DESIGN OF ACCELERATOR MAGNETS

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Abstract
Analytical and numerical field computation methods for the design of conventional and superconducting acceler- ator magnets are presented. The field in the aperture of these magnets is governed by the Laplace equation. The con- sequences for the field quality estimation and the ideal pole shapes (for conventional magnets) and ideal current distributions (for superconducting magnets) are described. Examples of conventional (LEP) and superconducting (LHC) dipoles and quadrupoles are given.

1 Guiding fields for charged particles
A charged particle moving with velocity \(\vec{v}\) through an electro-magnetic field is subjected to the Lorentz force

\[
\vec{F} = e(\vec{v} \times \vec{B} + \vec{E}).
\] (1)

While the particle moves from the location \(\vec{r}_1\) to \(\vec{r}_2\) with \(\vec{v} = d\vec{r}/dt\), it changes its energy by

\[
\Delta E = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \, d\vec{r} = e \int_{\vec{r}_1}^{\vec{r}_2} (\vec{v} \times \vec{B} + \vec{E}) \, d\vec{r}.
\] (2)

The particle trajectory \(d\vec{r}\) is always parallel to the velocity vector \(\vec{v}\). Therefore the vector \(\vec{v} \times \vec{B}\) is perpendicular to \(d\vec{r}\), i.e., \((\vec{v} \times \vec{B}) \, d\vec{r} = 0\). The magnetic field cannot contribute to a change in the particle’s energy. However, if forces perpendicular to the particle trajectory are needed, magnetic fields can serve for the guiding and the focusing of particle beams. At relativistic speed, electric and magnetic fields have the same effect on the particle trajectory if \(\vec{E} = c\vec{B}\). A magnetic field of 1 T is then equivalent to an electric field of strength \(E = 3 \cdot 10^8\) V/m. A magnetic field of one Tesla strength can easily be achieved with conventional magnets (superconducting magnets on an industrial scale can reach up to 10 T), whereas electric field strength in the Giga Volt / meter range are technically not to be realized. This is the reason why for high energy particle accelerators only magnetic fields are used for guiding the beam.

A charged particle forced to move along a circular trajectory looses energy by emission of photons according to

\[
\Delta E = \frac{1}{3\epsilon_0} \frac{e^2E^4}{(m_0c^2)^4R}
\] (3)

with every turn completed [24], where \(R\) is the curvature of the trajectory and \(E\) is the energy of the beam. A comparison between electron and proton beams of the same energy yields:

\[
\frac{\Delta E_p}{\Delta E_e} = \left(\frac{m_e c^2}{m_p c^2}\right)^4 = \left(\frac{0.511\text{ MeV}}{938.19\text{ MeV}}\right)^4 = 8.8 \cdot 10^{-14}.
\] (4)

For the heavier protons, synchrotron radiation is therefore not a limiting factor, however, the maximum energy is limited by the field in the dipole (bending) magnets. For a particle travelling on a circular closed orbit in an uniform bending field, the resulting Lorentz force \(e \, v \, B\) has to equal the centrifugal
force \( \frac{mv^2}{R} \). With the bending radius \( R \) given in meters and the magnetic flux density given in Tesla, the particle momentum
\[ p = \frac{mv}{0.2998 \cdot 10^9} = R \cdot B \cdot e \quad (5) \]
and therefore
\[ p = R \cdot B \cdot 0.2998. \quad (6) \]
The factor 0.2998 comes from the change of units \( \text{kgms}^{-1} \rightarrow \text{GeV/c} \). The maximum energy in circular lepton machines is limited by the synchrotron radiation and in linear colliders by the maximum achievable electric field in the accelerator structure. In circular proton machines the maximum energy is basically limited by the strength of the bending magnets.

2 Conventional and superconducting magnets

Figs. 1 - 3 show a “methamorphosis” between the conventional dipoles for the LEP (Large electron positron collider) and the single aperture dipole model used for testing the dipole coil manufacture for the LHC. All field calculations were performed using the CERN field computation program ROXIE. The field representations in the iron yokes are to scale, the size of the field vectors changes with the different field levels. Fig. 1 (left) shows the (slightly simplified) C-Core dipole for LEP. The advantage of C-Core magnets is an easy access to the beam pipe, but they have a higher fringe field and are less rigid than the H-Type magnets as shown in fig. 1 (right). Additional pole shims can be applied in order to improve the field quality in the aperture. The maximum field in the LEP dipoles is about 0.13 T. In order to reduce the effect of remanent iron magnetization, the yoke is laminated with a filling factor of only 0.27. It can be seen that the field is dominated by the shape of the iron yoke.

Fig. 1: Magnetic field strength in the iron yoke and field vector presentation of accelerator magnets. Left: C-Core dipole \((N \cdot I = 2 \times 5250 \text{ A}, B_1 = 0.13 \text{ T})\) with a filling factor of the yoke laminations of 0.27. Right: H-magnet \((N \cdot I = 12000 \text{ A}, B_1 = 0.3 \text{ T}, \text{Filling factor of yoke laminations } 0.98)\)

If the excitational current is increased above a density of about 10 A/m^2 one has to switch to superconducting coils. Neglecting the quantum-mechanical nature of the superconducting material, it is sufficient to notice that the maximum achievable current density in the superconducting coil is by the factor of 1000 higher than in copper coils. Magnets where the coils are superconducting, but the field shape is dominated by the iron pole are called super-ferric. Fig. 2 (left) shows the H-Type (super-ferric) magnet with increased excitation. The poles are starting to saturate and the field quality in the aperture is
Fig. 2: Magnetic field strength in the iron yoke and field vector presentation of accelerator magnets. Left: H-magnet with increased excitation current \(N \cdot I = 48000 \text{ A}, B_1 = 1.17 \text{ T}\). Right: Window frame geometry. \(N \cdot I = 180000 \text{ A}, B_1 = 2.28 \text{ T}\). Notice the saturation of the poles in the H-magnet.

decreased due to the increasing fringe field. This can be avoided by constructing so-called window-frame magnets as shown in fig. 2 (right).

The disadvantage of window-frame magnets is that the synchrotron radiation is partly absorbed in the (superconducting) coils and that access to the beam pipe is even more difficult. The advantage is a better field quality and that pole shims can be avoided.

It can be seen, that at higher field levels the field quality in the aperture is increasingly affected by the coil layout. Superconducting window-frame magnets are receiving considerable attention lately as high field (14-16 T) dipoles. As the coil winding is easier for window frame magnets than for the so-called cos \(\Theta\) magnets (shown in fig. 3, left), the application of the mechanically less stable materials with higher critical current density (e.g. \(\text{Nb}_3\text{Sn}\)) becomes feasible. The LHC superconducting magnets are of the cos \(n\Theta\) type. The advantage of the cos \(\Theta\) (dipole) magnets is that the field outside the coil drops with \(1/r^2\) and therefore the saturation effects in the iron yoke are reduced. Fig. 3 (right) finally shows the CTF (coil-test-facility) used for testing the manufacturing process of the LHC magnets. Notice that even with increased field in the aperture the field strength in the yoke is reduced in the cos \(\Theta\) magnet design.

Fig. 3: Magnetic field strength in the iron yoke and field vector presentation of accelerator magnets. Left: So-called cos \(\Theta\) magnet \(N \cdot I = 330000 \text{ A}, B_1 = 3.0 \text{ T}\). Right: LHC single aperture coil test facility \(N \cdot I = 480000 \text{ A}, B_1 = 8.33 \text{ T}\). Notice that even with increased field in the aperture the field strength in the yoke is reduced in the cos \(\Theta\) magnet design.
3 Field quality in accelerator magnets

The quality of the magnetic field is essential to keep the particles on stable trajectories for about $10^8$ turns. The magnetic field errors in the aperture of accelerator magnets can be expressed as the coefficients of the Fourier-series expansion of the radial field component at a given reference radius (in the 2-dimensional case). In the 3-dimensional case, the transverse field components are given at a longitudinal position $z_0$ or integrated over the entire length of the magnet. For beam tracking it is sufficient to consider the transverse field components, since the effect of the $z$-component of the field (present only in the magnet ends) on the particle motion can be neglected. Assuming that the radial component of the magnetic flux density $B_r$ at a given reference radius $r = r_0$ inside the aperture of a magnet is measured or calculated as a function of the angular position $\varphi$, we get for the Fourier-series expansion of the field

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$  

with

$$A_n(r_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(r_0, \varphi) \cos n\varphi d\varphi, \quad (n = 1, 2, 3, \ldots)$$  

$$B_n(r_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(r_0, \varphi) \sin n\varphi d\varphi. \quad (n = 1, 2, 3, \ldots)$$

If the field components are related to the main field component $B_N$ we get for $N = 1$ dipole, $N = 2$ quadrupole, etc.:

$$B_r(r_0, \varphi) = B_N(r_0) \sum_{n=1}^{\infty} (b_n(r_0) \sin n\varphi + a_n(r_0) \cos n\varphi).$$

The $B_n$ are called the normal and the $A_n$ the skew components of the field given in Tesla, $b_n$ the normal relative, and $a_n$ the skew relative field components. They are dimensionless and are usually given in units of $10^{-4}$ at a 17 mm reference radius. In practice the $B_n$ components are calculated in discrete points

$$\varphi_k = \frac{k\pi}{P} - \pi$$  

$k = 0, 1, 2, \ldots, 2P - 1$ in the interval $[-\pi, \pi)$ and a discrete Fourier transform is carried out:

$$A_n(r_0) \approx \frac{1}{P} \sum_{k=0}^{2P-1} B_r(r_0, \varphi_k) \cos n\varphi_k,$$  

$$B_n(r_0) \approx \frac{1}{P} \sum_{k=0}^{2P-1} B_r(r_0, \varphi_k) \sin n\varphi_k.$$  

The interpolation-error depends on the number of evaluation points and the amount of higher order multipole errors in the field. For the multipoles up to the order $n = 13$, 79 evaluation points ($P = 40$) are sufficient.
4 Field equations

Maxwell’s equations for the stationary case read in SI (MKS) units:

\[ \oint \vec{H} \cdot d\vec{s} = \int_A (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot d\vec{A}, \]  
(14)

\[ \oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_A \vec{B} \cdot d\vec{A}, \]  
(15)

\[ \int_A \vec{B} \cdot d\vec{A} = 0, \]  
(16)

\[ \int_A \vec{D} \cdot d\vec{A} = \int_V \rho dV. \]  
(17)

The vector fields \( \vec{E}, \vec{H}, \vec{D}, \vec{B} \) are called electric and magnetic field, and electric and magnetic induction (flux density), respectively. Eq. (14) is Ampère’s law as modified by Maxwell to include the displacement current distribution and eq. (15) is Faraday’s law of electromagnetic induction. Eq. (17) is Gauss’ fundamental theorem of electrostatics. The constitutive equations are:

\[ \vec{B} = \mu \vec{H} = \mu_0 (\vec{H} + \vec{M}), \]  
(18)

\[ \vec{D} = \varepsilon \vec{E} = \varepsilon_0 (\vec{E} + \vec{P}), \]  
(19)

\[ \vec{J} = \sigma \vec{E} + \vec{J}_{\text{imp}}, \]  
(20)

with the permeability of free space \( \mu_0 = 4\pi \cdot 10^{-7} \text{ H/m} \) and the permittivity of free space \( \varepsilon_0 = 8.8542 \cdot 10^{-12} \text{ F/m} \).

