1. **RF ACCELERATION**

1.1 **Energy gain and transit time factor**

Particles experience the effect of RF fields when they cross accelerating gaps that basically produce an electric field $\mathbf{E}$ parallel to their trajectories. The gap is the space between two electrodes provided with a beam pipe, which for simplicity we take as a circular cylinder of radius $a$.

Let $\hat{V}(r)$ be the amplitude of the RF voltage impressed across the two electrodes. When a particle with electric charge $e$ crosses the gap at a distance $r$ from the $s$-axis (see Fig. 1), it gains an energy $\Delta E = e \int \varepsilon_s(s,r,t) \, ds$.

![Fig. 1 - Longitudinal cross-section of an accelerating gap](image)

The time dependence of $\varepsilon_s$ is given by

$$\varepsilon_s(s,r,t) = \tilde{\varepsilon}_s(s,r) \sin (\omega_{RF} t)$$

Traditionally, for circular accelerators the origin of time is taken at the zero crossing of the RF voltage with positive slope. The phase $\phi$ of the RF voltage when a particle crosses the middle of the accelerating gap (at $s = 0$) is called the *phase of the particle* with respect to the RF voltage. On the other hand, for circular accelerators in the Russian literature and for linacs, the origin of time is taken at the crest of the RF voltage. The phase $\phi$ in that case is such that $\phi = \frac{\pi}{2} + \varphi$. (Strictly speaking, in the previous sentences, the term "RF voltage" should be understood as "RF voltage times the charge $e$ of the particles"). If we neglect the change in velocity of the particle when crossing the gap, the time $t$ when the particle is at position $s$ in the gap reads

$$t = \frac{\phi}{\omega_{RF}} + \frac{s}{v} \quad \text{and} \quad \omega_{RF} t = \phi + \frac{\omega_{RF}}{v} s$$

where $v$ is the particle velocity in the middle of the gap.
For simplicity's sake we assume that the gap is symmetric with respect to the plane $s = 0$; then

$$\Delta E = e \int \tilde{\varepsilon}_s(s,r) \sin \left( \phi + \frac{\omega_{RF}}{v} s \right) ds = e \sin \phi \int \tilde{\varepsilon}_s(s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) ds \quad (1-1)$$

By representing the fields for $r \leq a$ as Fourier integrals along $s$, one gets

$$\int_{-\infty}^{+\infty} \tilde{\varepsilon}_s(s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) ds = \frac{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{r}{v} \right)}{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{a}{v} \right)} \int_{-\infty}^{+\infty} \tilde{\varepsilon}_s(s,a) \cos \left( \frac{\omega_{RF}}{v} s \right) ds$$

With $\tilde{V}(r) = \int_{-\infty}^{+\infty} \tilde{\varepsilon}_s(s,r) ds = \tilde{V}(0) J_0 \left( \frac{\omega_{RF}}{c} r \right)$, this may be written as

$$\int_{-\infty}^{+\infty} \tilde{\varepsilon}_s(s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) ds = \tilde{V}(r) \cdot T(r) = \frac{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{r}{v} \right)}{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{a}{v} \right)} \cdot \tilde{V}(a) T(a)$$

where by definition

$$T(r) = \frac{\int_{-\infty}^{+\infty} \tilde{\varepsilon}_s(s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) ds}{\int_{-\infty}^{+\infty} \tilde{\varepsilon}_s(s,r) ds}$$

is the transit time factor at $r$; it is the ratio of peak energy gained by a particle with velocity $v$ to the same quantity if $v$ were infinite.

$T(r)$ is simplest at $r = a$, where $\tilde{\varepsilon}_s$ is zero outside the gap. In many practical cases, a good approximation is obtained when $\tilde{\varepsilon}_s(s,a)$ is considered to be constant in the gap; then

$$T(a) = \frac{\sin \left( \frac{\omega_{RF}}{v} \cdot \frac{g}{2} \right)}{\left( \frac{\omega_{RF}}{v} \cdot \frac{g}{2} \right)} \quad (1-2)$$

Finally,

$$\Delta E = eV \sin \phi \quad (1-3)$$
where

\[ V = \hat{V}(a) \cdot T(a) \cdot \frac{I_0\left(\frac{\omega_{RF}}{V} \cdot \frac{r}{\gamma}\right)}{I_0\left(\frac{\omega_{RF}}{V} \cdot \frac{a}{\gamma}\right)} = \hat{V}(0) \cdot J_0\left(\frac{\omega_{RF}}{c} \cdot \frac{a}{\gamma}\right) \cdot T(a) \cdot \frac{I_0\left(\frac{\omega_{RF}}{V} \cdot \frac{r}{\gamma}\right)}{I_0\left(\frac{\omega_{RF}}{V} \cdot \frac{a}{\gamma}\right)} \]

with \( eV > 0 \). Neglecting the second order variation in \( r \) due to \( I_0\left(\frac{\omega_{RF}}{V} \cdot \frac{r}{\gamma}\right) \), we are left with

\[ V = \hat{V}(a) \cdot \frac{T(a) \cdot \frac{a}{\gamma}}{I_0\left(\frac{\omega_{RF}}{V} \cdot \frac{a}{\gamma}\right)} \text{ for all particles} \quad (1-4) \]

It is seen that through the transit time factor and the Bessel function \( I_0\left(\frac{\omega_{RF}}{V} \cdot \frac{a}{\gamma}\right) \), the effective peak voltage \( V \) depends on the particle velocity \( v \). This effect will be neglected in what follows, so that all particles will be considered as experiencing the same peak voltage.

More precise (but more complicated) expressions for \( \Delta E \) can be found in Ref. 1.

1.2 Harmonic number

For some reference particle (also called synchronous particle), the phase \( \phi \) is kept unchanged (mod 2\( \pi \)) at a value \( \phi_s \) when the particle returns to the same accelerating gap after one revolution along the ring. This requires that \( \omega_{RF} = \omega_0 \), where \( \omega_0 = 2\pi/T_0 \) is the angular revolution frequency of the reference particle and \( h \) is an integer called harmonic number.

Then

\[ \omega_{RF} \cdot T_0 = 2\pi h \quad (1-5) \]

When the ring is large, \( \omega_0 \) is small and \( h \) may be quite a big number.

1.3 Finite difference equations

For simplicity, let us assume that RF acceleration takes place in \( N \) identical cavities evenly spaced along the synchrotron ring. Let \( n \) be the number of accelerating cavity traversals by a particle.

**Definition of variables** (see Fig. 2)

- \( p_s, v_s \), momentum and velocity of the reference (synchronous) particle
- \( t_n \), time of \( n \)th cavity traversal by the reference particle
- \( \delta p_n = p_n - p_s \)
- \( \delta \phi_n = \phi_n - \phi_s \)

In what follows, \( \delta \) represents a difference taken with respect to the reference particle at a given time; \( \delta \) represents an increment during acceleration.
Besides the general coordinates \((R, \theta)\) whose origin is the accelerator centre, each bending magnet has its own local coordinates \((r, \theta)\) whose origin is the centre of the reference particle orbit in the magnet. Any integral with respect to \(\theta\) is taken in the bending magnets only.

**Phase variation between adjacent cavities**

In order to keep the phase of the reference particle constant at every cavity traversal, the RF phase must be shifted by \(2\pi h/N\) between adjacent cavities. The phase of any particle with respect to the RF voltage is then given by

\[
\phi(t) = \int \omega_{RF} \, dt - h0(t) \tag{1-6}
\]

where \(\theta(t)\) is the azimuthal position of the particle. With this relation \(\phi\) is not only defined during cavity traversals, but it is defined at any time. In particular, for the reference particle,

\[
\frac{d\phi_s}{dt} = \omega_{RF} - h\omega_0 = 0 \tag{1-7}
\]

For a particle with an energy deviation \(\delta\gamma/\gamma_s\) with respect to the reference particle, the phase is compared to \(\phi_s\). If \(T_r\) is the revolution period, the variation of \(\delta\phi_n\) from one cavity to the next is

\[
\delta\phi_{n+1} - \delta\phi_n = \phi_{n+1} - \phi_n = \frac{\omega_{RF}}{N} - h \frac{2\pi}{N} = \frac{\omega_{RF}}{N} (T_r - T_0)
\]

But \(T_r = C/v\) where \(C\) is the orbit circumference and \(v\) is the particle velocity. For a relative momentum deviation \(\delta p/p\),

\[
\frac{\delta C}{C} = \alpha \frac{\delta p}{p} \text{ by definition of the momentum compaction } \alpha; \quad \alpha = \frac{1}{2\pi R} \int D_X(s) \, d\theta
\]

where \(D_X(s)\) is the radial dispersion.
\[
\frac{\delta v}{v} = \frac{1}{\gamma^2} \frac{\delta p}{p} \text{ by relativistic kinematics.}
\]

Therefore

\[
\frac{\delta T_T}{T_0} = \left( a - \frac{1}{\gamma^2} \right) \frac{\delta p}{p} = \frac{\delta p}{p} \quad \text{where} \quad \eta = a - \frac{1}{\gamma^2} \tag{1-8}
\]

If \( \eta \) vanishes for some \( \gamma \), this particular energy is called the \textit{transition energy} \( \gamma_{tr} \); when \( a \) is independent of \( \gamma \), \( a \) is equal to \( 1/\gamma_{tr}^2 \). Finally,

\[
\frac{\delta \phi_{n+1} - \delta \phi_n}{\gamma} = \frac{\omega_{RF}}{N} \frac{\delta T_T}{T_0} = \frac{\delta p_{n+1}}{N \eta} = \frac{2\pi h}{N} \frac{2\pi h}{N} \frac{\delta \gamma_{n+1}}{\gamma}
\]

since \( \frac{\delta \gamma}{\gamma} = g^2 \frac{\delta p}{p} \).

\textbf{Energy variation between adjacent cavities}

From now on, \( V \) will represent the total RF voltage produced by all accelerating cavities; the voltage produced by a single cavity will then be \( V/N \). With (1-3) we have

\[
\Delta E = E_{n+1} - E_n = \frac{eV}{N} \sin \phi_n - \frac{e}{N} \int \frac{\partial B_z}{\partial t} \cdot r \, d\theta \, dr
\]

In the righthand side, the first term represents an energy gain which is lumped in the accelerating cavities, whereas the second term represents an energy gain which is distribu-
ted all along the magnets. \textit{Although the second term is usually negligible with respect to the first one, its variation for particles with different energies must not be overlooked.}

For the reference particle

\[
\Delta E_s = E_{s,n+1} - E_{s,n} = \frac{eV}{N} \sin \phi_n - \frac{e}{N} \int \frac{\partial B_z}{\partial t} \cdot r_s \, d\theta \, dr
\]

To first order,

\[
\frac{N}{2\pi} (\Delta E - \Delta E_s) = \frac{eV}{2\pi} (\sin \phi_n - \sin \phi_s) - \frac{e}{2\pi} \int \frac{\partial B_z}{\partial t} \cdot r_s \, d\theta \, \delta x \tag{1-10}
\]

With

\[
\delta x = B_x(s) \cdot \frac{\delta p}{p} \quad \text{and} \quad p = -e B_z \cdot r = -e \langle B_x \rangle R \quad \text{(1-11)}
\]

where \( \langle B_x \rangle \) is the average magnetic field along a closed orbit, the last term of (1-10) becomes
because from (1-11),
\[
\frac{\dot{p}}{p} = \frac{1}{\langle B_z \rangle} \left[ \frac{\delta \langle B_z \rangle}{\delta t} + \frac{\delta \langle B_z \rangle}{\delta R} \frac{dR}{dt} \right] + \frac{1}{R} \frac{dR}{dt} = \frac{1}{\langle B_z \rangle} \cdot \frac{\delta \langle B_z \rangle}{\delta t} + \left[ \frac{R}{\langle B_z \rangle} \cdot \frac{\delta \langle B_z \rangle}{\delta R} + \frac{1}{R} \right]
\]

[average magnetic field index + 1] = \frac{1}{\alpha}.

Now we must remember that \( \Delta E, \Delta E_s \) are gained in different times \( T_r/N, T_0/N \). As next approximation, \( E \) and \( E_s \) are considered to be smooth functions of \( t \):
\[
\Delta E = \frac{T_r}{N} \frac{dE}{dt} \quad \Delta E_s = \frac{T_0}{N} \frac{dE_s}{dt}
\]

With (1-9),
\[
\Delta E - \Delta E_s = \frac{T_r}{N} \frac{dE}{dt} - \frac{T_0}{N} \frac{dE_s}{dt} = \frac{T_r}{N} \frac{d(E - E_s)}{dt} + \frac{T_r - T_0}{N} \frac{dE_s}{dt} = \frac{T_r}{N} \frac{d(\delta E)}{dt} + \frac{\delta p}{p} T_0 \frac{dE_s}{dt}
\]

Using the kinematic relations \( dE = v \cdot dp = \omega \cdot R \cdot dp \), \( \delta E = v \cdot \delta p = \omega_0 R \cdot \delta p \):
\[
\frac{N}{2\pi} (\Delta E - \Delta E_s) = \frac{1}{\omega_r} \frac{d(\delta E)}{dt} + \frac{\delta p}{p} \frac{1}{\omega_0} \frac{dE_s}{dt} = \frac{d}{dt} \left( \frac{\delta E}{\omega_0} \right) + \frac{\delta E}{\omega_0} + \frac{\delta p}{p} R_s \frac{\dot{R}}{R} + 2nd \text{ order terms}.
\]

which is exactly equal to the last term of (1-10). Finally (1-10) reduces to
\[
\frac{d}{dt} \left( \frac{E - E_s}{\omega_0} \right) = \frac{eV}{2\pi} (\sin \phi - \sin \phi_s)
\]
The corresponding finite difference equation reads

\[
\left( \frac{E - E_S}{\omega_0} \right)_{n+1} - \left( \frac{E - E_S}{\omega_0} \right)_n = \frac{eV}{N \omega_0, n} \left( \sin \phi_n - \sin \phi_s \right) \quad \text{where} \quad \frac{E - E_S}{\omega_0} = R_s (p - p_s)
\]

(1-13)

Finally, about the set of finite difference equations (1-9), (1-13), one should quote H. Hereward: "These equations are only roughly correct, and it is work to estimate how good they are" (Ref. 3, p. 11).

