Non-Factorizable Phases, Yukawa Textures and the Size of $\sin 2\beta$

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Abstract

We emphasize the crucial rôle played by non-factorizable phases in the analysis of the Yukawa flavour structure performed in weak bases with Hermitian mass matrices and with vanishing $(1,1)$ entries. We show that non-factorizable phases are important in order to generate a sufficiently large $\sin 2\beta$. A method is suggested to reconstruct the flavour structure of Yukawa couplings from input experimental data both in this Hermitian basis and in a non-Hermitian basis with a maximal number of texture zeros. The corresponding Froggatt–Nielsen patterns are presented in both cases.

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1 Introduction

The pattern of quark mass matrices and the implied flavour structure of Yukawa couplings may play a crucial rôle in providing evidence for the possible existence of an underlying family symmetry. The search for a family symmetry has become more urgent in view of the significant improvement of our knowledge \cite{1} of the Cabibbo–Kobayashi–Maskawa (\(V_{CKM}\)) matrix and the value of quark masses. Recent developments in the theory of heavy quarks, as well as in experimental techniques, have led to a more precise knowledge of the elements of \(V_{CKM}\), with \(|V_{us}|\) and \(|V_{cb}|\) known with errors at the 2\% level \cite{1}; the uncertainty on \(|V_{ub}|\) is of about 13\%, but there is hope to achieve a precision below the 10\% level in the near future \cite{2}. On the other hand, the measurements of \(\sin(2\beta)\) by B-factories at BaBar \cite{3} and Belle \cite{4} have reached the 6.5\% level \cite{5}, with more precise measurements expected in the near future. All these developments, combined with an improvement in our knowledge of quark masses arising from lattice QCD \cite{6}, constitute a great challenge for any theory of flavour.

The major difficulty one encounters when attempting to reconstruct the flavour structure of Yukawa couplings from experimental data stems from the fact that even a precise knowledge of the value of the six quark masses and the four physical parameters of \(V_{CKM}\) does not lead to a unique reconstruction of the Yukawa couplings. This reflects the freedom one has in the Standard Model (SM) of making weak basis (WB) transformations, which leave the charged currents diagonal and real, but change the flavour structure of Yukawa couplings. For a given set of precise experimental input data, there is an infinite set of Yukawa structures compatible with the data and related to each other through WB transformations. Even if it is assumed that there is indeed a family symmetry chosen by nature, the question remains of discovering in which WB that symmetry will be “transparent”.

In the literature, there are various approaches to the Yukawa puzzle \cite{7} including, for example, the assumption of a set of texture zeros \cite{8}–\cite{19} or the hypothesis of universality of the strength of Yukawa couplings (USY) \cite{20}. In all these approaches there is the implicit adoption of a specific WB, where the proposed feature of Yukawa couplings is manifest.

In this paper, we discuss possible ways of determining the flavour structure of Yukawa couplings from input experimental data, in spite of the difficulty described above. First we choose a WB where the quark mass matrices are Hermitian and hierarchical, with a vanishing (1,1) element both in the up and down quark mass matrices. It is worth emphasizing that this is a choice of WB, not implying any loss of generality. Indeed it has been shown \cite{13} that starting from arbitrary complex matrices \(M_u, M_d\) one can always make WB transformations that render both \(M_u, M_d\) Hermitian and with vanishing (1,1) elements. There are various motivations for considering this WB:

i) We will argue that although, this choice of WB by itself, has of course, no physical implications, it leads, when combined with a requirement of naturalness, to an understanding of the size of the Cabibbo mixing. This “naturalness” condition consists of considering that the smallness of \(|V_{ub}|\) results from the smallness of \(|U_{31}^u|\),
|U^d_{13}| and not from the cancellation of the contributions to \(V_{ab}\) arising from \(U^u\) and \(U^d\), the two unitary matrices that diagonalize \(M_u, M_d\).

ii) Most, if not all, of the ansätze considered in the literature, based on Hermitian and hierarchical quark mass matrices, have the above two texture zeros.

In our analysis, we classify the phases appearing in \(U^u\) and \(U^d\) into two categories, namely factorizable and non-factorizable phases. We show that, in the framework of our chosen WB, the existence of at least one non-factorizable phase is essential in obtaining a sufficiently large value of \(\sin(2\beta)\), to be consistent with experiment. This result is specially important in view of the fact that in most of the ansätze considered in the literature, based on Hermitian and hierarchical mass matrices, the non-factorizable phases can be rotated away.

The number of free parameters in this WB exceeds the number of measurable quantities and therefore we cannot reconstruct \(M_u, M_d\) just from input data. In order to achieve this task, we propose a set of well motivated naturalness criteria, which render the reconstruction possible. The resulting quark mass patterns then have only two texture zeros and it is possible to write the remaining small entries in terms of powers of a small parameter. The fact that we have only two texture zeros should not be interpreted as disfavouring the presence of a family symmetry. Indeed, the modern approach to texture zeros allows for a small deviation of the exact relations resulting from strict zero entries, since it is to be expected that family symmetries only require texture zeros to be approximate. Furthermore, renormalization group running, although usually leading to small effects, would not keep exact zeros at all scales. A possible mechanism producing structures with small but non-vanishing entries is the spontaneous breaking of a family symmetry through new scalar fields at high energies, leading to effective Yukawa couplings at low energies via the Froggatt–Nielsen mechanism [21]. The point of our analysis is deriving this Froggatt–Nielsen pattern from input data.