5 Maxwell’s equations in vector notation

The field equations can be written in differential form as follows:

\[ \text{curl} \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \]  
(21)

\[ \text{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \]  
(22)

\[ \text{div} \vec{B} = 0, \]  
(23)

\[ \text{div} \vec{D} = \rho. \]  
(24)

Eq. (21) - (24) are Maxwell’s equations in SI units and vector notation which is mainly due to O. Heaviside in the 1880s, who also eliminated the vector-potential and the scalar potential in Maxwell’s original set of equations. The link between Eq. (14) - (17) and (21) - (24) is given through the integral theorems:

\[ \int_V \text{div} \vec{g} dV = \int_A \vec{g} \cdot d\vec{A}, \]  
(25)

which is called Gauss’ (Ostrogradskii’s) divergence theorem and

\[ \int_A \text{curl} \vec{g} \cdot d\vec{A} = \oint \vec{g} \cdot d\vec{s}, \]  
(26)

which is Stokes’ surface integral theorem.
6 Maxwell’s equations for magnetostatic problems

For magnetostatic problems the time derivative can be set to zero, $\frac{\partial}{\partial t} = 0$, and Maxwell’s equations reduce to

\begin{align*}
\text{curl } \vec{H} &= \vec{J}, \\
\text{div } \vec{B} &= 0, \\
\vec{B} &= \mu(\vec{H})\vec{H} = \mu_0(\vec{H} + \vec{M}).
\end{align*}

(27) \hspace{2cm} (28) \hspace{2cm} (29)

7 Interface conditions

If we apply Ampère’s law in the integral form

\[ \oint \vec{H} \cdot d\vec{s} = \int_A \vec{J} \cdot d\vec{A}, \]

(30)

to the loop displayed in fig. 4 (left), and let $h \to 0$, then the enclosed current is zero, as in an infinitesimal small rectangle there cannot be a current flow. Therefore

\[ H_{t1} = H_{t2}, \]

(31)
i.e.,

\[ \vec{n} \times (\vec{H}_1 - \vec{H}_2) = 0. \]

(32)

Because of $\oint \vec{B} \cdot d\vec{A} = 0$ we get at the interface

\[ B_{n1} = B_{n2}, \]

(33)
i.e.,

\[ \vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0. \]

(34)

Now

\[ \frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\frac{B_{n1}}{B_{n2}}}{\mu_1 \vec{H}_{t1}} = \frac{\mu_1 H_{t1}}{\mu_2 H_{t2}} = \frac{\mu_1}{\mu_2}. \]

(35)

For $\mu_2 \gg \mu_1$ it follows that $\tan \alpha_1 \gg \tan \alpha_2$. Therefore for all angles $\frac{\pi}{2} > \alpha_2 > 0$ we get $\tan \alpha_1 \approx 0$, see also fig. 4 (right). The field exits vertically from a highly permeable medium into a medium with low permeability. We will come back to this point when we discuss ideal pole shapes of conventional magnets.
8 One-dimensional field computation for conventional magnets

Consider the magnetic (dipole) circuit shown in fig. 5 (left). With Ampère’s law in the integral form
\[ \oint \vec{H} \cdot d\vec{s} = \int_A \vec{J} \cdot d\vec{A}, \]
we can write
\[ H_{\text{iron}} s_{\text{iron}} + H_{\text{gap}} s_{\text{gap}} = NI, \tag{36} \]
\[ \frac{1}{\mu_0 \mu_r} B_{\text{iron}} s_{\text{iron}} + \frac{1}{\mu_0} B_{\text{gap}} s_{\text{gap}} = NI, \tag{37} \]

With \( \mu_r \gg 1 \) we get the easy relation
\[ B_{\text{gap}} = \frac{\mu_0 N I}{s_{\text{gap}}}. \tag{38} \]

![Figure 5: Magnetic circuit of a conventional dipole magnet (left) and a quadrupole magnet (right). Neglecting the magnetic resistance of the iron yoke, an easy relation between the air gap field and the required excitation current can be derived.](image)

For the quadrupole we can split up the integration path as shown in fig. 5 (right). From the origin to the pole (part 1), along an arbitrary path through the iron yoke (part 2), and back along the x-axis (part 3). Neglecting again the magnetic resistance of the yoke we get
\[ \oint \vec{H} \cdot d\vec{s} = \int_1 \vec{H}_1 \cdot d\vec{s} + \int_3 \vec{H}_3 \cdot d\vec{s} = NI. \tag{39} \]

As we will see later, in a quadrupole the field is defined by its gradient \( g \) with \( B_x = gy \) and \( B_y = gx \). Therefore the modulus of the field along the integration path 1 is
\[ H = \frac{g}{\mu_0} \sqrt{x^2 + y^2} = \frac{g}{\mu_0} r. \tag{40} \]

Along the x-axis the field integral is zero because \( \vec{H} \perp \vec{s} \). Therefore
\[ \int_0^{r_0} H dr = \frac{g}{\mu_0} \int_0^{r_0} r dr = \frac{g}{\mu_0} \frac{r_0^2}{2} = NI, \tag{41} \]
\[ g = \frac{2 \mu_0 N I}{r_0^2}. \tag{42} \]

Notice that for a given \( NI \) the field decreases linearly with the gap length of the dipole, whereas the gradient in a quadrupole magnet is inverse proportional to the square of the aperture radius \( r_0 \).
8.1 Permanent magnet excitation

For a magnetic circuit with permanent magnet excitation as shown in fig. 6 we can repeat the exercise with

\[ H_{\text{Iron}} s_{\text{iron}} + H_{\text{gap}} s_{\text{gap}} + H_{\text{mag}} s_{\text{mag}} = 0. \]  

(43)

Fig. 6: Dipole with permanent magnet excitation. Neglecting the magnetic resistance of the iron yoke, an easy relation between the air gap field and the required size of the permanent magnet can be derived.

In the absence of fringe fields we get with the pole surface \( A_{\text{gap}} \) and the magnet surface \( A_{\text{mag}} \):

\[ B_{\text{mag}} A_{\text{mag}} = B_{\text{gap}} A_{\text{gap}} = \mu_0 H_{\text{gap}} A_{\text{gap}}, \]  

(44)

For \( \mu_r \gg 1 \) we can again neglect the magnetic resistance of the yoke and from eq. 43 it follows that

\[ H_{\text{gap}} s_{\text{gap}} = -H_{\text{mag}} s_{\text{mag}}, \]  

(45)

\[ \frac{1}{\mu_0} B_{\text{mag}} \frac{A_{\text{mag}}}{A_{\text{gap}}} s_{\text{gap}} = -H_{\text{mag}} s_{\text{mag}}, \]  

(46)

\[ \frac{B_{\text{mag}}}{\mu_0 H_{\text{mag}}} = -\frac{s_{\text{mag}}}{s_{\text{gap}}} \frac{A_{\text{gap}}}{A_{\text{mag}}} = P, \]  

(47)

where \( P \) is called the permeance coefficient which (for \( A_{\text{gap}} = A_{\text{mag}} \)) becomes zero for \( s_{\text{gap}} \gg s_{\text{mag}} \) (open circuit) and becomes \( -\infty \) for \( s_{\text{mag}} \gg s_{\text{gap}} \) (short circuit). The case of \( A_{\text{mag}} > A_{\text{gap}} \) is usually referred to as the “flux concentration” mode. The permeance coefficient defines the point on the demagnetisation curve, i.e., the branch of the permanent magnet hysteresis curve in the second quadrant.

Fig. 7 shows a magnetic circuit with zero air gap, a circuit with \( s_{\text{gap}} = 2s_{\text{mag}} \) and an open circuit with a smarium cobalt magnet (remanent field 0.9T).

From eq. 44 and 45 we derive

\[ B_{\text{mag}} A_{\text{mag}} s_{\text{mag}} = \mu_0 H_{\text{gap}} A_{\text{gap}} \frac{-H_{\text{gap}} s_{\text{gap}}}{H_{\text{mag}}}. \]  

(48)

Therefore

\[ H_{\text{gap}} = \sqrt{\frac{(A_{\text{mag}} s_{\text{mag}})(-B_{\text{mag}} H_{\text{mag}})}{\mu_0 (A_{\text{gap}} s_{\text{gap})}}} = \sqrt{\frac{VOL_{\text{mag}} (-B_{\text{mag}} H_{\text{mag}})}{\mu_0 VOL_{\text{gap}}}}. \]  

(49)

For a given magnet volume, the maximum air gap field can be obtained by dimensioning the magnetic circuit in such a way that \( B_{\text{mag}} H_{\text{mag}} \) is maximum. It is usually said that this implies operating the permanent magnet with maximum energy density. At this point we have to be careful, however:
**The relation between the magnetization and the field is not linear (and not even unique). It depends not only on the external field but also on the history of how it was applied.**

**Because of the history dependence of the magnetization, the stored magnetic energy is not!** $W = 0.5BHV$ which holds only for linear material. We will have to study in detail the hysteresis effects of hard ferromagnetic material.

