WAVES IN PLASMAS
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ABSTRACT

A brief discussion of the relevance of linear waves in infinite uniform plasmas is given. The linear waves of a field-free plasma are then discussed in some detail in order to illustrate the methods which are used in more complicated problems. High-frequency waves in a magnetized plasma are then described. The main emphasis is to show how the magnetic field affects the wave phenomena. The two specific cases of propagation parallel and perpendicular to the magnetic field are examined. Next, low-frequency waves in a magnetized plasma are discussed from the point of view of ideal MHD. Finally, the problem of Raman scattering is discussed in some detail. This phenomenon illustrates how the nonlinear terms previously neglected can introduce additional effects. Finally, the preceding analysis is used to introduce the subject of the beat wave accelerator.

1. INTRODUCTION

A magnetized plasma is one of the richest wave media yet studied. It owes this richness to the long-range nature of the Coulomb interaction, and it is this long-range interaction which makes the many-body or collective effects in a plasma so subtle. One of the most important concepts arising from the collective effects is that of the self-consistent electromagnetic field, which is the field which a given plasma particle experiences due to the presence of all the other particles. This self-consistent field is crucial to the theory of waves in the plasma medium.

The idea is as follows. Suppose there is a small electromagnetic field present in the plasma. This produces forces on the plasma particles, resulting in currents and charge perturbations which act as source terms for further electromagnetic fields, which will then produce further plasma motions, and so on. This system of field perturbations and particle motions is iterated until the assumed electromagnetic field is itself produced by the resulting plasma motion. For the theory of linear waves in a plasma, the self-consistency condition gives the dispersion relation which contains all the information concerning the wave motion.

In this lecture we shall be mainly concerned with linear waves although an example of a nonlinear wave phenomenon will be discussed at the end. In order to discuss linear waves we must first of all define the equilibrium state. We shall confine ourselves to the idealization of an infinite, uniform plasma. Having specified the equilibrium, the linear waves describe the possible modes of oscillation when this equilibrium is subject to a small perturbation such that products of perturbations in the equations of motion can be
ignored. Physically, this means that the wave amplitude is small enough that effects such as the beating of two waves to produce sum and difference frequencies or modification of the equilibrium are negligible. Many of the most interesting problems in fact are concerned with these effects. However, we must first understand the linear modes of oscillation of the plasma medium. The theory of linear waves does not contain any criterion on how small the wave amplitude should be for the theory to be valid. To obtain such a criterion we must appeal to a nonlinear analysis.

Let us briefly consider the relevance of our idealized model, bearing in mind that most plasmas are either finite, non-uniform, varying in time or any combination of these. The primary justification is on the grounds of simplicity. A study of the simplest of all models serves to identify many of the basic phenomena and then forms a framework for more complicated situations. If the wavelength is less than the plasma radius \((\lambda \ll a)\) or the characteristic scale length \(L\) of some non-uniformity \((\lambda \ll L)\), or if the frequency of the wave is much greater than the inverse of the plasma lifetime \((\omega \gg \tau^{-1})\), then these complications will produce only small corrections to the simple theory.

Another noteworthy feature of the simple theory is that no modes are lost by the simplification — the more realistic models only modify the properties of existing modes of the simple model. Even though many of the wave motions observed in plasmas relate to a nonlinear saturated state it is usually the case that the frequencies and wavelengths are determined quite closely by the linear theory. A study of the properties of linear waves does therefore have relevance to physical problems.

Finally, it is worth mentioning that the plasma instabilities which can occur when there is a source of free energy result from one or two of these linear waves which can draw on this free energy. A knowledge of the linear waves in a plasma is therefore essential to the study of stability.

Let us now consider the simple example of linear waves in an infinite field-free, uniform plasma in some detail.

2. WAVES IN A FIELD-FREE PLASMA

In any problem in plasma physics there is always a choice of plasma model. For our present purposes the two-fluid (hydrodynamic) model is adequate to illustrate the properties of linear waves in a field-free plasma. By field-free we mean that there is no equilibrium magnetic field. A steady electric field would not change the nature of the waves but might cause some of them to become unstable. Here, we are only concerned with stable wave motions. As always the plasma is coupled to the electromagnetic field through Maxwell’s equations. We therefore require solutions to the following system of coupled partial-differential equations

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v_e) = 0
\]  

(2.1)
\[ \frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e + \gamma_e \frac{\kappa^T}{m_e} \frac{e}{m_e} \mathbf{v}_e = \mathbf{E} - \frac{e}{m_e} \mathbf{v}_e \times \mathbf{B} \quad (2.2) \]

\[ \frac{\partial n_e}{\partial t} + \mathbf{v}_e \cdot \nabla n_e = 0 \quad (2.3) \]

\[ \frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i = \frac{e}{m_i} \mathbf{E} + \frac{e}{m_i} \mathbf{v}_i \times \mathbf{B} \quad (2.4) \]

\[ \mathbf{J} \times \mathbf{H} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (2.5) \]

\[ \mathbf{J} \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad (2.6) \]

The above equations are written in MKS units where \( n_e, n_i, \mathbf{v}_e \) and \( \mathbf{v}_i \) are the electron and ion number densities and the electron and ion fluid velocities. We have assumed a zero pressure (cold) ion fluid and a finite pressure (temperature \( T_e \)) electron fluid where \( \gamma_e \) is the ratio of specific heats of the electron fluid. The current density in equation (2.5) is

\[ \mathbf{J} = n_i \mathbf{v}_i - n_e \mathbf{v}_e \quad (2.7) \]