1.4 Differential equations for an arbitrary RF voltage

If higher harmonics are added to the fundamental sinusoidal RF field, in Eq. (1-12) and (1-13) \( \sin \phi \) must be replaced by a more general function \( g(\phi) \) such that

\[
g(\phi + 2\pi) = g(\phi) \quad \text{and} \quad \int_{0}^{2\pi} g(\phi) \, d\phi = 0
\]

(1-14)

hence

\[
g(\phi) = \sum_{n=1}^{\infty} \left( a_n \sin n\phi + b_n \cos n\phi \right)
\]

(1-15)

where one can take \( a_1 = 1, b_1 = 0 \), since the values of \( a_1, b_1 \) are defined by normalization and by the choice of the origin of time. Eq. (1-12) then becomes

\[
\frac{d}{dt} \left( \frac{\delta E}{\omega_0} \right) = \frac{eV}{2\pi} \left[ g(\phi) - g(\phi_s) \right]
\]

(1-16)

which has to be combined with the differential form of (1-9):

\[
\frac{d}{dt} (\phi - \phi_s) = \hbar \omega_0 \frac{\hbar}{g^2} \frac{\delta E}{E}
\]

(1-17)

Instead of \( \phi \) and \( \delta E/\omega_0 \) as conjugate variables, we shall use \( \phi \) and \( \delta E/(\hbar \omega_0) \), so that the elementary phase space area will read

\[
\frac{\delta E}{\hbar \omega_0} \cdot \delta \phi = \frac{\delta E}{\hbar \omega_0} \delta(\omega_0 t) = \delta E \delta t
\]
With $\phi$ and $\delta E/(\hbar \omega_0)$ as conjugate variables, the system (1-16), (1-17) becomes

$$\frac{d}{dt} \left( \frac{\delta E}{\hbar \omega_0} \right) = \frac{eV}{\hbar} \frac{1}{2\pi} \left[ g(\phi) - g(\phi_s) \right]$$

(1-18)

$$\frac{d}{dt} (\phi - \phi_s) = \frac{\hbar^2}{2} \omega_0^2 \cdot \frac{n}{\hbar^2 E} \left( \frac{\delta E}{\hbar \omega_0} \right)$$

where $E = \gamma E_0 = \gamma m_0 c^2$

(1-19)

With (1-7), the system (1-18), (1-19) may be derived from the Hamiltonian

$$H = \frac{1}{2} \hbar^2 \omega_0^2 \frac{n}{\beta^2 \gamma E_0} \left( \frac{\delta E}{\hbar \omega_0} \right)^2 + \frac{eV}{\hbar^2} \left[ \Gamma \phi + G(\phi) \right]$$

(1-20)

where

$$\Gamma \equiv g(\phi_s) \quad \text{and} \quad G(\phi) = -\int g(\phi) d\phi = \sum_{n=1}^{\infty} \left( \frac{a_n}{n} \cos n \phi - \frac{b_n}{n} \sin n \phi \right)$$

(1-21)

$H$ depends explicitly on time through parameters which vary slowly during acceleration.

1.5 Hamiltonian with reduced variables

The study of particle motion can be simplified by using reduced variables $y$ and $t^*$ instead of $\delta E/(\hbar \omega_0)$ and $t$. Let

$$\begin{cases} \frac{dt}{dt^*} = K_1 & \text{where } K_1, K_2 \text{ are slowly varying parameters.} \\ \frac{\delta E}{\hbar \omega_0} = K_2 \cdot y \end{cases}$$

(1-22)

With the reduced variables and reduced Hamiltonian $H^*$, the equations of motion read

$$\frac{dy}{dt^*} = -\frac{K_1}{K_2} \frac{\partial H}{\partial \phi} = -\frac{\partial H^*}{\partial \phi}$$

$$\frac{d\phi}{dt^*} = \frac{K_1}{K_2} \frac{\partial H}{\partial y} = \frac{\partial H^*}{\partial y}$$

(1-23)

whence

$$H^* = \frac{K_1}{K_2} H = \frac{1}{2} \hbar^2 \omega_0^2 \frac{n}{\beta^2 \gamma E_0} K_1 K_2 y^2 + \frac{eV}{\hbar^2} \left[ \frac{K_1}{K_2} \left( \Gamma \phi + G(\phi) \right) \right]$$

By taking

$$\frac{\hbar^2 \omega_0^2}{\beta^2 \gamma E_0} K_1 K_2 = 1 \quad \text{and} \quad \frac{eV}{\hbar^2 \pi} \frac{K_1}{K_2} = \text{sgn}(\eta)$$
In the figure, $\Gamma = \frac{1}{2}$

$G(\phi) = \cos \phi$

$g(\phi) = \sin \phi$

*Fig. 3* - Potential energy as a function of $\phi$. 
Fig. 4 - Trajectories in synchrotron phase space, when $\eta > 0$; when $\eta < 0$, $\phi_s$ and $\phi_U$ are interchanged.

The complete phase space is wrapped around a cylinder $0 \leq \phi \leq 2\pi h$. 

Accelerating bucket ($\frac{\pi}{2} < \phi_s < \pi$)

Stationary bucket ($\phi_s = \pi$)

Vanishing bucket ($\phi_s = \frac{3\pi}{2}$)
\[ \frac{1}{K_1} = \text{sgn}(\gamma) \ h \omega_0 \left| \frac{n}{e^2 \gamma} \ \frac{e V}{h^2 \pi E_0} \right|^\frac{1}{2}, \quad K_2 = \frac{E_0}{h \omega_0} \left| \frac{\phi^2 \gamma}{n} \ \frac{e V}{h^2 \pi E_0} \right|^\frac{1}{2} \]

(1-24)

\[ H^*_s \text{ becomes} \]

\[ H^* = \frac{\gamma^2}{2} + \text{sgn}(\gamma) \left[ \Gamma \phi + G(\phi) \right] \]

(1-25)

**Fixed points.** From (1-23) they correspond to \[ \frac{\partial H^*}{\partial \phi} = 0 \]
\[ \frac{\partial H^*}{\partial \gamma} = 0 \]
i.e., with (1-21): \[ \gamma + G'(\phi) = 0 \quad \text{or} \quad \gamma \equiv g(\phi_s) = g(\phi) \text{ with } \gamma = 0. \]

Because of (1-14), beside \( \phi_s \) there will be in general another value of \( \phi \) satisfying the condition \( g(\phi) = \gamma \). Let \( \phi_0 \) be any one of them; for small \( \psi = \phi - \phi_0 \),

\[ \Gamma \phi + G(\phi) = \left[ \Gamma \phi_0 + G(\phi_0) \right] - g'(\phi_0) \frac{\phi^2}{2!} - g''(\phi_0) \frac{\phi^3}{3!} - \ldots \]

\[ H^* = \text{sgn}(\gamma) \left[ \Gamma \phi_0 + G(\phi_0) \right] + \frac{\gamma^2}{2} - \text{sgn}(\gamma) \cdot g'(\phi_0) \frac{\phi^2}{2} - \text{sgn}(\gamma) \cdot g''(\phi_0) \frac{\phi^3}{6} \]

(1-26)

If \( \text{sgn}(\gamma) \cdot g'(\phi_0) < 0 \), \( \phi_0 \) is an elliptic fixed point; this is the case of \( \phi_0 = \phi_s \) for \( \phi_s \) being a stable fixed point.

If \( \text{sgn}(\gamma) \cdot g'(\phi_0) > 0 \), \( \phi_0 \) is a hyperbolic fixed point; this is the case of the other fixed point \( \phi_0 = \phi_u \), which is unstable.

\( \phi_s \) is at a minimum of potential energy; \( \phi_u \) is at a maximum. When \( \gamma \) changes sign, the two points \( \phi_s \) and \( \phi_u \) are interchanged (see Figure 3). Therefore, when crossing the transition energy, the RF voltage must undergo a phase jump which puts the particles around the new stable fixed point.

**Separatrix.** The trajectory in phase space passing through the unstable fixed point \( (\phi_u, \gamma = 0) \) crosses the \( \phi \)-axis at another point \( (\phi_e, \gamma = 0) \). This trajectory is the boundary between trapped and untrapped motion (or between libration and rotation); it is called the *separatrix* (see Fig. 4). The phase space domain inside the separatrix is called *bucket*; its area \( A_S \) is the longitudinal acceptance of the accelerator.

From (1-25) the equation for the separatrix is

\[ \frac{\Gamma \phi + G(\phi) - \Gamma \phi_u - G(\phi_u)}{\text{sgn}(\gamma)} = 0 \]

(1-27)
Taking the derivative with respect to \( \Phi \):

\[
y \cdot \frac{dy}{d\Phi} + \text{sgn}(n) \left[ \Gamma - g(\Phi) \right] = 0
\]

This equation is satisfied with \( y = 0 \) at \( \Phi = \Phi_u \) and with \( \frac{dy}{d\Phi} = 0 \) at \( \Phi = \Phi_s \). Therefore \( y \) is maximum at \( \Phi_s \) (this is also the case for any trajectory.)

**Bucket width.** The bucket width is \((\Phi_e - \Phi_u)\) where \( \Phi_e \) is determined by the equation

\[
\Gamma \Phi_e + G(\Phi_e) = \Gamma \Phi_u + G(\Phi_u)
\]

(1-28)

**Bucket height.** With (1-22),

\[
\left( \frac{\delta E}{\delta y} \right)_{\text{ave}} = R \delta p = K_2 \gamma \quad \text{where} \quad \frac{\gamma^2}{\sqrt{2}} = \text{sgn}(n) \left[ \Gamma \Phi_u + G(\Phi_u) - \Gamma \Phi_s - G(\Phi_s) \right]
\]

(1-29)

**Bucket area** (per bunch)

\[
A_s = K_2A_s^* \quad \text{where} \quad A_s^* = 2 \int_{\Phi_u}^{\Phi_e} y \ d\Phi = 2\sqrt{2} \int_{\Phi_u}^{\Phi_e} d\Phi \left| \Gamma \Phi_u + G(\Phi_u) - \Gamma \Phi - G(\Phi) \right|^{\frac{1}{2}}
\]

(1-30)

It is not invariant during acceleration.

**Period \( T_s \) of (large) synchrotron oscillations around the stable fixed point**

The phase space trajectories are represented by (1-25) where \( H^* \) is constant. Let \( \Phi_1, \Phi_2 \) be the two phases where \( y = 0 \); then (1-25) may be written as

\[
\frac{\gamma^2}{\sqrt{2}} + \text{sgn}(n) \left[ \Gamma \Phi + G(\Phi) \right] = \text{sgn}(n) \left[ \Gamma \Phi_1 + G(\Phi_1) \right] = \text{sgn}(n) \left[ \Gamma \Phi_2 + G(\Phi_2) \right]
\]

From (1-23)

\[
\frac{d\Phi}{dt^*} = \frac{\partial H^*}{\partial y} = y
\]

hence

\[
dt^* = \frac{d\Phi}{y}
\]

and with (1-22),

\[
T_s = |K_1|T_s^* \quad \text{where} \quad T_s^* = 2 \int_{\Phi_1}^{\Phi_2} \frac{d\Phi}{y} = \sqrt{2} \int_{\Phi_1}^{\Phi_2} d\Phi \left| \Gamma \Phi_1 + G(\Phi_1) - \Gamma \Phi - G(\Phi) \right|^{\frac{1}{2}}
\]

(1-31)

For a general RF voltage this expression involves cumbersome elliptic integrals.
1.6 Small oscillations around the stable fixed point

From (1-26), the small amplitude trajectories around $\phi_s$ are represented by the ellipse equation

$$\frac{\dot{\psi}^2}{2} + \frac{g'(|\phi_s|)}{2} \cdot \dot{\psi}^2 = \frac{C}{2} > 0, \quad \dot{\psi} = \hat{\psi} - \dot{\phi}_s$$

(1-32)

It is apparent that all properties of small oscillations around $\phi_s$ involve the RF voltage only through its slope at $\phi_s$.

**Period $T_{so}$ of small synchrotron oscillations**

The subscript 0 refers here to vanishingly small amplitudes. With (1-32) the general formula (1-31) simplifies to

$$T_{so} = |K_1|T_{so}^*$$

where

$$T_{so}^* = 2 \int_{-\frac{\psi}{2}}^{\frac{\psi}{2}} \frac{d\psi}{\psi} = 2 \int_{-\frac{\psi}{2}}^{\frac{\psi}{2}} \frac{d\psi}{\sqrt{C - \left|g'(|\phi_s|)\right|^2 \sqrt{g'(|\phi_s|)}}} = \frac{2\pi}{T_{so}^*}$$

This is independent of the amplitude $\hat{\psi}$ as long as the $\psi^3$ and higher order terms are missing in (1-26), which means as long as $g(\psi)$, i.e. the RF voltage, is a linear function of $\dot{\phi}$.