**For larger gap sizes the fringe fields cannot be neglected.**

In the presence of magnetized domains the magnetic field can be calculated as in vacuum, if all currents (including the magnetization currents) are explicitly considered

$$\text{curl} \vec{B} = \mu_0 (\vec{J}_\text{free} + \vec{J}_\text{mag}) = \mu_0 \vec{J}_\text{free} + \mu_0 \text{curl} \vec{M}. \quad (50)$$

Hence

$$\text{curl} \left( \frac{\vec{B} - \mu_0 \vec{M}}{\mu_0} \right) = \vec{J}_\text{free}. \quad (51)$$

For the magnetized media we therefore have

$$\vec{H} = \frac{\vec{B} - \mu_0 \vec{M}}{\mu_0} \quad (52)$$

or

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}). \quad (53)$$

For linear material we get the relation between $\vec{B}$ and $\vec{H}$ as:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \chi_m \vec{H} = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H} = \mu \vec{H}, \quad (54)$$

where $\mu_r$ is the relative permeability and $\chi_m$ is called magnetic susceptibility. On the other hand, eq. 53 is valid for all non-linear media, e.g., permanent magnets, where the magnetization persists without...
exterior field. The magnetic induction $\vec{B}$ is always source free, i.e., $\text{div} \vec{B} = 0$, but the magnetic field $\vec{H}$ is not:

$$\text{div} \vec{H} = \text{div} \left( \vec{B} - \frac{\mu_0 \vec{M}}{\mu_0} \right) = - \text{div} \vec{M} = \rho_{\text{mag}}.$$  \hspace{1cm} (55)

In eq. 55 a fictitious magnetic charge density $\rho_{\text{mag}} = - \text{div} \vec{M}$ was introduced with the field starting and ending on these fictitious magnetic charges, c.f. fig. 8.

Coming back to the magnetic circuit as shown in fig. 6 the conclusions are formally correct, as long as the magnetization $\vec{M}$ is constant. This is indeed the case for so-called rare earth material like samarium cobalt SmCo$_5$ or neodymium iron boron Nd$_2$Fe$_{14}$B but not for iron alloys with aluminum and nickel (Alnico) or Ferrite (e.g., Fe$_2$O$_3$), see fig. 9.

Now we shall consider a torus of ferromagnetic material (fig. 10) with $N$ excitational windings that excite the field

$$H = \frac{NI}{2\pi r}.$$  \hspace{1cm} (56)

Fig. 8: Field, induction and magnetization in permanent magnets.

Fig. 9: Demagnetization curves for different permanent magnet materials.
The induced voltage in the pick-up coil is

\[ U_i = \frac{d\phi}{dt} = \frac{dB}{dt} A. \]  \hspace{1cm} (57)

Time integration (\( \int U_i dt = BA \)) yields the corresponding values of \( I \) and \( U \) and consequently the hysteresis curve for \( H \) and \( B \). The surface spanned by the hysteresis curve are the magnetization losses. For the torus in fig. 10, the power needed for exciting the field is

\[ \frac{dW}{dt} = IN \frac{d\phi}{dt} = INA \frac{dB}{dt} = \frac{IN}{2\pi r} A2\pi r \frac{dB}{dt} = HV \frac{dB}{dt} . \]  \hspace{1cm} (58)

Therefore

\[ \frac{1}{V} \frac{dW}{dt} = H \frac{dB}{dt}, \]  \hspace{1cm} (59)

\[ \frac{W}{V} = \int_{B_1}^{B_2} H dB. \]  \hspace{1cm} (60)

For a complete cycle we get

\[ \frac{W}{V} = \oint H dB = \oint H\mu_0 (dH + dM) = \mu_0 \oint H dM. \]  \hspace{1cm} (61)

As a consequence, the working point with maximal \( B_{mag} H_{mag} \) guarantees that the air gap induction is maximum for a given aperture and magnet volume. However, this is not the state with maximum energy density in the permanent magnet material.

9 Harmonic fields

We will now show that in the aperture of a magnet (two-dimensional, current free region) both the magnetic scalar-potential as well as the vector-potential can be used to solve Maxwell’s equations:

\[ \vec{H} = -\text{grad} \Phi = -\frac{\partial \Phi}{\partial x} \vec{e}_x - \frac{\partial \Phi}{\partial y} \vec{e}_y, \]  \hspace{1cm} (62)

\[ \vec{B} = \text{curl}(\vec{e}_z A_z) = \frac{\partial A_z}{\partial y} \vec{e}_x - \frac{\partial A_z}{\partial x} \vec{e}_y, \]  \hspace{1cm} (63)

and that both formulations yield the Laplace equation. These fields are called harmonic and the field quality can be expressed by the fundamental solutions of the Laplace equation. Lines of constant vector-potential give the direction of the magnetic field, whereas lines of constant scalar potential define the ideal pole shapes of conventional magnets.
9.1 Magnetic scalar potential

Every vector field can be split into a source free and a curl free part. In case of the magnetic field with

\[ \vec{H} = \vec{H}_s + \vec{H}_m \]  

(64)

the curl free part \( \vec{H}_m \) arises from the induced magnetism in ferromagnetic materials and the source free part \( \vec{H}_s \) is the field generated by the prescribed sources (can be calculated directly by means of Biot Savart's law). With \( \text{curl}\vec{H}_m = 0 \) it follows that

\[ \vec{H} = -\nabla \Phi_m + \vec{H}_s \]  

(65)

and we get:

\[ \text{div} \vec{B} = 0 \]  

(66)

\[ \text{div} \mu(-\nabla \Phi_m + \vec{H}_s) = 0 \]  

(67)

\[ \text{div} \mu \nabla \Phi_m = \text{div} \mu \vec{H}_s \]  

(68)

While a solution of eq. 68 is possible, the two parts of the magnetic field \( \vec{H}_m \) and \( \vec{H}_s \) tend to be of similar magnitude (but opposite direction) in non-saturated magnetic materials, so that cancellation errors occur in the computation. For regions where the current density is zero, however, \( \text{curl}\vec{H} = 0 \) and the field can be represented by a total scalar potential

\[ \vec{H} = -\nabla \Phi \]  

(69)

and therefore we get

\[ -\mu_0 \text{div} \nabla \Phi = 0, \]  

(70)

\[ \nabla^2 \Phi = 0. \]  

(71)

which is the Laplace equation for the scalar potential. The vector-operator Nabla (Hamilton operator) is defined as

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \]  

(72)

and the Laplace operator

\[ \Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \]  

(73)

The Laplace operator itself is essentially scalar. When it acts on a scalar function the result is a scalar, when it acts on a vector function, the result is a vector.

9.2 Vector-potential

Because of \( \text{div} \vec{B} = 0 \) a vector potential \( \vec{A} \) can be introduced: \( \vec{B} = \text{curl} \vec{A} \). We then get

\[ \text{curl} \vec{A} = \mu_0(\vec{H} + \vec{M}), \]  

(74)

\[ \vec{H} = \frac{1}{\mu_0} \text{curl} \vec{A} - \vec{M}, \]  

(75)

\[ \frac{1}{\mu_0} \text{curl} \text{curl} \vec{A} = \vec{J} + \text{curl} \vec{M} \]  

(76)

\[ \frac{1}{\mu_0} (-\nabla^2 \vec{A} + \text{grad} \text{div} \vec{A}) = \vec{J} + \text{curl} \vec{M}, \]  

(77)
Since the curl (rotation) of a gradient field is zero, the vector-potential is not unique. The gradient of any (differentiable) scalar field \( \psi \) can be added without changing the curl of \( \vec{A} \):
\[
\vec{A}_0 = \vec{A} + \text{grad}\psi.
\] (78)

Eq. (78) is called a gauge-transformation between \( \vec{A}_0 \) and \( \vec{A} \). \( \vec{B} \) is gauge-invariant as the transformation from \( \vec{A} \) to \( \vec{A}_0 \) does not change \( \vec{B} \). The freedom given by the gauge-transformation can be used to set the divergence of \( \vec{A} \) to zero
\[
\text{div}\vec{A} = 0,
\] (79)
which (together with additional boundary conditions) makes the vector-potential unique. Eq. (79) is called the Coulomb gauge, as it leads to a Poisson type equation for the magnetic vector-potential. Therefore, from eq. 77 we get after incorporating the Coulomb gauge:
\[
\nabla^2 \vec{A} = -\mu_0(\vec{J} + \text{curl}\vec{M}).
\] (80)

In the two-dimensional case with no dependence on \( z \), \( \frac{\partial}{\partial z} = 0 \) and \( J = J_z \), \( \vec{A} \) has only a \( z \)-component and the Coulomb gauge is automatically fulfilled. Then we get the scalar Poisson differential equation
\[
\nabla^2 A_z = -\mu_0 J_z.
\] (81)

For current-free regions (e.g. in the aperture of a magnet) eq. (81) reduces to the Laplace equation, which reads in Cartesian coordinates
\[
\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0.
\] (82)

and in cylindrical coordinates
\[
r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0.
\] (83)

### 9.3 Field harmonics

A solution of the homogeneous differential equation (83) reads
\[
A_z(r, \varphi) = \sum_{n=1}^{\infty} (C_{1n} r^n + C_{2n} r^{-n})(D_{1n} \sin n\varphi + D_{2n} \cos n\varphi).
\] (84)

Considering that the field is finite at \( r = 0 \), the \( C_{2n} \) have to be zero for the vector-potential inside the aperture of the magnet while for the solution in the area outside the coil all \( C_{1n} \) vanish. Rearranging eq. (84) yields the vector-potential in the aperture:
\[
A_z(r, \varphi) = \sum_{n=1}^{\infty} r^n (C_n \sin n\varphi - D_n \cos n\varphi),
\] (85)

and the field components can be expressed as
\[
B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} nr^{n-1}(C_n \cos n\varphi + D_n \sin n\varphi),
\] (86)
\[
B_\varphi(r, \varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} nr^{n-1}(C_n \sin n\varphi - D_n \cos n\varphi).
\] (87)
Each value of the integer \( n \) in the solution of the Laplace equation corresponds to a different flux distribution generated by different magnet geometries. The three lowest values, \( n=1,2, \) and 3 correspond to a dipole, quadrupole and sextupole flux density distribution. The solution in Cartesian coordinates can be obtained from the simple transformations

\[
B_x = B_r \cos \varphi - B_\varphi \sin \varphi, \tag{88}
\]
\[
B_y = B_r \sin \varphi + B_\varphi \cos \varphi. \tag{89}
\]

For the dipole field (\( n=1 \)) we get

\[
B_r = C_1 \cos \varphi + D_1 \sin \varphi, \tag{90}
\]
\[
B_\varphi = -C_1 \sin \varphi + D_1 \cos \varphi, \tag{91}
\]
\[
B_x = C_1, \tag{92}
\]
\[
B_y = D_1. \tag{93}
\]

This is a simple, constant field distribution according to the values of \( C_1 \) and \( D_1 \). Notice that we have not yet addressed the conditions necessary to obtain such a field distribution. For the pure quadrupole (\( n=2 \)) we get from eq. 86 and 87:

\[
B_r = 2r C_2 \cos 2\varphi + 2r D_2 \sin 2\varphi, \tag{94}
\]
\[
B_\varphi = -2r C_2 \sin 2\varphi + 2r D_2 \cos 2\varphi, \tag{95}
\]
\[
B_x = 2(C_2 x + D_2 y), \tag{96}
\]
\[
B_y = 2(-C_2 y + D_2 x). \tag{97}
\]

The amplitudes of the horizontal and vertical components vary linearly with the displacements from the origin, i.e., the gradient is constant. With a zero induction in the origin, the distribution provides linear focusing of the particles. It is interesting to notice that the components of the magnetic fields are coupled, i.e., the distribution in both planes cannot be made independent of each other. Consequently a quadrupole focusing in one plane will defocus in the other.

