A THEORETICAL APPROACH TO HIGH
ENERGY SCATTERING

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We want to sketch here in a very qualitative and rough way a line of research in which we are working — together with Stanghellini and Tonin — in order to understand some features of high energy scattering. Let us try to analyse some general experimental information available. We note at first that the energy region below the GeV is rich in peculiarities for the different processes (i.e. resonances in special isospin states for different partial waves in \( \pi N \), flatness of \( K^+p \) cross-section in energy and angle over a wide region of energy, bumps, peaks and threshold behaviour for different phenomena), as soon as the GeV region is surpassed, these peculiarities disappear; the cross-sections are smooth and the angular distribution is peaked in the forward direction. There are still many features that can distinguish different processes at these high energies. But what I want to stress is that in the bulk of information, we can find some qualitative behaviours that all processes share, at least up to the energies in which the present accelerators can provide us with accurate information. Let us try to find and list such "regularities".

1. All elastic cross-sections show the characteristic diffraction pattern consisting of a peak in the forward direction. The width of such a peak when plotted as a function of the transfer momentum is nearly independent of the process and of the energy, and of the order of magnitude of the pion mass. Another general indication is that besides the diffraction peak, elastic scattering is substantially small.

2. Total cross-sections tend to reach energy-independent behaviour (constants) satisfying more or less "Pomeranchuk's theorems".

3. If the energy is not too big, it seems that peripheral formulae work rather well for total cross-sections (for this point I refer to the review paper
by Drell to be published in Rev. of Mod. Phys.) as well as for many inelastic processes (mainly those for which there is a clear distinction between forward and backward "cones" in the c.m.s., which processes are rather dominant).

It is in some sense good luck for theoreticians that the regularities are present in elastic angular distributions and total cross-sections. Both features are related to the elastic scattering amplitudes (by using also the optical theorem) and, because just for elastic amplitudes we know some general theoretical properties (i.e. the location of singularities through the Mandelstam representation), we can have some hope to understand the regularities from a general theoretical standpoint.

Let us consider any elastic process (Fig. 1) where \( p_1 \) and \( p_2 \) (\( q_1 \) and \( q_2 \)) are the four momenta of the incident (scattered) particles; and let us define, as usual

\[
s = (p_1 + p_2)^2, \quad t = (p_1 - q_1)^2 = -2p_c^2 \cos \theta_c, \quad u = (p_1 - q_2)^2
\]

\[
s + t + u = 2m_A^2 + 2m_B^2
\]

\( p_c \) and \( \cos \theta_c \) are the momentum and scattering angle in the c.m.s.). In the physical region for our scattering process \( s \) is positive (and equal or bigger than \( (m_A + m_B)^2 \)), while \( t \) is smaller than or equal to zero. In our scattering process, the absorptive part \( A(st) \) of the scattering amplitude \( T(st) \) satisfies - by virtue of the Mandelstam representation -

\[
\text{Im } T(st) = A_{AB}(st) = \frac{1}{\pi} \int \frac{\rho(st')}{t' - t} \, dt' + \frac{1}{\pi} \int \frac{\rho(su')}{u' - u} \, du'
\]

where \( \mu \) is the pion mass.

The second integral in the r.h.s. of Eq. (2) shall contribute (if it does at all) only to the backward scattering, so we shall not deal with it here.
The integration over $t'$ in Eq. (2) starts at $4\mu^2$. (Fig. 2.)

Now we want to point out that the empirical "regularity" (1) implies that the bulk of the contribution to the integral over $t'$ in Eq. (2) must come from small values of $t'$. This means that $\rho(s,t')$ must contribute up to a certain value $t'_{\text{max}}$ and then either $\rho$ becomes very small or there is a mechanism (as for instance can be provided by an oscillating function) for which its contribution to the integral in Eq. (2) is small.

The value of $t'_{\text{max}}$, is roughly given by $t'_{\text{max}} = 4\mu^2 \approx \Gamma$ where $\Gamma$ is the width of the diffraction peak when represented as a function of $t$. This means, by the way, that the value of $t'_{\text{max}}$ is nearly independent of the energy $(s)$. This is the physical meaning of the strip approximation, i.e. the idea that the main contribution of $\rho(st')$ to the scattering amplitude must come from values of the variables lying in a narrow strip in the $s$-$t'$ plane, as soon as $s$ leaves the low energy region. (Fig. 3.)

