Shielding Particle Beams By Thin Conductors

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Abstract

We start with previous results [1] for the longitudinal impedance of a wire cage within a cylindrical pipe and generalise the outer boundary condition to include the case of a conducting pipe and of a ferrite pipe. The results are then interpreted by means of an equivalent circuit which makes self-evident the conditions for effective shielding of the outer wall by a cage of wires or wire stripes.

1 Introduction

In a previous report [1], we analyzed the shielding of a charged particle beam by a thin conducting layer or a cage of wires with finite conductivity. In the present report, we restate the results and present an equivalent circuit which allows for a more transparent interpretation of the final analytic result.

Our starting point are Eqs. (3.9) and (4.9) of [1]. Specifically, for a thin conducting layer of thickness \( \tau \ll b \) at \( r = b \) inside a pipe with radius \( d \), we showed that

\[
\frac{Z_{\parallel}(\omega)}{nZ_0} = -\frac{j}{\beta\gamma^2} \left[ \ln \frac{b}{a} + \frac{L}{1 - \eta L} \right],
\]

(1.1)
where \( a \) is the beam radius, \( Z_0 = \mu_0 c \) the free-space impedance, and

\[
\eta \equiv \frac{2j\tau b}{\beta^2 \gamma^2 \delta_b^2} \tag{1.2}
\]

\[
L \equiv \ln(d/b) - \xi_d \tag{1.3}
\]

\[
\xi_d \equiv (1 - j)\beta^2 \gamma^2 \delta_d / 2d \tag{1.4}
\]

Here \( \beta = v/c, \gamma = (1 - \beta^2)^{-1/2} \), and the skin depths \( \delta_b \) and \( \delta_d \) are defined by

\[
\delta_b^2 = \frac{2}{\omega \mu \sigma_b}, \quad \delta_d^2 = \frac{2}{\omega \mu \sigma_d}, \tag{1.5}
\]

where \( \sigma_b \) and \( \sigma_d \) are the conductivities of the thin shield and outer wall, respectively. The second term in Eq. (1.1) can be rewritten as

\[
-\frac{j}{\beta \gamma^2} \frac{L(1 - \eta^* L^*)}{|1 - \eta L|^2} \tag{1.6}
\]

so that

\[
\frac{\text{Re} \ Z_{\parallel}(\omega)}{nZ_0} = \frac{1}{|1 - \eta L|^2} \left[ \text{Im} \ L + |L|^2 \right. \left. \text{Im} \ \eta \right] \tag{1.7}
\]

Since one can show from Eqs. (1.2) and (1.4) that

\[
\text{Im} \ L = -\text{Im} \ \xi_d > 0 \quad \text{and} \quad \text{Im} \ \eta > 0, \tag{1.8}
\]

it is clear from Eq. (1.7) that \( \text{Re} \ Z_{\parallel}(\omega) > 0 \), as it must be to satisfy causality.

The corresponding result for a wire cage is given in Eq. (4.9) of [1], and can be written as

\[
\frac{Z_{\parallel}(\omega)}{nZ_0} = -\frac{j}{\beta \gamma^2} \left[ \ln \frac{b}{a} + \frac{L}{1 - \eta_w L} \right], \tag{1.9}
\]

where we have again assumed that \( \Delta \), defined in Eq. (4.10) of [1], is negligible compared with 1. For \( N \) equidistant wires of radius \( r_w \)

\[
\eta_w = \frac{jN r_w^2}{\beta^2 \gamma^2 \delta_b^2}. \tag{1.10}
\]

Since the cross sectional area of the conducting shield or cage wires is

\[
A_{\text{shield}} = 2\pi b \tau, \quad A_{\text{wires}} = N \pi r_w^2, \tag{1.11}
\]
we reached the important conclusion [1] that $\eta$ and $\eta_0$ were the same, implying that the effectiveness of the shield depended only on its cross sectional area. This clearly includes the alternative of regularly spaced conducting stripes. If the stripes are on a ceramic wall, this same conclusion should apply for frequencies below the resonances of the ceramic wall.

2 Equivalent Circuit

Consider the circuit shown in Fig. 1.

![Equivalent circuit](image)

Figure 1: Equivalent circuit for the total impedance.