Repeating the exercise for the case of the pure sextupole (\( n=3 \)) yields:

\[
B_r = 3r C_3 \cos 3\varphi + 3r D_3 \sin 3\varphi, \tag{98}
\]
\[
B_\varphi = -3r C_3 \sin 3\varphi + 3r D_3 \cos 3\varphi, \tag{99}
\]
\[
B_x = 3C_3 (x^2 - y^2) + 6D_3 xy, \tag{100}
\]
\[
B_y = -6C_3 xy + 3D_3 (x^2 - y^2). \tag{101}
\]

Along the x-axis (\( y=0 \)) we then get the expression for the y-component of the field:

\[
B_y = D_1 + 2D_2 x + 3D_3 x^2 + 4D_4 x^3 + \ldots \tag{102}
\]

If only the two lowest order elements are used for steering the beam, forces on the particles are either constant or vary linear with the distance from the origin. This is called a linear beam optic. It has to be noted that the treatment of each harmonic separately is a mathematical abstraction. In practical situations many harmonics will be present and many of the coefficients \( C_n \) and \( D_n \) will be non-vanishing. A successful magnet design will, however, minimize the unwanted terms to small values. It has to be stressed that the coefficients are not known at this stage. They are defined through the (given) boundary conditions on some reference radius or can be calculated from the Fourier series expansion of the (numerically) calculated field (ref. eq. 7) in the aperture using the relations

\[
A_n = nr_0^{n-1} C_n \quad \text{and} \quad B_n = nr_0^{n-1} D_n. \tag{103}
\]
10 Ideal pole shapes of conventional magnets

From the theory of electrostatics we remember that the potential difference between two points (close in space) is

$$d\varphi = -\vec{E} \cdot d\vec{s} = -(E_x dx + E_y dy + E_z dz) =$$

$$\left( \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz \right) = (\text{grad}\varphi) \cdot d\vec{s}. \tag{104}$$

Equipotentials are surfaces where \( \varphi \) is constant. For a path \( d\vec{s} \) along the equipotential it therefore results

$$d\varphi = \text{grad}\varphi \cdot d\vec{s} = 0 \tag{105}$$

i.e., the gradient is perpendicular to the equipotential. With the field lines (lines of constant vector potential) leaving highly permeable materials perpendicular to the surface (ref. chapter 7), the lines of total magnetic scalar potential define the pole shapes of conventional magnets. As in 2D (with absence of magnetization and free currents) the z-component of the vector potential and the magnetic scalar potential both satisfy the Laplace equation, we already have the solution for a bipolar field:

$$\Phi = C_1 x + D_1 y. \tag{106}$$

So \( C_1 = 0 \), \( D_1 \neq 0 \) gives a vertical (normal) dipole field, \( C_1 \neq 0 \), \( D_1 = 0 \) yields a horizontal (skew) dipole field. The equipotential surfaces are parallel to the x-axis or y-axis depending on the values of \( C \) and \( D \) and results in a simple flux density distribution used for bending magnets in accelerators. For the quadrupole:

$$\Phi = C_2 (x^2 - y^2) + 2D_2 xy \tag{107}$$

with \( C_2 = 0 \) giving a normal quadrupole field and with \( D_2 = 0 \) giving a skew quadrupole field (which is the above rotated by \( \pi/4 \)). The quadrupole field is generated by lines of equipotential having hyperbolic form. For the \( C_2 = 0 \) case, the asymptotes are the two major axes.

In practice, however, the magnets have a finite pole width (due to the need of a magnetic flux return yoke and space for the coil). To ensure a good field quality with these finite approximations of the ideal shape, small shims are added at the outer ends of each pole. The shim geometry has to be optimized (using numerical field computation tools) while considering, that with an increasing height of the shim saturation occurs at high excitation and leads to the field distribution being strongly dependent on the magnet excitation level. On the other hand, with a very thin and long shim the nature of the field generated will change and different harmonics are being generated. Fig. 11 shows the pole shape of a conventional dipole and quadrupole magnet, with magnetic shims.

Fig. 12 shows the cross-section of the LEP dipole and quadrupole magnets with iso-surfaces of constant vector-potential (for the dipole) and magnetic field modulus in the iron yoke for the quadrupole. The field quality in the dipole was improved by adding shims on the pole surface. In case of the quadrupole, however, the pole shape is defined as a combination of a hyperbola, a straight section and an arc. The points at which the segments are connected was found in an optimization process not only considering the multipole components in the cross-section, but also to provide for a part compensation of the end-effects.

11 Coil field of superconducting magnets

For coil dominated superconducting magnets with fields well above one Tesla the current distribution in the coils dominate the field quality and not the shape of the iron yoke, as it is the case in conventional magnets. The problem therefore remains how to calculate the field harmonics from a given current
Fig. 11: Pole shape of a conventional dipole and quadrupole magnet.

Fig. 12: Cross-section of the LEP dipole and quadrupole magnets with iso-surfaces of constant vector-potential (left) and magnetic field modulus (right).

distribution. It is reasonable to focus on the fields generated by line-currents, since the field of any current distribution over an arbitrary cross-section can be approximated by summing the fields of a number of line-currents distributed within the cross-section. For a set of $n_k$ of these line-currents at the position $(r_i, \Theta_i)$ carrying a current $I_i$, the multipole coefficients are given by [15]

$$B_n(r_0) = \sum_{i=1}^{n_k} \frac{\mu_0 I_i r_0^{n-1}}{2\pi r_i^n} \left( 1 + \frac{\mu_r - 1}{\mu_r + 1} \frac{r_i}{R_{\text{yoke}}} \right)^{2n} \cos n\Theta_i,$$  \hspace{1cm} (108)

$$A_n(r_0) = \sum_{i=1}^{n_k} \frac{\mu_0 I_i r_0^{n-1}}{2\pi r_i^n} \left( 1 + \frac{\mu_r - 1}{\mu_r + 1} \frac{r_i}{R_{\text{yoke}}} \right)^{2n} \sin n\Theta_i,$$  \hspace{1cm} (109)

where $R_{\text{yoke}}$ is the inner radius of the iron yoke with the relative permeability $\mu_r$. The field of any current distribution over an arbitrary cross-section can be approximated by summing the fields of a number of line-currents distributed within the cross-section. As superconducting cables are composed of single strands with a diameter of about 1 mm, a good computational accuracy can be obtained by representing each cable by two layers of equally spaced line-currents at the strand position. Thus the grading of the current density in the cable due to the different compaction on its narrow and wide side is automatically considered.
With equations (108) and (109), a semi-analytical method for calculating the fields in superconducting magnets is given. The iron yoke is represented by image currents (the second term in the parentheses). At low field level, when the saturation of the iron yoke is low, this is a sufficient method for optimizing the coil cross-section. Under that assumption some important conclusions can be drawn:

- For a coil without iron yoke the field errors scale with $1/r^n$ where $n$ is the order of the multipole and $r$ is the mid radius of the coil. It is clear, however, that an increase of coil aperture causes a linear drop in dipole field. Other limitations of the coil size are the beam distance, the electromagnetic forces, yoke size, and the stored energy which results in an increase of the hot-spot temperature during a quench.

- For certain symmetry conditions in the magnet, some of the multipole components vanish, i.e. for an up-down symmetry in a dipole magnet (positive current $I_0$ at $(r_0, \Theta_0)$ and at $(r_0, -\Theta_0)$) no $A_n$ terms occur. If there is an additional left-right symmetry, only the odd $B_1, B_3, B_5, B_7, ..$ components remain.

- The relative contribution of the iron yoke to the total field (coil field plus iron magnetization) is for a non-saturated yoke ($\mu_r \gg 1$) approximately $(1 + (\frac{R_{\text{yoke}}}{r})^{2n})^{-1}$. For the main dipoles with a mean coil radius of $r = 43.5$ mm and a yoke radius of $R_{\text{yoke}} = 89$ mm we get for the $B_1$ component a 19% contribution from the yoke, whereas for the $B_5$ component the influence of the yoke is only 0.07%.

It is therefore appropriate to optimize for higher harmonics first using analytical field calculation, and include the effect of iron saturation on the lower-order multipoles only at a later stage.

12 The generation of pure multipole fields

Consider a current shell $r_1 < r < r_e$ with a current density varying with the azimuthal angle $\Theta$, $J(\Theta) = J_0 \cos m\Theta$, then we get for the $B_n$ components

$$B_n(r_0) = \int_{r_1}^{r_e} \int_0^{2\pi} \frac{\mu_0 J_0 r_0^{n-1}}{2\pi r^n} \left(1 + \frac{\mu_1 - 1}{\mu_1 + 1} \left(\frac{r}{R_{\text{yoke}}}\right)^{2n}\right) \cos m\Theta \cos n\Theta \ r d\Theta \ dr.$$ (110)
With $\int_0^{2\pi} \cos m\Theta \cos n\Theta d\Theta = \pi \delta_{m,n}(m, n \neq 0)$ it follows that the current shell produces a pure $2m$-polar field and in the case of the dipole ($m = n = 1$) one gets

$$B_1(r_0) = -\frac{\mu_0 J_0}{2} \left( (r_e - r_i) + \frac{\mu_e - 1}{\mu_t + 1} \frac{1}{R_{yoke}^2} \frac{1}{3} (r_e^3 - r_i^3) \right). \quad (111)$$

Obviously, since $\int_0^{2\pi} \cos m\Theta \sin n\Theta d\Theta = 0$, all $A_n$ components vanish. A shell with $\cos \Theta$ and $\cos 2\Theta$ dependent current density is displayed in figure 15. Because of $|B| = \sum_n \sqrt{B_n^2 + A_n^2}$ the modulus of

![Fig. 14: Shells with $\cos \Theta$ (left) and $\cos 2\Theta$ (right) dependent current density.](image)

the field inside the aperture of the shell dipole without iron yoke is given as

$$|B| = \frac{\mu_0 J_0}{2} (r_e - r_i). \quad (112)$$

### 12.1 Coil-block arrangements

Usually the coils do not consist of perfect cylindrical shells because the conductors itself are either rectangular or keystoneed with an insufficient angle to allow for perfect sector geometries. Therefore the shells are subdivided into coil-blocks, separated by copper wedges. The field generated by this coil layout has to be calculated with the line-current approximation of the superconducting cable. Real coil-geometries with one and two layers of coil-blocks are shown in fig. 15.
13 Numerical field computation

For the calculation of the saturation-induced field errors of the lower-order ($b_2 - b_5$), which vary as a function of the excitation field, numerical techniques such as the finite-element method (FEM) have to be applied. Fig. 16 (right) shows the variation of the lower-order multipoles as a function of the excitation current in the two-in-one main dipoles of the LHC. It can be seen that already the $b_4$ component is hardly influenced by saturation effects. Fig. 16 (left) shows the transfer function $B/I$ as a function of the excitation from injection to nominal field level including the saturation effects and the persistent current multipoles. Fig. 17 shows the relative permeability of the iron yoke for both injection and nominal field level.

