SUSY Relics in One-Flavor QCD from a New 1/N Expansion

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We suggest a new large $N_c$ limit for multi-flavor QCD. Since fundamental and two-index antisymmetric representations are equivalent in SU(3), we have the option to define SU($N_c$) QCD keeping quarks in the latter. We can then define a new 1/$N_c$ expansion (at fixed number of flavors $N_f$) that shares appealing properties with the topological (fixed $N_f/N_c$) expansion while being more suitable for theoretical analysis. In particular, for $N_f = 1$, our large-$N_c$ limit gives a theory that we recently proved to be equivalent, in the bosonic sector, to $N = 1$ supersymmetric gluodynamics.

Using known properties of the latter, we derive several qualitative and semi-quantitative predictions for $N_f = 1$ massless QCD that can be easily tested in lattice simulations. Finally, we comment on possible applications for pure SU(3) Yang-Mills theory and real QCD.

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Very few techniques are available for analytical study of non-perturbative properties of non-supersymmetric gauge theories such as QCD. Among the most promising ones, large-$N_c$ expansions play a special role, in particular because of their conjectured connection to string theories.

The simplest and the oldest 1/$N_c$ expansion in QCD is that suggested by ’t Hooft [1]. The ’t Hooft limit assumes $N_c \to \infty$ while keeping the ’t Hooft coupling $g^2 N_c$ fixed. If the number of quark flavors is fixed too (i.e. does not scale with $N_c$) then each quark loop is suppressed by 1/$N_c$. Only quenched planar diagrams survive in the leading order. Non-planar diagrams with “handles” are suppressed by 1/$N_c^2$ per handle. Thus, the corrections to the leading approximation run in powers of ($N_f/N_c$) and 1/$N_c^2$. The ’t Hooft expansion led to a number of notable successes in such issues as the Zweig rule, the $\eta'$ mass formula (see below), and so on. Unfortunately, nobody succeeded in fully solving QCD even to the leading order in the ’t Hooft expansion.

In the range of questions where the quark loops are important, a better approximation is provided by the topological expansion (TE) in which $N_f/N_c$ is kept fixed in the large-$N_c$ limit, rather than $N_f$. Then, in the leading order, TE preserves all planar diagrams, including quark loops. This is easily seen by slightly modifying the ’t Hooft double-line notation — adding a flavor line to the single color line for quarks. In the leading (planar) diagrams the quark loops are “empty” inside, since gluons do not attach to the flavor line. Needless to say, obtaining analytic results in TE is even harder than in the ’t Hooft case.

In this letter we propose a new large-$N_c$ expansion that shares some advantages of TE while retaining a significant predictive power. The results obtained in the new large-$N_c$ limit are complementary to those derived in the ’t Hooft limit.

Our basic idea is as follows. Let us start from $N_c = 3$ QCD with $N_f$ quark flavors, which may or may not be massless. (For definiteness we will consider the massless case.) The quark can be described by a Dirac field transforming in the fundamental representation of SU(3)$_c$ or, equivalently, in the two-index antisymmetric representation (plus their complex conjugates). In extrapolating from $N_c = 3$ to arbitrary $N_c$, the former alternative leads to the ’t Hooft limit. We will explore the latter alternative, representing the quark of a given flavor by a Dirac field in the two-index antisymmetric representation. Now, taking the limit $N_c \to \infty$ at $g^2 N_c$ and $N_f$ fixed does not decouple the quark loops since, for large $N_c$, the number of degrees of freedom in the anti-symmetric field scales as $N_c^2$. This is the starting element of our new 1/$N_c$ expansion. For reasons explained in [2], it will be referred to as the orientifold large-$N_c$ limit. The leading order of this new expansion corresponds to the sum of all planar diagrams, in the same way as in TE, but with the crucial difference that quark loops are now “filled”, because the second line in the fermion propagator is now also a color line.