With (1-24) the synchrotron tune

$$Q_s = \frac{1}{\omega_0} \frac{2\pi}{\hat{T}_S} = \frac{1}{\omega_0 |K_1|} \frac{2\pi}{T_{so}^*}$$

is given by

$$Q_{so}^2 = \frac{\text{hn} eV g'(|\phi_s|)}{\gamma^2 E_0} = -\frac{1}{2\pi} \left|\frac{\text{hn} eV g'(|\phi_s|)}{\gamma^2 E_0}\right|$$

where $g'(|\phi_s|) = \frac{d}{d\phi}$ (RF voltage) at $\phi_s$

(1-33)

**Height of a trajectory in synchrotron phase space.**

For a trajectory of half width $\hat{\psi}$, the height is obtained from (1-32) as $\hat{\psi}^2 = \left|g'(|\phi_s|)\right|^2 \hat{\psi}^2$ whence, with (1-22) and (1-24),

$$\left(\frac{\delta E}{\hbar \omega_0}\right) = K_2 \hat{\psi} \left|\begin{array}{c}
\text{hn} \\
\gamma^2 \\
\text{hn} \left|\frac{eV g'(|\phi_s|)}{2\pi E_0}\right| \hat{\psi}
\end{array}\right|$$

**Longitudinal emittance. Bunch matching.**

The area of a bunch in synchrotron phase space is its longitudinal emittance $E_s$; it is an invariant by Liouville's theorem. If we call "emittance of a single particle" the area $2\pi J$ in phase space which is enclosed by the particle trajectory,
The action $J$ is an adiabatic invariant, i.e. it stays constant if the parameters in $H$ are varied infinitely slowly (Ref. 4, p. 154; Ref. 5, p. 110; Ref. 6, p. 234). If at some time a bunch is matched (which means that its border in phase space is just the closed trajectory of the outermost particles) then its emittance $E_s$ is equal to the single particle emittance of its outermost particles. After a change of the parameters in $H$, the emittance $E_s$ is unchanged but the action of the outermost particles has changed slightly and differently for each particle, which means that the bunch is no longer matched exactly: therefore the matching of a bunch can only be preserved in the adiabatic sense, i.e. if the parameters in $H$ are varied very slowly.

### 7. Stationary bucket with an harmonic cavity

When the beam is not accelerated but is simply kept bunched at a fixed energy, $\gamma = 0$ and the bucket is called stationary. In this case, which corresponds to collider operation, the frequency of synchrotron oscillations as a function of phase amplitude $\hat{\psi}$ is given by

$$\omega_s = \frac{\omega_s^*}{|K_1|}$$

where as a first approximation

$$\omega_s^* = - \text{sgn}(\eta) \cdot \sum_{n=1}^{\infty} a_n \cos n \psi_s \cdot \frac{2}{\hat{\psi}} J_1(n \hat{\psi}) + o(\hat{\psi})$$ (1-35)

when

$$g(\psi) = \sum_{n=1}^{\infty} a_n \sin n \psi, \quad \Gamma = g(\psi_s) = 0, \quad \text{sgn}(\eta) \cdot g'(\psi_s) = \text{sgn}(\eta) \cdot \sum_{n=1}^{\infty} n a_n \cos n \psi_s < 0.$$ 

Assuming that $a_1 = 1$ is the dominant term,

$$\psi_s = 0 \quad \text{if} \quad \eta < 0$$

$$\psi_s = \pi \quad \text{if} \quad \eta > 0.$$ 

With (1-35), the synchrotron frequency for vanishingly small amplitudes is

$$\omega_{s0}^* = -\text{sgn}(\eta) \cdot \sum_{n=1}^{\infty} n a_n \cos n \psi_s = -\text{sgn}(\eta) \cdot g'(\psi_s) = |g'(\psi_s)|$$ (1-36)

in agreement with (1-33).

*) $J_v(x)$ is the Bessel function of the first kind of order $v$ and argument $x$. 
The relation (1-35) allows shaping the variation of $\omega_s^2$ with $\hat{\phi}$. For example, if $\omega_s^2$ is to be proportional to $\hat{\phi}^2$, it is sufficient to take $a_1 = 1$ and $\text{sgn}(n) \cdot n \cos n \phi_s = 1$ for some $n > 1$, then

$$\omega_s^2 = \frac{2}{\hat{\phi}} J_1(\hat{\phi}) - \frac{2}{n \hat{\phi}} J_1(n \hat{\phi}) + 0(\hat{\phi}^4) = \frac{n^2 - 1}{2} \left( \frac{\hat{\phi}}{2} \right)^2 + 0(\hat{\phi}^4) \quad (1-37)$$

If the sign of $a_n$ is reversed so that $\text{sgn}(n) \cdot n \cos n \phi_s = -1$, then

$$\omega_s^2 = \frac{2}{\hat{\phi}} J_1(\hat{\phi}) - \frac{2}{n \hat{\phi}} J_1(n \hat{\phi}) + 0(\hat{\phi}^4) = 2 \left[ 1 - \frac{n^2 - 1}{4} \left( \frac{\hat{\phi}}{2} \right)^2 + 0(\hat{\phi}^4) \right] \quad (1-38)$$

The latter case (1-38) corresponds to the usual operation of a Landau harmonic cavity, which increases the relative spread of synchrotron frequencies as a function of $\hat{\phi}$. In both cases the amplitude of the harmonic voltage is such that

$$\left| \frac{a_n}{a_1} \right| = \frac{1}{n}.$$  

1.8 Formulae for a sinusoidal RF voltage

Formulae are simpler when using the phase $\varphi$ measured from the crest of the RF voltage:

$$\phi = \frac{\pi}{2} + \varphi,$$

$$\phi_s = \frac{\pi}{2} + \varphi_s, \quad \phi_u = \frac{\pi}{2} - \varphi_s, \quad \text{sgn}(\varphi_s) = \text{sgn}(n), \quad \sin \phi_s = \cos \varphi,$$

From (1-15) and (1-21), $g(\phi) = \sin \phi = \cos \varphi$, $\Gamma = g(\phi_s) = \sin \phi_s = \cos \varphi_s$,

$$G(\phi) = \cos \phi = -\sin \varphi.$$  

**Bucket width** ($\phi_e - \phi_u$) where

$$\tan(\theta_e - \theta_u - 3 \varphi_s) = \frac{1}{10} \tan^3 \varphi_s \left[ 1 + \frac{1}{140} \tan^2 \varphi_s + 0(\tan^4 \varphi_s) \right]^{-1} \quad (1-39)$$

**Bucket height** From (1-29), $\frac{\pi^2}{2} = 2|\sin \varphi_s - \varphi_s \cos \varphi_s|$ and, using (1-24):

$$\frac{\delta E}{h\nu_0} = \frac{R}{h} \delta \phi = \frac{E_0}{h\nu_0} 2 \left[ \delta^2 \gamma \frac{eV \sin \varphi_s}{h} \left[ 1 - \varphi_s \cot \varphi_s \right]^{1/2} = \frac{E_0}{h\nu_0} 2 \left[ \frac{e^2 \gamma}{h} Q \sqrt{1 - \varphi_s \cot \varphi_s} \right] \quad [eV \cdot s] \quad (1-40)$$

**Bucket area** From (1-30),

$$A^*_s = \sqrt{2} \int_{\phi_u}^{\phi_e} d\psi \sqrt{|\psi_u \sin \phi_u + \cos \phi_u - \phi \sin \phi \cos \phi|^{1/2}}$$
\( A_s^* \) is steadily increasing with \(|\varphi_s|\). Its maximum value is obtained for \( \varphi_s = \pi/2 \) (stationary bucket):

\[
A_s^{\ast \text{ max}} = 2\sqrt{2} \int_0^{2\pi} \text{d} \phi \sqrt{1 - \cos \phi} = 16
\]

Let

\[
a(\gamma) = \frac{A_s^*}{A_s^{\ast \text{ max}}} = \frac{A_s}{A_s^{\ast \text{ max}}} ; \quad a(\gamma) = \frac{3}{10} \left| \varphi_s \right|^{\frac{5}{2}} \left[ 1 + \frac{1}{60} \varphi_s^2 + 0(\varphi_s^4) \right]^{-1}
\]

With (1-24)

\[
A_s = 16 \frac{E_m}{\hbar n_0} \left| \frac{\delta \gamma}{\hbar n} \cdot \frac{\text{eV}}{2\pi E_0} \right|^\frac{1}{2} a(\gamma) \text{ per bunch} \quad \text{[eV.s]} \quad (1-42)
\]

Remark: Instead of (1-40), the dimensionless quantity \( \hat{\delta} p/m_0 c \) is often used. In these coordinates, we have:

**Bucket height**

\[
\frac{\hat{\delta} p}{m_0 c} = 2 \left| \gamma \frac{\text{eV sin } \varphi_s}{\hbar n} \right|^\frac{1}{2} \left[ 1 - \varphi_s \cotg \varphi_s \right]^\frac{1}{2} \left| \frac{\delta p}{m_0 c} \right| \quad (1-43)
\]

**Bucket area**

\[
A_s \cdot \frac{\hbar}{m_0 c R} = 16 \left| \frac{\gamma}{\hbar n} \cdot \frac{\text{eV}}{2\pi E_0} \right|^\frac{1}{2} a(\gamma) \text{ per bunch} \quad \left| \frac{\delta p}{m_0 c} \right| \cdot \text{rad} \quad (1-44)
\]

**Period of (large) synchrotron oscillations in a stationary bucket**

Besides the trivial case of a linear RF voltage, a sinusoidal RF voltage is the only case where it is possible to compute simply the synchrotron period for any amplitude.

With \( \gamma = 0 \) and \( G(\psi) = \cos \phi \), the general equation (1-31) reduces to

\[
T_s^* = \sqrt{2} \int_{\varphi_1}^{\varphi_2} \text{d} \phi \left| \cos \phi_1 - \cos \phi \right|^{-\frac{1}{2}} = \sqrt{2} \int_{-\psi}^{\psi} \text{d} \psi \left| \cos \psi - \cos \hat{\psi} \right|^{-\frac{1}{2}} = 4K \left( \sin \frac{\hat{\psi}}{2} \right)
\]

where \( \psi = \phi - \phi_s; K(k) \) is the complete elliptic integral of the first kind with modulus \( k \).

Therefore

\[
\omega_s^2 = \left[ \frac{2}{\pi} K \left( \sin \frac{\hat{\psi}}{2} \right) \right]^{-2} = 1 - \frac{1}{2} \sin^2 \frac{\hat{\psi}}{2} - \frac{3}{32} \sin^4 \frac{\hat{\psi}}{2} - \ldots \quad (1-45)
\]

\[
= 1 - \frac{1}{2} \left( \frac{\hat{\psi}}{2} \right)^2 + \frac{7}{8.12} \left( \frac{\hat{\psi}}{2} \right)^4 - \ldots
\]
The approximate Eq. (1.35) would yield in this case

\[ \omega_s^2 = \frac{2}{\phi} J_1(\tilde{\phi}) = 1 - \frac{1}{2} \sin^2 \frac{\tilde{\phi}}{2} - \frac{1}{12} \sin^4 \frac{\tilde{\phi}}{2} - \ldots = 1 - \frac{1}{2} \left(\frac{\tilde{\phi}}{2}\right)^2 + \frac{1}{12} \left(\frac{\tilde{\phi}}{2}\right)^4 - \ldots \]

which shows that the error on the \( \hat{u}^4 \) term in Eq. (1-35) is rather small.

1.9 Back to finite difference equations. Stochasticity

For an arbitrary RF voltage, the finite difference equations (1-13) and (1-9) read in case of a synchrotron \( (R_s = \text{constant}) \):

\[
\begin{align*}
\delta p_{n+1} - \delta p_n &= \frac{eV}{N_{ps}} \left[ g(\phi_n) - g(\phi_s) \right] \quad (1-46) \\
\phi_{n+1} - \phi_n &= \frac{2\pi n}{N_{ps}} \cdot \delta p_{n+1} \quad (1-47)
\end{align*}
\]

where \( n, v_s, p_s \) are slowly varying parameters.

This mapping preserves area in the \( \delta p, \phi \) plane.