Magnets for particle accelerators have always been a key application of numerical methods in electromagnetism. Hornsby [11], in 1963, developed a code based on the finite difference method for...
the solving of elliptic partial differential equations and applied it to the design of magnets. Winslow [21] created the computer code TRIM (Triangular Mesh) with a discretization scheme based on an irregular grid of plane triangles by using a generalized finite difference scheme. He also introduced a variational principle and showed that the two approaches lead to the same result. In this respect, the work can be viewed as one of the earliest examples of the finite element method applied to the design of magnets. The POISSON code which was developed by Halbach and Holsinger [10] was the successor of this code and was still applied for the optimization of the superconducting magnets for the LHC during the early design stages. Halbach had also, in 1967, [9] introduced a method for optimizing coil arrangements and pole shapes of magnets based on the TRIM code, an approach he named MIRT. In the early 1970’s a general purpose program (GFUN) for static fields had been developed by Newman, Turner and Trowbridge that was based on the magnetization integral equation and was applied to magnet design. Nevertheless, for the superconducting magnet design, it was necessary to find more appropriate formulations which do not require the modeling of the coils in the finite-element mesh. The integral equation method of GFUN would be appropriate, however, it leads to a very large (fully populated) matrix if the shape of the iron yoke requires a fine mesh.

The method of coupled boundary-elements/finite-elements (BEM-FEM), developed by Fetzer, Haas, and Kurz at the University of Stuttgart, Germany, combines a FE description using incomplete quadratic (20-node) elements and a gauged total vector-potential formulation for the interior of the magnetic parts, and a boundary element formulation for the coupling of these parts to the exterior, which includes excitational coil fields. This implies that the air regions need not to be meshed at all.

The principle steps in numerical field computation are:

- Formulation of the physical laws by means of partial differential equations.
- Transformation of these equations into an integral equation with the weighted residual method.
- Integration by parts in order to obtain the so-called weak integral form. Consideration of the natural boundary conditions.
- Discretization of the domain into finite elements.
- Approximation of the solution as a linear-combination of so-called shape functions.
In case of the finite element (Galerkin) method, the shape functions in the elements and the weighting functions of the weak integral form are identical.

In case of the boundary-element method, another partial integration of the weak integral form results in an integral equation. Using the fundamental solution of the Laplace operator as weighting functions yields an algebraic system of equations for the unknowns on the domain boundary. In case of the coupled boundary-element/finite-element method, the two domains are coupled through the normal derivatives of the vector-potential on their common boundary.

Consideration of the essential boundary conditions in the resulting equation system.

Numerical solution of the algebraic equations. A direct solver with Newton iteration is used in the 2D case; the domain decomposition method with a $M(B)$ iteration [8] is applied in the 3D case.

In order to understand the special properties of these methods and the reasoning which leads to their application in the design and optimization of accelerator magnets, it is sufficient to concentrate on some aspects of the formulations. The function approximation with finite or boundary elements and the solution techniques will only be explained very briefly, as this report cannot replace a lecture on finite-element techniques. Nevertheless, as the magnetostatic problem is one of the most simple cases, the basic concepts of numerical field computation can be explained by means of the most commonly used method with the total vector-potential formulation, and triangular elements with linear shape functions.

14 Total vector-potential formulation

Consider the elementary model problem, fig. 18, consisting of two different domains: $\Omega_i$ the iron region with permeability $\mu$ and $\Omega_a$ the air region with the permeability $\mu_0$. The regions are connected to each other at the interface $\Gamma_{ai}$. Furthermore, each volume is bounded by a surface $\Gamma$ (sometimes denoted $\partial \Omega$) itself consisting of two different parts $\Gamma_H$ and $\Gamma_B$ with their outward normal vector $\vec{n}$. The elementary model problem as shown in fig. 18 is a mixed boundary value problem. The non-conductive air region $\Omega_a$ may also contain a certain number of conductor sources $\vec{J}$ which do not intersect the iron region $\Omega_i$.

Subsequently, the Cartesian coordinate system is always used, as only in this case the vector Laplace operator, decomposes into three scalar Laplace operators acting on the three components of the vector-potential.

As the divergence of the magnetic flux is zero, the application of a total $\vec{A}$-formulation in the domain $\Omega = \Omega_a \cup \Omega_i$ automatically satisfies eq. (28). Ampère’s law (27) then takes the form

$$\nabla \times \frac{1}{\mu} \nabla \times \vec{A} = \vec{J} \quad \text{in } \Omega. \quad (113)$$

Because of the iron saturation, the permeability $\mu$ depends on the magnetic field and therefore $\vec{B} = \mu(\vec{H})\vec{H}$. The boundary conditions read:

$$\vec{H} \times \vec{n} = \frac{1}{\mu}(\nabla \times \vec{A}) \times \vec{n} = 0 \quad \text{on } \Gamma_H, \quad (114)$$

$$\vec{B} \cdot \vec{n} = \nabla \times \vec{A} \cdot \vec{n} = 0 \quad \text{on } \Gamma_B. \quad (115)$$

Eq. (114) is the homogeneous Neumann boundary condition on $\Gamma_H$ where $\vec{H}$ is normal to the boundary. Surface current densities do not appear as long as finite conductivity and continuous time dependency is assumed. Eq. (115) is the homogeneous Dirichlet boundary condition ($\vec{B}$ is parallel to the boundary $\Gamma_B$, no fictitious magnetic surface charge density). The far-field boundary is also part of $\Gamma_B$. The condition (115) is equivalent to $\vec{A}_t$, i.e., $\vec{A} \times \vec{n} = 0$ on $\Gamma_B$ [?]. At the interface $\Gamma_{ai}$ between $\Omega_i$ and $\Omega_a$ interface
conditions have to be satisfied (continuity of $\vec{B}_n$ and $\vec{H}_t$):

$$\vec{B}_i \cdot \vec{n}_i + \vec{B}_a \cdot \vec{n}_a = 0 \quad \text{on } \Gamma_{ai},$$

$$\vec{H}_i \times \vec{n}_i + \vec{H}_a \times \vec{n}_a = 0 \quad \text{on } \Gamma_{ai},$$

where $\vec{n}_i$ and $\vec{n}_a$ are the outer normals associated with the respective subregions. The normal component of the magnetic flux density $B_n$ is continuous due to the chosen shape functions of the nodal finite-elements. For the tangential component of the magnetic field intensity $\vec{H}_t$, the equation (117) written in terms of the vector-potential is:

$$\frac{1}{\mu} (\text{curl } \vec{A}_i) \times \vec{n}_i + \frac{1}{\mu_0} (\text{curl } \vec{A}_a) \times \vec{n}_a = 0 \quad \text{on } \Gamma_{ai}. \quad (118)$$

The normal vector $\vec{n}_i$ on the boundary between iron and air is pointing out of the iron domain $\Omega$ and $\vec{n}_a$ is pointing out of the air domain $\Omega_a$. The solution of the 3D (vector) boundary value problem is not unique, however. Introducing a penalty term subtracted from eq. (113), yields

$$\text{curl } \frac{1}{\mu} \text{curl } \vec{A} - \text{grad } \frac{1}{\mu} \text{div } \vec{A} = \vec{J} \quad \text{in } \Omega. \quad (119)$$
With the additional boundary conditions

\[ \vec{A} \cdot \vec{n} = 0 \quad \text{on } \Gamma_H, \quad (120) \]
\[ \vec{A} \times \vec{n} = 0 \quad \text{on } \Gamma_B, \quad (121) \]
\[ \frac{1}{\mu} \text{div } \vec{A} = 0 \quad \text{on } \Gamma_B, \quad (122) \]
\[ \frac{1}{\mu_0} \text{div } \vec{A}_a = \frac{1}{\mu} \text{div } \vec{A}_i \quad \text{on } \Gamma_{ai}, \quad (123) \]

it can be proved that the boundary value problem has a unique solution satisfying the Coulomb gauge

\[ \frac{1}{\mu} \text{div } \vec{A} = 0 \quad \text{in } \Omega. \quad (124) \]

The complete formulation for the vector-potential reads

\[ \text{curl } \frac{1}{\mu} \text{curl } \vec{A} - \text{grad } \frac{1}{\mu} \text{div } \vec{A} - \vec{J} = \vec{R} \quad \text{in } \Omega, \quad (125) \]
\[ \vec{A} \cdot \vec{n} = 0 \quad \text{on } \Gamma_H, \quad (126) \]
\[ \frac{1}{\mu} \text{div } \vec{A} = 0 \quad \text{on } \Gamma_B, \quad (127) \]
\[ \vec{A} \times \vec{n} = 0 \quad \text{on } \Gamma_B, \quad (128) \]
\[ \frac{1}{\mu} \text{(curl } \vec{A}) \times \vec{n} = 0 \quad \text{on } \Gamma_H, \quad (129) \]
\[ \frac{1}{\mu_0} \text{div } \vec{A}_a - \frac{1}{\mu} \text{div } \vec{A}_i = 0 \quad \text{on } \Gamma_{ai}, \quad (130) \]
\[ \frac{1}{\mu} \text{(curl } \vec{A}_i) \times \vec{n}_i + \frac{1}{\mu_0} \text{(curl } \vec{A}_a) \times \vec{n}_a = 0 \quad \text{on } \Gamma_{ai}, \quad (131) \]
\[ \vec{A} \text{ continuous on } \Gamma_{ai}. \quad (132) \]