Most of the analysis presented in this article is made in the framework of the above described Hermitian WB. For completeness, we will also consider another possible approach for reconstructing \(M_u, M_d\) from input experimental data. This proposal arises as an attempt at answering the following question: Starting from arbitrary quark mass matrices \(M_u, M_d\), what is the maximal number of zeros that can be achieved, by making WB transformations? We will show that if we do not require \(M_u, M_d\) to be Hermitian, the maximal number of zeros is nine. In this WB, \(M_u, M_d\) contain a total of ten free parameters, nine real numbers and one physical phase. Since the number of free parameters equals the number of measurable quantities (i.e., the six quark masses and the four physical parameters contained in \(V_{CKM}\)), it is possible to fully reconstruct \(M_u, M_d\) from experiment. This is an interesting feature of this WB. Its possible drawback is the fact that \(M_u, M_d\) are not Hermitian and furthermore that they are not treated on an equal footing. An interesting consequence of our analysis is that, in this WB, both quark mass matrices are of the nearest-neighbour-interaction (NNI) form, but the experimental data require a strong deviation from hermiticity. This result was to be expected, since the NNI basis [22] with hermiticity leads to the Fritzsch ansatz [23], which is
known to be ruled out by experiment.

2 Factorizable and non-factorizable phases

2.1 Definitions

It is well known that one can choose, without loss of generality, a weak basis (WB) where both the up and down quark mass matrices $M_{u,d}$ are Hermitian and therefore can be parametrized in the form:

$$M_u = U^u D_u U^u \dagger; \quad M_d = U^d D_d U^d \dagger,$$

where $D_{u,d}$ denote real diagonal matrices and the unitary matrices $U^{u,d}$ can be written as:

$$U^u = P_u U^u_o K_u; \quad U^d = P_d U^d_o K_d \tag{2.2}$$

with $P_u \equiv \text{diag}[e^{i\phi_1}, 1, e^{i\phi_3}]$, $K_u \equiv \text{diag}[e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}]$, with analogous expressions for $P_d$, $K_d$. The matrices $U^{u,d}_o$ are unitary with only one phase each and can be parametrized using, for example, the standard parametrization [24]. These phases, which we denote by $\sigma_u$, $\sigma_d$ are non-factorizable phases, in the sense that they cannot be removed by redefinitions of $P_{u,d}$, $K_{u,d}$. The CKM matrix is then given by:

$$V_{CKM} = U^u_o \dagger P U^d_o \tag{2.3}$$

with $P \equiv \text{diag}[e^{i\phi_1}, 1, e^{i\phi_3}]$ where $\phi_1 = \phi_1^u - \phi_1^d$, $\phi_3 = \phi_3^d - \phi_3^u$. We have omitted in $V_{CKM}$ the matrices $K_u$, $K_d$ since their phases can be eliminated through redefinitions of up and down quark fields. We call the phases $\phi_1$ and $\phi_3$ factorizable phases in contrast with the non-factorizable phases $\sigma_u$, $\sigma_d$ contained in $U^u_o$, $U^d_o$. Needless to say, neither $U^u_o$ nor $U^d_o$ are measurable quantities, since only $V_{CKM}$ appears in the charged weak currents. Of course, for three fermion generations $V_{CKM}$ will contain only one physical phase, which is a complicated function of the factorizable and non-factorizable phases. CP violation through the KM mechanism can be generated even in the limit where only the factorizable phases $\phi_1$, $\phi_3$ are present, as well as in the limit where only the non-factorizable phases are non-vanishing.

Although the unitary matrices $U^u_o$, $U^d_o$ are not measurable, they are useful in the discussion of specific flavour structures for $M_u$, $M_d$. In the following, we will show that in most of the flavour structures considered in the literature, the structure of $M_u$, $M_d$ are such that there are no non-factorizable phases. This means that for these textures, the unitary triangles corresponding to $U^u_o$, $U^d_o$ collapse to a line. For these flavour structures, CP violation arises exclusively from the factorizable phases $\phi_1$ and $\phi_3$. We will see that this feature plays a crucial rôle in preventing these flavour textures from generating a sufficiently large value for $\sin(2\beta)$, recently measured with significant accuracy by BaBar and Belle.
2.2 A class of ansätze without non-factorizable phases

Next, we will show that there is a large class of ansätze for $M_u$, $M_d$, where there are no non-factorizable phases. This includes all five texture zeros classified by Ramond, Roberts and Ross [8], as well as the four texture zeros considered in [12], [16]. In order to see how this result comes about, note that from Eq. (2.1) one readily obtains:

$$|\text{Im}[(M_u)_{12}(M_u)_{23}(M_u)_{13}^*]| = (m_u^3 - m_u^2)(m_u^2 - m_u^1)(m_u^3 - m_u^1) |\text{Im}(Q_u)|,$$

where $Q_u$ denotes any rephasing-invariant quartet constructed with the elements of $U_u$. Obviously, an analogous result holds for $M_d$. From Eq. (2.4), it follows that if at least one of the off-diagonal elements of $M_{u,d}$ vanishes then $\text{Im}(Q_{u,d}) = 0$, which in turn implies the absence of non-factorizable phases in $U_{u,d}$. It is seen from Table 1 of Ref. [8] that all five texture zeros classified there and the four texture zeros analysed in [12], [16] have at least one vanishing off-diagonal element and that therefore there are no non-factorizable phases in these textures.