The orientifold large-$N_c$ limit is, therefore, unquenched, and its ’t Hooft diagrams look precisely as those of QCD with $N_f$ Majorana fields in the adjoint representation, modulo reversal of some arrows in the fermion loops. Let us call the SU($N_c$) Yang–Mills theory with $N_f$ Majorana fields in the adjoint representation adjoint QCD. Trivially extending our original arguments, one can actually prove [3] that the orientifold large-$N_c$ limit of QCD is equivalent, in its bosonic sector, to the adjoint QCD just mentioned. This can be shown to be the case even in the presence of quark masses.

Adjoint QCD can be seen, in turn, as a softly-broken version of supersymmetric Yang–Mills theory (SYM)
with $N_f - 1$ additional adjoint chiral superfields. This class of models is currently under intense scrutiny in connection with the works of Refs. [2, 3]. Soft-breaking mass terms are ascribed to the scalar fields, so that the scalar sector decouples. We will briefly comment on the $N_f > 1$ case later, focusing now on the particular case $N_f = 1$.

One-flavor QCD in the orientifold large-$N_c$ limit becomes SU($N_c$) Yang–Mills theory with one Dirac spinor in the antisymmetric representation. As was shown in Ref. [3], this theory is planar-equivalent to adjoint QCD with $N_f = 1$, which is nothing but SU($N_c$) super Yang–Mills theory. This result was a development of the previously formulated Strassler’s conjecture [2] and studies of brane configurations of type 0 string theory [4]. The planar equivalence between $\mathcal{N} = 1$ SYM theory and orientifold field theories amounts to the following statement: the SU($N_c$) gauge theory with two Weyl fermions in the two-index antisymmetric representation,

$$\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$$

(i.e. one Dirac anti-symmetric fermion) at $N_c \to \infty$ is equivalent, in a bosonic subsector, to $\mathcal{N} = 1$ gluodynamics. Not only were the planar $\mathcal{N} = 1$ SYM theory and the planar orientifold field theory prove to be equivalent at the perturbative level: we argued that the full (perturbative and non-perturbative) partition functions of the two theories become equivalent after integration over the respective fermions, at the planar level.

It is worth noting that the two theories are not fully identical. In particular the color-singlet spectrum of the orientifold theory consists only of bosons and does not include composite fermions at $N_c \to \infty$. Some consequences of the (partial) equivalence were discussed in detail in [2, 4].

Now we will use the above planar equivalence to make predictions for one-flavor QCD, keeping in mind that they are expected to be valid up to corrections of the order of $1/N_c = 1/3$ (barring large numerical coefficients):

(i) Confinement with a mass gap. Here we assume that large-$N_c$ $\mathcal{N} = 1$ gluodynamics is a confining theory with a mass gap. Alternatively, if we start from the statement that one-flavor QCD confines, we arrive at the statement that so does $\mathcal{N} = 1$ SYM theory, while the mass gaps are dynamically generated in both theories.

(ii) Degeneracy in the color-singlet bosonic spectrum. Even/odd parity mesons (typically mixtures of fermionic and gluonic color-singlet states) are expected to be degenerate. In particular,

$$\frac{m_{\eta_T}^2}{m_{\eta_0}^2} = 1 + O(1/N_c), \quad \text{one-flavor QCD},$$

where $\eta_T$ and $\sigma$ stand for $0^-$ and $0^+$ mesons, respectively. This follows from the exact degeneracy in $\mathcal{N} = 1$ SYM theory. Note that the $\sigma$ meson is stable in this theory, as there are no pions. The prediction (1) should be taken with care, (i.e. a rather large numerical coefficient in front of $1/N_c$ may occur), since the $\eta_T$ mass is given by the anomaly (the WV formula [3, 11]), whereas the $\sigma$ mass is more “dynamical.” The degeneracy among the even/odd parity mesons should improve at higher levels of the expected Regge trajectory.