Fixed points (mod 2\( n \)): If \( k \) is any integer,

\[
\begin{align*}
\phi_n &= \phi_s + 2\pi kn, \quad \frac{hn}{N_{ps}} \cdot \delta p_n = k \quad \text{is a stable fixed point} \\
\phi_n &= \phi_u + 2\pi kn, \quad \frac{hn}{N_{ps}} \cdot \delta p_n = k \quad \text{is an unstable fixed point.}
\end{align*}
\]

The \( k \neq 0 \) case corresponds to working with the same RF frequency, but with an harmonic number \( (h + Nk) \). Indeed, for a given \( w_{RF} \) the synchronous revolution frequencies are such that

\[
\omega_0 = \frac{w_{RF}}{h}, \quad \frac{\delta \omega_0}{\omega_0} = -\frac{\delta h}{h}
\]

to which correspond the synchronous momenta

\[
\eta \frac{\delta p}{p_s} = \frac{\delta \omega_0}{\omega_0} = \frac{\delta h}{h} = Nk
\]

This means that the vertical distance between neighbouring buckets is \( \frac{\delta p}{p_s} = \frac{N}{|hn|} \). In order to prevent stochastic effects (Ref. 7), the ratio \( \ell \) between the full bucket height and the vertical distance between neighbouring buckets should be less than 1 (Chirikov's criterion). From (1-45),
Fig. 5 - Phase space enlarged to several harmonic numbers at a fixed RF frequency (for \( n > 0 \))

\[
\xi = \frac{4}{\beta \gamma} \left| \frac{\mathcal{R}_{\text{RF}}}{\hbar n} \frac{eV \sin \phi_s}{2 \pi E_0} \right| \frac{1}{\sqrt{1 - \varphi_s \cotg \varphi_s}} = \frac{4}{N} \left| \frac{\hbar n}{\beta \gamma} \frac{eV \sin \phi_s}{2 \pi E_0} \right| \frac{1}{\sqrt{1 - \varphi_s \cotg \varphi_s}}
\]

so that Chirikov's criterion reads, with \((1.33)\):

\[
\xi = \frac{Q_{s0}}{N} \cdot \frac{1}{1 - \varphi_s \cotg \varphi_s} < 1
\]

Therefore, differential equations are valid only when \(Q_{s0}/N \ll 1\). For a finite \(Q_{s0}\), the motion near the bucket border becomes chaotic, making the bucket area shrink (Ref. 8); in practice, this effect is still very small for \(Q_{s0}/N < 0.1\).

**Period of small synchrotron oscillations**

In the close vicinity of \(\phi_s\), the system \((1.46)\), \((1.47)\) reduces to

\[
\begin{align*}
\frac{p_{n+1} - p_n}{\hbar N \phi_s} &= -K_0 (\phi_n - \phi_s) \\
\phi_{n+1} - \phi_n &= p_{n+1}
\end{align*}
\]

where we have put

\[
p_n = \frac{2 \pi n}{N \phi_s} \cdot \delta p_n
\]

and

\[
K_0 = -\frac{2 \pi \hbar}{N^2} \frac{eV g'(\phi_s)}{p_s \phi_s} > 0
\]
With \( \mu \) defined by

\[ 4 \sin^2 \frac{\mu}{2} = K_0, \quad (1-51) \]

the system (1-48) admits of the general solution

\[ \phi_n - \phi_s = \text{Re} \left[ a e^{i \mu n} \right], \quad p_n = \text{Re} \left[ i a 2 \sin \frac{\mu}{2} e^{i(n-j)\mu} \right] \quad (1-52) \]

This represents small synchrotron oscillations with angular frequency

\[ \omega_{so} = \frac{N}{T_0} \mu = N \omega_0 \frac{\mu}{2\pi}. \]

From (1-50) and (1-51) the synchrotron tune for small oscillations is given by

\[ 4 N^2 \sin^2 \left( \frac{\omega_{so}}{N} \right) = N^2 K_0 = -2\pi \hbar n \frac{eV g'(\phi_s)}{p_s \gamma} = -2\pi \frac{\hbar n}{\beta \gamma} \frac{eV g'(\phi_s)}{E_0} \quad (1-53) \]

The analogous formula (1-33) obtained with differential equations appears to be the limiting case of (1-53) when \( N \to \infty \), i.e. when particle acceleration is evenly distributed all around the ring.

1.10 Phase displacement acceleration

For a fixed harmonic number, the RF frequency determines the synchronous revolution frequency or, because

\[ \frac{\eta \delta p}{p_s} = \frac{\delta \omega_0}{\omega_0}, \]

the synchronous momentum of the particles. Let \( \omega_1 \) and \( \omega_2 \) be two revolution frequencies located on both sides of the central revolution frequency \( \omega_0 \) of the stack, well outside the stack, with \( \omega_1 \) corresponding to a higher momentum than the stack (Fig. 6).

![Fig. 6 - Sweeping an empty bucket through the stack, from \( \omega_1 \) to \( \omega_2 \) (\( \eta > 0 \)). With \( \gamma = \sin \phi_s < 0 \), particles move upwards around the bucket (when \( \gamma \) changes sign, all \( \phi \)'s change sign).](image-url)
If the RF frequency is varied from $\omega_1$ to $\omega_2$ (Fig. 7), an empty bucket is moved completely through the stack in the direction of decreasing $p$; because phase space is incompressible, the average position of the stack is moved upwards by a quantity equal to (Bucket area/Horizontal axis period). Therefore, according to (1-44), the average momentum of the stack is increased by

$$\frac{\Delta p}{m_0c} = \frac{16}{2\pi} \left| \frac{\gamma}{h_n} \frac{eV}{2\pi E_0} \right|^\frac{1}{2} \cdot \alpha(\tau) \text{ per empty bucket sweep} \quad (1-54)$$

This method of phase displacement (Ref. 9) allows acceleration of a stack by an empty bucket with a momentum height which is much smaller than the momentum spread of the stack; this is in contrast with normal acceleration, where the momentum height of the bucket is necessarily larger than the momentum spread of the bunch. Phase displacement acceleration has been successfully used in the ISR to accelerate coasting beams from 26.6 to 31.4 GeV/c (Ref. 10).
2. **RF NOISE**

2.1 **Statistical properties of random variables**

We shall consider only random variables which are stationary in time, i.e. random variables for which some statistical properties (the mean and the autocorrelation) are independent of time. Obviously, a random variable \( f(t) \) which is stationary in time does not vanish at \( t = \infty \). Therefore its Fourier transform can only be computed on a finite time interval \( T \):

\[
\hat{f}(\omega, T) = \int_0^T e^{-i\omega t} f(t) dt, \quad f(t) = \int_{-\infty}^{+\infty} e^{i\omega t} \hat{f}(\omega, T) \cdot \frac{d\omega}{2\pi} \quad \text{for } 0 < t < T \tag{2-1}
\]

Let brackets represent an ensemble average over all noises of the same kind; it is assumed that

\[
\langle f(t) \rangle = 0 \quad \text{whence} \quad \langle \hat{f}(\omega, T) \rangle = 0
\]

The time average over \( T \) of the power of a particular noise sample reads

\[
\frac{1}{T} \int_0^T [f(t)]^2 dt = \int_{-\infty}^{+\infty} \frac{1}{\pi} |\hat{f}(\omega, T)|^2 \cdot \frac{d\omega}{2\pi}
\]

and the ensemble average of the noise power is

\[
\frac{1}{T} \int_0^T \langle [f(t)]^2 \rangle dt = \int_{-\infty}^{+\infty} \frac{1}{\pi} \langle |\hat{f}(\omega, T)|^2 \rangle \cdot \frac{d\omega}{2\pi} \tag{2-2}
\]

If \( f(t) \) is stationary in time, \( \langle [f(t)]^2 \rangle \) is independent of \( t \) (by definition) and the integrand in the right-hand side of (2-2) should approach a definite limit when \( T \to \infty \); therefore it is also assumed that the following limit exists:

\[
\lim_{T \to \infty} \frac{1}{T} \langle [\hat{f}(\omega, T)]^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \langle \left| \int_0^T e^{-i\omega t} f(t) dt \right|^2 \rangle = S_F(\omega) \geq 0 \quad S_F(-\omega) = S_F(\omega) \tag{2-3}
\]

\( S_F(\omega) \) is called the spectral power density of the random variable \( f \) at frequency \( \omega \).

If \( S_F(\omega) \) contains a Dirac peak at \( \omega_n \), the variable \( f(t) \) has a discrete spectral line at frequency \( \omega_n \). Pure noise is characterized by a continuous spectrum with a power density \( S_F(\omega) \) free of Dirac peaks. The existence of \( S_F(\omega) \) as an ordinary function implies that

\[
|\hat{f}(\omega, T)| = \left| \int_0^T e^{-i\omega t} f(t) dt \right| = \text{0 for } (\omega \cdot T) \quad \text{when } T \to \infty \tag{2-4}
\]
However, for a particular noise of the ensemble, the quantity

\[ \frac{1}{T} |\bar{f}(\omega, T)|^2 \]  

(2-5)
does not tend to a definite limit when $T \to \infty$ (Ref. 11, p. 222; Ref. 12, p. 305). The quantities (2-5) take different values for each particular (large) $T$ and each particular noise. They are statistically distributed around the ensemble average (2-3), with a standard deviation which is at least equal to the ensemble average itself; this means that fluctuations around the ensemble average (2-3) are quite large. These fluctuations can be reduced by smoothing the spectrum (i.e., replacing each point of the spectrum by some average taken over the neighbouring points) at the cost of reduced resolution (Ref. 12, p. 380; Ref. 13, p. 72).

Another way of estimating the power spectrum $S_F(\omega)$ is to use the autocorrelation (or autocovariance) function $R(\tau)$, which is defined as

\[ R(\tau) = (f(t) \cdot f(t + \tau)) \]  

(2-6)

For a stationary process, $R(\tau)$ is independent of $t$ and $R(-\tau) = R(\tau)$.

It is easy to show (Wiener-Khintchine theorem) that the spectral power density of the random variable $f(t)$ is the Fourier transform of its autocorrelation function:

\[ S_F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \tau} R(\tau) \, d\tau = 2 \int_0^{\infty} R(\tau) \cos(\omega \tau) \, d\tau \]  

(2-7)

whence

\[ R(\tau) = \int_{-\infty}^{\infty} e^{i\omega \tau} S_F(\omega) \frac{d\omega}{2\pi} = 2 \int_0^{\infty} S_F(\omega) \cos(\omega \tau) \cdot \frac{d\omega}{2\pi} \]  

(2-8)

If $R(\tau)$ is appreciable only for $\tau \leq \tau_C$, $S_F(\omega)$ extends at least up to frequencies $\omega_C$ where

\[ 2 \frac{\omega_C}{2\pi} \cdot 2 \tau_C \geq 1 \]

(uncertainty principle).

By combining either (2-2) and (2-3) or (2-6) and (2-8) we get

\[ \langle \left[ \bar{f}(t) \right]^2 \rangle = R(0) = 2 \int_0^{\infty} S_F(\omega) \cdot \frac{d\omega}{2\pi} \]  

(2-9)

Very often a stationary process having a continuous power spectrum is assumed to be ergodic, which means that ensemble averages are equal to time averages taken over any single sample of the process. Then the autocorrelation function may also be computed as
\[ R(t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \cdot f(t + \tau) \, d\tau \quad (2-10) \]

2.2 Fokker-Planck equation

Here we follow the derivation given in Ref. 14, p. 98. Let \( x \) be a generalized coordinate describing the motion of particles; in the most general case \( x \) should be replaced by a complete set of coordinates in phase space. Let \( \rho(x_0, x, t) \, dx \) be the probability of a particle starting at \((x_0, t=0)\) to arrive somewhere between \((x, t)\) and \((x + dx, t)\); obviously \( \int \rho \, dx = 1 \).

When \( t \) increases by \( \Delta t \), \( x \) increases by a \( \Delta x \) which is different for each particle. Let \( \psi(\Delta x, x, t) \cdot d(\Delta x) \) be the probability of finding \( \Delta x \) between \( \Delta x \) and \( \Delta x + d(\Delta x) \). The basic assumption is that \( \psi \) is independent of \( x_0 \), which means (by definition) that the stochastic process is a Markoff process. Writing \( x_1 = x + \Delta x \) we get

\[
\rho(x_0, x_1, t + \Delta t) = \int \rho(x_0, x_1 - \Delta x, t) \cdot \psi(\Delta x, x_1 - \Delta x, t) \, d(\Delta x) \quad (2-11)
\]

The integrand is expanded as a Taylor series in \( \Delta x \) about \( x_1 \):

\[
\rho(x_0, x_1, t) \psi(\Delta x, x_1, t) = \Delta x \frac{\partial}{\partial x_1} \left[ \rho(x_0, x_1, t) \psi(\Delta x, x_1, t) \right] + \frac{(\Delta x)^2}{2!} \frac{\partial^2}{\partial x_1^2} \left[ \rho(x_0, x_1, t) \psi(\Delta x, x_1, t) \right] -
\]

With

\[
\int \psi(\Delta x, x_1, t) \, d(\Delta x) = 1 \quad \text{and} \quad \int (\Delta x)^m \psi(\Delta x, x_1, t) \, d(\Delta x) = ((\Delta x)^m),
\]

Eq. (2.11) becomes

\[
\rho(x_0, x_1, t + \Delta t) = \rho(x_0, x_1, t) - \frac{\partial}{\partial x_1} \left[ \rho(x_0, x_1, t) \psi(\Delta x) \right] + \frac{1}{2} \frac{\partial^2}{\partial x_1^2} \left[ \rho(x_0, x_1, t) \psi(\Delta x)^2 \right] -
\]

Let

\[
A_1(x_1, t) = \lim_{\Delta t \to 0} \frac{\langle \Delta x \rangle}{\Delta t} \quad A_2(x_1, t) = \lim_{\Delta t \to 0} \frac{\langle (\Delta x)^2 \rangle}{\Delta t} \quad (2-12)
\]

and assume

\[
A_k(x_1, t) = \lim_{\Delta t \to 0} \frac{\langle (\Delta x)^k \rangle}{\Delta t} = 0 \quad \text{for} \quad k > 2 \quad (2-13)
\]

then the last equation reduces to

\[
\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} (\rho A_1) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (\rho A_2) \quad (2-14)
\]

which is the Fokker-Planck equation in one dimensional space.
In the limits (2-12), (2-13), $\Delta t$ must include many periods of the unperturbed motion (in case of RF noise, many synchrotron oscillations). The limit $\Delta t \to 0$ means that the time scale which is considered in the Fokker-Planck equation is much longer than $\Delta t$, and that on the intermediate time scale $\Delta t$, $\langle \Delta x \rangle$ and $\langle (\Delta x)^2 \rangle$ are small quantities of the order of $\Delta t$. In fact, in equations (2-35) and (2-39), $\Delta t$ will be considered as infinite (with respect to the period of synchrotron oscillations).