3 The search for an appropriate Hermitian weak basis

In this section, we address the question of discovering an appropriate Hermitian WB for reconstructing the quark mass matrices $M_u$, $M_d$ from input experimental data. The hope is to get from these reconstructed matrices a hint of the underlying symmetry. Of course, we have to define what is meant by an “appropriate” WB. A criterion for the choice of WB could be, for example, the requirement of having a number of free parameters equal to the number of measurable quantities. This would mean a total of ten free parameters, to be determined from the knowledge of the six quark masses and the four physical parameters contained in $V_{CKM}$. Actually, there are two well known Hermitian WBs that satisfy the above criterion, namely the WB where $M_u$ is diagonal real and $M_d$ Hermitian, and the one where it is $M_d$ that is diagonal real and $M_u$ Hermitian. It is clear that each one of these two WBs has ten free parameters. For example, in the WB where $M_u$ is diagonal real, one would have as parameters the three up-quarks, the six real parameters contained in the Hermitian $M_d$, and the physical phase $\text{arg}[(M_d)_{12}(M_d)_{23}(M_d)_{13}^*]$, which is the only phase that cannot be rephased away. The reconstruction of $M_d$ from experimental data is then straightforward, since one has $M_d = V_{CKM} \cdot \text{diag}[m_d, m_s, m_b] \cdot V_{CKM}^\dagger$. The obvious disadvantage of these two WBs is the fact that in each one of them, $M_u$, $M_d$ are not treated on an equal footing.

We will argue that the WB where $M_u$, $M_d$ are Hermitian and furthermore $(M_u)_{11} = (M_d)_{11} = 0$ is an appropriate basis for reconstructing the quark mass matrices from input data. This WB treats $M_u$, $M_d$ on an equal footing, but it has, of course, the disadvantage of having more free parameters than measurable quantities, thus rendering the reconstruction somewhat ambiguous. We argue below that such a reconstruction is possible, provided some naturalness criteria are introduced.
3.1 The \((M_u)_{11} = (M_d)_{11} = 0\) weak basis

Starting with arbitrary quark mass matrices, it has been shown that one can always make WB transformations such that \((M_u)_{11} = (M_d)_{11} = 0\). From Eqs. (2.1), we obtain in this basis:

\[
m_u |U_{11}^u|^2 - m_c |U_{12}^u|^2 + m_s |U_{13}^u|^2 = 0
\]

\[
m_d |U_{11}^d|^2 - m_s |U_{12}^d|^2 + m_b |U_{13}^d|^2 = 0,
\]

where we have made the identification \(m_1^u \equiv m_u\), \(m_2^u \equiv -m_c\), \(m_3^u \equiv m_t\) and analogously for the down quark sector. Since the \((1,1)\) zeros only reflect a choice of basis, they do not have, by themselves, any physical implications. However, when combined with a reasonable assumption on the smallness of \(|m_1^u\), \(|m_1^d|\), namely the choice of the \((1,1)\) zeros only, one may arrive at Eq. (3.9) with much weaker assumptions, the point of the above discussion is that one may arrive at Eq. (3.9) with much weaker assumptions, namely the choice of the \((1,1) = 0\) WB (we emphasize that this is always possible to achieve starting with arbitrary \(M_u, M_d\), together with some reasonable qualitative assumptions on the size of \(|U_{13}^u|, |U_{13}^d|\). The fact that Eq. (3.9) is in good agreement with experiment can be interpreted as an indication that the \((1,1) = 0\) WB is a good basis to derive the structure of \(M_u, M_d\) and thus that of Yukawa couplings, from experimental input. This question will be addressed in Section 4.

\[
\frac{|U_{12}^u|}{|U_{11}^u|} \approx \sqrt{\frac{m_u}{m_c}}; \quad \frac{|U_{12}^d|}{|U_{11}^d|} \approx \sqrt{\frac{m_d}{m_s}}.
\]

In order to qualify what we mean above by “reasonable assumption”, let us recall that:

\[
V_{ub} = U_{11}^{uu} U_{13}^{dd} + U_{21}^{uu} U_{23}^{dd} + U_{31}^{uu} U_{33}^{dd}.
\]

The experimental value of \(|V_{ub}|\) tells us that \(|V_{ub}|\) is of order \(m_d/m_b\) and therefore from Eq. (3.7), assuming \(|U_{11}^u| = O(1)\) and barring unnatural cancellations in the three terms contributing to \(|V_{ub}|\), one is led to the conclusion that \(|U_{13}^d|\) cannot be much larger than \(m_d/m_b\). Assuming that \(M_u, M_d\) have analogous flavour structures, one is led to the following order of magnitude for \(|U_{13}^u|, |U_{13}^d|\):

\[
|U_{13}^d| = O(m_d/m_b); \quad |U_{13}^u| = O(m_u/m_t)
\]