(iii) Bifermion condensate. $\mathcal{N} = 1$ SU($N_c$) gluodynamics has a bifermion condensate [1] that can take $N_c$ distinct values:

$$\langle \lambda \lambda \rangle_k \sim M_\text{YM}^3 e^{-\tau/N_c} e^{2\pi k/N_c} = c \lambda^3 e^{\Theta(\theta + 2\pi k)/N_c},$$

$$k = 0, 1, \ldots, N_c - 1,$$

(2)

with

$$\tau = \frac{8\pi^2}{g^2} - i \theta$$

and $c$ a calculable numerical coefficient. The finite-$N_c$ orientifold field theory is non-supersymmetric, and here we expect (taking account of pre-asymptotic $1/N_c$ corrections) $N_c - 2$ degenerate vacua with

$$\langle \overline{\Psi}_L \Psi_R \rangle_{k'} \sim M_\text{YM}^3 \exp \left\{ -\frac{8\pi^2}{g^2(N_c + 4/9)} + i \frac{\theta + 2\pi k'}{N_c - 2} \right\},$$

$$\sim c' \lambda^3 \exp \left\{ i \frac{\theta + 2\pi k'}{N_c - 2} \right\},$$

$$k' = 0, 1, \ldots, N_c - 3.$$

(3)

The term $4/9$ in $b^3$ is due to the one-loop $\beta$ function of the orientifold field theory, $b = 3N_c + \frac{4}{3}$, while $N_c - 2$ in $b^3$ is twice the dual Coxeter number of the antisymmetric representation (fixing the coefficient of the axial anomaly),

$$\text{tr} \overline{\psi} \gamma^\mu \psi = \frac{1}{2} (N_c - 2) \delta^{ab}.$$  

Finally, $c'$ is a normalization factor. For a suitably normalized renormalization-group-invariant quark bilinear we predict

$$\langle \overline{\Psi}_L \Psi_R \rangle = c' \lambda^3 e^{\theta}, \quad c' = c(1 + O(1/N_c)),$$

(4)

i.e. a non-vanishing condensate and a single true vacuum. Note that the naive prediction of three degenerate vacua from the $\mathcal{N} = 1$ SYM theory is useless because in this case the $1/N_c$ correction is large ($N_c \to N_c - 2$ in passing from $c$ to $c'$). We expect, however, the absolute value of the condensate to be in reasonable agreement with the supersymmetry-based prediction.

(iv) The accuracy of “dynamical” predictions, namely those results that are not saturated by one loop (i.e. the anomalies), can be inferred from perturbative arguments. In one-flavor QCD the first coefficient of the $\beta$ function is $b = 31/3$, while in adjoint QCD with $N_f = 1$ it becomes $b = 27/3$.

In fact, in the very same approximation it can be asserted that the $\beta$ function of the one-flavor QCD should coincide with the exact NSVZ beta function [12]:

$$\beta = \frac{1}{2\pi} \frac{9\pi^2}{1 - 3\alpha/2\pi}.$$  

(5)
Thus, for the (relative) value of the 2-loop $\beta$-function coefficient, we predict $+\frac{3\alpha}{2\pi}$, to be compared with the exact value in one-flavor QCD:

$$\frac{134}{31} \frac{\alpha}{2\pi} \approx 4.32 \frac{\alpha}{2\pi}.$$  

Note that our $1/N_c$ expansion, unlike 't Hooft's, overemphasizes the quark-loop contributions, and thus makes the theory less asymptotically free than in reality—the opposite of what happens in 't Hooft's expansion. Parametrically, the error in our way of treating one-flavor QCD is $1/N_c$ rather than $1/N_c^2$. This is because there are $N_c^2 - 1$ gluons and $N_c^2 - N_c$ fermions in the orientifold field theory.

Now we will comment on possible applications for the pure SU(3) Yang–Mills theory. As was explained, one-flavor QCD is approximated (to $O(1/N_c)$) by a supersymmetric theory. On the other hand, a different $1/N_c$ expansion—the 't Hooft one—connects one-flavor QCD to the pure SU(3) gauge theory. Therefore, the pure SU(3) theory is also approximated, in a sense, by a supersymmetric theory. Although we suspect that in this case the two approximations accumulate, so that errors are large, it is reasonable to ask whether there are relics of SUSY in the SU(3) Yang-Mills theory.

