, their operation will not be affected by phase-shifters in front of them.

As defined by Eqn (3.27), a single 50:50 symmetric beam-splitter transforms the input creation operators \(\hat{b}^\dagger\) and \(\hat{c}^\dagger\) in accord with

\[
\hat{S}_1 \hat{b}^\dagger \hat{S}_1^\dagger = 2^{-1/2} (\hat{b}^\dagger + i\hat{c}^\dagger) \tag{5.29}
\]

\[
\hat{S}_1 \hat{c}^\dagger \hat{S}_1^\dagger = 2^{-1/2} (i\hat{b}^\dagger + \hat{c}^\dagger) \tag{5.30}
\]

where \(\hat{S}_1\) is the unitary operator for the action of the single beam-splitter. By using this relation successively, it is not difficult to show that the input creation operators for the eight-port interferometer, including the dotted phase-shifters, are transformed as

\[
\hat{S} \hat{a}_i^\dagger \hat{S}^\dagger = \exp(i\gamma) \sum_{j=0}^{3} U_{ij} \hat{a}_j^\dagger \tag{5.31}
\]

where \(\hat{S}\) represents the total unitary transformation of all the optical elements and

\[
U_{ij}^* = \frac{\omega^{ij}}{2} \tag{5.32}
\]

with \(\omega = \exp(-i\pi/2)\), provided we set the phase-shifter in input mode 1 and 2 to shift the phase by \(\pi/2\), that is to attach a value \(-i\) to them. Expressions (5.31) and (5.32) are in agreement with (5.12) for \(N = 3\) apart from the phase factor \(\exp(i\gamma)\), which depends on the difference between the distance between beam-splitters and an integer number of wavelengths. This phase factor does not affect the photocount probabilities and can be ignored. So we see that the eight-port interferometer and the associated photodetectors used by Noh et al. to measure operational phase is precisely equivalent to the apparatus needed to measure the canonical phase distribution for the weak state \(|\psi\rangle\) given by Eqn (5.23), with \(N + 1 = 4\). The difference between the operational phase measurement and the proposed canonical phase measurement is in the reference states present at the input of the interferometer.
Binomial reference state

For the probabilities associated with the photo-detection events to correspond to canonical phase measurements, it is necessary for the reference state in input mode 1 of the interferometer to have the first four terms in its number state expansion proportional to the square root of the binomial coefficients,

$$|0⟩ + \sqrt{2}|1⟩ + \sqrt{2}|2⟩ + |3⟩,$$

as well as there being vacuum states present in input modes 2 and 3. Such an arrangement is then consistent with the single-shot apparatus introduced in Section 5.1.2 with $N + 1 = 4$. Thus we see that the eight-port apparatus, without modification, can be used to synthesize the projection of the state to be measured onto one of four truncated phase states $|θ_n⟩$, with $θ_n = 0, \pi/2, \pi, 3\pi/2$. Specifically the probability of the measurement event $(0, 1, 1, 1)$, that is the detection of zero photocounts at detector $D_0$ and one at each of $D_1$, $D_2$ and $D_3$, is given by Eqn (5.7) and is proportional to the square of the modulus of the projection of the measured state onto the truncated phase state

$$|θ_0⟩ = 2^{-1}(|0⟩ + |1⟩ + |2⟩ + |3⟩)$$

while the probability of the event $(1, 0, 1, 1)$ is proportional the square of the modulus of the projection of the measured state onto the truncated phase state

$$|θ_1⟩ = 2^{-1}(|0⟩ + i|1⟩ − |2⟩ − i|3⟩)$$

and so on, in accord with Eqn (5.18) with $N + 1 = 4$. Repeating the experiment a number of times with a reproducible state will allow a probability $Pr(θ_n)$ with $n = 0, 1, 2, 3$ to be measured for each of the four events $(0, 1, 1, 1)$, $(1, 0, 1, 1)$, $(1, 1, 0, 1)$ and $(1, 1, 1, 0)$ respectively. To plot the canonical phase distribution, it is necessary to obtain more probabilities associated with different phase angles. This can be achieved by repeating the experiment with a $φ$ phase shift applied to the binomial reference state. We see from Eqn (5.18) that such a phase shift will shift the phase of all four retrodictive phase states from $θ_n$ to $θ_n + φ$. Figure 5.3 shows a simulated measured distribution containing sixteen points obtained by shifting the reference phase three times.