From Eq. (3.8) it then follows that \(m_t \ |U_{13}^u|^2 \approx m_u \left(\frac{m_s}{m_t}\right)\), \(m_b \ |U_{13}^d|^2 \approx m_d \left(\frac{m_c}{m_b}\right)\), thus implying that the third terms in Eqs. (3.5) are entirely negligible, which in turn leads to Eq. (3.6). From this equation, one then obtains:

\[
|V_{us}| = \left| \sqrt{\frac{m_d}{m_s}} + e^{i\alpha} \sqrt{\frac{m_u}{m_c}} \right|,
\]

with \(\alpha \equiv \text{arg}(U_{11}^u U_{22}^d U_{21}^{uu} U_{12}^{dd})\). It is well known that Eq. (3.9) can be obtained [7] in the framework of models with a specific set of texture zeros for \(M_u, M_d\). The imposition of these sets of zeros goes beyond a choice of WB. The point of the above discussion is that one may arrive at Eq. (3.9) with much weaker assumptions, namely the choice of the \((1,1) = 0\) WB (we emphasize that this is always possible to achieve starting with arbitrary \(M_u, M_d\), together with some reasonable qualitative assumptions on the size of \(|U_{13}^u|, |U_{13}^d|\). The fact that Eq. (3.9) is in good agreement with experiment can be interpreted as an indication that the \((1,1) = 0\) WB is a good basis to derive the structure of \(M_u, M_d\) and thus that of Yukawa couplings, from experimental input. This question will be addressed in Section 4.
3.2 The experimental value of $\sin(2\beta)$ and the need for non-factorizable phases

We point out that in schemes where the $(1, 1) = 0$ basis is used and if one adopts some naturalness requirements, at least one non-factorizable phase is needed in order to achieve a sufficiently large value of $\sin(2\beta)$. In order to show how this result is obtained, let us first consider the case where $U^{u}_o, U^{d}_o$ in Eq. (2.3) do not contain any non-factorizable phases. This is equivalent to assuming that:

$$\text{Im}(Q_u) = \text{Im}(Q_d) = 0,$$

(3.10)

where $Q_u, Q_d$ are any of the rephasing invariant quartets constructed from $U^{u}_o, U^{d}_o$. As we have previously emphasized, the conditions of Eq. (3.10) are satisfied in any scheme where $M_u, M_d$ are Hermitian and there is at least one off-diagonal texture zero in both $M_u$ and $M_d$. In this case, the matrices $U^{u}_o, U^{d}_o$ are orthogonal real matrices, which we denote by $O^{u}, O^{d}$. The CKM matrix can then be written in terms of $O^{u}, O^{d}$ and the factorizable phases $\phi_1, \phi_3$ leading, for example, to:

$$V_{cd} = e^{i\phi_1}O^{u}_{12}O^{d}_{11} + O^{u}_{22}O^{d}_{21} + e^{i\phi_3}O^{u}_{32}O^{d}_{31},$$

$$V_{cb} = e^{i\phi_1}O^{u}_{12}O^{d}_{13} + O^{u}_{22}O^{d}_{23} + e^{i\phi_3}O^{u}_{32}O^{d}_{33},$$

$$V_{td} = e^{i\phi_1}O^{u}_{13}O^{d}_{11} + O^{u}_{23}O^{d}_{21} + e^{i\phi_3}O^{u}_{33}O^{d}_{31},$$

$$V_{tb} = e^{i\phi_1}O^{u}_{13}O^{d}_{13} + O^{u}_{23}O^{d}_{23} + e^{i\phi_3}O^{u}_{33}O^{d}_{33}.$$  

(3.11)

In order to estimate the size of

$$\beta \equiv \arg(-V_{cd}V_{tb}V^{*}_{cb}V^{*}_{td}),$$

(3.12)

we have to make a set of assumptions:

(i) From experiment, we know that $V_{CKM} \equiv U^{u\dagger} \cdot U^{d} \approx \mathbb{I}$. We assume that this results from having both $U^{u} \approx \mathbb{I}, U^{d} \approx \mathbb{I}$ rather than from unnatural cancellations in the contributions of $U^{u}, U^{d}$ to $V_{CKM}$.

(ii) We assume that the magnitude of the off-diagonal matrix elements of $U^{u}$ are significantly smaller than the corresponding ones in $U^{d}$, i.e. that $|U^{u}_{ij}| < |U^{d}_{ij}|$ for $i \neq j$. This is certainly true for all flavour models with analogous texture zeros for $M_u, M_d$, where the mixing angles are expressable in terms of quark mass ratios. Since the hierarchy of quark masses are stronger in the up-quark sector (e.g. $m_u/m_t \ll m_d/m_b, m_u/m_c \ll m_d/m_s$) this leads to smaller mixing angles in that same sector.