The total impedance is given by

$$Z^T = Z^{(1)} + \frac{1}{Z^{(2)}} + \frac{1}{(Z^{(3)} + Z^{(4)})}.$$  

(2.1)

Comparison with Eq. (1.1) suggests the identification

$$Z^{(1)} = -\frac{jnZ_0}{\beta\gamma^2} \ln \frac{b}{a}, \quad (2.2)$$

$$Z^{(2)} = \frac{jnZ_0}{\beta\gamma^2\eta}, \quad (2.3)$$

$$Z^{(3)} = -\frac{jnZ_0}{\beta\gamma^2} \ln \frac{d}{b}, \quad (2.4)$$

$$Z^{(4)} = \frac{jnZ_0}{\beta\gamma^2} \zeta_d. \quad (2.5)$$
The term $Z^{(1)}$ is simply the direct space charge impedance of a perfectly conducting cylinder of radius $b$, and $Z^{(3)}$ is the supplement to this space charge impedance in the absence of a conducting shield. We re-express $Z^{(2)}$ and $Z^{(4)}$ as follows:

$$n \equiv \omega/\omega_0.$$  \hspace{1cm} (2.6)

Then, using Eqs. (1.2) and (1.5),

$$Z^{(2)} = j(\omega/\omega_0)\mu c\beta^2\gamma^2 \frac{2}{\beta\gamma^2 j\tau b\omega b \sigma_b} = \frac{v}{\omega_0 \tau b \sigma_b}$$  \hspace{1cm} (2.7)

and

$$Z^{(4)} = j(\omega/\omega_0)\mu c(1-j)\beta^2\gamma^2 \frac{\delta_d}{\gamma^2 \cdot 2d} = \frac{2}{\omega_o} \frac{v \omega \mu \delta_d}{d}.$$  \hspace{1cm} (2.8)

If we denote the impedance of the conducting pipe at $r = d$ by

$$Z_d \cong \sqrt{j\omega \mu / \sigma_d} = \frac{1 + j}{2} \omega \mu \delta_d,$$  \hspace{1cm} (2.9)

we can write

$$Z^{(4)} = \frac{v}{\omega_0 d} Z_d.$$  \hspace{1cm} (2.10)

Considering a ring of radius $R = \mathcal{L}/2\pi$, we can rewrite $Z^{(2)}$ and $Z^{(4)}$ as

$$Z^{(2)} = \frac{\mathcal{L}}{2\pi b \tau \sigma_b} = R_b,$$  \hspace{1cm} (2.11)

$$Z^{(4)} = \frac{\mathcal{L}}{2\pi d} Z_d,$$  \hspace{1cm} (2.12)

where $R_b$ is the resistance of a cylinder of radius $b$, thickness $\tau$, length $\mathcal{L}$ and conductivity $\sigma_b$.

The equivalent circuit in Fig. 1 is now easily interpreted:

1. If the shield is totally effective, $Z^{(2)}$ is small and the impedance consists of the space charge impedance of the pipe of radius $b$ in series with the resistance of the shield.

2. If the shield is ineffective and $Z^{(2)}$ is large, the impedance consists of the space charge impedance of the outer pipe of radius $d$ in series with an effective wall impedance.
An important point to note: $Z^{(2)}$ depends only on the properties of the shield, and $Z^{(4)}$ depends only on the properties of the outer wall.

In our previous results [1], we have calculated $Z_{\parallel}(\omega)/nZ_0$, where we assumed that $Z_0 = \sqrt{\mu/\epsilon}$ referred to the vacuum values of $\mu$ and $\epsilon$. For materials for which $\mu$ differs from its vacuum value it is clear that $\mu$ should correspond to the medium to which it refers. Specifically the $\mu$ used to evaluate Eq. (2.7) is the $\mu$ of the shielding material and the $\mu$ used in Eq. (2.9) is the $\mu$ of the outer wall. We avoid ambiguity by presenting results for the impedance itself rather than for $Z_{\parallel}(\omega)/Z_0$.

An additional observation is that $Z^{(3)}$ depends on $\sigma_b$ and $Z^{(4)}$ on $\sigma_d$. If $\omega\epsilon/\sigma$ is not small compared to 1, one must include the displacement current by the replacement

$$\sigma \rightarrow \sigma + j\omega\epsilon.$$  

(2.13)

### 3 Derivation of Eq. (1.9)

In this section we present some of the details in the derivation of Eq. (4.9) of [1]. We do this for general boundary conditions at $r = d$ which we will later use for a ferrite boundary.

We start with Eqs. (4.1) and (4.2) of [1]. In the absence of the wires, we have

$$E_z(r, z) = A e^{-jkz}\{G_0(\nu n) \equiv [K_0(\nu r) + \alpha I_0(\nu r)]\},$$  

(3.1)

$$H\theta(r, z) = \frac{j\beta\gamma}{Z_0} A e^{-jkz}\{G'_0(\nu n) \equiv [K'_0(\nu r) + \alpha I'_0(\nu r)]\},$$  