The magnetic field appearing in the above equations is, of course, an oscillating field due to the presence of a wave whose properties are still to be determined. Equations (2.1)-(2.7) are highly nonlinear and in order to describe the linear wave motions we must linearize this system of equations. To do this we must first define the equilibrium which, in this case, is particularly simple. The equilibrium configuration consists of a uniform density plasma with no average electron or ion flows. There is therefore no equilibrium electric current and since the electron and ion fluid densities are uniform, there is no equilibrium space charge and hence no electric field. All variables are now separated into an equilibrium part and a part which represents a small perturbation, varying both in space and time. The linearization is carried out by substituting the variables expressed in this way into equations (2.1)-(2.7) and neglecting all quantities which involve products of perturbed variables. Equations (2.1)- (2.7) are thus transformed into a set of linear equations in which the variables are the perturbed-field variables and the coefficients of the equations are constants due to the uniform equilibrium. The equations can therefore be Fourier analyzed in space and time. Since there is no preferred direction it is sufficient to assume that all the perturbed fields vary as

\[ \exp i(kz - \omega t) \]

The set of simultaneous, linear, partial differential equations are therefore reduced to the following set of algebraic equations

\[ -i\omega \mathbf{v}_e + ikn_0 \mathbf{v}_e \_z = 0 \quad (2.8) \]

\[ -i\omega \mathbf{v}_e + \gamma_e \frac{k^T}{m_e} \frac{e}{m_e} \mathbf{v}_e = -\frac{e}{m_e} \mathbf{E}_z \quad (2.9) \]

\[ -i\omega \mathbf{v}_i + ikn_0 \mathbf{v}_i \_z = 0 \quad (2.10) \]
\[
-i \omega y_{11} = \frac{e}{m_i} E_1 \\
(2.11)
\]
\[
\kappa \times \underline{H}_1 = n_0 e (v_{x1} - v_{e1}) - i \omega \nu_0 \underline{E}_1 \\
(2.12)
\]
\[
\kappa \times \underline{E}_1 = i \omega \nu_0 \underline{H}_1 \\
(2.13)
\]

where perturbed variables carry a subscript 1.

The self-consistency condition on the fields and particle motions is now obtained by requiring that the above set of equations be simultaneously satisfied. This condition is given by the vanishing of the determinant of the coefficients and results in the dispersion relation for the system which is usually written as

\[
D(\omega, \kappa) = 0. \\
(2.14)
\]

The solutions \(\omega(\kappa)\) of this equation then give all the properties of the waves of the system.

Let us now obtain the dispersion relation from equations (2.8)-(2.13). First, solving for the fluid velocities in terms of the components of the electric field \(\underline{E}_1\) and substituting into equation (2.7) we obtain the conductivity tensor defined by

\[
\underline{v}_1 = g(\omega, \kappa) \underline{E}_1 \\
(2.15)
\]

where

\[
g(\omega, \kappa) = \begin{bmatrix}
\frac{i n_0 e^2}{\omega m_1} + \frac{i n_0 e^2}{\omega m_e} & 0 & 0 \\
0 & \frac{i n_0 e^2}{\omega m_1} + \frac{i n_0 e^2}{\omega m_e} & 0 \\
0 & 0 & \frac{i n_0 e^2}{\omega m_1} + \frac{i n_0 e^2}{\omega m_e} \left[1 - \nu_e \frac{k^2 \nu_e^2}{\omega^2} \right]^{-1}
\end{bmatrix}
\]

and \(v_e^2 \equiv kT_e/m_e\). With the aid of equations (2.12), (2.13) and (2.15) we obtain

\[
\kappa \times (\kappa \times \underline{E}_1) = -\frac{\omega^2}{c^2} \underline{g}(\omega, \kappa) \underline{E}_1 \\
(2.16)
\]

where

\[
\underline{g}(\omega, \kappa) = \underline{I} + \frac{1}{\omega \nu_0} g(\omega, \kappa) \\
(2.17)
\]

is the dielectric tensor of the plasma. The dispersion relation follows immediately from equation (2.16) and is
\begin{align}
-k^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} &= 0 \\
\frac{\omega^2}{c^2} \varepsilon_{yy} &= 0 \\
0 &= 0 .
\end{align} 
(2.18)

It is clear that the transverse \((\mathbf{E} \perp \mathbf{k})\) modes are decoupled from the longitudinal \((\mathbf{E} \parallel \mathbf{k})\) modes.

2.1 Longitudinal modes

The dispersion relation for these modes is

\begin{equation}
\varepsilon_{zz} = 0 
\end{equation} 
(2.19)

ignoring the \(\omega^2 = 0\) solutions. There are two solutions of equation (2.19), a high frequency branch

\begin{equation}
\omega^2 = \frac{\omega_{pe}^2}{c^2} + \gamma_e k_2 v_e^2 
\end{equation} 
(2.20)

involving only the electrons, where \(\omega_{pe}^2 = n_e e^2 / \varepsilon_0 m_e\), known as the LANGMUIR wave and a low frequency branch involving both electrons and ions

\begin{equation}
\omega^2 = \frac{k^2 c_s^2}{1 + \gamma_e k^2 \lambda_{pe}^2} 
\end{equation} 
(2.21)

known as the ION ACOUSTIC wave. For long wavelengths the ion acoustic wave is non-dispersive and travels with phase and group velocity both equal to \(c_s \equiv (\gamma_e \kappa T_e / m_i)^{1/2}\).

At shorter wavelengths \(k \lambda_{pe} \gtrsim 1\), where \(\lambda_{pe} \equiv v_e / \omega_{pe}\), the upper limit of frequency of the ion acoustic wave is the ion plasma frequency \(\omega_{pi}\), where \(\omega_{pi}^2 \equiv n_i e^2 / \varepsilon_0 m_i \).