Taking into account the above reasonable assumptions, we obtain to an excellent approximation:

$$\arg V_{tb} \simeq \phi_3 \quad \arg V_{td}^{*} \simeq -\phi_3 + \epsilon_{td}$$

$$\arg V_{cb} \simeq \epsilon_{cb} \quad \arg(-V_{cd}) \simeq \arctan\left(\frac{O^{u}_{21} \sin \phi_1}{O^{u}_{21}}\right),$$

(3.13)

where

$$\epsilon_{td} = \arctan\left[\frac{O^{u}_{23}O^{d}_{21} \sin \phi_3}{O^{u}_{32}O^{d}_{31} + O^{u}_{32}O^{d}_{21} \cos \phi_3}\right],$$

(3.14)

$$\epsilon_{cb} = \arctan\left[\frac{-O^{u}_{32}O^{d}_{33} \sin \phi_3}{O^{u}_{22}O^{d}_{23} + O^{u}_{32}O^{d}_{33} \cos \phi_3}\right].$$

(3.15)
From Eq. (3.13), it follows that:

\[ \beta \simeq \arctan \left( \frac{O_{12}^u \sin(\phi_1)}{O_{21}^d} \right) + \epsilon_{td} + \epsilon_{cb} . \]  

(3.16)

In the framework of our assumptions (i), (ii), it can readily be seen that \( \epsilon_{td}, \epsilon_{cb} \) are small. For example, one expects \( O_{31}^d \) to give the dominant contribution to \( V_{td} \) and therefore to be of order \( \lambda^2 \) (\( \lambda \) denoting as usual the Cabibbo angle). On the other hand, one expects \( O_{21}^d \simeq \lambda \), but \( O_{23}^u \) significantly smaller than \( \lambda^2 \), because of the strong up quark hierarchy. As a result one has \( \epsilon_{td} \ll 1 \). Similar arguments apply to \( \epsilon_{cb} \) so that \( \epsilon_{cb} \ll 1 \). Therefore, the dominant contribution to \( \beta \) is given by the first term in Eq. (3.16), which arises from \( \arg(\ -V_{cd}) \). In order to evaluate \( \arg(\ -V_{cd}) \), one has to know the value of \( \phi_1 \). This is essentially fixed by the magnitude of \( V_{us} \), which is given by:

\[ |V_{us}| = \left| e^{i\phi_1} O_{11}^u O_{12}^d + O_{21}^u O_{22}^d + e^{i\phi_3} O_{31}^u O_{32}^d \right| \simeq \left| e^{i\phi_1} O_{11}^u O_{12}^d + O_{21}^u O_{22}^d \right| . \]  

(3.17)

Using Eq. (3.6), together with the central values for the quark mass ratios and the experimental value of \( |V_{us}| \):

\[ \frac{m_d}{m_s} = \frac{1}{20} , \quad \frac{m_u}{m_c} = \frac{1}{325} , \quad |V_{us}| = 0.2196 \pm 0.0026 , \]  

(3.18)

one obtains:

\[ \phi_1 = -1.77 \text{ rad } (-101^\circ) . \]  

(3.19)

Note that \( |V_{us}| \) only fixes \( |\phi_1| \). We have opted for a negative value for \( \phi_1 \), so that the leading contribution to \( \beta \) in Eq. (3.16) be positive (recall that \( O_{21}^d \) is negative in the standard parametrization). From Eqs. (3.16) and (3.19) one obtains for the leading contribution to \( \beta \):

\[ \beta_{\text{leading}} \approx 0.24 \text{ rad } (14^\circ) . \]  

(3.20)

Note that at this stage we have neglected the contributions to \( \beta \) arising from \( \epsilon_{td}, \epsilon_{cb} \). In order to evaluate the size of these contributions we write:

\[ O_{13}^d = k_1 \frac{m_d}{m_b} ; \quad O_{23}^d = k_2 \frac{m_s}{m_b} , \]  

(3.21)

where we assume \( k_i \) to be of order 1. We use analogous expressions for \( O_{13}^u \) and \( O_{13}^d \), keeping the same \( k_i \) factors for simplicity. We take the quark mass ratios in the ranges [24]:

\[ 0.21 \leq \left( \frac{m_d}{m_s} \right)^{\frac{1}{2}} \leq 0.24 \quad ; \quad 3.8 \times 10^{-2} \leq \left( \frac{m_u}{m_c} \right)^{\frac{1}{2}} \leq 7.2 \times 10^{-2} \]  

\[ 2.1 \times 10^{-2} \leq \frac{m_s}{m_b} \leq 4.2 \times 10^{-2} \quad ; \quad 3.3 \times 10^{-3} \leq \frac{m_c}{m_t} \leq 4.1 \times 10^{-3} \]  

(3.22)

Allowing \( k_i \) to vary independently in the range

\[ 0.8 \leq k_i \leq 1.4 \quad i = 1,2 , \]  

(3.23)
and keeping the factorizable phases \( \phi_1, \phi_3 \) as free parameters, we evaluate \( \beta \) in Eq. (3.12) from Eqs. (3.11). We plot in Fig.1 the allowed values for \( \beta \) and the CP violation invariant \( I = |\text{Im}[V_{ij}V_{kl}V_{kj}^*V_{il}^*]| \). Note that all points included in Fig. 1 correspond to values of \( |V_{us}|, |V_{cb}| \) and \( |V_{ub}|/|V_{cb}| \) within the range allowed by experiment, and with positive \( \beta \). It is clear from Fig. 1 that, without non-factorizable phases, the central value for \( \beta \) is around 18° and the following bound holds:

\[
\beta \leq 21°; \quad \sin 2\beta \leq 0.67. \tag{3.24}
\]