(3.2)

where $\alpha$ will be determined later to satisfy the appropriate boundary condition at $r = d$. When we add the wires, we replace $G_0(\nu r)$ and $G'_0(\nu r)$, as outlined in Eqs. (4.4) and (4.5) of [1], by

$$G_0(\nu r) \rightarrow G_0(\nu r) + \sum_{p=0}^{N-1} 2I \sum_{n=1}^{\infty} \left\{ \begin{array}{cc} I_n(\nu r) & G_n(\nu b) \\ I_n(\nu b) & G_n(\nu r) \end{array} \right\} \cos n(\theta - \theta_p) + I \sum_{p=0}^{N-1} \left\{ \begin{array}{cc} I_0(\nu r) & G_0(\nu b) \\ I_0(\nu b) & G_0(\nu r) \end{array} \right\},$$  

(3.3)

$$G'_0(\nu r) \rightarrow G'_0(\nu r) + \sum_{p=0}^{N-1} 2I \sum_{n=1}^{\infty} \left\{ \begin{array}{cc} I'_n(\nu r) & G'_n(\nu b) \\ I'_n(\nu b) & G'_n(\nu r) \end{array} \right\} \cos n(\theta - \theta_p)$$
\[ I_N = I_0^r G_0^b \left\{ \begin{array}{l}
I_0^r G_0^b \\
I_0^b G_0^r
\end{array} \right\}, \]

where the upper and lower terms in each brace \( \{ \} \) correspond to \( r \leq b \) and \( r \geq b \) respectively. Here \( I_0^e \) is a normalized current in each wire of the cage whose relation to \( I_{\text{wire}} \) can be specified by referring to Eq. (4.3) of [1]. Specifically, the magnetic field near a wire is given by

\[ H_\theta \sim \frac{j \beta \gamma}{Z_0} A e^{-j \varepsilon} \varepsilon K_0^r (\nu \| \mathbf{r} - \mathbf{r}_p \|) \approx -\frac{j \beta \gamma}{Z_0 \nu r_w} A e^{-j \varepsilon} \varepsilon \]

using

\[ H_\theta = \frac{I_{\text{wire}}}{2 \pi r_w} \]

leads to

\[ I_{\text{wire}} = -\frac{2 \pi j \beta \gamma}{Z_0 \nu} A e^{-j \varepsilon} \varepsilon = -\frac{2 \pi j \beta \gamma^2}{\omega \mu} A e^{-j \varepsilon} \varepsilon. \]

The analysis is continued in the Appendix.

### 4 Conducting Wall at \( r = d \)

When there is a conducting wall at \( r = d \), we have

\[ Z_d \approx \frac{j \omega \mu_d}{\sigma_d} = \frac{1 + j}{2} \omega \mu_d \sigma_d. \]

In the Appendix, we show in Eq. (A.19) that \( Z^{(4)} \) is equivalent to the parallel combination of circuit elements shown in Fig. 2. If the wall is a good conductor, the impedance of the capacitor on the right hand side of Fig. 2 can be considered infinite, except at very high frequencies.

### 5 Ferrite Wall at \( r = d \)

If we have a ferrite wall at \( r = d \), we have

\[ Z_d = \frac{\mu_d}{\varepsilon_d} = \frac{\mu_f}{\varepsilon_f}, \]
where, for a lossy ferrite, \( \mu_f = \mu_f' - j\mu_f'' \) becomes complex. The fields in the ferrite are proportional to

\[
E_z = B e^{-jkz} H_0^{(2)}(\kappa r)
\]

\[
H_\theta = -\frac{j\omega \epsilon_f}{\kappa} B e^{-jkz} H_0^{(2)\prime}(\kappa r)
\]

\[
E_r = -\frac{jk}{\kappa} B e^{-jkz} H_0^{(2)\prime}(\kappa r),
\]

corresponding to a wall impedance

\[
\frac{E_z}{H_\theta} = \frac{\kappa}{j\omega \epsilon_f} \frac{H_0^{(2)}(\kappa d)}{H_0^{(2)\prime}(\kappa d)},
\]

where \( H^{(2)} \) is the Hankel function of the second kind, and where

\[
\kappa^2 = \omega^2 \mu_f \epsilon_f - \omega^2/c^2.
\]

For \( |\mu_f \epsilon_f| \gg 1 \),

\[
\kappa = \omega \sqrt{\mu_f \epsilon_f}.
\]

For \( |\kappa|d \gg 1 \), one uses the asymptotic form for the Hankel function, leading to

\[
\frac{E_z}{H_\theta} = \frac{\kappa}{\omega \epsilon_f}.
\]
For smaller $\omega$ this must be modified according to Eqs. (5.5) and (5.6). This applies as long as the frequency is not so low that the skin depth exceeds the thickness of the ferrite.

The corresponding equivalent circuit for $Z^{(4)}$ is therefore the one shown in Fig. 3. The capacitance in parallel to the resistive wall impedance is again very small as it is divided by the length $L$.

\begin{align*}
Z^{(4)} &\equiv (\frac{\mu_f}{\epsilon_f})^{1/2} \quad \text{C} = \frac{\pi \epsilon_o d^2}{L} 
\end{align*}

Figure 3: Equivalent circuit for $Z^{(4)}$.