2.2 Transverse modes

The dispersion relation, equation (2.18), shows that there are two independent transverse modes, both linearly polarized. The dispersion relations of these modes are

\begin{align}
-k^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} &= 0 \\
-k^2 + \frac{\omega^2}{c^2} \varepsilon_{yy} &= 0 .
\end{align} 
(2.22)

Since \(\varepsilon_{xx} = \varepsilon_{yy}\) the two transverse waves have the same propagation properties, one being polarized with its electric field in the \(x\)-direction and the other with its electric field in the \(y\)-direction. With the aid of equation (2.17) and the expression for the
conductivity tensor we obtain

\[ \varepsilon_{xx} = 1 - \frac{\omega_{pe}^2}{\omega^2}. \]  (2.24)

Using (2.24) in equations (2.22) and (2.23) we finally obtain the dispersion relation for the two transverse modes

\[ \omega^2 = \omega_{pe}^2 + c^2 k^2. \]  (2.25)

This is also a high-frequency wave involving only the electrons. Transverse electromagnetic waves do not propagate in a plasma for frequencies below the plasma frequency, i.e. for \( \omega < \omega_{pe} \), \( k^2 < 0 \) corresponding to an evanescent wave. The longitudinal Langmuir wave also has this property being cut off at \( \omega = \omega_{pe} \). The properties of the linear waves in a field free plasma are conveniently summarized with the aid of an \( \omega-k \) diagram shown in figure 1.

One final point worth making which is brought out by figure 1 concerns the Langmuir wave. It can be seen that the phase velocity \( (\omega/k) \) can vary widely ranging from values much larger than the velocity of light \( c \) down to values of the order of the electron thermal speed.

3. WAVES IN A MAGNETIZED PLASMA

Let us now consider the effect of an equilibrium magnetic field on the linear waves in a plasma. We shall not go into as much detail as in the previous section but will point out the differences introduced by an equilibrium magnetic field and how the wave properties are altered. We shall keep things as simple as possible by considering only high-frequency waves which do not involve the ions. The only change to the system of equations given by (2.1), (2.2), (2.5) and (2.6) is that the linearized momentum equation for the electron fluid now becomes
where the additional term due to the presence of the equilibrium magnetic field \( B_0 \) appears on the right hand side of equation (3.1). We assume that the magnetic field is uniform in space, constant in time and points in the z-direction, i.e. \( B_0 = (0,0,B_0) \). Since the magnetic field now defines a preferred direction in the plasma we must allow for perturbations to the equilibrium which vary as 

\[
\exp(i(\vec{k} \cdot \vec{x} - \omega t))
\]

where, without loss of generality we may assume \( \vec{k} = (0,k_y,k_z) \). We shall not go through the derivation of the dielectric tensor in this case. Even with the neglect of ion motion the algebra is still rather tedious. Instead, we simply note the important fact that due to the presence of the equilibrium magnetic field in equation (3.1) the \( x \)- and \( y \)-components of the fluid velocity are now coupled together, in contrast to the field free case. This results in some of the off-diagonal elements of the dielectric tensor being non-zero. Once we have obtained the dielectric tensor \( \xi(\omega,\vec{k}) \) we can again write down the dispersion relation from equation (2.16). This time however all the elements of \( \xi(\omega,\vec{k}) \) are non-zero for the most general case. The dispersion relation can be written formally as

\[
\begin{vmatrix}
-k^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} & \frac{\omega^2}{c^2} \varepsilon_{xy} & \frac{\omega^2}{c^2} \varepsilon_{xz} \\
\frac{\omega^2}{c^2} \varepsilon_{yx} & -k^2 + \frac{\omega^2}{c^2} \varepsilon_{yy} & k_y k_z + \frac{\omega^2}{c^2} \varepsilon_{yz} \\
\frac{\omega^2}{c^2} \varepsilon_{zx} & k_y k_z + \frac{\omega^2}{c^2} \varepsilon_{yz} & -k^2 + \frac{\omega^2}{c^2} \varepsilon_{zz}
\end{vmatrix} = 0 .
\]  

(3.2)

In order to see the consequences of the presence of the equilibrium magnetic field we do not need to analyze the most general case. Instead we shall consider two special cases.

### 3.1 Propagation parallel to the magnetic field

For propagation parallel to the magnetic field \( k_y = 0 \). It is then the case that

\[
\begin{align*}
\varepsilon_{xz} &= 0 = \varepsilon_{zx} \\
\varepsilon_{yz} &= 0 = \varepsilon_{zy}
\end{align*}
\]

with the result that the dispersion relation, equation (3.2) reduces to

\[
(\frac{k^2}{c^2} \varepsilon_{xx} - \frac{\omega^2}{c^2} \varepsilon_{yy}) (\frac{k^2}{c^2} \varepsilon_{xx} - \frac{\omega^2}{c^2} \varepsilon_{yy}) - \frac{\omega^2}{c^2} \varepsilon_{xy} \varepsilon_{yx} \frac{\omega^2}{c^2} \varepsilon_{zz} = 0 .
\]

(3.3)
We therefore find that, for parallel propagation the longitudinal modes are again decoupled from the transverse modes and in fact are unaffected by the magnetic field. This was to be expected since \( E_1 \parallel B_0 \). The longitudinal dispersion relation is again

\[
\varepsilon_{zz} = 0
\]

which yields the Langmuir waves previously discussed.