With the definitions (2-12), the Fokker-Planck equation (2-14) remains invariant when $x$ is replaced by $kx$, where $k$ is any constant. Moreover, since $A_1$ and $A_2$ are ensemble averages over all noises with the same power spectrum, the Fokker-Planck equation does not represent the evolution of $\rho(x,t)$ under the influence of a particular noise sample; it rather represents an average evolution of $\rho(x,t)$ under the influence of all noises of the ensemble. However, any non-linearity in the RF voltage produces a variation of synchrotron frequency with amplitude, which entails filamentation and mixing of particles in phase space; this mixing is ultimately sufficient to make the description by Eq. (2-14) adequate, even for a single noise sample. The only exception is the case of a linear RF voltage with a noise whose spectrum does not extend so far as the RF frequency; in that particular case, the bunch is shaken (by phase noise) or deformed (by amplitude noise) as a whole but it does not diffuse in phase space (Ref. 19).

The Fokker-Planck equation has been used to describe momentum stochastic cooling in the ISR, in the form (Ref. 15, p.93)

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial E} (F \rho) + \frac{\partial}{\partial E} \left( D \frac{\partial \rho}{\partial E} \right) \tag{2-15}$$

where

- $E$ is the particle energy
- $F$ is a cooling coefficient due to feedback
- $D$ is a diffusion coefficient due to amplifier noise and particle Schottky noise.

In 1977, an experiment was performed at the ISR (Ref. 16) in order to decide between $\frac{\partial^2}{\partial E^2} (D \rho)$ or $\frac{\partial}{\partial E} (D \frac{\partial \rho}{\partial E})$ in the right-hand side of Eq. (2-15); it confirmed the validity of the diffusion equation as written in (2-15). In fact, it will be shown later that for trapped motion (i.e. libration), the Fokker-Planck equation (2-14) reduces to a diffusion equation (2-41) when the generalized coordinate $x$ in (2-14) is taken as being proportional to the action $J$. For a coasting beam (i.e. rotation), to first order $\delta J = \delta E/\omega_\perp$ is proportional to $\delta E$; this is the reason why (2-14) reduces to the diffusion equation (2-15) when $E$ is taken as the generalized coordinate.

2.3 Differential equations for a stationary bucket with amplitude and phase noise

With noise, the finite difference equations (1-46), (1-47) become
\[
\begin{align*}
\delta P_{n+1} - \delta P_n &= \frac{eV(1 + a_n)}{Nv_s} \left[ g(\phi_n) - g(\phi_s) \right] \quad \text{where } a_n \text{ is the relative amplitude noise} \\
\phi_{n+1} - \phi_n &= \frac{2n\eta}{Np_s} \cdot \delta P_{n+1} + \psi_{n+1} - \psi_n \quad \text{where } \psi_n \text{ is the phase noise} \\
\delta \phi &= \phi_n - \phi_s \quad \text{is the actual phase with noise}
\end{align*}
\]

For a stationary bucket, \( g(\phi_s) = 0 \).

**Reduced variables**

Let us define

\[
g_s(\delta \phi) = \frac{g(\delta \phi + \delta \phi)}{g'(\phi_s)} \quad \text{so that} \quad g'_s(0) = 1
\]

and, by analogy with (1-21),

\[
G_s(\delta \phi) = -\int g_s(\delta \phi) \, d(\delta \phi) = \frac{G(\delta \phi + \delta \phi)}{g'(\phi_s)}
\]

From now on, we shift the origin of phases to \( \phi_s \), which means that \( \phi_n \) will represent \( \delta \phi = \phi_n - \phi_s \); similarly \( \psi_n \) will represent \( (\psi_n - \phi_s) \).

For a linear RF voltage,

\[
g_s(\phi) = \phi \quad \text{and} \quad G_s(\phi) = -\frac{\phi^2}{2}
\]

For a sinusoidal RF voltage,

\[
g_s(\phi) = \sin \phi \quad \text{and} \quad G_s(\phi) = \cos \phi
\]

With (1-49), (1-50) and (2-17), the system (2-16) reads

\[
\begin{align*}
\dot{P}_{n+1} - P_n &= -K_0(1 + a_n)g_s(\phi_n) \\
\dot{\phi}_{n+1} - \phi_n &= \dot{P}_{n+1} + \psi_{n+1} - \psi_n
\end{align*}
\]

The effect of noise will be considered as a perturbation of the motion without noise. With finite difference equations, the unperturbed motion is known only for a linear RF voltage (see paragraph 2.6). For any other RF voltage, we must again use differential equations:

\[
\begin{align*}
\dot{P} &= -\frac{N^2}{T_0} K_0 \left[ 1 + a(t) \right] g_s(\phi) \\
\dot{\phi} &= P + \dot{\phi}(t)
\end{align*}
\]
where

\[ P = \frac{N}{T_0} P_n \]  

(2-23)

We also put

\[ \frac{N^2}{T_0^2} K_0 = \Omega^2 \]  

(2-24)

where, with (1-53),

\[ \Omega = \frac{2N}{T_0} \sin \left( \frac{\omega_{so} T_0}{2N} \right) \]

Let us note that \( \dot{\phi}(t) = f(t) \) is the frequency noise, and that \( \omega_{so} \) is the frequency of small synchrotron oscillations for finite difference equations; for differential equations this frequency is \( \Omega \).

The system (2-22) can be derived from the time-dependent Hamiltonian

\[ H = \frac{1}{2} P^2 + \Omega^2 \left[ 1 + a(t) \right] \left[ G_s(0) - G_s(\phi) \right] + P f(t) \]  

(2-25)

**Action-angle variables**

For the unperturbed motion,

\[ H_0 = \frac{1}{2} P^2 + U(\phi) \quad \text{where} \quad U(\phi) = \Omega^2 \left[ G_s(0) - G_s(\phi) \right] \]  

(2-26)

The action \( J \) is defined as

\[ 2\pi J = \int P \, d\phi = 2 \int_{\phi_1}^{\phi_2} d\phi \sqrt{2H_0 - 2U(\phi)} \]  

(2-27)

where at \( \phi_1 \) and \( \phi_2 \),

\[ 2H_0 - 2U(\phi) = 0. \]

Using (2-27) we get

\[ \frac{dJ}{dH_0} = \frac{1}{2\pi} \int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{2H_0 - 2U(\phi)}} + \frac{\sqrt{2H_0 - 2U(\phi_2)}}{0} \cdot \frac{1}{U'(\phi_2)} \]

\[ - \frac{\sqrt{2H_0 - 2U(\phi_1)}}{0} \cdot \frac{1}{U'(\phi_1)} \]

hence

\[ \frac{dJ}{dH_0} = \frac{1}{2\pi} \int P \, d\phi = \frac{T_s}{2\pi} = \frac{1}{\omega_s} \quad \text{and} \quad H_0 = \int \omega_s \, dJ \]  

(2-28)

We now seek a canonical transformation from the variables \((P, \phi)\) to other canonical variables \((J, \theta)\). This is achieved (Ref. 18, p.239-241) by using the generating function
\[ F_2(\phi, J) = \int d\phi \sqrt{2 \int \omega_s \, dJ - 2 \, U(\phi)} \]

which is defined in such a way that \( P = \frac{\partial F_2}{\partial \dot{\phi}} \); the variable \( \theta \) is then given by

\[ \theta = \frac{3F_2}{\dot{J}} = \omega_s \int \frac{d\phi}{\sqrt{2 \int \omega_s \, dJ - 2 \, U(\phi)}} = \omega_s \int dt \text{ in unperturbed motion} \quad (2-29) \]

while the new Hamiltonian reads \( H + \frac{3F_2}{\dot{\theta}} = H \).

From (2-29) it is seen that \( \theta \) has period \( 2\pi \) in the unperturbed motion; hence its name "angle variable". (This angle variable should not be confused with the azimuthal variable \( \theta \) defined in paragraph 1.3).

For the motion with noise, from (2-25), (2-26) and (2-28):

\[ H = \int \omega_s \, dJ + P \, f(t) + U \, a(t) \quad (2-30) \]

In (2-30) \( P \) and \( U \) must be expressed in terms of \( J \) and \( \theta \). Since the unperturbed motion has period \( 2\pi \) in \( \theta \), we may write

\[
\begin{align*}
P = \frac{\partial H}{\partial \dot{\theta}} &= \sum_{m=-\infty}^{+\infty} b_m(J) e^{im\theta} \quad \text{where} \quad b_m = b_m^* \\
U = \frac{\partial H}{\partial \dot{a}} &= \sum_{m=-\infty}^{+\infty} c_m(J) e^{im\theta} \quad \text{where} \quad c_m = c_m^* 
\end{align*}
\quad (2-31)
\]
2.4 Computation of the coefficients $A_1$, $A_2$ in the Fokker-Planck equation

The equations of motion are canonical:

$$\frac{dJ}{dt} = -\frac{\partial H}{\partial \theta}, \quad \frac{d\theta}{dt} = \frac{\partial H}{\partial J}$$

For computing $A_1$, $A_2$ as defined in (2-12), we have first to compute (with $\Delta t = T$)

$$A_J = J(T) - J(0) = -\int_0^T \frac{\partial H}{\partial \theta} \, dt$$

In order to simplify the writing, from now on we consider only the phase noise; the derivation would be identical for the amplitude noise. From (2-30) and (2-31):

$$J(T) - J(0) = -\int_0^T f(t) \, dt \sum_{m=-\infty}^{+\infty} \text{im} b_m(t) \cdot e^{i\omega t}$$

where

$$b_m(t) = b_m(0) + \frac{db_m}{dT}(0) \cdot [J(t) - J(0)] + \ldots,$$

whereas

$$\frac{d\theta}{dt} = \omega_s + f(t) \sum_{n=-\infty}^{+\infty} \frac{db_n}{dT}(t) \cdot e^{i\omega t}$$

where

$$\omega_s(t) = \omega_s(0) + \frac{d\omega_s}{dT}(0) \cdot [J(t) - J(0)] + \ldots$$

The relation (2-4) shows that in order to compute quantities of order $T$, we must keep quantities up to 2nd order in $f(t)$. Moreover, for computing $\Delta J$ up to 2nd order in $f(t)$, we must retain quantities up to first order when using $b_m(t)$ and $\theta$.

Integrate (2-33) up to first order in $f(t)$:

$$\theta(t) = \theta_s + \omega_s(0) \cdot t + \frac{d\omega_s}{dT}(0) \cdot \int_0^t dt'[J(t') - J(0)] + \int_0^t f(t') \, dt' \sum_{n=-\infty}^{+\infty} \frac{db_n}{dT}(0) \cdot e^{i\delta_s + i\omega_s(0)t'}$$

$$- \int_0^t dt' \int_0^t f(t'') \, dt'' \sum_{n=-\infty}^{+\infty} \text{im} b_n(0) \cdot e^{i\delta_s + i\omega_s(0)t''}$$
and insert into (2-32):

\[
J(T) - J(0) = - \sum_{m=\infty}^{+\infty} \int_0^T f(t) dt \left( 1 + \text{im} \sum_{n=-\infty}^{+\infty} e^{in\theta_0} \int_0^t f(t') dt' \int_0^{t''} f(t'') dt'' \text{in} \, b_n(0) e^{in\omega_s(t''-t')} \right)
\]

\[
= \sum_{m=\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} e^{im\theta_0} \cdot \text{im} b_m(0) \cdot \text{in} b_n(0) \cdot \int_0^T f(t) dt \, e^{in\omega_s(t)}.
\]

Up to 2nd order in \( f(t) \):

\[
[J(T) - J(0)]^2 = \left[ - \sum_{m=\infty}^{+\infty} e^{im\theta_0} \cdot \text{im} b_m(0) \int_0^T f(t) dt \, e^{im\omega_s(t)} \right]^2
\]

\[
= \sum_{m=\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} e^{i(m+n)\theta_0} \cdot \text{im} b_m(0) \cdot \text{in} b_n(0) \cdot \int_0^T f(t) dt \, e^{i(n+1)\omega_s(t)}
\]
When taking the ensemble average of this expression we must also average over all initial conditions \( \theta_0 \); since particles are assumed to be uniformly distributed in \( \theta_0 \), the only terms which will be left are those with \((m+n) = 0\):

\[
(\mathbb{J}(T)-\mathbb{J}(0))^2 = \sum_{m=-\infty}^{+\infty} |\text{im } b_m(0)|^2 \cdot \left( \int_0^T f(t) dt e^{i\omega_S(0)t} \right)^2
\]

With (2-3):

\[
A_2 = \lim_{T \to \infty} \frac{(\mathbb{J}(T)-\mathbb{J}(0))^2}{T} = \sum_{m=-\infty}^{+\infty} |\text{im } b_m|^2 \cdot S_f(\omega_S)
\]  