In fact, it has been known for a long time [13] that such relics do exist in pure gauge theory, although at that time they were not interpreted in terms of supersymmetry. Indeed, it was shown [13] that one can use an approximate holomorphic formula

$$\langle \text{tr } F^2 + i \text{ tr } \tilde{F}^a \rangle = M^4 \text{exp}(-\gamma/3).$$  

(For this equation to be holomorphic in $\tau$, renormalization-group-invariant and, in addition, to have the correct $\theta$ dependence, we must use $b = 12$ rather than the actual value $b = 11$ in SU(3) Yang–Mills theory. The decomposition $11 = 12 - 1$ has a deep physical meaning: $-1$ presents a unitary contribution in the one-loop $\beta$ function, while 12 is a $\text{bona fide}$ antiscreening, see e.g. [13].) The holomorphic dependence is a relic of supersymmetry. Equation (6) implies, in particular, that the topological susceptibility $\chi$ is expressible in terms of the expectation value of $\text{tr } F^2$:

$$12\chi \equiv -i \int d^4x \left( \frac{\sqrt{3}}{16\pi^2} F^a \tilde{F}^a(x) \right) \left( \frac{\sqrt{3}}{16\pi^2} F^a \tilde{F}^a(0) \right)_{\text{conn}} \approx \left\langle \frac{1}{8\pi^2} F^a F^a \right\rangle.$$  

The numerical value of the left-hand side is known, either from the WV formula [8, 11] or from lattice measurements [13], to be $\approx 1.3 \times 10^{-2}$ GeV$^4$. The gluon condensate on the right-hand side is that of pure Yang–Mills theory and was estimated to be approximately twice larger (see Ref. [13], Sect. 14) than the gluon condensate in actual QCD,

$$\left\langle \frac{1}{4\pi^2} F^a F^a \right\rangle_{\text{QCD}} \approx 0.012 \text{ GeV}^4,$$

see Ref. [10]. If so, the numerical value of the right-hand side is $\approx 1.2 \times 10^{-2}$ GeV$^4$. Note that the phenomenological estimate of the gluon condensate and the factor 2 enhancement mentioned above are valid up to $\sim 30\%$.

Finally, we want to comment on real QCD. Let us assume for a moment that the $\sigma$ mass is not very sensitive to the number of flavors. On the other hand, according to the WV formula [9, 10] and neglecting quark masses, the $\eta'$ mass scales like $\sqrt{N_f}$; we can therefore extrapolate the relation (1) to obtain a prediction for real QCD

$$m_{\eta'} \sim \sqrt{3} m_\sigma.$$  

(8)

Although $\sigma$ is very broad in real QCD, it is amusing that the above relation is indeed in qualitative agreement with the position of the enhancement in the $\pi\pi$ channel.

The connection between three-flavor QCD and adjoint QCD with $N_f = 3$ can be used to relate the patterns of the chiral symmetry breaking in these two theories, see Ref. [5].

In 't Hooft's large $N_c$ limit the width of $q\bar{q}$ mesons scales as $1/N_c$, as compared to $1/N_c^2$ for glueballs, i.e. the latter are expected to be relatively more narrow. This is quite weird, given that no glueball was identified so far, in spite of decades of searches.

Our $N_c$ counting, instead, predicts that the widths of both quarkonic and gluonic mesons scale as $1/N_c^2$. Given that glueballs are expected to be heavier than their quark counterparts, this might explain why many narrow quarkonia have been found but no comparably-narrow glueballs.

In conclusion, we have argued that the orientifold theory generalization of QCD to arbitrary $N_c$ can be used (through its SYM limit) in order to make predictions for one-flavor QCD. Hopefully, these will be tested in lattice simulations in the near future.

Note added Meanwhile, the calculation of the quark condensate in one-flavor QCD, starting from SUSY gluodynamics, as outlined in the text, has been completed [17] and has exhibited a remarkable agreement with the currently accepted “phenomenological” value.

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