So we find that the canonical phase distribution can be obtained from the existing apparatus used by Noh et al. to measure their operational phase. The only changes that need
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\[
\langle n \rangle = 0.076 \\
\eta = 1.0
\]

\[
\frac{1}{2\pi} - \pi \theta \quad (\text{rad})
\]

\[
\langle n \rangle = 0.5 \\
\eta = 1.0
\]

\[
\frac{1}{2\pi} - \pi \theta \quad (\text{rad})
\]

Figure 5.3: Canonical phase probability distribution \(P(\theta)\) for a coherent state field with, on the left, a mean photon number of 0.076 and, on the right, a mean photon number of 0.5. The full line is the theoretical result and the dots are simulated measured results with ideal detectors and a squeezed reference state.

to be made to the experiment is to replace the low intensity coherent reference state present in input mode 1 of the operation phase measurement device with a binomial reference state.

Squeezed reference state

The above analysis and suggested procedure assumes that the reference field is in a perfect binomial state. If, instead, we use the squeezed state approximation to the binomial state as derived in Appendix D, then the vacuum state coefficient differs from the ideal value and the measured state is no longer projected onto the truncated phase state \(|\theta_m\rangle\) but is instead projected onto a state proportional to

\[
|0\rangle + \exp(i\theta_m)|1\rangle + \exp(2i\theta_m)|2\rangle + 1.0146 \exp(3i\theta_m)|3\rangle.
\]

We would expect that this would lead to some small errors when the procedure suggested above is applied. In practice, if we are measuring a state, such as a coherent or squeezed state, that does not have a truncated photon number distribution, the error caused by the modulus of the three-photon coefficient in expression (5.36) differing from unity may in general be smaller that the error caused by assuming the input state can be sufficiently well approximated by the truncated expression of Eqn. (5.23) with \(N + 1 = 4\). In Figure 5.3 we show the canonical phase distribution histogram, with points rather than bars for clarity, obtained from a simulated experiment for a coherent state with a mean photon number of 0.076, which is comparable to the field strength of interest in Ref. [72], using a squeezed
reference state. The close agreement with the canonical distribution is apparent. For weaker fields, for example the other field of interest in Ref. [72] with a mean photon number of 0.047, the agreement is even closer. Agreement is still good for stronger coherent state fields with mean photon numbers of 0.139 and 0.23, as used in Ref. [73], with divergence from the canonical distribution becoming apparent for mean photon numbers of around 0.4. The histogram on the right in Figure 5.3 shows simulated results for a coherent state with a mean photon number of 0.5. The error here is almost entirely due to the truncation of the coherent state after the three photon component rather than to the non-unit coefficient of the fourth term in Eqn (5.36). A mean photon number of 0.5 represents the approximate limit to the field strength for a coherent state for which this measurement technique is suitable.

Coherent reference states

The alternative method to the above approach for measuring the phase distribution is the projection synthesis method outlined in Chapter 3. This technique, as discussed, requires three coherent reference states $|\alpha_1\rangle_1$, $|\alpha_2\rangle_2$ and $|\alpha_3\rangle_3$ in the input modes 1, 2 and 3, respectively, of the interferometer. By appropriately choosing the coherent amplitudes $\alpha_1$, $\alpha_2$ and $\alpha_3$, we can engineer the retrodictive state conditioned on observing the photo-detection sequence $(0, 1, 1, 1)$ at photo-detectors $D_0$, $D_1$, $D_2$ and $D_3$ to be proportional to the truncated phase state

$$|\theta_0\rangle = 2^{-1}(|0\rangle + |1\rangle + |2\rangle + |3\rangle).$$ (5.37)

To find the amplitudes of the coherent reference states, we solve the characteristic polynomial for $N = 3$ with $\psi_n = 1$ to give the three complex roots

$$\beta_1 = -0.2168 + i1.3563$$
$$\beta_2 = -1.2984$$
$$\beta_3 = -0.2168 - i1.3563.$$
Then from Eqn (3.32), with $U_{ij}$ defined by (5.32), we find the necessary amplitudes of the coherent reference states to be

\[ \alpha_1 = 1.4358 \]
\[ \alpha_2 = 0.2168 \]
\[ \alpha_3 = 0.0795 \]

The probability of observing the specific photocount sequence \((0, 1, 1, 1)\) over repeated trials is then found from Eqns (3.30) and (2.12) with $\hat{\Gamma} = \hat{1}$ to be $\Pr(0, 1, 1, 1) = 0.0425|\langle \theta_0 | \psi \rangle|^2$. So the value of the projection of the input state onto the truncated phase state of Eqn (5.37) is $(0.0425)^{-1}$, or approximately 23.5, times greater than the probability of observing the photocount sequence \((0, 1, 1, 1)\). Measuring the occurrence frequency for this specific observation gives one value which can be plotted on the phase probability histogram. To obtain additional points corresponding to different values of phase the experiment needs to be repeated with a phase-shift applied to the state to be measured in input mode 0. The distribution histogram can then be built up in this way by directly measuring the value of the phase distribution for different values of phase.