The first change brought about by the magnetic field is shown by the transverse dispersion relation. Since \( \varepsilon_{xy} \) and \( \varepsilon_{yx} \) are non-zero, \( E_{1x} \) and \( E_{1y} \) are now coupled. This was also to be expected from our observation concerning the electron momentum equation. The symmetry of the present problem is such that

\[
\varepsilon_{yx} = -\varepsilon_{xy}
\]

and

\[
\varepsilon_{xx} = \varepsilon_{yy}
\]

The transverse dispersion relation can then be written

\[
\left( k_z^2 - \frac{\omega^2}{c^2} \varepsilon_{xx} \right)^2 = -\frac{\omega^4}{c^4} \varepsilon_{xy}^2
\]

so that the two transverse modes are given by

\[
k_z^2 - \frac{\omega^2}{c^2} \varepsilon_{xx} = \pm i \frac{\omega^2}{c^2} \varepsilon_{xy}.
\]  \( (3.5) \)

Equation (3.5) can now be used in equation (2.16) to obtain the wave polarization giving

\[
E_{1x} = \pm i E_{1y}.
\]  \( (3.6) \)

We therefore obtain the result that transverse electromagnetic waves propagating parallel to the magnetic field are right or left hand circularly polarized. The dispersion relation for these transverse modes can now be obtained by substituting the following explicit expressions for \( \varepsilon_{xx} \) and \( \varepsilon_{xy} \),

\[
\varepsilon_{xx} = 1 - \frac{\omega^2}{\omega^2 - \Omega_e^2}
\]  \( (3.7) \)

\[
\varepsilon_{xy} = \frac{i \omega^2 \Omega_e}{\omega (\omega^2 - \Omega_e^2)}
\]  \( (3.8) \)

into equation (3.5). The result is

\[
\frac{c^2 k_z^2}{\omega^2} = 1 - \frac{\omega^2}{\omega (\omega^2 - \Omega_e^2)}
\]  \( (3.9) \)

where \( \Omega_e = e B_0/m_e > 0 \). The cut-off frequencies, \( \omega_{1,2} \), of the right and left hand
modes are no longer given by $\omega_{\text{pe}}$ but now depend on the magnetic field as well as the density

$$\omega_{1,2} = \frac{Q_e}{2} \left[ 1 + \left( 1 + 4 \frac{\omega^2}{\omega_{\text{pe}}^2} \right)^{1/2} \right]. \quad (3.10)$$

However, the most striking effect of the magnetic field is the existence of a propagating branch of the right hand mode for $\omega \ll Q_e$ and $\omega_{\text{pe}}^2 >> \omega Q_e$ giving

$$\frac{c^2 k_z^2}{\omega^2} = \frac{\omega_{\text{pe}}^2}{\omega Q_e}. \quad (3.11)$$

The left hand wave is evanescent but the right hand wave propagates. This branch is known as the WHISTLER due to the dependence of the group velocity on $\omega^{1/2}$. The received signal has therefore a falling tone. Whistlers are electromagnetic waves which can propagate with phase velocities, $v_{\text{ph}} < c$ and have been observed in the ionosphere, laboratory discharges, and in a rod of indium at liquid-helium temperatures. The properties of the waves propagating along the magnetic field can again be displayed in the $\omega-k$ diagram shown in figure 2.

![Diagram](image)

**Fig. 2** Dispersion relation for the electromagnetic waves propagating along the equilibrium magnetic field (RH - right hand circular polarization, LH - left hand circular polarization) where $\omega_{1,2}$ are defined by equation (3.10)

### 3.2 Propagation perpendicular to the magnetic field

For this case $k_z = 0$ and again we find

$$\varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0.$$ 

Under these conditions the general dispersion relation, equation (3.2) reduces to

$$\left( \varepsilon_{xx} \right) \frac{\omega^2}{c^2} \varepsilon_{yy} + \left( \varepsilon_{xy} \right) \frac{\omega^2}{c^2} \varepsilon_{yx} \left( \varepsilon_{xx} \right) = 0. \quad (3.12)$$

Equation (3.12) contains two factors. The second one gives
\[ k_y^2 = \frac{\omega^2}{c^2} \varepsilon_{zz} \]  
(3.13)

For a cold plasma \( \varepsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2} \) so that equation (3.13) becomes

\[ \frac{c^2 k_y^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} \]  
(3.14)

which is the same result as for transverse waves in a field-free plasma. This is due to the fact that this wave is linearly polarized with its electric field aligned along the equilibrium magnetic field. This wave is therefore called the ordinary mode.

Now consider the other factor of equation (3.12). This illustrates another effect of the magnetic field which is to couple the longitudinal field \( E_y \) to the transverse field \( E_x \). For a cold plasma the dielectric tensor elements are again given by equations (3.7) and (3.8). Substituting these equations into the first factor of equation (3.12) we obtain

\[ \frac{c^2 k_y^2}{\omega^2} = \frac{\left(\omega^2 - \omega_{pe}^2\right)^2 - \omega^2 Q_e^2}{\left(\omega^2 - \omega_{pe}^2 - Q_e^2\right)} \]  
(3.15)

The cut-off frequencies are the same as for the previous case but now the wave has a resonance at \( \omega_{ph} = \omega_{pe} + Q_e \) where \( \omega_{ph} \) is called the upper hybrid frequency. Equation (2.16) can again be used to obtain the wave polarization resulting in

\[ \frac{E_{1y}}{E_{1x}} = \frac{\varepsilon_{yx}}{\varepsilon_{yy}} = \frac{Q_e}{\omega} \frac{\omega_{pe}}{(\omega^2 - \omega_{ph}^2)} \]  
(3.16)

When \( \omega \gg Q_e \), \( E_{1y} \ll E_{1x} \) and the wave is almost transverse whereas for \( \omega \approx \omega_{ph} \), \( E_{1y} \gg E_{1x} \) and the wave is almost longitudinal. The \( \omega-k \) diagram for the case of propagation perpendicular to \( B_0 \) is shown in figure 3.

![Fig.3 Dispersion relation for waves propagating perpendicular to the equilibrium magnetic field (O denotes the O-mode and X the X-mode)](image-url)
The hybrid wave has two branches, one of which has a resonance at $\omega = \omega_{\text{num}}$. It is worth noting that compared with the field-free case, the magnetic field has removed the degeneracy between the transverse and longitudinal waves which occurred at $\omega = \omega_{\text{pe}}$.