In the future we shall restrict our integrations to the strip and we shall see that this procedure shall allow us to construct a solution for the scattering amplitude and, therefore, of its spectral function $\rho$. At the end of this programme we must determine whether the solution we obtain is consistent with the strip approximation (i.e. $\rho$ outside the strip does not contribute to the integrals). If this is the case we should have obtained a physical solution for the scattering amplitude, if this is not the case, we shall be in conflict both with nature and with ourselves. Since this is a programme in progress more than an accomplished work, we are not yet able to prove the consistency but we have good reasons to hope for the best; and this is owing to the fact that we already see in our preliminary solution the mechanism that shall kill the integrals outside the strip.

Let us now try to show how the solution is constructed. First of all, we note that in the strip we have a definite expression for $\rho$ and, therefore, for $A(st)$ from Eq. (2) (that was already given by Mandelstam three years ago); i.e.

$$A_{AB}(st) = \int ds_1 \int ds_2 \int_{t_0(s,s_1,s_2)}^{t_{\text{max}}} dt' \frac{A_{A^*}(s_1,t') A_{B}(s_2,t')f(s,s_1,s_2)}{(t'-t)} \sqrt{t'(t'-t_0(s,s_1,s_2))}$$  \hspace{1cm} (3)
where \( f(s_1, s_2) \) and \( t_0(s_1, s_2) \) are well known functions for every definite process and \( \pi^A(s_1, t') \), \( \pi^B(s_2, t') \) are themselves absorptive amplitudes for the \( \pi A \to \pi A \) and \( \pi B \to \pi B \) processes, respectively.

For instance in \( \pi\pi \) scattering (the simplest process) the \( A \)'s inside the integral in Eq. (3) are again absorptive amplitudes for \( \pi\pi \) scattering (as \( A \) itself) and \( f \) and \( t_0 \) are given by

\[
f(s_1, s_2) = \left[ s^2 + s_1^2 + s_2^2 - 2(s_1 + s_2 + s_1 s_2) \right]^{-1/2}
\]

(4)

\[
t_0 = 4\mu^2 + 4s_1 s_2 f^2(s_1, s_2)
\]

(5)

For \( \pi N \), or \( NN \), \( f \) and \( t \) would have slightly more complicated expressions.

We note immediately that we cannot consider Eq. (3) as a solution of our problem; in fact Eq. (3) gives us a scattering amplitude in terms of other scattering amplitudes and so we would have needed to solve already a problem in order to use Eq. (3) for solving others. As is already seen in the \( \pi\pi \pi \) case, Eq. (3) is really an integral equation and one belonging to a bad family (non-linear and singular). The solution of such integral equations (and therefore the solution of both low- and high-energy \( \pi\pi \pi \) scattering) is a problem that Chew and Frautschi are dealing with. We shall not attempt here to solve Eq. (3), but to try to use some physical knowledge for \( \pi\pi^A \) and \( \pi\pi^B \) in Eq. (3) so as to obtain new physical information about \( \pi A \). Let us first ask: Can we try to take \( \pi\pi^A(s_1, t') \) (or \( \pi\pi^B(s_1, t') \)) directly from experiments? The answer is clearly no, and the reason is that we need \( t' > 4\mu^2 \) while the physical information is given for \( t' \leq 0 \). So we must try to "push" the physical information from negative \( t' \) up to \( 4\mu^2 \leq t' \leq t_{\text{max}} \). We can now ask ourselves if such an extrapolation can be done, and this time the answer is, yes, provided \( s_1 \) and \( s_2 \) are rather small; and the procedure is just to expand \( A \) in Legendre polynomials of \( \cos \Theta' = 1 + \frac{2t'}{s_1} \) in the usual way familiar to experimentalists, and simply use this expression for the values of \( t' \) we are interested in.
The theoretical reason of why such a continuation breaks down when \( s_1 \) is too big is that singularities of \( A_{\pi A}(s_1t') \) (in \( t' \)) enter at their turn in the strip so that the Legendre polynomial expansion fails to converge. Or, more physically, as soon as you have too many partial waves you begin to have the typical constructive interference in the forward direction and destructive in others so that such a simple reasoning wave by wave is not more true. The Legendre polynomial continuation can be done in a straightforward manner; because the values of \( t' \) in which we are interested are very small, let us simplify a little bit more by taking just their value at \( t' = 0 \). This means that for small \( s_1 \) and \( s_2 \) (some hundreds MeV of kinetic energy) we shall take