5.3 Some practical considerations

In addition to the statistical error inherent in obtaining probabilities from measured relative frequencies, in a practical experimental situation errors can arise from collection inefficiencies, non-unit quantum efficiencies for one, two and multiple photon detection, dead times and accidental counts arising from dark counts and background light. The fact, however, that Noh et al. [37, 41] have performed successful experiments involving the measurement of joint detection probabilities with an eight-port interferometer, by means of photon counting, for states with field strengths similar to those of interest in this paper is an encouraging indication that there should be no insurmountable difficulties for the method proposed here arising from such errors. It is still however worth considering some specific aspects of the sources of error. In the experiments of Noh et al. [37, 41] photon count rates were of the order of $10^4$ per second with a counting interval of about 5 $\mu$s, to give the required small mean photon number, and dead-time effects were negligible. In the present proposed experiment dead times are even less important because it is only necessary to discriminate
among zero, one and many counts rather than among general numbers of counts. Dark counts can be reduced to about 200 per second or even to 20 per second by cooling the detectors and background light can be reduced by appropriate shielding. In the event that the residual dark and background counts are not negligible, the measured joint probabilities of the four photocount events can be corrected by a deconvolution procedure using the data obtained by blocking the input signals.

Concerning detector efficiencies, even if collection efficiencies are made to approach unity by, for example, suitable geometry and reflection control, there will still be some detector inefficiency due to non-unit quantum efficiency, so some correction for detector inefficiency may be needed. Conventional single-photon counting module detectors can have an efficiency of around 0.7, while visible light photon counters that distinguish between single-photon and two-photon incidence can have quantum efficiencies of about 0.9 with some sacrifice of smallness of dark count rate. We denote the one-photon detection efficiency, that is the probability of recording one photocount if one photon is present, by \( \eta \). Then, as dead times are not important, the general multiple detection efficiency is such that the probability of recording \( n \) photocounts if \( N \) photons are present is \( \binom{N}{n} \eta^n (1-\eta)^{N-n} \) where the first factor is the binomial coefficient. If \( \eta \) is the same for all four detectors the probability for the joint four-count detection event \((m, n, p, q)\) is given by

\[
P_c(m, n, p, q) = \sum_{s=m}^{\infty} \sum_{t=n}^{\infty} \sum_{u=p}^{\infty} \sum_{v=q}^{\infty} \binom{s}{m} \binom{t}{n} \binom{u}{p} \binom{v}{q} \times \eta^{m+n+p+q} (1-\eta)^{s+t+u+v-m-n-p-q} P_I(s, t, u, v)
\]

where \( P_I(s, t, u, v) \) is the probability that an ideal detector would have detected the joint four-count event \((s, t, u, v)\). The relation can be inverted by use of the four-function Bernoulli transform, which is straightforward to derive in a similar manner to that of the one-function transform stated in the Appendix or the two-function transform derived in Ref. 46. This allows us to calculate the ideal probabilities from the measured probabilities and thus correct for non-unit efficiencies.

Although we can correct for non-unit efficiencies an analysis shows that the effect of not correcting for them is not as serious as it may first appear. Essentially this is because

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3In the case where dead times are significant their effect can be substantially reduced by use of beam-splitters. See, for example, 42.
5.3. SOME PRACTICAL CONSIDERATIONS

\[ \pi_0 - \pi \theta (\text{rad}) \]

\[ \frac{1}{2\pi} P(\theta) \]

\[ \langle n \rangle = 0.076 \]

\[ \eta = 0.6 \]

Figure 5.4: Uncorrected simulated measurements (dots) and the theoretical canonical phase distribution (full line) for a coherent state field with a mean photon number of 0.076 where the photodetectors have an efficiency \( \eta = 0.6 \).

The four measured probabilities are always normalized so their sum is unity. The major effect of \( \eta \) not being unity is, as can be seen from (5.38), that the probabilities for the four events \((0,1,1,1), (1,0,1,1), (1,1,0,1)\) and \((1,1,1,0)\) to be actually recorded are reduced by a factor \( \eta^3 \). As this affects the event probabilities uniformly, however, the effect vanishes upon normalization. The next order effect is that some ideal four-count events, such as \((1,1,1,1)\) and \((0,2,1,1)\), are registered, for example, as \((0,1,1,1)\) because of the inefficiency. The effect of this is only partially removed by the normalization. Ideal higher-count events also contribute to the error, but for weak fields the probability of ideal high-count events is not large. A numerical calculation of the total effect of non-unit efficiency, including the effect of normalization, shows that the proposed procedure is not highly sensitive to detector inefficiency, provided the efficiency is reasonable, for the weak states of interest. More precisely, for coherent states with a mean photon number up to 0.5 photons, as discussed above, the error in the final normalized probabilities is less than 2% for \( \eta \geq 0.9 \). For a mean photon number of 0.076, the error is less than 0.5% for such efficiencies. In Figure 5.4 we show the effect of a poorer efficiency of \( \eta = 0.6 \) for a mean photon number of 0.076.