4. LOW-FREQUENCY WAVES IN A MAGNETIZED PLASMA

For frequencies in the vicinity of the ion cyclotron frequency the two-fluid model is still required. However, for $\omega \ll Q_i (Q_i \equiv \frac{eB_0}{m_i})$ and wavelengths long compared with the ion Larmor radius the ions and electrons move together maintaining charge neutrality. Under these circumstances it is a good approximation to describe the plasma as a single fluid. This is the MHD model (ideal or non-ideal depending on whether the plasma resistivity is zero or non-zero). We shall restrict ourselves to the ideal model which is the simplest model to describe the three low-frequency waves of a magnetized plasma. The equations of ideal MHD are

\begin{align}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \times \mathbf{B} \right) &= -\nabla P + \mathbf{J} \times \mathbf{B} \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mathbf{J}
\end{align}

where $\rho$, $\mathbf{v}$ and $P$ are the mass density, velocity and pressure of the fluid. The first of these equations expresses the continuity of mass flow and the second is the momentum equation. Notice that there is no force due to the electric field due to the absence of space charge. Apart from the force due to pressure gradients the only other force is due to the flow of current in the presence of a magnetic field. Equation (4.3) results from the assumption of infinite conductivity so that there can be no electric field parallel to the magnetic field. The magnetic field is "frozen in" to the plasma which always moves so that in the plasma rest frame the electric field is zero. The final pair of equations are of course Maxwell's equations but with the displacement current neglected due to the low frequency assumption. The neglect of the displacement current is also related to the absence of space charge.

It is convenient to reduce the above set of five equations to three by substituting for $\mathbf{J}$ from (4.5) into (4.2) and eliminating $\mathbf{E}$ from (4.4) with the aid of (4.3). The linearized version of these equations can then be written

\begin{align}
\frac{\partial \rho_0}{\partial t} &= -\nabla P_0 + (\mathbf{v}_0 \times \mathbf{B}_0) \times \frac{\mathbf{B}_0}{\rho_0} \\
\frac{\partial \rho_1}{\partial t} &= \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) \\
\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) &= 0
\end{align}
In order to close this set we must add an equation of state relating $P_1$ and $\rho_1$. We can use the adiabatic or isothermal model and in both cases obtain the relation

$$P_1 = c_s^2 \rho_1,$$

(4.9)

where, for the isothermal model $c_s^2 = P_0/\rho_0$. Assuming a constant uniform magnetic field pointing in the $z$-direction we look for wave solutions varying as

$$\exp i(k^z x - \omega t)$$

and again take $k = (0, k^y, k^z)$. We will not go through the details but simply state the main conclusions. First, on writing equations (4.6)-(4.8) in their cartesian components we find that $v_{1x}$ and $B_{1x}$ are independent of all other variables. Solving for these two variables we obtain the dispersion relation

$$\omega^2 = \frac{c_s^2 k^2}{A^2 z^2},$$

(4.10)

where

$$c_s^2 = B_0^2/\rho_0 \mu_0.$$ 

This is the SHEAR ALFVÉN wave which propagates at the Alfvén speed $c_A$. It will not propagate perpendicularly to the magnetic field and will only transport energy along the field.

Solving for the remaining variables we obtain the dispersion relation for the other two modes

$$(\omega^2 - k_s^2 c_s^2)(\omega^2 - k_c^2 c_s^2 - k_s^2 c_A^2) - k_s^2 k_c^2 c_A^2 = 0.$$ 

(4.11)

For most laboratory plasmas $c_s^2/c_A^2 \ll 1$. With this assumption the solutions of equation (4.11) are

$$\omega^2 = \frac{k_c^2 c_A^2 (1 + \frac{c_s^2}{c_A^2})}{c_s^2 A},$$

(4.12)

$$\omega^2 = \frac{k_s^2 c_s^2 (1 + \frac{c_s^2}{c_A^2})}{c_s^2 A}.$$ 

(4.13)

The first solution is called the fast magnetosonic wave (sometimes the compressional Alfvén wave) and the second one the slow magnetosonic wave. In general both these waves are hybrid modes but for the parallel propagation the fast wave is a transverse mode and the slow wave a longitudinal mode. The fast wave is the only one of the three low-frequency waves which can propagate perpendicularly to the magnetic field. Apart from its lower velocity the propagation behaviour of the slow wave is similar to the shear wave.
5. RAMAN SCATTERING

Let us now go beyond the linear theory discussed so far and consider the effect of keeping the nonlinear terms contained in equations (2.1)–(2.7). We shall discuss one of the simplest nonlinear effects which is the interaction of two light waves to produce a Langmuir wave. Since all three waves are high frequency only the electrons are involved. Furthermore, the coupling can take place in one dimension.

In order to describe this effect mathematically we return to the electron fluid equations (2.1) and (2.2), Maxwell's equations (2.5) and (2.6) and the equation for the current but with the ion contribution neglected. In addition, we also add a dissipation term $v_e v_e$ to the left hand side of equation (2.2) where $v_e$ can be thought of either as the electron-ion collision frequency or more loosely as a term which simulates both collisional and collisionless dissipation for the electrons.

Before proceeding with the analysis we make one important observation. The energy density of an electromagnetic wave in a plasma with an electric field amplitude $E$ is

$$\varepsilon = \frac{\varepsilon_0 |E|^2}{2}.$$  

The ratio of this quantity to the energy density $n_0 k_T$ of the plasma is a measure of the size of perturbation the wave produces. It is fortunate from the analytical point of view that for the threshold fields of many interesting nonlinear effects this ratio is very much less than unity and hence such interactions can be described with the aid of perturbation analysis.