\[
A(s_1t') \sim A(s_1, 0) \propto s_1 \sigma_{\text{tot}}(s_1).
\] (6)

We have still not investigated what role \( t_0(s_1, s_2) \) has as limit of integration in Eq. (3). What it does is to cut out the contribution of big \( s_1 \) and \( s_2 \); in fact it is easy to verify from Eqs. (5) and (6) that \( t_0 \) is nearly \( 4\mu^2 \) for small \( s_1 \) and \( s_2 \) and grows with growing \( s_1 \) and \( s_2 \). As soon as \( s_1 \) and \( s_2 \) are sufficiently big so that \( t_0(s_1, s_2) \) reaches \( t_{\text{max}} \) no contribution to the integral in Eq. (3) is left.

In \(^{N-N}\) scattering, for instance, for \( s = 16 \text{ GeV}^2 \) (4 GeV of kinetic energy in the lab. system), \( \sqrt{s_1} \) and \( \sqrt{s_2} \) that are total energy for the \(^{\pi N}\) systems in their c.m.s. are restricted to be lower than \( \approx 1.6 \text{ GeV} \). This means a rather small energy, some hundreds MeV of kinetic energy of the pion. This fact is very fortunate just for what we said before about the possibility of continuing \( A_{\pi A}(s_1t') \) and \( A_{\pi B}(s_2t') \) from the physical region. There is therefore a region of energy \( s \), going up to several GeV, for which \( A_{\pi A}(s_1t') \) and \( A_{\pi B}(s_2t') \) in Eq. (3) can still be approximated by Eq. (6). In such a case, the dependence on \( t \) in Eq. (3) is explicit, and we obtain for \( A(st) \)

\[
A(st) = \int ds_1 ds_2 s_1 \sigma(s_1) s_2 \sigma(s_2) f(s_1, s_2) F_1(t t_0)
\] (7)

where

*) \( \text{The integration over } t' \text{ in such a case being convergent, can be extended up to } \infty \text{ with almost no change in the result. The important thing to keep in mind is that the } s_1 \text{ and } s_2 \text{ integrations in Eq. (7) are restricted by the "strip".} \)
\[ F_1 = \frac{2}{\sqrt{t(t-t_0)}} \log \left[ 1 + \frac{-t + \sqrt{t(t-t_0)}}{t_0/2} \right] \quad (8) \]

Because \( s_1 \) and \( s_2 \) are restricted to be small, as we said before, \( t_0 \sim 4\mu^2 \) in which case \( F_1 \) can be written as

\[ F_1 = \frac{1}{X} \frac{P_0}{\mu} \sin \frac{\Theta}{2} \log \left( \sqrt{1 + X^2} + X \right) \quad (9) \]

where \( X = \frac{P_0}{\mu} \sin \frac{\Theta}{2} \).

The whole angular dependence of \( A \) is in \( F \); this means that \( F^2 \) is our theoretical prediction for the diffraction pattern of every process in the several GeV (c.m.s.) region. The comparison with experiments is encouragingly good.

For \( t = 0 \)

\[ A(s, 0) \propto s \sigma_{\text{tot}}(s) \quad (10) \]

so that Eq. (7) provides us with an expression for the total cross-section. By putting at their place coefficients \( \pi, 2, \) etc., after a little bit of inspection it is easy to realize that such results coincide with the prediction of the peripheral formulae (as obtainable from the work of Salzmann\textsuperscript{1}) for total cross-sections.