To produce the squeezed state required as an approximate binomial state, we note that squeezed vacuum states can be transformed into various types of squeezed states, the squeezing axis can be rotated, coherent amplitude can be added and the squeezing can be controlled independently of the coherent amplitude. The degree of squeezing needed here is of a magnitude that is a realistic expectation either now or in the near future [7].
5.4 Summary

We have shown in this chapter how a single-shot measurement of canonical phase can be performed. We emphasize that this is not a deterministic measurement, that is, only some of the experiments provide successful measurements. The results of other measurements are discarded. This happens in a significant number of measurement techniques, for example, that used by Noh et al. \cite{37, 38, 41} to measure their operational phase. We have also shown how the eight-port interferometer used by Noh et al. can also be used to measure the canonical phase distribution given by Eqn (5.1), where the canonical phase is defined as the complement of photon number. The procedure is applicable for weak fields in the quantum regime, by which we mean explicitly states for which number state components for photon numbers greater than three are negligible. For coherent states, this requirement translates to a mean photon number of a half a photon or less. This is precisely the quantum regime in which large differences between the operational phase and the canonical phase distributions are most apparent. For example fields of interest in Refs \cite{72, 73} are coherent states with mean photon numbers of 0.23, 0.139, 0.076 and 0.047. The success of the experiments in the foregoing references indicates that the procedure proposed in this paper should be viable, given a reliable source of the required reference state.

The procedures in this chapter have advantages over the original projection synthesis method proposed for measuring the canonical phase distribution. The most significant of these is that the procedures require reference states that are either coherent states, or states that can be derived from coherent states by reliable procedures such as squeezing. Another advantage is that we require only photodetectors that can distinguish among zero, one and more than one photocounts. The measurements are not particularly sensitive to photodetector inefficiency and, for reasonably good detector efficiencies, no corrections should be needed. Overall, we feel that the proposal in this thesis brings the measurement of the canonical phase distribution for weak optical fields closer to reality.
Chapter 6

Conclusion

The generation of retrodictive quantum optical states was the focus of this thesis. For this purpose we investigated in particular the use of the lossless optical multiport. The motivation for this was to try to introduce simple experimental techniques capable of extending the current range of measuring apparatuses in the field of quantum optics. In Chapter 3 I showed that a lossless optical multiport, constructed from an array of beam-splitters and phase-shifters, a coherent reference state and photodetection is all that is needed to generate any retrodictive state with a finite number of number-state components. Such an apparatus has applications in measurement, particularly projection synthesis, and quantum state preparation. We showed in Chapter 4 how some measurement techniques involving predictive optical probe states can be converted to experiments involving retrodictive probe states which are far easier to generate in practice. Another finding of this thesis was the single-shot measuring device for canonical phase presented in Chapter 5. With the theoretical description of canonical phase already well formulated, the introduction of such an apparatus removes the final distinction separating canonical phase from all other observables in quantum optics, that is, canonical phase can now be measured, at least in principle.

Just as there exists an intimate relationship between the process of preparation and a predictive state in the conventional predictive formalism of quantum mechanics, there exists a symmetric relationship between the process of measurement and a retrodictive state in the retrodictive formalism. One is said to be the cause of the other. In Chapter 2 I review a formalism of quantum mechanics which did not discriminate against either process. From such a formalism, both the predictive and retrodictive formalisms were derived. We found,
consistent with the original conclusion of Aharonov et al. [3], that the difference between
preparation and measurement is not an intrinsic property of quantum mechanics. It is
reasonable to assume, therefore, that it results from the macroscopic laws of the universe
as a whole and must be introduced into quantum mechanics as an additional postulate. By
introducing causality in the form of the postulate that messages can be sent only forwards
in time, we regained the usual normalisation conditions which do not apply symmetrically
to the predictive density operator and the measurement POM elements. Provided the
asymmetric normalisation conditions are maintained, causality allows a symmetry in the
use of predictive and retrodictive states. That is causality does not lie in the time direction
of propagation of the states as may have been thought at first glance as mentioned in
the Introduction. Using this symmetric formalism, we found that any physical ensemble of
preparation and measurement devices can always be reduced to either a single measurement
device or a single preparation device. We used this result to investigate linear optical
multiport devices as a general measurement device.