Let us now write the dynamical equations with this in mind. Again splitting the fields into an equilibrium part and an oscillating part we write the equations with the linear terms on the left and the nonlinear terms on the right.

\[
\begin{align*}
\frac{\partial v_e}{\partial t} + v_e \frac{\varepsilon_0}{n_0 e} u_1 v_e - 1 + \frac{v_e}{m} E_1 &= -e \frac{v_e}{m} e_1 \times B_1 - (v_e v_e)v_e \\
&+ \frac{\varepsilon_0}{n_0 m} n_0 v_e^n e_1 \\
\frac{\partial n_1}{\partial t} + n_0 \varepsilon_0 v_e v_e &= -\varepsilon_0 (n_0 v_e v_e) \\
\frac{\partial B_1}{\partial t} - \frac{1}{c^2} \frac{\partial E_1}{\partial t} + e \mu_0 n_0 v_e &= -e \mu_0 n_0 v_e e_1 \\
\frac{\partial E_1}{\partial t} + \frac{\partial B_1}{\partial t} &= 0.
\end{align*}
\]

(5.1)

(5.2)

(5.3)

(5.4)

We know that the linear terms on the left hand side will describe the linear, high frequency waves of a field-free plasma so that the quadratic terms on the right hand sides which are to be treated as perturbations will provide coupling between the modes. We shall begin by assuming that, initially, there is an incident, finite-amplitude electromagnetic
wave present whose electric field is $E_{0}$, magnetic field $B_{0}$, and wave vector $k_{0}$.

We choose the following linear polarization

$$E_{0} = (0, E_{0}, 0)$$
$$B_{0} = (0, 0, B_{0})$$
$$k_{0} = (k_{0}, 0, 0)$$

The frequency $\omega_{0}$ is then given by

$$\omega_{0}^{2} = \omega_{e}^{2} + c^{2}k_{0}^{2}$$

The incident wave is a travelling wave varying as

$$\exp i(k_{0}x - \omega_{0}t)$$

The electron fluid velocity associated with this wave is obtained from the linearized version of equation (5.1) and is

$$v_{0} = eE_{0}/i\omega_{0}$$

(5.5)

where we have neglected $v_{e}$ (the effect of $v_{e}$ will come in later). So far this is just the usual linear theory.

We must now obtain the modified equations for a Langmuir wave and another electromagnetic wave of a frequency different from $\omega_{0}$ due to the presence of the finite amplitude wave $(\omega_{0}, k_{0})$. We shall find that these waves are no longer independent but are coupled together due to the presence of the finite amplitude wave.

First let us obtain the equation of a transverse perturbation which we assume to have the same polarization as the incident wave. Taking the $y$-components of (5.1) and (5.3) and the $z$-component of (5.4) and assuming the perturbation varies as $\exp i(kx - \omega t)$, we obtain

$$(\omega^{2} - \omega_{e}^{2} - c^{2}k^{2})E_{y} = -iv_{e}E_{0} \frac{\partial E_{y}}{\partial y} - \omega_{e}^{2}v_{e}B_{1y} + \frac{en_{0}}{\varepsilon_{0}}v_{e}B_{0y} + \frac{ie\omega}{\varepsilon_{0}}n_{0}E_{1y}$$

(5.6)

The collisional term can now be seen to produce a linear damping on the transverse wave and the three quadratic terms on the right hand side of (5.6) are coupling terms. We shall treat the damping and coupling terms as small perturbations to the linear waves whose dispersion relation, in the absence of the pump wave, is given by the left hand side of
[5.6]. Let us now write the electric field of the transverse perturbation as $E_{T1}$ to distinguish it from the incident wave. Since the transverse perturbation is subject to coupling to other modes we cannot assume its amplitude will remain constant. However, since the coupling is assumed to be weak we may assume that $E_{T1}$ can be written as

$$E_{T1}(x,t) = \text{Re}\{A_{T1}(t) \exp \{i(k_{T1} x - \omega_{T1} t)\}\}$$

where

$$\omega_{T1}^2 = \omega^2 + c^2 k_T^2$$

and $A_{T1}(t)$ varies slowly in time in comparison with the rapidly varying linear phase. We must also be careful to take only real parts since the wave equation now contains products of complex amplitudes.

It is convenient to express all transverse quantities in terms of the electric field. Since we are performing a perturbation analysis we can relate $B_T$ and $v_T$ to $E_T$ by means of the linear equations. We then obtain

$$B_T = \frac{k_T}{\omega_T} E_T$$

$$v_T = -\frac{i e E_T}{\omega_T}$$

where $B_T$ and $v_T$ are written instead of $B_{1x}$ and $v_{1y}$ and may represent either the incident or perturbed transverse wave. Since we are describing a process in which an incident transverse wave $(T)$ is transformed into a scattered transverse wave $(T')$ and a Langmuir wave $L$

i.e. $T = T' + L$

we shall refer to the perturbed electromagnetic wave as the scattered wave. The coupling terms on the right hand side of [5.6] must consist of products of a transverse field and a Langmuir field. It is clear that the Langmuir fields are $v_{1x}$ and $n_{e1}$. We shall represent the Langmuir wave amplitude by its electric field $E_{1x}$ which we denote by $E_L$ and again expressing $v_{1x}$ and $n_{e1}$ in terms of $E_L$ by means of the linear equations we obtain

$$v_L = -\frac{i \omega c}{en_0} E_L$$

$$n_L = -\frac{ik c}{e} E_L$$

We may now obtain a non-linear differential equation for the scattered electromagnetic wave by expanding [5.6] about the linear solution and identifying $\omega$ with $i\partial/\partial t$ to obtain
\[ \exp \left\{ i(k_{T0}x - \omega_{T0}t) \right\} \frac{\partial \epsilon_{T1}}{\partial t} = -i \nu \frac{\omega^2}{\omega_{T1}^2} e_{T1} + \frac{i}{2} \frac{\epsilon_{T0} k_{L}}{\omega_{0} m e T0} \epsilon_{L1} \epsilon_{T0} \]  \hfill (5.12)