This is to be compared with the combined BaBar and Belle results [5]:

\[
\sin(2\beta) = 0.739 \pm 0.048. \tag{3.25}
\]

It is clear that such low values for \( \beta \) are already strongly disfavoured (more than 1\( \sigma \) deviation) and will be ruled out by a more precise measurement of \( \sin(2\beta) \), provided the central value does not decrease significantly. These low values of \( \beta \) were to be expected from our previous evaluation of the size of the leading contribution, given in Eq. (3.20). The values obtained for \( I \) in Fig. 1 are also low with respect to the experimental range given in [24] \( (I = 3.0 \pm 0.3) \times 10^{-5} \). It follows from our discussion that if one works in a WB where the Yukawa matrices are Hermitian, with vanishing \((1,1)\) elements, and conforms to some naturalness requirements, one cannot obtain a sufficiently large value of \( \beta \), without allowing for non-factorizable phases. This
point is specially relevant since, as previously pointed out, in a large class of anzätze considered in the literature, non-factorizable phases cannot be introduced. The difficulty of obtaining a sufficiently large value of \( \sin(2\beta) \) in a specific ansatz of the above type was recently pointed out in [19].

We consider now the general case, allowing for the presence of non-factorizable phases. The values of the elements of \( V_{CKM} \) are now given by expressions analogous to those in Eqs. (3.11), with the orthogonal real matrices \( O^{u,d} \) substituted by the unitary matrices \( U^{u,d}_o \), as defined in Eq. (2.2). Note that \( U^{u,d}_o \) now contain non-factorizable phases \( \sigma_u, \sigma_d \). For simplicity, we adopt the “standard parametrization” [24] for \( U^u_o, U^d_o \). It is then easy to understand why, with the presence of at least one non-factorizable phase, one can obtain a sufficiently large size of \( \sin 2\beta \). The reason is that, in the case of no non-factorizable phases, one has in leading order \( \arg(V_{td}) \simeq \arg(V_{tb}) \simeq \phi_3 \), and therefore the leading contribution of \( V_{td} \) to \( \beta \equiv \arg(-V_{cd}V_{tb}V^*_{cb}V^*_{td}) \) just cancels that of \( V_{tb} \). In the presence of non-factorizable phases, this no longer happens since \( (U^d_o)_{31} \) can have a significant phase. Figures 2 and 3 clearly illustrate this point. In these figures, once again, we allow the \( k_i \) to vary within the range given by Eq. (3.23), and keep \( \phi_1 \) as a free parameter while fixing now \( \phi_3 = 0 \) for simplicity. Furthermore the phases \( \sigma_u \) and \( \sigma_d \) are taken to be equal to each other and allowed to vary freely. The difference between Fig. 2 and Fig. 3 is due to the fact that, in plotting Fig. 2, we did our calculations with the quark mass ratios fixed in their central values as given by Eq. (3.18), whilst in Fig. 3 we covered the whole allowed range as given by Eq. (3.22). Although we have set \( \phi_3 = 0 \), it should be pointed out that \( \beta \) is almost unaffected by variations in the value of \( \phi_3 \). Comparing Figs. 2 and 3, we conclude that in both cases sufficiently large values of \( \beta \) can be obtained. The choice of \( \sigma_u \) equal to \( \sigma_d \) follows our rationale of treating on an equal
Figure 3: \( \beta \) as a function of \( I \), with non-factorizable phases and quark masses varying within the allowed range footing the up and down quark sectors as was already done for the variables \( k_i \). However, the value of \( \beta \) is not sensitive to this choice, since it is \( \sigma_d \) that plays the crucial rôle in generating, together with \( \phi_1 \), a sufficient large value for \( \beta \).

The reconstruction of the mass matrices from the experimental data is then possible and one obtains, to leading order:

\[
|M_u| \sim m_t \begin{pmatrix} 0 & \sqrt{m_u m_c} & k_1 m_u/m_t \\ \sqrt{m_u m_c} k_2 m_u/m_t & m_c/m_t & \sqrt{m_u m_c} k_1 m_u/m_t \\ m_c/m_t & k_2 m_c/m_t & 1 \end{pmatrix} ; \\
|M_d| \sim m_b \begin{pmatrix} 0 & \sqrt{m_d m_s} & k_1 m_d/m_b \\ \sqrt{m_d m_s} k_2 m_d/m_b & m_s/m_b & \sqrt{m_d m_s} k_1 m_d/m_b \\ m_s/m_b & k_2 m_s/m_b & 1 \end{pmatrix}.
\]

(3.26)

The structure of these mass matrices in terms of powers of \( \varepsilon \) and \( \varepsilon' \) is given by:

\[
|M_u| \sim m_t \begin{pmatrix} 0 & \varepsilon^3 & \varepsilon^4 \\ \varepsilon^3 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^2 & 1 \end{pmatrix} ; \\
|M_d| \sim m_b \begin{pmatrix} 0 & \varepsilon'^3 & \varepsilon'^4 \\ \varepsilon'^3 & \varepsilon'^2 & \varepsilon'^2 \\ \varepsilon'^4 & \varepsilon'^2 & 1 \end{pmatrix},
\]