This investigation, although similar to the original projection synthesis proposal, had
two important differences. First, we generalised the optical element, a single beam-splitter,
to include an arbitrary array of beam-splitters and phase-shifters. In doing so we found
that we can synthesise the same arbitrary set of projections with classical, that is coherent,
reference states as achieved by the original projection synthesis with non-classical reference
states. So by limiting ourselves to coherent reference states, linear optics and photodetection
we can construct a general apparatus that is capable of producing any retrodictive quantum
state with a finite number of photon number state coefficients. That is, a measuring appa-
tratus that can project an initial predictive state onto any state vector of a finite dimensional
Hilbert space. We found, quite surprisingly, that it is less demanding practically to generate
a wide class of retrodictive states that it is to produce the predictive counterparts. This
asymmetry originates from the ease in which simultaneous single photons can be observed
as opposed to created.

A procedure for the optimum measurement technique, defined in the text as the one
which will produce the desired retrodictive state with maximum likelihood, was derived.
We found that, although the amplitudes and phases of the coherent reference states are
constrained, the elements of the transformation matrix are not. This corresponds to ad-
justing the transmission to reflection ratio of the beam-splitters, the basic elements of the
multiport. To illustrate such a protocol, we considered three apparatuses which are capable of projecting onto a truncated phase state of $N + 1 = 3$ dimensions, one of which is optimal.

In Chapter 4 we exploited the fact that non-classical retrodictive states are simpler to produce than the predictive equivalent. This was done by redesigning two experimental proposals with a need for such non-classical probe states, so that the necessary probe state could be a retrodictive state. On the whole, both experiments are simpler, involving only a single coherent reference state, two beam-splitters and photodetection. This example serves to demonstrate the advantage in viewing measurement in the retrodictive formalism as it is computationally much easier to evolve backwards the single photon number states conditioned on detection, than it is to evolve forward all possible superposition states that may be at the input of such an apparatus.

Another notable result of this thesis was presented in Chapter 5. We found that by replacing just one of the coherent inputs in the general measuring apparatus of Chapter 3, it was possible to produce a single-shot measurement of canonical phase. The price paid for obtaining a single-shot measuring apparatus is a tradeoff with ease of construction, since the necessary reference state is a binomial state. However, we were able to show that such a state can be sufficiently well approximated by a suitably squeezed state. Such a measurement scheme has long been sought after as it was the last difficulty associated with the challenging concept of phase. Quite ironically, the apparatus used to measure “measured phase”, a concept introduced because of the difficulties in measuring canonical phase, can in fact be used to measure the canonical phase distribution, without alteration. By expressing the canonical phase distribution in terms of the Fourier coefficients, it was demonstrated that only a finite number of points need be sampled from the continuous distribution in order to reconstruct the whole canonical phase distribution for a state possessing a finite number of photons. Alternatively, more probabilities can be measured to produce the canonical phase distribution histogram directly. With the introduction of such phase measurement schemes, both the single-shot method and the linear optical apparatus, there seems little need to define phase dependent observables based upon a measurement scheme which is simple to implement.

In addition to the application to measurement discussed in this thesis, it has already been recognized that the retrodictive formalism of quantum mechanics should have applications such as quantum communication [15]. A potential application of retrodictive quantum state
engineering that we have mentioned but not explored in this thesis is for predictive state engineering. If a two-mode entangled state can be prepared with a significant number of non-negligible photon-number-state coefficients then sending a suitably engineered retrodictive state into one of these modes will result in an associated predictive state in the other mode. As it is easier to generate exotic retrodictive states than exotic predictive states, such a technique could prove useful.

Overall, we can conclude that the retrodictive formalism of quantum mechanics is not just a curiosity of philosophical value only. Instead, it has the potential to be of real practical value.
Appendix A

Corrections for imperfect photon detection

There are many physical processes which manifest themselves as an imperfection in realistic photodetectors. Practical photodetectors suffer from non-unit quantum efficiencies that reduce the number of counts, the presence of dark counts not associated with the absorption of a photon and a non-zero dead time following a count during which no other counts are registered. Using weak fields, in which we are particularly interested, and sufficiently long gating times reduces the effect of the dead time. Dark counts are related to thermal excitations within the detector and are independent of the number of photons incident upon the detector. Generally, sufficient cooling of the detectors can minimise this effect. If the remaining dark counts are not negligible, then a deconvolution of the measured data with the counts obtained when the detector is blocked from the light source can remove the majority of the remaining counts [38]. Detector inefficiencies is a collective term including the effects of such processes as quantum efficiency, mode mismatching and coupling efficiency. All these inefficiencies can be represented by one parameter, $\eta$, the detector efficiency.