In order to obtain the final form of this equation we put

\[ E_{T0} = \text{Re} \left[ \epsilon_{T0} \exp i(k_{T0} - \omega_{T0} t) \right] \]  \hfill (5.13)

and write \( E_L \) as the product of a slowly varying amplitude and the linear phase

\[ E_L(x,t) = \text{Re} \left[ \epsilon_L(t) \exp i(k_L x - \omega_L t) \right] \]  \hfill (5.14)

The only significant nonlinear coupling terms will be those which satisfy the frequency and wave number matching conditions

\[ \omega_{T0} = \omega_{T1} + \omega_L \]  \hfill (5.15)

\[ k_{T0} = k_{T1} + k_L \]  \hfill (5.16)

whose physical interpretation is the conservation of energy and momentum. Using equations (5.13)-(5.16) in equation (5.12) we obtain the equation for \( \epsilon_{T1} \)

\[ \frac{\partial \epsilon_{T1}}{\partial t} + \frac{\nu}{2} \frac{\omega^2}{\omega_{T1}^2} \epsilon_{T1} = \frac{1}{4} \frac{ek_L}{\omega_{T0}^2 \omega_0 m} e_L \epsilon_{T0}^{*} e L^{*} e^{-i\psi t} \]  \hfill (5.17)

where we have imposed perfect k-matching but have allowed for a small frequency mismatch \( \psi = \omega_{T0} - \omega_{T1} - \omega_L \).

The equation for \( E_L \) is obtained in a similar way. Again assuming \( \exp i(kx - \omega t) \) dependence and taking the x-components of (5.1) and (5.3) together with equation (5.2) we obtain

\[ \left( \omega^2 - \omega^2_{pe} - v e k^2 \nu_2 \right) E_L = -i \nu \omega \epsilon_L + \omega^2 \nu \chi B_{Lz} \]  \hfill (5.18)

In this case the only coupling term comes from the \( \nu \times B \) force in the momentum equation since this is the only term which consists of a product of transverse fields. Expanding (5.18) about the linear solution, using (5.10) and (5.11) and imposing the matching relations (5.15) and (5.16) we obtain the equation for \( \epsilon_{L} \)

\[ \frac{\partial \epsilon_{L}}{\partial t} + \frac{\nu}{2} \epsilon_{L} = \frac{1}{4} \frac{e \omega^2}{\omega_{T0}^2 \omega_0} \frac{k_L}{\omega_{T1}} e L^{*} e^{-i\psi t} \]  \hfill (5.19)

Equations (5.17) and (5.19) describe the coupling between high frequency transverse and longitudinal perturbations in the presence of a finite amplitude transverse wave. Under these conditions we have \( \epsilon_{T0} \gg \epsilon_{T1} \) and \( \epsilon_{T0} \gg \epsilon_{L} \). We may therefore linearize...
equations (5.17) and (5.19) by assuming that the incident wave amplitude remains constant. Using $A_{T1}$ and $A_L^{-1}\psi t$ as the amplitudes the coupled equations then have constant coefficients so that a solution proportional to $\exp(-i\Omega t)$ may be assumed. The dispersion relation for the coupled waves is

$$\left(\Omega + i\gamma_L\right)(\Omega + i\gamma_T - \Omega) + C_{0L}C_{01}A_{T0}\bigg|_{T0}^2 = 0$$

(5.20)

where

$$\gamma_L = \frac{v_e}{2}, \quad \gamma_T = \frac{v_e^2}{2}\frac{\omega_T}{\omega_L^2}$$

$$C_{01} = \frac{e\omega_e^2 k_L}{4m_e \omega_T 0^2 \omega_L^1 \omega_L^1}$$

and

$$C_{0L} = \frac{e\kappa_L}{4m_e \omega_T^0}$$

If we neglect the damping terms and put $\psi = 0$ we obtain a growing solution with growth rate

$$\gamma = \left(C_{0L}C_{01}\right)^{1/2} A_{T0}$$

(5.21)

Putting $C_{0L}C_{01}A_{T0}\bigg|_{T0}^2 \equiv K$ we obtain the threshold condition for instability

$$K = \gamma_T \gamma_L + \frac{\phi^2 \gamma_T \gamma_L}{(\gamma_T + \gamma_L)^2}$$

(5.22)

Clearly, the minimum threshold occurs, as expected, for perfect frequency matching. For an incident wave whose amplitude exceeds the above threshold a scattered electromagnetic wave and a Langmuir wave would grow out of the background noise. In terms of the plasma parameters the threshold can be written as

$$\frac{\nu_0^2}{v_T^2} = \frac{4}{\nu_T^2} \frac{v_e^2 v_e}{k_L^2 \omega_L^1 \omega_T^1 \omega_L^1}$$

(5.23)

where $\lambda_{De} \equiv \nu_e / \omega_p e$ and $v_0$ is the velocity of the electron fluid in the field of the incident wave. For parameters typical of laser plasmas $v_0^2 / v_T^2 \ll 1$ in keeping with our perturbation analysis.