This is in some way a Mandelstam representation support to peripheralism, but we remember that this is true only when the energy is not too high. When \( s \) increases too much (around 8 or 10 GeV in the c.m.s.), Eq. (6) breaks down and therefore the rest of the argumentation. Can we still hope to say something for higher energies? Fortunately, yes, because the breaking down of Eq. (6) is given by the appearance of diffraction in the \( nA \) process and therefore a strong \( t' \) dependence of \( A_{nA}(s, t') \). But the \( t' \) dependence of \( A_{nA}(s, t') \) for energies \( s_1 \) not very high, but in which diffraction is present, is just given by our previous results (by Eq. (7)); so we can put this first
solution for $A_{NA}(s,t')$ in Eq. (3), obtain again an explicit $t'$ dependence and integrate over it. This shall give us a result that shall look roughly as

$$\int s_1 \sigma(s_1) s_2 \sigma(s_2) s_3 \sigma(s_3) F_2(t)$$ (11)

where $F_2$ is a definite function a little bit more complicated than $F_1$.

The angular distribution of such a contribution is somewhat more peaked forward than Eq. (7); it is still difficult to say if the very preliminary indication on elastic angular distribution at high energies shows such a shrinking of the diffraction peak or not.

For the contribution of Eq. (11) to the total cross-section ($t = 0$), we note that Eq. (11) contains three $\sigma$ in the integrand. As Eq. (7) was representable by the usual peripheral diagram (Fig. 4), Eq. (11) shall be represented by Fig. 5. We see that following the procedure indicated, when $s$ grows we shall construct $A(st)$ by iterating further and further our first solution. For the time being, we arrived only at the second iteration; we are, however, pushing forward this programme hoping, by the way, that after a few iterations (energies of the order of some tenths of GeV in the c.m.s.) we shall already be able to visualize the qualitative asymptotic behaviours.

Let us summarize which properties of high energy scattering (among the "regularities") can be understood from this theoretical approach.

1. First it appears clearly why the shape of the diffraction peak (including its width) is rather independent of the process; it is mainly controlled by the $\sqrt{t'(t' - t_0)}$ in Eq. (3) for small values of $t'$. This square root is something similar to a phase space quantity that tells us roughly how two pions can share the transfer momentum and is therefore nearly independent of the process provided the particles which take part in it interact strongly with pions (strong interactions).

2. For energies of some GeV in the c.m.s. the diffraction peak is predicted to be given by Eq. (9); we note that such a formula contains no other parameter besides the pion mass. This form could be further checked with more refined experimental data. It is expected that at higher energies, the form of the diffraction peak shall be somewhat narrower.
3. For total cross-sections a simple one boson exchange formula is justified if the c.m. energy of every one of the two bubbles is sufficiently low so that for such energy the processes in question are still non-diffractive. If this is not the case the bubble must be "split" in two, so that the one boson exchange picture is still valid but the picture would look as Fig. 5. For higher energies this "splitting" shall go on so we shall find chains of bubbles indicated in Fig. 6.

4. The question arises: can we hope that this simple peripheral picture which we found for total cross-sections is still valid for partial inelastic cross-sections? It would be tempting to answer, yes, even if there is not much theoretical justification. If one believed in such a yes, one would have an open way for investigating inelastic processes; this is actually in our plans, as well as in those of other physicists. The first very qualitative indications of such a model for multiplicities, nature of secondary particles and spectra of them, seem to point in a good direction. So it is, perhaps, reasonable to expect that from such a description of inelastic processes it would be possible to understand some characteristic features pointed out by high energy accelerators and cosmic rays; for instance, the large amount of pions among the secondaries, as compared to K mesons and baryons, their multiplicity as function of the incident primary energy, the small and rather constant mean transverse momentum, as well as the spectra of secondary pions showing a reasonable high energy tail.

REFERENCE

Fig. 1 Kinematics of the discussed scattering process.

Fig. 2 Physical domain in the momentum transfer variable $t'$ and the integration region of the Mandelstam representation.

Fig. 3 The "strip" in the $t'$-$s$ plane.
Fig. 4 The "peripheral" diagram representing the first iteration step.

Fig. 5 The second iteration step.

Fig. 6 The "bubble-chain" necessary for higher energies.