(2-35)

In the ensemble average of (2-34), the only terms which are left is the \( m = 0 \) term in \( \sum_{m=-\infty}^{+\infty} \), and the \((m+n) = 0\) terms in \( \sum_{m=-\infty}^{+\infty} \); therefore

\[
(\mathbb{J}(T)-\mathbb{J}(0))^2 = \sum_{m=-\infty}^{+\infty} \left| \text{im } \frac{d b_m}{dt}(0) \right|^2 \left| \text{im } b_m(0) \right|^2 \cdot \left( \int_0^T f(t) dt e^{i\omega_S(0)t} \right) \left( \int_0^T f(t') dt' e^{-i\omega_S(0)t'} \right)
\]

\[
+ |\text{im}|^2 b_m(0) \cdot \left( \int_0^T f(t) dt e^{i\omega_S(0)t} \left[ \text{im } b_m^*(0) \frac{d b_m}{dt}(0) \cdot \right] \right)
\]

\[
\cdot \left( \int_0^t dt' \left( \int_0^{t'} f(t'') dt'' e^{-i\omega_S(0)t''} + \frac{d b_m^*}{dt}(0) \cdot \int_0^t f(t') dt' e^{-i\omega_S(0)t'} \right) \right)
\]

\[
= \frac{1}{2} \sum_{m=-\infty}^{+\infty} \left| \text{im } \frac{d b_m}{dt}(0) \right|^2 \left| b_m(0) \right|^2 \cdot \left( \int_0^T f(t) dt e^{i\omega_S(0)t} \int_0^T f(t') dt' e^{-i\omega_S(0)t'} + \text{complex conjugate} \right)
\]

\[
+ |\text{im } b_m(0)|^2 \cdot \left( \int_0^T f(t) dt e^{i\omega_S(0)t} \right)
\]

\[
\cdot \left( \int_0^t dt' \left( \int_0^{t'} f(t'') dt'' e^{-i\omega_S(0)t''} - \text{complex conjugate} \right) \right)
\]

(2-36)

The complex conjugate terms are obtained by changing \( m \) into \(-m\).
Now, with (2-1):

\[
\begin{align*}
\int_0^T f(t) dt e^{im_0 s(0)t} &= \int_0^t f(t') dt' e^{-im_0 s(0)t'} + \text{complex conjugate} \\
&= \int_0^T d\left[\tilde{f}(m_0 s(0), t)\right]^* \tilde{f}(m_0 s(0), t) + \text{complex conjugate} = |\tilde{f}(m_0 s(0), T)|^2 \\
\end{align*}
\]

whereas

\[
\begin{align*}
\int_0^T f(t) dt e^{im_0 s(0)t} &= \int_0^t dt' \int_0^{t'} f(t'') dt'' e^{-im_0 s(0)t''} - \text{complex conjugate} \\
&= \int_0^T d\left[\tilde{f}(m_0 s, t)\right]^* \int_0^t dt' \tilde{f}(m_0 s, t') - \text{complex conjugate} \\
&= \int_0^T d\left[\tilde{f}(m_0 s, t)\right]^* t \tilde{f}(m_0 s, t) - \left[\tilde{f}(m_0 s, T)\right]^* \int_0^T dt' \tilde{f}(m_0 s, t') \\
&\quad + \int_0^T \left[\tilde{f}(m_0 s, t)\right]^* t \tilde{f}(m_0 s, t) \right\} - \text{complex conjugate} \\
&= \left\{ \int_0^T t d|\tilde{f}(m_0 s, t)|^2 - \left[\tilde{f}(m_0 s, T)\right]^* \int_0^T t e^{-im_0 s t} f(t) dt \right\} - \text{complex conjugate} \\
&= \frac{1}{i} \left[\tilde{f}(m_0 s, T)\right]^* \left[\frac{\partial}{\partial \omega} \tilde{f}(\omega, T)\right]_{\omega=m_0 s(0)} - \text{complex conjugate} \\
&= \frac{1}{i} \left[\frac{\partial}{\partial \omega} |\tilde{f}(\omega)|^2\right]_{\omega=m_0 s(0)} \\
\end{align*}
\]

(2-38)

In this derivation, \(m_0 s\) has been written shortly for \(m_0 s(0)\).
With (2-37) and (2-38), (2-36) reads

\[
\begin{align*}
\langle [J(T) - J(0)] \rangle &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} \left\{ \frac{d}{dT} |\text{im} \ b_m(0)|^2 \cdot (|\hat{f}(\omega, T)|^2) \\
&\quad + |\text{im} \ b_m(0)|^2 \cdot m \frac{d}{d\omega} S_{\omega}(0) \left[ \frac{d}{d\omega} \langle |\hat{f}(\omega, T)|^2 \rangle \right]_{\omega=\omega_0} \right\} \\
&= \lim_{T \to \infty} \langle [J(T) - J(0)] \rangle = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \left\{ \frac{d}{dT} |\text{im} \ b_m|^2 \cdot S_{\omega}(\omega_0) + |\text{im} \ b_m|^2 \cdot m \frac{d}{d\omega} S_{\omega}(\omega_0) \right\}
\end{align*}
\]

Using (2-3) again:

\[
A_1 = \lim_{T \to \infty} \frac{\langle [J(T) - J(0)] \rangle}{T} = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \left\{ \frac{d}{dT} |\text{im} \ b_m|^2 \cdot S_{\omega}(\omega_0) + |\text{im} \ b_m|^2 \cdot m \frac{d}{d\omega} \right\}
\]

From (2-34),

\[
\langle [J(T) - J(0)] \rangle = 0 \langle \hat{f}(t) \rangle^2
\]

and

\[
A_2 = \lim_{T \to \infty} \frac{\langle [J(T) - J(0)]^2 \rangle}{T} = 0 \langle \hat{f}(t) \rangle^2 \quad \text{for } \kappa > 2.
\]

Therefore, the conditions (2-13) are satisfied up to 2nd order in \( f(t) \).

Comparing (2-39) with (2-35) we get the remarkable relation

\[
A_1 = \frac{1}{2} 3A_2
\]

Because of this relation, if the variable \( x \) is taken as the action \( J \) in (2-14), the Fokker-Planck equation contains a single coefficient \( A_2 \) and reduces to

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( A_2 \frac{\partial \rho}{\partial x} \right)
\]

which is the general one-dimensional form of a diffusion equation

\[
\frac{\partial \rho}{\partial t} = \text{div} (D \text{ grad } \rho)
\]

where \( A_2/2 = D \) is the diffusion coefficient.

If amplitude noise is also present, (2-35) becomes

\[
A_2 = \sum_{m=-\infty}^{+\infty} |\text{im} \ b_m|^2 \cdot S_{\omega}(\omega_0) + \sum_{m=-\infty}^{+\infty} |\text{im} \ c_m|^2 \cdot S_{\omega}(\omega_0)
\]

In this expression, we have assumed that frequency and amplitude noises are uncorrelated; in that case the ensemble average of the crossed terms over all possible noises is zero. From (2.31), it appears that the coefficient of \( S_{\omega}(\omega_0) \) is the square of the modulus of the Fourier
component of $3^2 H/\theta H$ at $m_\omega$; similarly the coefficient of $S_a(m_\omega)$ is the square of the modulus of the Fourier component of $3^2 H/\theta H$. It should be noticed that noise plays a role only at those frequencies which are present in the unperturbed motion of the particles; moreover, as expected, the zero-frequency component of the noise never plays any role.

Since $S_f(\omega) = \omega^2 S_f(\omega)$ where $S_f(\omega)$ is the spectral density of phase noise, we have

$$A_2 = \sum_{m=-\infty}^{+\infty} (m\omega)^2 |b_m|^2 S_f(m\omega) + \sum_{m=-\infty}^{+\infty} |c_m|^2 S_a(m\omega) \quad (2-44)$$

**Remark:** The relation (2-40) was first derived in 1980 for the case of finite difference equations and a linear RF voltage (Ref. 17); it was then extended to the case of differential equations and a sinusoidal RF voltage (Ref. 19). In 1982 it was proven to hold for any RF voltage (Ref. 20).

2.5 Case of a sinusoidal RF voltage

We only need to determine the coefficients $b_n(J), c_n(J)$ which appear in (2-31) for the unperturbed motion. With (2-20), the unperturbed Hamiltonian (2-26) reads

$$H_0 = \frac{1}{2} P^2 + \Omega^2 \cdot 2 \sin^2 \frac{\phi}{2} = 2 \Omega^2 \left[ \frac{P^2}{2\Omega} + \sin^2 \frac{\phi}{2} \right]$$

We define polar coordinates $k, \psi$ through the relations

$$\frac{P}{2\Omega} = k \cos \psi, \quad \sin \frac{\phi}{2} = k \sin \psi, \quad k = \sin \frac{\phi}{2} \leq 1 \quad (2-45)$$

The unperturbed motion corresponds to $H_0 = 2 \Omega^2 \cdot k^2$ being a constant. The action is given by

$$J = \frac{1}{2\pi} \oint P \, d\phi = \frac{4\Omega k^2}{2\pi} \int_0^{2\pi} \cos \psi \, \frac{\cos \psi \, d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = 2 \Omega k^2 \frac{4}{\pi} B(k) \quad \text{where} \quad \frac{\pi}{4} \leq B(k) \leq 1 \quad (2-46)$$

$$J = 2\Omega \cdot \frac{4}{\pi}$$

when $0 \leq k \leq 1$

The notations for elliptic integrals are those of Jahnke and Emde (Ref. 21, p. 73). From (2-22) and (2-45) we get for the unperturbed motion

$$t = \int_{\phi_0}^{\phi} \frac{d\phi}{P} = \frac{1}{\Omega} (u-u_0) \quad \text{where} \quad u = \int_0^{\psi} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} \quad (2-47)$$

By definition,

$$sn(u,k) = \sin \psi, \quad cn(u,k) = \cos \psi, \quad dn(u,k) = \sqrt{1 - k^2 \sin^2 \psi} \quad (2-48)$$

To the period $2\pi$ in $\psi$ corresponds the period $4K$ in $u$ and $4K/\Omega$ in time; therefore, the angular frequency of synchrotron oscillations is

$$\omega_s = \Omega \cdot \frac{\pi}{2K(k)} \quad (2-49)$$
From (2-29), (2-47) and (2-49),
\[ \theta - \theta_s = \omega_s t = \frac{\pi}{2K} (u-u_s) \]
so that we may take
\[ \theta = \frac{m \pi}{2K} \quad \text{(2-50)} \]

Finally, with (2-45) and (2-48), (2-31) becomes
\[
\begin{align*}
\sum_{m=-\infty}^{+\infty} b_m e^{im\theta} &= P = 2 \Omega \ k \ cn \ u \\
\sum_{m=-\infty}^{+\infty} c_m e^{im\theta} &= U = \Omega^2 \cdot 2 \sin^2 \frac{\theta}{2} = 2 \Omega^2 \ k^2 \ sn^2 \ u
\end{align*}
\quad \text{(2-51)}
\]

These Fourier series are found in treatise on elliptic functions. For the first one, we have (Ref. 22, p. 256; Ref. 23, p. 911)
\[ k \frac{2K}{\pi} \ cn \ u = \sum_{m \ odd=-\infty}^{+\infty} \frac{e^{im\theta}}{ch(mv)} \quad \text{where} \ v = \frac{\pi}{2} \frac{k'(k)}{K(k)} \quad \text{(2-52)} \]

Therefore
\[ b_m = 2 \Omega \frac{\pi}{2K} \frac{1}{ch(mv)} = \frac{2\omega_s}{ch(mv)} \quad \text{for} \ m \ odd, \quad b_m = 0 \quad \text{for} \ m \ even \quad \text{(2-53)} \]

The Fourier series of \( k^2 \ sn^2 \ u \) may be obtained through Jacobi's Zeta function (Ref. 22, p. 292, 295):
\[
Z(u) = \left(1 - \frac{E}{K}\right) u - \int_0^u k^2 \ sn^2 \ u \cdot du = \frac{\pi}{2K} \sum_{m \ even=-\infty}^{+\infty} \frac{e^{im\theta}}{sh(mv)} \quad \text{(without} \ m = 0) \]

whence
\[
\frac{\partial Z}{\partial u} = \left(1 - \frac{E}{K}\right) - k^2 \ sn^2 \ u \cdot \left(\frac{\pi}{2K}\right)^2 \sum_{m \ even=-\infty}^{+\infty} \frac{m}{sh(mv)} \frac{e^{im\theta}}{sh(mv)} \quad \text{(without} \ m = 0) \quad \text{(2-54)}
\]

Therefore,
\[
\begin{align*}
c_s &= 2 \Omega^2 \left(1 - \frac{E}{K}\right), \quad c_m = -2 \Omega^2 \left(\frac{\pi}{2K}\right)^2 \frac{m}{sh(mv)} = -2 \omega_s \frac{m}{sh(mv)} \quad \text{for} \ m \ even \neq 0 \\
c_m &= 0 \quad \text{for} \ m \ odd \quad \text{(2-55)}
\end{align*}
\]

It remains to introduce (2-53) and (2-55) into (2-44) in order to get:
\[
\frac{A_2}{2} = 4 \sum_{m=1,3}^{\infty} \frac{(m_0)^n}{\sinh^2(mv)} S_n(m_0) + 4 \sum_{m=2,4}^{\infty} \frac{(m_0)^n}{\sinh^2(mv)} S_n(m_0)
\] 