(3.27)

with \( \varepsilon = 0.06 \) and \( \varepsilon' = 0.2 \). This Froggatt–Nielsen pattern coincides with the ansatz in [12] and also with that in [15]. The above analysis illustrates the importance of a precise measurement of \( \sin(2\beta) \) in obtaining restrictions on the allowed patterns for Yukawa couplings. A measurement of \( \gamma \) will also have a significant impact in obtaining further constraints on the Yukawa structures, specially since \( 3 \times 3 \) unitarity leads to the exact relation [25]:

\[
\frac{|V_{ub}|}{|V_{cb}|} = \frac{|V_{cd}|}{|V_{ud}|} \frac{\sin(\beta)}{\sin(\beta + \gamma)},
\]

(3.28)

which will further restrict the allowed range for the ratio \( |V_{ub}|/|V_{cb}| \).
4 Weak basis with nine texture zeros

In this section, we address the question of finding the maximal number of zeros that can be obtained starting with arbitrary complex mass matrices $M_u$, $M_d$ and making WB transformations, without imposing the requirement that mass matrices be Hermitian. It is important to recall that, in the SM, Yukawa couplings are not required to be Hermitian. In fact, within the framework of the SM (or $SU(5)$) it is very difficult (if not impossible) to have a symmetry that automatically constrains the Yukawa matrices to be Hermitian. On the other hand, it is well known that Hermitian (or symmetric) Yukawa matrices can be obtained in the framework of left–right-symmetric theories or within $SO(10)$ GUTs.

Here, we show that there is a WB with nine texture zeros and this is the maximal number of zeros that can be obtained. In this WB, the number of free parameters (nine real numbers and one phase) equals the number of physical quantities (six quark masses plus the four physical parameters contained in $V_{CKM}$) and thus it is possible to obtain a full reconstruction of $M_u$, $M_d$ from the experimental input. In order to prove our result we start with a basis where $M_u$ is diagonal real and $M_d$ is a general matrix. We then proceed through the following steps:

i) We first perform a unitary transformation on the right of $M_d$, leading to two zeros:

$$M_d \rightarrow M_d \cdot U = \begin{pmatrix} 0 & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}; \quad (4.1)$$

this is achieved by choosing the first column of $U$ orthogonal to the first and third rows of $M_d$. Since this is a transformation on the right, it is not felt by $M_u$.

ii) The second step is again a unitary transformation on the right of $M_d$, leading to an additional zero in the (22) entry:

$$M_d \rightarrow M_d \cdot U^{(23)} = \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}, \quad (4.2)$$

where $U^{(23)}$ denotes a three-by-three unitary matrix acting only in the (23) space. Of course the first column of $M_d$ remains unchanged.

iii) The third step is a unitary transformation on the left which, of course, will also have to be applied to $M_u$, in order to keep charged weak currents diagonal real

$$M_d \rightarrow U^{(13)} \cdot M_d = \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}. \quad (4.3)$$

The first row of the unitary matrix is chosen in such a way that a zero in the (13) entry of $M_d$ is generated. Applying the same unitary transformation on the left of the diagonal matrix $M_u$, we obtain:

$$M_u \rightarrow \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \quad (4.4)$$
which can be put in the form

$$ M_u = \begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ 0 & \times & \times \end{pmatrix} $$

through a permutation of columns that corresponds to a unitary transformation applied on the right.

iv) Finally an additional zero in (13) of $M_u$ can be generated by a unitary transformation on its right:

$$ M_u \rightarrow M_u \cdot U^{(23)} = \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & \times & \times \end{pmatrix}. \quad (4.6) $$

The final texture after following these steps is of the form

$$ M_d = \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}; \quad M_u = \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}. \quad (4.7) $$

It is clear that phase redefinitions allow for the elimination of all phases in $M_d$, while the freedom left to perform phase redefinitions on the right of $M_u$ leaves this matrix with a single phase in the second column. This results in a total of nine real parameters and one phase, and thus equals the number of physical measurable quantities. This fact implies the possibility of fully reconstructing the quark mass matrices from experimental data. In order to do so, a convenient strategy is to start with

$$ M_u = \text{diag} \left( \frac{m_u}{m_t}, \frac{m_c}{m_t}, 1 \right) m_t $$

$$ M_d = V_{CKM} \cdot \text{diag} \left( \frac{m_d}{m_b}, \frac{m_s}{m_b}, 1 \right) m_b, $$

where both $M_u$ and $M_d$ are already written in terms of physical quantities and then to perform the necessary transformations following step-by-step the path outlined above.