In this appendix I discuss the most dominate processes relevant to the proposals presented in this thesis; detector inefficiency. Most importantly, I review how the measured probabilities obtained from the photocount distribution of a single photodetector can be corrected to obtain the photon number probability distribution that would be observed if the detector had unit efficiency.
A.1 Detector inefficiency

Real photodetectors are imperfect. The probability that \( m \) photocounts are registered, \( P(m) \), is related to the probability that \( n \) photos where present, \( Q(n) \), by

\[
P(m) = \sum_n p(m|n)Q(n)
\]

(A.1)

where \( p(m|n) \) is the conditional probability that there are \( m \) photocounts if there was \( n \) photons present. For a photodetector with finite efficiency \( \eta \), it has been shown \[30, 35\] that the conditional probability can be expressed as,

\[
p(m|n) = \binom{n}{m} (1 - \eta)^{m-n} \eta^n.
\]

(A.2)

In general, a distribution of the form

\[
P(m) = \sum_{n=m}^{\infty} Q(n) \binom{n}{m} (1 - \eta)^{m-n} \eta^m
\]

(A.3)

is known as a Bernoulli distribution, and can be inverted to give \[46\]

\[
Q(n) = \sum_{m=n}^{\infty} P(m) \binom{m}{n} (\eta - 1)^{m-n} \eta^{-m}.
\]

(A.4)

The two distributions \( P(m) \) and \( Q(n) \) form a Bernoulli transform pair. For the case considered here, Equation (A.4) gives a way in which the true photon number distribution \( Q(n) \) can be extracted from the observed photocount distribution \( P(m) \). For situations where there is a joint probability distribution corresponding to the photocount of two photodetectors, then a two-function Bernoulli transform pair can be derived \[46\]. Such a derivation can readily be extended to a multi-function Bernoulli transform pair which is necessary to extract the photon number distribution from the photocount distribution obtained from the multiport devices considered in this thesis.
Appendix B

Raising operator

It is the purpose of this appendix to derive an expression for the non-unitary operator $1\langle 0|\hat{S}^\dagger|N\rangle_1$, governing the evolution of a retrodictive state backwards in time through a beam-splitter. The evolution is conditioned on measuring $N$ photons in the output mode 1 of the beam-splitter, while a vacuum field is present in the input mode 1. We represent the (backward time) unitary evolution operator of the beam-splitter as $\hat{S}^\dagger$.

Writing the identity operator for the field mode 0 as $\sum_m |m\rangle_0 \langle m|$, where $|m\rangle_0$ is a photon number state, and expressing the kets as raising operators acting on the vacuum as $(\hat{b}^\dagger)^n/\sqrt{n!}|0\rangle$, we obtain

$$1\langle 0|\hat{S}^\dagger|N\rangle_1 = \sum_m 1\langle 0| (\hat{S}^\dagger\hat{b}_0^\dagger\hat{S})^m (\hat{S}^\dagger\hat{b}_1^\dagger\hat{S})^N \hat{S}^\dagger|0\rangle_1 |0\rangle_0 \langle m|,$$  

(B.1)

where the identity, $\hat{S}\hat{S}^\dagger = \hat{1}$, has been used repeatedly. The backward-time evolution is now represented by a mode transformation of the output mode operators to the input mode operators. The set of operators are related by the unitary transformation of Eqn (3.9). After substituting this relation into (B.1) and taking the inner product with the vacuum state it follows that

$$1\langle 0|\hat{S}^\dagger|N\rangle_1 = \sum_m \frac{(t\hat{a}_0^\dagger)^m (ir\hat{a}_0^\dagger)^N}{\sqrt{m!N!}} |0\rangle_0 \langle m|,$$  

(B.2)

where $t = \cos \theta$ and $r = \sin \theta$ are the transmission and reflection coefficients. After some simple algebra, this can be written as

$$1\langle 0|\hat{S}^\dagger|N\rangle_1 = \frac{(ir\hat{a}_0^\dagger)^N t\hat{a}}{\sqrt{N!}},$$  

(B.3)
where

\[ t^n = \sum_m t^m |m\rangle_{00} \langle m| \]  

(B.4)

is, in general, a non-unitary operator since \( t = \cos \theta \) can take on non-unit values. It is interesting to consider briefly the effect of the transformation \[ t^n \] when the transmission coefficient takes on the two extreme values 0 and 1. In the limit as \( t \to 1 \), it is straightforward to see that \( t^n \) approaches the identity operator. Since in the same limit the reflectance goes to zero, \[ t^n \] is only non-zero for \( N = 0 \), in which case it is just the identity, implying that the field propagates unchanged. This is consistent with what one would expect of the transformation operator for a beam-splitter which is totally transmitting as there is no coupling between the two fields.