(2-56)

This expression can be put in the form of a weighted average over the noise spectral density by using the relations [see Appendix, eq. (A-6) and (A-8)]

\[
4 \sum_{m=1,3}^{\infty} \frac{m^4}{\sinh^2(mv)} = \left[ k^2 \frac{4}{\pi} \right] \frac{2K}{B} \left( 1-k^2 \right) \alpha
\]

and

\[
4 \sum_{m=2,4}^{\infty} \frac{m^4}{\sinh^2(mv)} = \left[ k^2 \frac{4}{\pi} \right] \frac{2K}{B} \left( 1-k^2 \right) \alpha
\]

(2-57)

where, from eq. (A-4),

\[\alpha(k^2) = \frac{4}{15} \left[ \frac{2K}{B} - \left( 1-k^2 \right) \frac{C}{B} \right]\]

We have

\[1 \geq \alpha \geq \frac{8}{15}\]

when

\[0 \leq k^2 \leq 1\]

If instead of \( J \) we use \( x = J/\tilde{J} \) as independent variable in the Fokker-Planck equation, the corresponding diffusion coefficient reads

\[
D = \frac{A_2}{2\tilde{J}^2} = x S_1 + x^2 S_2
\]

(2-58)

where

\[
S_1 = \frac{\omega_s^2}{4} \left( \frac{2K}{\pi} \right)^3 \frac{\pi}{4} \left( 1-k^2 \right) \alpha \sum_{m=1,3}^{\infty} \frac{m^n}{\sinh^2(mv)} S_n(m_0),
\]

\[
S_2 = \frac{\omega_s^2}{4} \left( \frac{2K}{\pi} \right)^3 \frac{\pi}{4B} \sum_{m=2,4}^{\infty} \frac{m^n}{\sinh^2(mv)} S_n(m_0)
\]

(2-59)

This result has been given in Ref. 19, where the independent variable was taken as \( J/2\tilde{J} \); the only difference is that with \( J/\tilde{J} \) as independent variable, \( S_1 \) contains an extra factor \( \pi/4 \).

Let us notice that the positive quantities \( S_1, S_2 \) are slowly varying functions of \( k = \sin \left( \frac{\Theta}{2} \right) \).

Remark: In terms of frequency noise, eq. (2-56) reads:
Fig. 8 - Levels of frequency noise for equal rate of diffusion.
The dots are the results of numerical simulations;
the continuous curves are the theoretical estimates (2-61).
\[
\frac{A}{2} = 4 \sum_{m=1,3}^{\infty} \frac{(m\omega_s)^2}{c \hbar^2 (m\nu)} S_F(m\omega_s) + 4 \sum_{m=2,4}^{\infty} \frac{(m\omega_s)^4}{c \hbar^2 (m\nu)} S_d(m\omega_s)
\] (2-60)

Using bandpass filters with the same bandwidth centered on the harmonics \( m = 1, 2, 3, 5 \) of \( \omega_s \), G. Gurov (Ref. 24) compared the levels of frequency noise \( \sqrt{\mathcal{T}(m\omega_s)} \) which are necessary to produce the same diffusion of particles at a given synchrotron amplitude \( \hat{\phi} \). Since \( \mathcal{T}(m\omega_s) \) is then proportional to \( S_F(m\omega_s) \), \( \sqrt{\mathcal{T}(m\omega_s)} \) should be inversely proportional to its weight \( m/c \hbar (m\nu) \) in (2-60), which means that one expects to have

\[
\frac{\sqrt{\mathcal{T}(m\omega_s)}}{\sqrt{\mathcal{T}(\omega_s)}} = \frac{c \hbar (m\nu)}{m \hbar v} \quad \text{where} \quad v = \frac{\pi}{2} \frac{K'(k)}{K(k)} \text{ is determined by } k = \sin \frac{\phi}{2}.
\] (2-61)

In figure 8, the results of Gurov's numerical simulations are compared to the theoretical estimate (2-61) at \( m = 3 \) and 5; Gurov also verified that the frequency noise at \( m = 2 \) has no effect on the particles.

### 2.6 Finite difference equations

Even without noise, the system (2-21) of finite difference equations is integrable only in the case of a linear RF voltage. In this simple case it reduces to

\[
\begin{align*}
\phi_{n+1} - \phi_n &= -K_e (1+a_n) \phi_n \\
\phi_{n+1} - \phi_n &= \phi_{n+1} + \phi_{n+1} - \phi_n
\end{align*}
\] (2-62)

#### Motion without noise

With \( \nu \) defined as in (1-51), the unperturbed motion is given by (1-52):

\[
\phi_n = \text{Re} \left[ a e^{i \nu n} \right], \quad \phi_n = \text{Re} \left[ i a \sin \frac{\nu}{2} e^{i(n-\nu)\mu} \right]
\]

where "a" is a complex amplitude. It is seen that

\[
\phi_n - i \frac{p_{n+1} + p_n}{2 \sin \nu} = a e^{i \nu n} \quad \text{and} \quad \frac{\phi_n + \phi_{n-1}}{2 \cos \frac{\nu}{2}} - i \frac{p_n}{2 \sin \frac{\nu}{2}} = a e^{i(n-\nu)\mu}
\] (2-63)

From (2-63) we derive a constant for the unperturbed motion:

\[
\phi_n^2 + \left( \frac{p_n + p_{n+1}}{2 \sin \nu} \right)^2 = \left( \frac{\phi_n + \phi_{n-1}}{2 \cos \frac{\nu}{2}} \right)^2 + \left( \frac{p_n}{2 \sin \frac{\nu}{2}} \right)^2 = |a|^2
\] (2-64)

Using (1-51) and (2-62), both expressions are converted into a single one:

\[
\phi_n^2 + \left( \frac{p_n}{\sin \nu} - \frac{p_n}{\cos \frac{\nu}{2} \cdot \phi_n} \right)^2 = \left( \frac{\phi_n}{\cos \frac{\nu}{2}} - \frac{p_n}{2 \cos \frac{\nu}{2}} \right)^2 + \left( \frac{p_n}{2 \sin \frac{\nu}{2}} \right)^2 = \frac{\phi_n^2 - p_n \phi_n}{\cos^2 \frac{\nu}{2}} + \frac{p_n^2}{4 \sin^2 \frac{\nu}{2} \cos^2 \frac{\nu}{2}} = |a|^2
\] (2-65)

In the \( P, \phi \) phase plane where \( P \) is defined as in (2-23), all successive points of a particle trajectory lie on an ellipse with area

\[
2\pi J = \frac{N}{t \phi} |a|^2 \sin \nu
\]
Therefore, if one takes \( x_0 = |a/2|^2 = (\xi/2)^2 \) as generalized coordinate (as done in Ref. 17), \( x_0 \) is proportional to the action \( J \) and by (2-45), it is very close, for small amplitudes, to \( k^2 = \sin^2 (\xi/2) \) or to \( [k^2 (4/n) B(k)] \). In terms of \( x_0 \), the action reads

\[
J = x_0 \cdot \frac{2N}{T_0} \sin \left( \frac{\omega_s T_0}{N} \right) \quad \text{with} \quad \frac{\omega_s T_0}{N} = \mu \tag{2-66}
\]

It should be noticed that in the case of a nonlinear RF voltage, this relation only applies for small amplitudes.

**Motion with noise**

For finite difference equations with a linear RF voltage, it can be verified that the remarkable relation (2-40) still applies (Ref. 17). Therefore, with the action \( J \) as generalized coordinate, the Fokker-Planck equation still reduces to a diffusion equation; with (2-66) the diffusion coefficient reads

\[
D = \frac{A^2}{2} = \left( \frac{2N}{T_0} \right)^2 \sin^2 \left( \frac{\omega_s T_0}{N} \right) \cdot [x_0 S_1 + x_0^2 S_2]
\]

where (Ref. 17)

\[
S_1 = \left( \frac{N}{T_0} \right)^2 \sum_{n=-\infty}^{+\infty} S_\varphi (\omega_s + nN \omega_0), \quad S_2 = \left( \frac{N}{T_0} \right)^2 \sum_{n=-\infty}^{+\infty} S_a (2\omega_s + nN \omega_0)
\]

Finally,

\[
D = \frac{A^2}{2} = \left( \frac{2N}{T_0} \sin \frac{\omega_s T_0}{2N} \right)^4 \cdot \left[ x_0 \sum_{n=-\infty}^{+\infty} S_\varphi (\omega_s + nN \omega_0) + x_0^2 \sum_{n=-\infty}^{+\infty} S_a (2\omega_s + nN \omega_0) \right] \tag{2-68}
\]

In order to compare (2-68) to the corresponding expression (2-56) for differential equations with a sinusoidal RF voltage, we must take in (2-56) the limit of vanishingly small amplitudes, which means that the infinite series in \( m \) both reduce to their first term; by (2-57) these terms coincide with the \( n = 0 \) terms in (2-68), provided that in (2-56), \( \omega_s^m \) is replaced by the first factor of (2-68).

For small synchrotron tune \( Q_s / N \), the coarse motion is still well represented by differential equations; therefore, a reasonable generalization of (2-56) for finite difference equations with a sinusoidal RF voltage would involve replacing

\[
\omega_s^m \text{ by } \left( \frac{2N}{T_0} \sin \frac{\omega_s T_0}{2N} \right)^m, \quad S_\varphi (m\omega_s) \text{ by } \sum_{n=-\infty}^{+\infty} S_\varphi (m\omega_s + nN \omega_0), \quad S_a (m\omega_s) \text{ by } \sum_{n=-\infty}^{+\infty} S_a (m\omega_s + nN \omega_0)
\]

The main effect of a finite \( N \) is that the spectral density of noise has to be taken into account not only at frequencies \( m\omega_s \), but at all frequencies \( (m\omega_s + nN \omega_0) \).

The last term is written
2.7 Diffusion equation

Taking \( x = J / \bar{J} \) as independent variable in the Fokker-Planck equation, the diffusion equation (2-42) reads, with (2-58):

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( x S_1 + x^2 S_2 \right) \frac{\partial \rho}{\partial x} \tag{2-70}
\]

where \( x = 1 \) on the separatrix; \( S_1 \) and \( S_2 \) are slowly varying functions of \( x \).

**Boundary conditions**

At \( x = 0 \) the flux of particles must vanish:

\[
\lim_{x \to 0} \left( x S_1 + x^2 S_2 \right) \frac{\partial \rho}{\partial x} = 0 \tag{2-71}
\]

At \( x = 1 \) particles are lost, which entails that

\[
\rho(x = 1, t) = 0 \tag{2-72}
\]

**Normalization condition**

At \( t = 0 \),

\[
\int_0^1 \rho(x, 0) \, dx = 1 \tag{2-73}
\]

Let us restrict to phase noise. Since the action \( J \) is an area in phase space, we may put \( x = r^2 \) where \( r, \phi \) are polar coordinates. The diffusion equation becomes

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( x S_1 \frac{\partial \rho}{\partial x} \right) = \frac{1}{4} \frac{\partial}{\partial r} \left( r S_1 \frac{\partial \rho}{\partial r} \right) \tag{2-74}
\]

It is solved by separation of variables, putting \( \rho(r, t) = R(r) \cdot T(t) \). Then

\[
\frac{1}{T} \frac{dT}{dt} = \frac{1}{4R} \frac{d}{dr} \left( r S_1 \frac{dR}{dr} \right) = -\frac{\lambda^2}{4} \quad \text{where} \quad \lambda^2 \quad \text{is some constant.}
\]

\( T \) varies according to

\[
T = e^{-\left(\lambda^2/4\right) t} \tag{2-75}
\]

whereas \( R \) must satisfy

\[
\frac{d}{dx} \left( x S_1 \frac{dR}{dx} \right) + \frac{\lambda^2}{4} R = 0 \quad \text{with} \quad \lim_{x \to 0} x S_1 \frac{dR}{dx} = 0 \quad \text{and} \quad R(1) = 0
\]

or

\[
\frac{d}{dr} \left( r S_1 \frac{dR}{dr} \right) + \lambda^2 R = 0 \quad \lim_{r \to 0} r S_1 \frac{dR}{dr} = 0 \quad R(1) = 0 \tag{2-76}
\]

The conditions (2-76) define \( \lambda^2 \) and \( R(r) \) as eigenvalue and eigenfunction of the operator

\(-1/r \cdot d/dr ( r S_1 \, d/dr)\), we anticipate here that all eigenvalues \( \lambda_n^2 \) are positive. The general solution of (2-70) then reads:
\[ \rho(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{3}{2}/4\right) t} R_n(x) \text{(2-77)} \]

The smallest eigenvalue \( \lambda_1^2 \) determines the ultimate lifetime of the particles under noise diffusion; it can be computed easily in a few cases only, when \( S_1(x) \) is simple enough.

**Case 1:** \( S_1(x) = d_o^2 \) constant

\[ R(r) = J_0\left(\frac{\lambda}{d_o} r\right) \]

with

\[ \frac{\lambda}{d_o} = j_{s,n}, \quad \text{\( n \)th root of Bessel function } J_0. \]

The smallest eigenvalue is thus given by

\[ \frac{\lambda_1^2}{d_o^2} = j_{0,1} = 2.404826 \]

**Case 2:** \( S_1(x) = d_o^2 (1-x)^{-\nu} \). The solution satisfying the boundary condition (2-76) when \( x \to 0 \) is of the form

\[ R(x) = \sum_{m=0}^{\infty} \alpha_m x^m \]

where the coefficients \( \alpha_m \) can be computed numerically by a recurrence relation derived from the differential equation (2-76); \( \lambda^2 \) is then adjusted until \( R(1) = 0 \).