In the final result, we then further interchange the second and third columns in $M_u$:

$$ M_d = m_b \begin{pmatrix} 0 & m_{12}^d & 0 \\ m_{21}^d & 0 & m_{23}^d \\ 0 & m_{32}^d & m_{33}^d \end{pmatrix}; \quad M_u = m_t \begin{pmatrix} 0 & 0 & m_{13}^u \\ m_{21}^u & 0 & 0 \\ 0 & m_{32}^u & m_{33}^u \end{pmatrix}. \quad (4.9) $$

Thus, we ensure that the diagonal entry (33) is larger, or of the same order, as the non-zero off-diagonal entries. In a leading order approximation, we obtain:

$$ m_{12}^d = -a \frac{m_u}{m_t} |V_{us}| $$

$$ m_{21}^d = b \frac{m_d}{m_b m_t} $$

$$ m_{23}^d = -a' \frac{m_c}{m_t} \quad a = \sqrt{1 + \frac{(m_s/m_b)}{|V_{cb}|}} $$

$$ m_{32}^d = 1/a' \quad a' = \sqrt{1 + \frac{|V_{cb}|/(m_s/m_b)}{m_d/m_s}} $$

$$ m_{33}^d = 1/a \quad b = (m_d/m_s)^{1/2}/|V_{us}| $$

(4.10)
and
\[
\begin{align*}
m_{13}^u &= p \frac{m_s}{m_b} |V_{us}| (1 - q e^{-i\delta}) \\
m_{21}^u &= \frac{m_s}{m_t} [1 + q^2 - 2 q \cos(\delta)]^{-1/2} \\
m_{32}^u &= \frac{m_s}{m_t} \left[1 + \frac{q^2}{|V_{us}| \sqrt{m_s/m_b}} \right] \\
m_{33}^u &= 1
\end{align*}
\]

The structure of the matrices thus reconstructed in terms of powers of \( \lambda = 0.212 \) is of the form:
\[
M_d \sim \frac{m_b}{\sqrt{2}} \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & 0 & \lambda^2 \\ 0 & 1 & 1 \end{pmatrix} ; \quad M_u \sim m_t \begin{pmatrix} 0 & 0 & \lambda^3 \\ \lambda^3 & 0 & 0 \\ 0 & \lambda^7 & 1 \end{pmatrix} .
\]

A few comments are in order at this stage. The matrix \( M_u M_u^\dagger \) is block-diagonal and is thus diagonalized by a two-by-two rotation with only (13) mixing. As a result, in this WB, (12) and (23) mixing in \( V_{CKM} \) arises only from \( M_d \). Notice that we have chosen a phase convention such that the single CP-violation phase was placed in \( (M_u)^{13} \). Furthermore, one can choose a WB, given by Eq. (4.7), where the quark mass matrices have the NNI form: apart from the \( (3,3) \) element, the only non-vanishing entries are \( (1,2) \), \( (2,1) \), \( (2,3) \), \( (3,2) \). It has been shown [22] that starting with arbitrary matrices \( M_u \), \( M_d \), one can always make WB transformations such that \( M_u \), \( M_d \), are put in the NNI form. Here, we conclude, that within the NNI weak basis, it is possible to obtain an extra zero entry. It is clear from Eq. (4.12) that the experimental data lead to strong deviations of hermiticity in \( M_d \). This result was to be expected, taking into account that hermiticity, together with the NNI form, leads to the Fritzsch ansatz, which was ruled out once it was found that the top quark is very heavy, with \( m_c/m_t \ll m_s/m_b \). Recall that the Fritzsch ansatz predicts \( |V_{cb}| = |\sqrt{m_s/m_b} - e^{i\alpha} \sqrt{m_s/m_s}| \) and thus the experimental value of \( |V_{cb}| \) could only be reproduced with a strong cancellation of the contributions to \( V_{cb} \) arising from \( (m_s/m_b)^{1/2} \) and \( (m_c/m_t)^{1/2} \). With a large top quark mass, these cancellations became impossible. The interesting point of the above derivation of \( M_d \) from experiment is that it shows that deviations from hermiticity have to be strong in the \( (2,3) \) sector with \( |m_{23}^d|/|m_{32}^d| \sim \lambda^2 \), while one has \( |m_{12}^d| \sim |m_{21}^d| \). The fact that the maximal number of zeros is odd prevents the treatment of \( M_u \) and \( M_d \) on an equal footing.

5 Conclusions

Using an Hermitian WB for Yukawa couplings, we have parametrized the up and down quark mass matrices through Eqs. (2.1), (2.2), separating the phases appearing in \( M_u \), \( M_d \) into factorizable and non-factorizable. We have then chosen to work, without loss of generality, in a WB where the \( (1,1) \) elements of \( M_u \), \( M_d \) both vanish. It was then pointed out, that in the framework of this WB, the existence of at least one non-factorizable phase is crucial to generate, in a natural way, a value of \( \sin(2\beta) \) sufficiently large to be in agreement with experiment. This result is specially relevant.
in view of the fact that many of the Yukawa textures proposed in the literature have the \((1,1)\) zero, together with another off-diagonal texture zero in both \(M_u, M_d\) thus implying the absence of non-factorizable phases in a large class of ansätze. From our analysis we conclude that the present experimental data and in particular the value of \(\sin(2\beta)\) does not favour the existence of simultaneous off-diagonal texture zeros in \(M_u, M_d\), in the \((1,1) = 0\) WB. Allowing for the presence of non-factorizable phases one can, of course, obtain a value of \(\sin(2\beta)\) in agreement with experiment. The reconstruction of the Yukawa flavour structure from input data is then possible in the \((1,1) = 0\) basis, provided one adopts some naturalness requirements. From the experimental data, we also derive a Froggatt–Nielsen pattern for the quark mass matrices. Finally it is worth emphasizing the important rôle that a more precise measurement of \(\sin(2\beta)\) and a measurement of \(\gamma\) will have in narrowing down the allowed Yukawa textures.

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