The above interaction is just one example of a whole class. We can use this example to illustrate some important conservation relations which such interactions satisfy. To do this we must first obtain the nonlinear equation for the incident wave $A_{T0}$ whose amplitude is no longer assumed constant as we must relax the constraint that $A_{T0} \gg A_T$ and $A_L$. The equation for $A_{T0}$ can be obtained from equation (5.6) and is
\[
\frac{\partial a_{T0}}{\partial t} + \frac{v_e}{2} \frac{\omega_p^2}{\omega_{T0}^2} a_{T0} = \frac{1}{4} \frac{e k_e}{\omega_{T1}^2 m_e} A T L T1 e^{i \omega t}. \tag{5.24}
\]

Equations (5.17), (5.19) and (5.24) now form a closed system of coupled nonlinear differential equations for the three waves, two transverse and one Langmuir. Using the result that the wave energy densities of transverse and Langmuir waves are given by

\[
\rho_T = \frac{1}{2} \epsilon_0 \left| E_T \right|^2
\]
\[
\rho_L = \frac{1}{2} \epsilon_0 \left| E_L \right|^2 \frac{\omega_L^2}{\omega_{pe}^2}
\]

we normalize the wave amplitudes to the total energy density carried by each mode as follows

\[
a_{T0,1} = \left( \frac{\epsilon_0}{2} \right)^{1/2} \frac{\epsilon_0}{\omega_{T0,1}} a_{T0,1}
\]
\[
a_{L} = \left( \frac{\epsilon_0}{2} \right)^{1/2} \frac{\omega_L}{\omega_{pe}} a_L
\]

Neglecting the damping terms and assuming perfect matching the equations for the interacting waves are

\[
\frac{\partial a_{T0}}{\partial t} = -\Gamma a_{T0} a_{T1} a_L
\]
\[
\frac{\partial a_{T1}}{\partial t} = \Gamma a_{T1} a_{T0} a_L^*
\]
\[
\frac{\partial a_{L}}{\partial t} = \Gamma a_{L} a_{T0} a_{T1}^*
\]

where

\[
\Gamma = \left( \frac{2}{\epsilon_0} \right)^{1/2} \frac{ek_e \omega_{pe}}{4m \omega_{T1} \omega_{T0} \omega_L}
\]

With the aid of equations [5.29] - [5.31] it is straightforward to show that

\[
\frac{\partial}{\partial t} \left( \left| a_{T0} \right|^2 + \left| a_{T1} \right|^2 + \left| a_L \right|^2 \right) = 0
\]

which corresponds to the conservation of energy for the interacting waves.

A more revealing result concerns the wave action density \( |a_n|^2 / \omega_n \). Calculating the rate of change of this quantity from equations (5.29)-(5.31) we obtain the Manley-Rowe relations (first discussed in the field of electronics)
which shows in what proportion the energy from one wave is divided between the other two - the lowest-frequency wave receives the least energy. The coupled equations can also be used, with the aid of perfect wave-number matching, to demonstrate the conservation of momentum in the interaction. These general conservation relations are characteristic of all three wave interactions.

5.1 The heat-wave accelerator

To conclude this discussion we will consider the application of this interaction to the acceleration of electrons. We noted earlier that the longitudinal Langmuir wave could propagate with a wide range of phase velocities both greater and much less than the velocity of light. Since the Langmuir wave is longitudinal it can accelerate electrons provided it has a finite amplitude since an electron travelling close to the phase speed of the wave will be trapped in the potential well of the wave. An electron with the correct phase will then be accelerated from the bottom of the potential well to the top gaining energy from the wave. If two laser signals are launched into a plasma such that they differ in frequency by the Langmuir wave frequency their beating will generate a Langmuir wave. The energy transferred to the Langmuir wave will be subject to the constraint given by the Manley-Rowe relation. The three wave interaction can also be represented on an ω-k diagram which is shown in figures 4a,b.

![Fig. 4 Nonlinear coupling of two transverse waves with Langmuir wave](image)

(a) \( \omega_{T1} = \omega_{pe} \), (b) \( \omega_{T1} >> \omega_{pe} \)

Figure 4a is for the case where the lower frequency laser is comparable to the plasma frequency and figure 4b where the lower laser frequency is much larger than the plasma frequency. These two cases correspond to high or low-density plasmas respectively. In the first case more energy would be coupled to the Langmuir wave but it would be excited at phase velocities below the speed of light whereas for the second case the Langmuir wave would travel at or near the speed of light but would receive proportionately less of the laser energy and so would require more power to generate the same strength Langmuir wave.
Despite the constraint imposed by the Manley-Rowe relation the second case is the favoured one for the beat wave accelerator since all three waves travel close to the speed of light. In any case the practical situation is more complicated since both laser fields can act as pumps thus giving rise to further waves. In addition, since $\omega_L \ll \omega_T$, not only the down-shifted wave but also the up-shifted one, $\omega_T + \omega_L$, $k_T + k_L$, must be included. The division of energy is now more complicated. However, no matter how many additional couplings are included the basic interaction is still given by the previous discussion of Raman scattering.

To give some idea of the possibilities offered by the beat wave accelerator we use present neodymium-glass lasers as an example. These lasers can presently deliver $5 \times 10^{13}$ watts. For two such lasers with beam widths ~ 1 mm the intensity ~ $10^{16}$ watts/cm$^2$. The generated Langmuir wave can then have longitudinal electric fields ~ $10^8$ volts/cm which assumes a perturbed electron density $n/n_0$ ~ 0.1 in a plasma where $n_0$ is in the range $10^{16} - 10^{18}$ cm$^{-3}$. If the beat wave could be maintained over a length ~ 5 m then an electron injected with 1 MeV energy could, in principle, be accelerated up to 50 GeV in the above distance. However, present experiments are only in the very early stages. In particular, Langmuir wave electric fields ~ $10^8$ volts/cm have been generated but only over a distance of 1 mm. The reader is referred to the original work for a detailed discussion of the above points.

* * *

REFERENCES

The following are a selection of references on waves in plasmas:
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The original idea of the beat wave accelerator is due to:

Other references on the subject are:
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