If \( \nu \leq -1 \), the condition \( R(1) = 0 \) cannot be met. For \( -2 < \nu \leq -1 \), \( \lambda^2 \) is determined by requiring that \( R(1) \) remains finite; for \( \nu < -2 \) even the latter condition is only met when \( \lambda^2 = 0 \), and the \( \lambda^2 \) spectrum is continuous. In all these cases the flux of particles at the separatrix is zero, which means that no particles are lost; when \( t \to \infty \), \( \rho(x,t) \) approaches an equilibrium value (corresponding to \( \lambda^2 = 0 \)) which is independent of \( x \).

Unfortunately, the condition \( \nu \leq -1 \) implies that the RF noise goes fast enough to an absolute zero (i.e. \( \to 0 \) dB) close to the separatrix; this cannot be realized in practice.

If \( \nu = -1 \), the eigenvalues are given by

\[ \frac{\lambda_{n+1}^2}{4d_o^2} = n(n + 1) \]

where \( n = 0, 1, 2, \ldots; \) the corresponding eigenfunctions are the Legendre polynomials \( P_n(1-2x) \), and \( R_n(1) = (-1)^n \). The smallest eigenvalue \( \lambda_1^2 \) is zero, with eigenfunction \( P_0(1-2x) = 1 \) independent of \( x \).

If \( \nu > -1 \), numerical computations yield the following results:
\[ \nu \text{ or } \left( \frac{\lambda}{4d_0^2} \right) \]

\[ -1 + \varepsilon \quad \frac{1 + 2(\nu+1)}{1 + (\nu+1)} \]

obtained by a Rayleigh-Ritz variational method

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( \frac{\lambda}{4d_0^2} )</th>
<th>( \frac{1 + 2(\nu+1)}{1 + (\nu+1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.658 424</td>
<td>1.445 796</td>
</tr>
<tr>
<td>0.5</td>
<td>2.264 778</td>
<td>3.094 169</td>
</tr>
<tr>
<td>1</td>
<td>4.763 840</td>
<td>5.77 (\nu + \frac{5}{6})</td>
</tr>
</tbody>
</table>

obtained by a Rayleigh-Ritz variational method

**Case 3:** \( S_1(x) = d_0^2 x^\nu \). The solution satisfying (2-76) is

\[ R = z^{-\mu} J_\mu (z) \quad \text{where} \quad \mu = \frac{\nu}{1-\nu}, \quad z = \frac{\lambda}{d_0} \frac{r^{1-\nu}}{1-\nu} \]  

(2-78)

The condition \( R(1) = 0 \) reads

\[ J_\mu \left( \frac{\lambda}{d_0} \frac{1}{1-\nu} \right) = 0 \]  

(2-79)

From properties of zeros of Bessel functions, one gets the following results:

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( \mu )</th>
<th>( \frac{\lambda}{4d_0^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty)</td>
<td>-1</td>
<td>1.5 - \nu - \ldots</td>
</tr>
<tr>
<td>-1</td>
<td>-\frac{1}{3}</td>
<td>2.467 401</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1.445 796</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>0.917 623</td>
</tr>
<tr>
<td>1 - \varepsilon</td>
<td>\infty</td>
<td>0.25 + 0.927 879 (1-\nu) + \ldots</td>
</tr>
<tr>
<td>\geq 1</td>
<td>&lt; -1</td>
<td>0 &lt; \nu &lt; \infty \text{ continuous spectrum}</td>
</tr>
</tbody>
</table>

In the case of a continuous spectrum, the diffusion equation can be solved by using Laplace transforms.

In practice, the noise level increases from the centre of the bucket towards the separatrix, which corresponds to \( \nu \) being positive in cases 2 and 3.

**Bunch current \( I(\phi) \)**

With the normalization (2-73) we have, if \( q \) is the total charge in the bunch at \( t = 0 \):

\[ I(\phi) \, dt = q \int \rho \frac{dJ}{\partial \phi} \, d\phi \]  

(2-80)

But \( dJ \cdot d\phi = dP \cdot d\phi \) since it is the elementary area in phase space. From (2-45) we may write

\[ k^2 = \sin^2 \frac{\phi}{2} + y^2 \quad \text{where} \quad y = \frac{P}{2\pi} \]  

(2-81)

so that (2-80) reads

\[ I(\phi) \, dt = q \frac{2}{2\pi J} \int \rho \, dP \, d\phi = q \frac{2\pi}{2\pi J} \int \rho \, dy \, \omega_{RF} \, dt \]
With (2-46) and (2-81):

\[
I(\phi) = q \frac{\mu RF}{4} \cos \frac{\phi}{2} \int_0^\infty \phi(x,t) \, dy
\]

Using (2-77) we finally get

\[
I(\phi) = q \frac{\mu RF}{4} \sum_{n=1}^{\infty} a_n e^{-\left(\chi^2/4\right)t} \int_0^\infty R_n(x) \, dy \quad \text{where } x = k_x B(k)
\]

and \( k_x^2 = \sin^2 \frac{\phi}{2} + y^2 \) \hspace{1cm} (2-82)

\[
\cos \frac{\phi}{2} + (\text{Function of } \sin^2 \frac{\phi}{2})
\]

This relation yields the shape of the instantaneous bunch current as a function of time.

2.8 Appendix: Computation of \( \sum_{m=1,3,5} m^b \chi^2(m\nu) \) and \( \sum_{m=2,4} m^b \chi^2(m\nu) \)

If \( h(\theta) = \frac{1}{4} \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \), then

\[
\sum_{m=-\infty}^{\infty} |a_m|^2 = \frac{1}{2\pi} \int_0^{2\pi} |h(\theta)|^2 \, d\theta \quad \text{ (A-1)}
\]

From (2-52) and (2-50),

\[
\sum_{m \, \text{odd}}^{\infty} \frac{\cos(\theta)}{\chi(m\nu)} = k \left( \frac{2K}{\pi} \right) \text{cn}(u,k) \quad \text{where } \theta = \frac{\mu u}{2K}
\]

Therefore,

\[
\sum_{m \, \text{odd}}^{\infty} \left( \frac{\cos \theta}{\chi(m\nu)} \right)^2 = k \left( \frac{2K}{\pi} \right)^3 \left( \frac{3}{\mu^2} \right) \text{cn}(u,k)
\]

and

\[
\sum_{m \, \text{odd}}^{\infty} \frac{m^b}{\chi^2(m\nu)} = \frac{1}{2\pi} 2K \int_0^\infty \left[ k \left( \frac{2K}{\pi} \right)^3 \frac{3}{\mu^2} \text{cn} u \right]^2 \, du = \frac{2}{\pi} k^2 \left( \frac{2K}{\pi} \right)^5 \int_0^\infty \left[ \frac{3}{\mu^2} \text{cn} u \right]^2 \, du \quad \text{(A-2)}
\]

Now

\[
\frac{3}{\mu^2} \text{cn} u = - \frac{3}{\mu} \left[ \text{sn} u \cdot \text{dn} u \right] = - \text{cn} u \cdot \text{dn}^2 u + \text{cn}^2 u - \text{sn}^2 u \cdot \text{cn} u = -\text{cn} u \left( \text{dn}^2 u - k^2 \text{sn}^2 u \right) = -\text{cn} u \left( 2\text{dn}^2 u - 1 \right)
\]

hence

\[
\left| \frac{3}{\mu^2} \text{cn} u \right|^2 = \text{cn}^2 u \left( 1 - 4 \text{dn}^2 u + 4 \text{dn}^4 u \right) = \text{cn}^2 u + 4 \text{cn}^2 u \text{dn}^2 u \left( \text{dn}^2 u - 1 \right)
\]

\[
= \text{cn}^2 u - 4 k^2 \text{sn}^2 u \text{cn}^2 u \text{dn}^2 u
\]
The integral with respect to $u$ is most easily computed by reverting to (2-47), (2-48):

$$\int_0^K \left( 3 \frac{\partial^2}{\partial u^2} \operatorname{cn} u \right)^2 \, du = \int_0^K \operatorname{cn} u \cdot d\Lambda - 4k^2 \int_0^K \operatorname{sn}^2 u \cdot \operatorname{cn}^2 u \cdot d\Lambda$$

$$= \int_0^{\pi/2} \frac{\cos^2 \psi}{\sqrt{1 - k^2 \sin^2 \psi}} \, d\psi - 4k^2 \int_0^{\pi/2} \sin^2 \psi \cdot \cos^2 \psi \cdot \sqrt{1 - k^2 \sin^2 \psi} \, d\psi \quad \text{(A-3)}$$

The first integral in (A-3) is $B(k)$ (Ref. 21). The second integral is given by (Ref. 23, p. 386)

$$\frac{2}{\pi} \int_0^{\pi/2} \sin^2 \psi \cdot \cos^2 \psi \cdot \sqrt{1 - k^2 \sin^2 \psi} \, d\psi = \frac{1}{\pi} B \left[ \frac{3}{2}, \frac{3}{2}; 3, k^2 \right] = \frac{1}{8} \pi \left[ \frac{1}{2}, \frac{3}{2}; 3, k^2 \right]$$

By repeated application of Gauss' relations for contiguous hypergeometric functions, the above hypergeometric function can be expressed as a linear combination of

$$F \left[ \frac{1}{2}, \frac{1}{2}; 1; k^2 \right] = \frac{2K}{\pi} \quad \text{and} \quad F \left[ \frac{1}{2}, \frac{1}{2}; 1; k^2 \right] = \frac{2E}{\pi};$$

there results

$$\int_0^K \operatorname{sn}^2 u \cdot \operatorname{cn}^2 u \cdot \operatorname{dn}^2 u \, d\Lambda$$

$$= \int_0^{\pi/2} \sin^2 \psi \cdot \cos^2 \psi \cdot \sqrt{1 - k^2 \sin^2 \psi} \, d\psi = \frac{1}{15k^4} \left[ (k^2 - 2)(E - k^2 B) + 2(1 - k^2 + k^4)E \right]$$

With the notations of Ref. 21, this is transformed into

$$\frac{1}{15k^4} \left[ (k^2 - 2)(E - k^2 B) + 2(1 - k^2 + k^4)E \right] = \frac{1}{15k^4} \left[ (1 + k^2)B + E(2k^2 - 1) \right]$$

$$= \frac{1}{15k^4} \left[ (E - k^2 k^4 C) + E(2k^2 - 1) \right] = \frac{1}{15} \left[ 2E - k^4 C \right] = \frac{B}{4} \alpha(k^2) \quad \text{(A-4)}$$

where $\alpha(k^2)$ is defined by the last relation; it decreases from 1 to 8/15 when $k^2$ increases from 0 to 1.

Finally,

$$\frac{K}{4} \int_0^K \operatorname{sn}^2 u \cdot \operatorname{cn}^2 u \cdot \operatorname{dn}^2 u \, d\Lambda = B \cdot \alpha(k^2) \quad \text{(A-5)}$$
and (A-3) becomes

\[ \int_0^K \left( \frac{3}{\sin^3 u} \right)^2 \ du = B(1-k^2\alpha) \]

hence, from (A-2):

\[ 2 \sum_{m=1,3}^\infty \frac{m^2}{\sinh^2(m\varnothing)} = k^2 \frac{2}{\pi} B \left( \frac{2K}{\varnothing} \right)^5 (1-k^2\alpha) \tag{A-6} \]

For the second series, we start from (2-54):

\[ \sum_{m \text{ even} = \infty}^{\infty} \frac{m}{\sinh(m\varnothing)} e^{im\varnothing} = \left( \frac{2K}{\varnothing} \right)^2 \left[ 1 - \frac{E}{K} - k^2 \sin^2 u \right] \]

This can be rewritten as:

\[ \sum_{m \text{ even} = \infty}^{\infty} i\varnothing e^{im\varnothing} \frac{m^2}{\sinh(m\varnothing)} = -k^2 \left( \frac{2K}{\varnothing} \right)^2 \frac{\partial}{\partial u} \sin^2 u \]

Therefore,

\[ \sum_{m \text{ even} = \infty}^{\infty} i\varnothing \frac{m^2}{\sinh(m\varnothing)} e^{im\varnothing} = -k^2 \left( \frac{2K}{\varnothing} \right)^2 \frac{\partial}{\partial u} \sin^2 u \]

and, with (A-1):

\[ \sum_{m \text{ even} = \infty}^{\infty} \frac{m^2}{\sinh^2(m\varnothing)} \int_0^K \left( \frac{2K}{\varnothing} \right)^3 \frac{\partial}{\partial u} \sin^2 u \ du = \frac{2}{\varnothing} K k^4 \left( \frac{2K}{\varnothing} \right)^5 \int_0^K \left| \frac{\partial}{\partial u} \sin^2 u \right|^2 \ du \tag{A-7} \]

Now \( \frac{\partial}{\partial u}(\sin^2 u) = 2 \sin u \cos u \) and the integral in (A-7) is the same as (A-5); thus

\[ 2 \sum_{m=2,4}^\infty \frac{m^2}{\sinh^2(m\varnothing)} = k^2 \frac{2}{\pi} B \left( \frac{2K}{\varnothing} \right)^5 \alpha \tag{A-8} \]
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