In the other limit as \( t \to 0 \), the beam-splitter is totally reflecting acting like a double-sided mirror. In such case \( t^n \) goes to the vacuum state projector \( |0\rangle_{00} \langle 0| \), giving, as the non-unitary transformation operator \( t^n \),

\[ 1 \langle 0| S^\dagger |N\rangle_1 = i^N |N\rangle_{00} \langle 0|. \]  

(B.5)

Remembering that this is a backward-time-evolving operator, it is most natural to analyse this in the retrodictive picture. In this picture, the photodetector in output mode 1 acts like a photon source. Because the beam-splitter acts as a double-sided mirror, the photons are reflected into mode 0. Considering just the dynamics of mode 0, this would look as though the photon number of the state originally in that mode, that is the vacuum, suddenly was raised by \( N \) photons. This is consistent with the operator of \( t^n \). So we find that this operator does conform with our expectations in these simple limits.
Appendix C

Fourier coefficients of the phase distribution

It is the aim of this appendix to show that the Fourier coefficients introduced in Chapter 5,

\[ \alpha_q = \int P(\theta) \exp(-iq\theta) \, d\theta \quad (C.1) \]

associated with the state \(|\psi\rangle = \sum_{n=0}^{N} \psi_n |n\rangle\) are zero for \(|q| > N\).

Following on from Eqn (5.3), the probability density \(P(\theta)\) can be expressed as the trace of the product of the state to be measured \(\hat{\rho}\) and the POM element \(|\theta\rangle \langle \theta|\) as

\[ P(\theta) = \text{Tr} \left[ \hat{\rho} |\theta\rangle \langle \theta| \right] , \quad (C.2) \]

where

\[ |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(in\theta) |n\rangle \quad (C.3) \]

is the phase state introduced in Eqn (5.2). Substituting Eqn (C.2) into the Eqn (C.1) allows the Fourier coefficient \(\alpha_q\) to be associated with an operator \(\hat{\alpha}_q\) as

\[ \alpha_q = \text{Tr}[\hat{\rho} \hat{\alpha}_q] \quad (C.4) \]

where the operator

\[ \hat{\alpha}_q = \int |\theta\rangle \langle \theta| \exp(-iq\theta) \, d\theta \quad (C.5) \]

is defined. Substituting the expression for the phase state, Eqn (C.3), allows the operator of (C.5) to be written in the photon number basis as

\[ \hat{\alpha}_q = \sum_{n=0}^{\infty} |n + q\rangle \langle n| \quad (C.6) \]
for \( q = 0, 1, \ldots \), where the definition of the Kronecker delta

\[
\delta_{n,m} = \int \exp[i(n-m)\theta] \, d\theta
\] (C.7)

has been used to simplify. By taking the conjugate transpose of Eqn (C.5) it is straightforward to derive the operator relation \( \hat{\alpha}^\dagger_q = \hat{\alpha}_{-q} \), which gives, from Eqn (C.6), an expression for the operator \( \hat{\alpha}_q \), for \( q = -1, -2, \ldots \), in the photon number basis. We note as an aside that the operator \( \hat{\alpha}_q \) is equivalent to the non-unitary Susskind-Glogower “exponential” phase operator \([70]\) introduced as an attempt to represent phase as a non-Hermitian operator. As this derivation does not depend on phase being represented as an operator, the problems associated with the Susskind-Glogower formalism are not of concern here.

The most general description for the state of a system is to assign a density operator,

\[
\hat{\rho} = \sum_{n,m=0}^{\infty} \rho_{n,m} |n\rangle \langle m|
\] (C.8)

expressed here in the photon number basis. For such a state, we see from Eqns (C.4) and (C.6) that the Fourier coefficients \( \alpha_q \) can be expressed in term of the off-diagonal matrix elements \( \rho_{n,n+q} \) as

\[
\alpha_q = \sum_{n=0}^{\infty} \rho_{n,n+q},
\] (C.9)

with \( \alpha_{-q} = \alpha_q^* \). It is straightforward then to see that for a system containing at most \( N \) photons, that is the density matrix can be represented by a \((N+1) \times (N+1)\) dimensional matrix, that the coefficients \( \alpha_q \) are zero for \( q > N \). Since \( \alpha_{-q} = \alpha_q^* \), then it must follow that the coefficient are also zero for \( q < N \).