### 14.1 The weighted residual

The domain \( \Omega = \Omega_a \cup \Omega_i \) is discretized into finite-elements in order to solve this problem numerically. For the approximate solution of \( \vec{A} \) defined on the nodes of the finite element mesh, the differential equation (125) is only approximately fulfilled:

\[ \text{curl } \frac{1}{\mu} \text{curl } \vec{A} - \text{grad } \frac{1}{\mu} \text{div } \vec{A} - \vec{J} = \vec{R} \quad (133) \]

with a residual (error) vector \( \vec{R} \). A linear equation system for the unknown nodal values of the vector potential \( \vec{A}^{(k)} \) can be obtained by minimizing the weighted residuals \( \vec{R} \) in an average sense over the domain \( \Omega \), i.e.,

\[ \int_\Omega \vec{w}_a \cdot \vec{R} \, d\Omega = 0, \quad a = 1, 2, 3 \quad (134) \]

with the vector weighting functions

\[ \vec{w}_1 = \begin{pmatrix} w_1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0 \\ w_2 \\ 0 \end{pmatrix}, \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \\ w_3 \end{pmatrix}, \quad (135) \]
where \( w_1, w_2, w_3 \) are arbitrary (but known) weighting functions. The vector weighting functions \( \vec{w}_a \) obey the homogeneous boundary conditions
\[
\vec{w}_a \cdot \vec{n} = 0 \quad \text{on } \Gamma_H, \\
\vec{w}_a \times \vec{n} = 0 \quad \text{on } \Gamma_B.
\]  
(136)
(137)

Forcing the weighted residual to zero yields
\[
\int_{\Omega} \vec{w}_a \cdot \left( \text{curl} \frac{1}{\mu} \text{curl} \vec{A} - \text{grad} \frac{1}{\mu} \text{div} \vec{A} \right) \, d\Omega = \int_{\Omega} \vec{w}_a \cdot \vec{J} \, d\Omega, \quad a = 1, 2, 3.
\]  
(138)

14.2 The weak form

With the boundary conditions (136) and (137) for \( \vec{w}_a \) and the identities
\[
\int_{\Omega} \left( \text{curl} \frac{1}{\mu} \text{curl} \vec{A} \right) \cdot \vec{w}_a \, d\Omega = \int_{\Omega} \frac{1}{\mu} \text{curl} \vec{A} \cdot \text{curl} \vec{w}_a \, d\Omega - \int_{\Gamma} \frac{1}{\mu} \left( \text{curl} \vec{A} \times \vec{n} \right) \cdot \vec{w}_a \, d\Gamma,
\]  
(139)
\[
\int_{\Omega} \left( -\text{grad} \frac{1}{\mu} \text{div} \vec{A} \right) \cdot \vec{w}_a \, d\Omega = \int_{\Omega} \frac{1}{\mu} \text{div} \vec{A} \cdot \text{div} \vec{w}_a \, d\Omega - \int_{\Gamma} \frac{1}{\mu} \text{div} \vec{A} \cdot \vec{w}_a \, d\Gamma,
\]  
(140)

the weighted residual of (125) can be transformed to
\[
\int_{\Omega} \frac{1}{\mu} \text{curl} \vec{A} \cdot \text{curl} \vec{w}_a \, d\Omega - \int_{\Gamma_H} \frac{1}{\mu} \left( \text{curl} \vec{A} \times \vec{n} \right) \cdot \vec{w}_a \, d\Gamma_H + \int_{\Omega} \frac{1}{\mu} \text{div} \vec{A} \cdot \text{div} \vec{w}_a \, d\Omega - \\
\int_{\Gamma_B} \frac{1}{\mu} \text{div} \vec{A} \cdot \vec{w}_a \, d\Gamma_B - \int_{\Gamma_{ai}} \frac{1}{\mu} \left( \text{div} \vec{A}_i \cdot \vec{n}_i \cdot \vec{w}_a \right) + \frac{1}{\mu_0} \text{div} \vec{A}_a \cdot \vec{n}_a \cdot \vec{w}_a \) \, d\Gamma_{ai} - \\
\int_{\Gamma_{ai}} \left( \frac{1}{\mu} \left( \text{curl} \vec{A}_i \times \vec{n}_i \right) + \frac{1}{\mu_0} \left( \text{curl} \vec{A}_a \times \vec{n}_a \right) \right) \cdot \vec{w}_a \, d\Gamma_{ai} = \int_{\Omega} \vec{w}_a \cdot \vec{J} \, d\Omega,
\]  
(141)

with \( a = 1,2,3 \). Due to the boundary conditions (127),(129)-(129), all the boundary integrals in eq. (141) vanish and therefore
\[
\int_{\Omega} \frac{1}{\mu} \text{curl} \vec{w}_a \cdot \text{curl} \vec{A} \, d\Omega + \int_{\Omega} \frac{1}{\mu} \text{div} \vec{w}_a \cdot \text{div} \vec{A} \, d\Omega = \int_{\Omega} \vec{w}_a \cdot \vec{J} \, d\Omega
\]  
(142)

with \( a = 1,2,3 \). Eq. (142) is called the weak form of the vector-potential formulation because the second derivatives have been removed and the continuity requirements on \( \vec{A} \) have been relaxed at the expense of an increase in the continuity conditions of the weighting functions. Only this makes possible the use of elements with linear shape functions. Inside this elements the first derivative of the shape functions is a constant and the second derivative vanishes. On the element boundary we find a jump in the first derivative and a Dirac \( \delta \)-function for the second. Thus there would be a problem in eq. (142) if the second derivative was present. This level of continuity is termed as \( C_0 \) continuous. The boundary value problem (125)-(132) is identical with the weak formulation (142) and the additional boundary conditions (126),(128) and (132) which have to be considered when the matrix of the linear equation system is assembled. Therefore these boundary conditions are also called essential, in contrast to the boundary conditions (127),(129)-(132) which are incorporated in the weak formulation and are called natural boundary conditions. The natural boundary conditions are only satisfied in the integral average sense over the domain \( \Omega \), i.e., in the weak sense.

In two dimensions, with \( \frac{\partial}{\partial z} = 0 \), the Coulomb gauge is automatically fulfilled and eq. (142) further reduces to
\[
\int_{\Omega} \frac{1}{\mu} \text{curl} \vec{w}_3 \cdot \text{curl} \vec{A}_z \, d\Omega = \int_{\Omega} \vec{w}_3 \cdot \vec{J}_z \, d\Omega.
\]  
(143)
With the relation \( \text{curl} \vec{G}_z = \text{grad} G_z \times \vec{e}_z \) it follows:

\[
\int_{\Omega} \frac{1}{\mu} \text{grad} w_3 \cdot \text{grad} A_z \, d\Omega = \int_{\Omega} \vec{w}_3 \cdot \vec{J}_z \, d\Omega.
\] (144)

The essential boundary condition \( \vec{A} \times \vec{n} = 0 \) on \( \Gamma_B \) takes the easy form \( A_z = 0 \) on \( \Gamma_B \) and the boundary condition on \( \Gamma_H, \vec{A} \cdot \vec{n} = 0 \) is automatically fulfilled as \( \vec{n} \perp \vec{e}_z \).

The current density \( \vec{J} \) appears on the right hand side of the differential equations (144) or (142). In consequence, when using the FE-method for the solution of this problem the relatively complicated shape of the coils must be modeled in the FE-mesh, c.f. fig. 19.

![Finite element mesh of the LHC main dipole coil](image)

Fig. 19: Finite element mesh of the LHC main dipole coil. The mesh required for the accurate modeling of the coil is very dense, resulting in large number of unknowns in particular if the surrounding iron yoke geometry has to be considered. Simplifications of the coil geometry yield inaccurate field quality estimates.

15 Coupled BEM-FEM method

The disadvantage of the finite-element method is that only a finite domain can be discretized, and therefore the field calculation in the magnet coil-ends with their large fringe-fields requires a large number of elements in the air region. The relatively new boundary-element method is defined on an infinite domain and can therefore solve open boundary problems without approximation with far-field boundaries. The disadvantage is that non-homogeneous materials are difficult to consider. The BEM-FEM method couples the finite-element method inside magnetic bodies \( \Omega = \Omega_{\text{FEM}} \) with the boundary-element method in the domain outside the magnetic material \( \Omega_{\text{m}} = \Omega_{\text{BEM}} \), by means of the normal derivative of the vector-potential on the interface \( \Gamma_{\text{m}} \) between iron and air. The application of the BEM-FEM method to magnet design has the following intrinsic advantages:

- The coil field can be taken into account in terms of its source vector potential \( \vec{A}_s \), which can be obtained easily from the filamentary currents \( I_s \) by means of Biot-Savart type integrals without the meshing of the coil.
• The BEM-FEM coupling method allows for the direct computation of the reduced vector potential $\vec{A}_{r}$ instead of the total vector potential $\vec{A}$. Consequently, errors do not influence the dominating contribution $\vec{A}_{s}$ due to the superconducting coil.

• Because the field in the aperture is calculated through the integration over all the BEM elements, local field errors in the iron yoke cancel out and the calculated multipole content is sufficiently accurate even for very sparse meshes.

• The surrounding air region need not be meshed at all. This simplifies the preprocessing and avoids artificial boundary conditions at some far-field-boundaries. Moreover, the geometry of the permeable parts can be modified without regard to the mesh in the surrounding air region, which strongly supports the feature based, parametric geometry modeling that is required for mathematical optimization.

• The method can be applied to both 2D and 3D field problems.

The elementary model problem for a single aperture model dipole (featuring both Dirichlet and Neumann bounds on the iron yoke) is shown in fig. 20.

Fig. 20: Elementary model problem for the numerical field calculation of a superconducting (single aperture) model magnet. In the iron domain the total vector potential is displayed. The non-conductive air region $\Omega_a$ contains a certain number of conductor sources $\vec{J}$ which do not intersect the iron region $\Omega_i$. The finite-element method inside the magnetic body $\Omega_i = \Omega_{FEM}$ is coupled with the boundary-element method in the domain outside the magnetic material $\Omega_a = \Omega_{BEM}$, by means of the normal derivative of the vector-potential on the interface $\Gamma_{ai} = \Gamma_{BEMFEM}$ between iron and air.
15.1 The FEM part

Inside the magnetic domain $\Omega_i$ a gauged vector-potential formulation is applied. Starting from eq. (76) a different (but equivalent) formulation is obtained:

$$\frac{1}{\mu_0} \text{curl} \text{curl} \vec{A} = \vec{J} + \text{curl} \vec{M} \quad \text{in } \Omega_i, \quad (145)$$

$$\frac{1}{\mu_0} (-\nabla^2 \vec{A} + \text{grad} \text{div} \vec{A}) = \vec{J} + \text{curl} \vec{M} \quad \text{in } \Omega_i. \quad (146)$$

Using the Coulomb gauge $\text{div} \vec{A} = 0$, the complete formulation of the problem reads

$$-\frac{1}{\mu_0} \nabla^2 \vec{A} = \vec{J} + \text{curl} \vec{M} \quad \text{in } \Omega_i, \quad (147)$$

$$\vec{A} \cdot \vec{n} = 0 \quad \text{on } \Gamma_H, \quad (148)$$

$$\frac{1}{\mu_0} \text{div} \vec{A} = 0 \quad \text{on } \Gamma_B, \quad (149)$$

$$\vec{A} \times \vec{n} = 0 \quad \text{on } \Gamma_B, \quad (150)$$

$$\frac{1}{\mu} (\text{curl} \vec{A}) \times \vec{n} = 0 \quad \text{on } \Gamma_H, \quad (151)$$

$$\frac{1}{\mu_0} \text{div} \vec{A}_a - \frac{1}{\mu_0} \text{div} \vec{A}_i = 0 \quad \text{on } \Gamma_{ai}, \quad (152)$$

$$\frac{1}{\mu_0} (\text{curl} \vec{A}_i - \mu_0 \vec{M}_i) \times \vec{n}_i + \frac{1}{\mu_0} (\text{curl} \vec{A}_a) \times \vec{n}_a = 0 \quad \text{on } \Gamma_{ai}. \quad (153)$$

Eq. (153) is the continuity condition of $H_{li} = H_{ia}$ on the interface between iron and air. Forcing the weighted residual to zero yields

$$-\int_{\Omega_i} \frac{1}{\mu_0} \nabla^2 \vec{A} \cdot \vec{w}_a \, d\Omega_i = \int_{\Omega_i} (\vec{J} + \text{curl} \vec{M}) \cdot \vec{w}_a \, d\Omega_i \quad a = 1, 2, 3. \quad (154)$$

$$\vec{w}_1 = \begin{pmatrix} w_1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0 \\ w_2 \\ 0 \end{pmatrix}, \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \\ w_3 \end{pmatrix}. \quad (155)$$

The weighting functions $\vec{w}_a$ obey again the homogeneous boundary conditions

$$\vec{w}_a \cdot \vec{n} = 0 \quad \text{on } \Gamma_H, \quad (156)$$

$$\vec{w}_a \times \vec{n} = 0 \quad \text{on } \Gamma_B. \quad (157)$$

With Green’s first theorem

$$\int_{\Omega_i} \nabla^2 \vec{A} \cdot \vec{w}_a \, d\Omega_i = -\int_{\Omega_i} \text{grad}(\vec{A} \cdot \vec{e}_a) \cdot \text{grad} w_a \, d\Omega_i + \oint_{\Gamma} \frac{\partial \vec{A}}{\partial n_i} \cdot \vec{w}_a \, d\Gamma \quad (158)$$

and the identity

$$\int_{\Omega_i} \text{curl} \vec{M} \cdot \vec{w}_a \, d\Omega_i = \int_{\Omega_i} \vec{M} \cdot \text{curl} \vec{w}_a \, d\Omega_i - \oint_{\Gamma} (\vec{M} \times \vec{n}_i) \cdot \vec{w}_a \, d\Gamma \quad (159)$$
we get for the weak form
\[
\frac{1}{\mu_0} \int_{\Omega_i} \text{grad}(\vec{A} \cdot \vec{e}_a) \cdot \text{grad} w_a \, d\Omega_i - \frac{1}{\mu_0} \int_{\Gamma_B} \left( \frac{\partial \vec{A}}{\partial n_i} - (\mu_0 \vec{M} \times \vec{n}_i) \right) \cdot \vec{w}_a \, d\Gamma_B
\]
\[- \frac{1}{\mu_0} \int_{\Gamma_B} \left( \frac{\partial \vec{A}}{\partial n_i} - (\mu_0 \vec{M} \times \vec{n}_i) \right) \cdot \vec{w}_a \, d\Gamma_B - \frac{1}{\mu_0} \int_{\Gamma_{ai}} \left( \frac{\partial \vec{A}}{\partial n_i} - (\mu_0 \vec{M} \times \vec{n}_i) \right) \cdot \vec{w}_a \, d\Gamma_{ai} =
\]
\[
\int_{\Omega_i} \vec{M} \cdot \text{curl} \vec{w}_a \, d\Omega_i + \int_{\Omega_i} \vec{w}_a \cdot \vec{J} \, d\Omega_i
\]
with $a = 1,2,3$. With the boundary conditions (148)-(151), and taking into account that the current density in the iron domain is zero, equation (160) further reduces to
\[
\frac{1}{\mu_0} \int_{\Omega_i} \text{grad}(\vec{A} \cdot \vec{e}_a) \cdot \text{grad} w_a \, d\Omega_i - \frac{1}{\mu_0} \int_{\Gamma_{ai}} \left( \frac{\partial \vec{A}}{\partial n_i} - (\mu_0 \vec{M} \times \vec{n}_i) \right) \cdot \vec{w}_a \, d\Gamma_{ai} =
\]
\[
\int_{\Omega_i} \vec{M} \cdot \text{curl} \vec{w}_a \, d\Omega_i
\]
with $a = 1,2,3$. The continuity condition of $\vec{H}$, eq. (153), i.e.,
\[
\frac{1}{\mu_0} (\text{curl} \vec{A}_i - \mu_0 \vec{M} \times \vec{n}_i) + \frac{1}{\mu_0} (\text{curl} \vec{A}_a) \times \vec{n}_a = 0 \quad \text{on } \Gamma_{ai}
\]
on the boundary between iron and air is equivalent to
\[
\frac{\partial \vec{A}_i}{\partial n_i} - (\mu_0 \vec{M} \times \vec{n}_i) + \frac{\partial \vec{A}_a}{\partial n_a} = 0,
\]
where $\vec{n}_i$ is the normal vector on $\Gamma_{ai}$ pointing out of the FEM domain $\Omega_i$, and $\vec{n}_a$ is the normal vector on $\Gamma_{ai}$ pointing out of the BEM domain $\Omega_a$. The boundary integral term on the boundary between iron and air $\Gamma_{ai}$ in (161) serves as the coupling term between the BEM and the FEM domain. Let us now assume that the normal derivative on $\Gamma_{ai}$
\[
\vec{Q}_{ai} = - \frac{\partial \vec{A}_{ai}^{\text{BEM}}}{\partial n_a}
\]
is given. If the domain $\Omega_i$ is discretized into finite-elements $\Omega_j$ ($C^0$-continuous, isoparametric 20-noded hexahedron elements are used) and the Galerkin method is applied to the weak formulation, then a nonlinear system of equations is obtained
\[
\left( \begin{matrix} [K_{\Omega_i \Omega_i}] & [K_{\Omega_i \Gamma_{ai}}] & 0 \\ [K_{\Gamma_{ai} \Omega_i}] & [K_{\Gamma_{ai} \Gamma_{ai}}] & [T] \end{matrix} \right) \left( \begin{matrix} \{ \vec{A}_{\Omega_i} \} \\ \{ \vec{A}_{\Gamma_{ai}} \} \\ \{ \vec{Q}_{\Gamma_{ai}} \} \end{matrix} \right) = \left( \begin{matrix} 0 \\ 0 \end{matrix} \right)
\]
with all nodal values of $\vec{A}_{\Omega_i}$, $\vec{A}_{\Gamma_{ai}}$ and $\vec{Q}_{\Gamma_{ai}}$ grouped in arrays
\[
\{ \vec{A}_{\Omega_i} \} = (\vec{A}_{1,\Omega_i}, \vec{A}_{2,\Omega_i}, \ldots), \quad \{ \vec{A}_{\Gamma_{ai}} \} = (\vec{A}_{1,\Gamma_{ai}}, \vec{A}_{2,\Gamma_{ai}}, \ldots), \quad \{ \vec{Q}_{\Gamma_{ai}} \} = (\vec{Q}_{1,\Gamma_{ai}}, \vec{Q}_{2,\Gamma_{ai}}, \ldots).
\]
The subscripts $\Gamma_{ai}$ and $\Omega_i$ refer to nodes on the boundary and in the interior of the domain, respectively. The domain and boundary integrals in the weak formulation yield the stiffness matrices $[K]$ and the boundary matrix $[T]$. The stiffness matrices depend on the local permeability distribution in the nonlinear material. All the matrices in (165) are sparse.
15.2 The BEM part

By definition, the BEM domain $\Omega_a$ contains no iron, and therefore $\vec{M} = 0$ and $\mu = \mu_0$. Eq. (145) then reduces to

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad (167)$$

As Cartesian coordinates are used, eq. (167) decomposes into three scalar Poisson equations to be solved. For an approximate solution of these equations and the weighted residual forced to zero yields:

$$\int_{\Omega_a} \nabla^2 A \, w \, d\Omega_a = -\int_{\Omega_a} \mu_0 J \, w \, d\Omega_a. \quad (168)$$

Employing Green’s second theorem

$$\int_{\Omega} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d\Omega = \oint_{\Gamma} (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) \, d\Gamma \quad (169)$$

yields

$$\int_{\Omega_a} A \nabla^2 w \, d\Omega_a = -\int_{\Omega_a} \mu_0 J \, w \, d\Omega_a + \int_{\Gamma_{ai}} A \frac{\partial w}{\partial n_a} \, d\Gamma_{ai} - \int_{\Gamma_{ai}} \frac{\partial A}{\partial n_a} \, w \, d\Gamma_{ai}. \quad (170)$$

In eq. (170) it is already considered that all the boundary integrals on the far field boundary $\Gamma_{BEM\infty}$ vanish. Now the weighting function is chosen as the fundamental solution of the Laplace equation, which is in 3D

$$w = u^* = \frac{1}{4\pi R} \quad (171)$$

With

$$\frac{\partial w}{\partial n_a} = q^* = -\frac{1}{4\pi R^2} \quad (172)$$

and

$$\nabla^2 w = -\delta(R) \quad (173)$$

we get the Fredholm integral equation of the second kind:

$$\frac{\Theta}{4\pi} A + \int_{\Gamma_{ai}} Q_{\Gamma_{ai}} u^* \, d\Gamma_{ai} + \int_{\Gamma_{ai}} A_{\Gamma_{ai}} q^* \, d\Gamma_{ai} = \int_{\Omega_a} \mu_0 J u^* \, d\Omega_a. \quad (174)$$

As it is common practice in literature on boundary element techniques, e.g. [7], the notation $\vec{A}$ for weighting functions is used in eq. (174). The right hand side of eq. (174) is a Biot-Savart-type integral for the source vector potential $A_s$.