So we find, since \( |\psi\rangle \langle \psi| \) is a specific example of such a density matrix, that \( \alpha_q = 0 \) for all integer \( |q| > N \), as required.
Appendix D

Squeezed states as approximate binomial states

In this Appendix we show how the required binomial reference of Section 5.2.2 state can be approximated by a suitably squeezed state. The particular binomial state of interest to us is given by

\[
|B\rangle = \sum_{n=0}^{N} b_n |n\rangle = 2^{-N/2} \sum_{n=0}^{N} \left( \begin{array}{c} N \\ n \end{array} \right)^{1/2} |n\rangle
\]  

(D.1)

where \( \left( \begin{array}{c} N \\ n \end{array} \right) \) is the binomial coefficient. The binomial state derived in Ref. [59] with alternating signs for the number state coefficients can be obtained by phase shifting this state by \( \pi \).

The general form for a squeezed state is [35]

\[
|\alpha, \zeta\rangle = \sum_{n=0}^{\infty} \alpha_n |n\rangle = (\cosh |\zeta|)^{-1/2} \exp\left\{-\frac{1}{2}(|\alpha|^2 + t(\alpha^*)^2)\right\} \\
\times \sum_{n=0}^{\infty} \frac{(t/2)^{n/2}}{(n!)^{1/2}} H_n \left[ \frac{\alpha + t\alpha^*}{(2t)^{1/2}} \right] |n\rangle
\]  

(D.2)

where \( \zeta = |\zeta| \exp(i\phi) \) with \( |\zeta| \) being the squeezing parameter, \( t = \exp(i\phi) \tanh |\zeta| \) and \( H_n(x) \) is a Hermite polynomial of order \( n \). \( \alpha \) is the complex amplitude of the coherent state obtained in the limit of zero squeezing.

The first case we study is where we are interested in finding a squeezed state whose coefficients \( \alpha_n \) are proportional to the coefficients \( b_n \) of binomial state for the early terms,
that is for \( n \ll N \). In this case we can approximate the binomial coefficient by

\[
\binom{N}{n}^{1/2} = \frac{N^{n/2}}{\sqrt{n!}} \sqrt{(1 - \frac{1}{N})(1 - \frac{2}{N}) \ldots (1 - \frac{n-1}{N})} \\
\approx \frac{N^{n/2}}{\sqrt{n!}} \left[ 1 - \frac{n(n-1)}{4N} \right] \quad (D.3)
\]

We can approximate the Hermite polynomial for large \( x \) by its leading terms:

\[
H_n(x) \approx (2^n x^n - n(n-1)(2x)^{n-2}) \quad (D.4)
\]

We find, remarkably, that choosing \( t = 0.5 \) and \( \alpha = (2/3)N^{1/2} \) allows us to write

\[
\binom{N}{n}^{1/2} \approx \frac{(t/2)^{n/2}}{(n!)^{1/2}} H_n \left( \frac{\alpha + t\alpha^*}{(2t)^{1/2}} \right) \quad (D.5)
\]

for \( n \ll N \). Thus the first \( n \) number state coefficients of a squeezed state with these values of \( t \) and \( \alpha \) will be proportional to the required binomial coefficients to a good approximation.

With this degree of squeezing, the squeezed quadrature variance is 1/3 that of the vacuum level, that is, 4.77 dB below the standard quantum limit.

The opposite case to the above is where we require a small number of coefficients \( \alpha_n \) for \( n = N, N-1, N-2 \ldots \) to be proportional to \( b_n \). It is not as easy to obtain as general a relationship as the above so we look at each case individually. In this paper we are interested in the particular case with four values of \( b_n \), that is, \( N = 3 \). By using the explicit form of the Hermite polynomials in Eqn (D.2) and setting \( \alpha_2/\alpha_3 = b_2/b_3 \) and \( \alpha_1/\alpha_3 = b_1/b_3 \) we find that the values \( t = 0.5 \) and \( \alpha = (2 + 2^{1/2})/3 \) satisfy the two simultaneous equations obtained. We note that the required squeezing parameter \( \tanh^{-1} 0.5 \) is the same as for the first case above but the value 1.138 of \( \alpha \) varies slightly from 1.155, the value of \( (2/3)N^{1/2} \) with \( N = 3 \), which is required to make the first few coefficients of \( |\alpha, \zeta\rangle \) proportional to binomial coefficients. We also note that with perfect matching of the last three coefficients the ratio \( \alpha_0/\alpha_3 \) becomes 1.0146, a mismatch of only 1.5% with the corresponding binomial coefficient.
Bibliography