The components of the vector potential $\vec{A}$ at arbitrary points $\vec{r}_0 \in \Omega_a$ (e.g. on the reference radius for the field harmonics) have to be computed from (174) as soon as the components of the vector potential $\vec{A}_{\Gamma_{ai}}$ and their normal derivatives $Q_{\Gamma_{ai}}$ on the boundary $\Gamma_{ai}$ are known. $\Theta$ is the solid angle enclosed by the domain $\Omega_a$ in the vicinity of $\vec{r}_0$.

For the discretization of the boundary $\Gamma_{ai}$ into individual boundary elements $\Gamma_{ai,j}$, again $C^0$-continuous, isoparametric 8-noded quadrilateral boundary elements (in 3D) are used. In 2D 3-noded line elements are used. They have to be consistent with the elements from the FEM domain touching this
boundary. The components of $\vec{A}_{\Gamma_{ai}}$ and $\vec{Q}_{\Gamma_{ai}}$ are expanded with respect to the element shape functions, and (174) can be rewritten in terms of the nodal data of the discrete model, 

$$\frac{\Theta}{4\pi} \vec{A} = \vec{A}_s - \{\vec{Q}_{\Gamma_{ai}}\} \cdot \{g\} - \{\vec{A}_{\Gamma_{ai}}\} \cdot \{h\}. \tag{175}$$

In (175), $g$ results from the boundary integral with the kernel $\kappa$, and $h$ results from the boundary integral with the kernel $\kappa^*$. The discrete analogue of the Fredholm integral equation can be obtained from (175) by successively putting the evaluation point $\vec{r}_j$ at the location of each nodal point $\vec{r}_i$. This procedure is called point-wise collocation and yields a linear system of equations,

$$[G] \{\vec{Q}_{\Gamma_{ai}}\} + [H] \{\vec{A}_{\Gamma_{ai}}\} = \{\vec{A}_s\}. \tag{176}$$

In (176), $\{\vec{A}_s\}$ contains the values of the source vector potential at the nodal points $\vec{r}_j$, $j = 1, 2, \ldots$. The matrices $[G]$ and $[H]$ are unsymmetrical and fully populated.

### 15.3 The BEM-FEM Coupling

An overall numerical description of the field problem can be obtained by complementing the FEM description (165) with the BEM description (176) which results in

$$\begin{pmatrix} [K_{\Omega_i,\Omega_i}] & [K_{\Omega_i,\Gamma_{ai}}] & 0 \\ [K_{\Gamma_{ai},\Omega_i}] & [K_{\Gamma_{ai},\Gamma_{ai}}] & [T] \\ 0 & [H] & [G] \end{pmatrix} \begin{pmatrix} \{\vec{A}_{\Omega_i}\} \\ \{\vec{A}_{\Gamma_{ai}}\} \\ \{\vec{Q}_{\Gamma_{ai}}\} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \{\vec{A}_s\} \end{pmatrix}. \tag{177}$$

Equation (176) gives exactly the missing relationship between the Dirichlet data $\{\vec{A}_{\Gamma_{ai}}\}$ and the Neumann data $\{\vec{Q}_{\Gamma_{ai}}\}$ on the boundary $\Gamma_{ai}$. It can be shown [12] that this procedure yields the correct physical interface conditions, the continuity of $\vec{n} \cdot \vec{B}$ and $\vec{n} \times \vec{H}$ across $\Gamma_{ai}$.

### 16 Field calculation of the LHC main dipole using BEM-FEM coupling

#### 16.1 Cross-section

The coils of the LHC dipole magnets are wound of Rutherford-type cable of trapezoidal (keystoned) shape. The coil consists of two layers with cables of the same height but of different width. Both layers are connected in series so that the current density in the outer layer, being exposed to a lower magnetic field, is about 40% higher. The conductor for the inner layer consists of 28 strands of 1.065 mm diameter, the outer layer conductor has 36 strands of 0.825 mm diameter. The strands are made of thousands of filaments of NbTi material embedded in a copper matrix which serves for stabilizing the conductor and to take over the current in case of a quench.

The keystoning of the cable is not sufficient to allow the cables to build up arc segments. Therefore copper wedges are inserted between the blocks of conductors. The size and shape of these wedges yield the necessary degrees of freedom for optimizing the field quality produced by the coil. The coil must be shaped to make the best use of the superconducting cable (limited by the peak field to which it is exposed) while producing a dipole field with a highest possible field homogeneity. As the beam pipe has to be kept free, the coil is wound in a so-called constant perimeter shape, around saddle shaped end-spacers.

The size and elastic modulus of each coil layer is measured to determine pole and coil-head shimming for the collaring. The required shim thickness is calculated such that the compression under the collaring press is about 120 MPa. After the collaring rods are inserted and external pressure is released, the residual coil pre-stress is about 50-60 MPa on both layers. The collars (made of stainless steel) are surrounded by an iron yoke which not only enhances the magnetic field by about 10% but also reduces the stored energy and shields the fringe field. The dipole magnet, its connections, and the bus-bars are
enclosed in the stainless steel shrinking cylinder closed at its ends and form the dipole cold-mass, a containment filled with static pressurized superfluid helium at 1.9 K. The cold-mass, weighing about 24 tons, is assembled inside its cryostat, which comprises a support system, cryogenic piping, radiation insulation, and thermal shield, all contained within a vacuum vessel.

The version (04.2000) cross-section of the magnet cold-mass in its cryostat is shown in fig. 21.

![Fig. 21: Cross-section of the dipole magnet and cryostat. 1. Heat exchanger 2. Bus bar 3. Superconducting coil, 4. Cold-bore and beam-screen, 5. Cryostat, 6. Thermal shield (55 to 75 K), 7. Shrinking cylinder, 8. Super-insulation, 9. Collars, 10. Yoke Fig. 22 and 23 show the quadrilateral (higher order) finite element mesh of collar and yoke for the LHC main dipole, the magnetic field strength, the magnetic vector potential and the relative permeability in the iron yoke at nominal field level.

Table 1 shows the field errors calculated for the 2D magnet cross-section as shown in Fig. 22 and 23. As can be expected from the analytical estimation, the iron yoke hardly influences the the higher order multipoles $b_7 - b_{11}$. These values are therefore a measure for the accuracy of the method.
Fig. 22: Left: Quadrilateral (higher order) finite element mesh of collar and yoke for the LHC main dipole. Right: Magnetic field strength in collar and yoke

Fig. 23: Left: Magnetic vector potential; Right: Relative permeability in the iron yoke at nominal field level

16.2 Magnet extremities

The coil must so be shaped as to make the best use of the superconducting cable (limited by the peak field to which it is exposed), while producing a dipole field with a highest possible field homogeneity. As the beam pipe has to be kept free, the coil is wound on a winding mandrel around saddle-shaped end-spacers such that the two narrow sides follow curves of equal arc length.

This is called a constant-perimeter end, cf. fig. 24, and it allows a more appropriate shaping of the coil in the straight section than would be possible with race-track coils. The geometric models used for the calculation of the magnetic fields are displayed in fig. 25 where the 3D coil representation, and the full 3D model are shown.

In order to reduce the peak field in the coil-end and thus increase the quench margin in the region with a weaker mechanical structure, the magnetic iron yoke ends approximately 100 mm from the onset of the ends. The BEM-FEM coupling method is therefore used for the calculation of the end-fields. The computing time for the 3D calculation is in the order of 5 hours on a DEC Alpha 5/333 workstation. The iterative solution of the linear equation system converges better in the case of a high excitational field...
Table 1: Field errors in the LHC main dipole (pre-series magnets) in units of $10^{-4}$ at 17 mm reference radius. For the analytical field computation, the iron yoke is considered by means of the imaging method, assuming an inner radius of 98 mm and a constant relative permeability of $\mu_r = 2000$. The two-in-one configuration creates additional $b_2, b_4, \ldots$ field errors which also vary as a function of the excitation from injection to nominal field level.

Fig. 24: Race-track coil (left) and constant perimeter type coil (right) as used for the LHC dipoles. Only one coil-block is displayed; connections are not shown. Constant perimeter ends allow a more appropriate shape of the coil in the cross-section, while keeping the space for the beam pipe.

than in the case of the injection field with its non-saturated iron yoke. It is therefore still impossible to apply mathematical optimization techniques to the 3D field calculation with iron yoke. However, as the additional effect from the fringe field on the field quality is low, it is sufficient to calculate the additional effect and then partially compensate with the coil design, if necessary. It has already been explained that the BEM-FEM coupling method allows the distinction between the coil field and the reduced field from the iron magnetization. Fig. 26 shows the field components along a line in the end-region of the twin-aperture dipole prototype magnet (MBP2), 43.6 mm above the beam-axis in aperture 2 (on a radius between the inner and outer layer coil) from $z = -200$ mm inside the magnet yoke to $z = 200$ mm outside the yoke. The iron yoke ends at $z = -80$ mm, the onset of the coil-end is at $z = 0$. 

![Image of coils](image-url)
Fig. 25: Geometric model of the dipole coil-end and full 3D model of coils and two-in-one iron yoke.

Fig. 26: Magnetic flux density at nominal current along a line at $x = 97\text{mm}$, $y = 43.6\text{ mm}$ (above the beam-axis of aperture 2 on a radius between the inner and the outer layer coil) from $z = -200\text{ mm}$ inside the magnet yoke to $z = 200\text{ mm}$ outside the yoke. The iron yoke ends at $z = -80\text{ mm}$, the onset of the coil-end is at $z = 0$. Left: coil field, Middle: reduced field from iron magnetization, Right: total field. Notice the different scales and the relatively small contribution from the yoke.
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