Once again, the capacitive elements in Figs. 2 and 3 can be considered open, and can thus be neglected except at very high frequencies.

6 Conclusions

We have derived an equivalent circuit diagram for the impedance of a wire cage within an outer pipe, which may be either conducting or made of Ferrite. This diagram can be used to understand the effectiveness of shielding the beam fields from the region outside the cage.
References

Appendix

We here continue the detailed analysis starting in Section 3.

The functions $G_n(\nu b)$ are given by

$$G_n(\nu b) = K_n(\nu b) + \alpha_n I_n(\nu b), \quad (A.1)$$

where the $\alpha_n$ are to be chosen so that each harmonic term satisfies the appropriate boundary condition at $r = d$.

The impedance is directly related to the bracket $[\ ]$ in Eq. (3.1) evaluated at $r = a$. In this case, all terms with $n \neq 0$ can be neglected and we obtain

$$\frac{Z_{\parallel}(\omega)}{nZ_0} = -\frac{j}{\beta \gamma^2}[G_0(\nu a) + N I G_0(\nu b)], \quad (A.2)$$

where we take $I_0(\nu a) = I_0(\nu b) = 1$ for $\nu b = \omega b / \beta \gamma c \ll 1$. Using

$$G_0(\nu a) - G_0(\nu b) \cong \ln b/a, \quad (A.3)$$

we find

$$\frac{Z_{\parallel}(\omega)}{nZ_0} \cong -\frac{j}{\beta \gamma^2} \left[ \ln(b/a) + G_0(\nu b)(1 + NI) \right]. \quad (A.4)$$

In order to determine $I$, we require

$$I_{\text{wire}} = \pi r_w^2 \sigma_b E_z(r = b, z, \theta = 0) \quad (A.5)$$

in each wire. From Eqs. (3.1), (3.3), and (3.7) we find

$$-\frac{2j \beta^2 \gamma^2 I}{r_w^2 \omega \mu \sigma_b} = G_0(\nu b) + 2I \sum_{n=1}^{\infty} I_n(\nu b) G_n(\nu b) \sum_{p=1}^{N-1} \cos n\theta_p + (N - 1) I G_0(\nu b), \quad (A.6)$$

where we have excluded the wire $p = 0$ in evaluating $E_z(r = b, z, \theta = 0)$. The contribution of the current in the wire $p = 0$ has been excluded in obtaining $E_z(r = b, z, \theta = 0)$ since our model consists of infinitesimally thin wires located at $r = b$, $\theta_p = 2\pi p/N$, and the wire at $p = 0$ contributes only to $E_r$ and not to $E_z$ at $\theta = 0$. The sum over $p$ in Eq. (A.6) is therefore taken from $p = 1$ to $p = N - 1$, and is

$$\sum_{p=1}^{N-1} \cos n\theta_p \left\{ \begin{array}{ll} N, & n = \ell N, \quad \ell = 1, 2, \cdots \\ 0, & \text{all other } n \end{array} \right\} - 1. \quad (A.7)$$
Since
\[ I_n(\nu b)G_n(\nu b) \to \frac{1}{2n} = \frac{1}{2N\ell} \] (A.8)
for large \( N \), the term involving \( p \) in Eq. (A.6) is negligible compared to the last term, and we get
\[ I \approx -\frac{G_0(\nu b)}{j\beta^2\gamma^2 \delta \nu^2 / \eta^2 + NG_0(\nu b)} \] (A.9)

We therefore find, from Eqs. (A.4), (A.9), and (1.10),
\[ \frac{Z_{\parallel}(\omega)}{nZ_0} \approx -\frac{j}{\beta\gamma^2} \left[ \ln \frac{b}{a} + \frac{G_0(\nu b)}{1 - \eta_0 G_0(\nu b)} \right] \] (A.10)

If we write
\[ G_0(\nu b) = K_0(\nu b) + \alpha I_0(\nu b) \cong \ln(d/b) + G_0(\nu d) \] (A.11)
we obtain the equivalent circuit shown in Fig. 1 with \( Z^{(1)} \), \( Z^{(2)} \) and \( Z^{(3)} \) given in Eqs. (2.2), (2.11), (2.4) and with Eq. (2.5) replaced by
\[ Z^{(4)} = -\frac{jnZ_0}{\beta\gamma^2} G_b(\nu d) \] (A.12)

From Eqs. (3.1) and (3.2) we obtain the impedance at \( r = d \) as
\[ Z_d = \frac{E_z(d, z)}{H_\theta(d, z)} = -\frac{Z_0}{j\beta\gamma} \frac{K_0(\nu d) + \alpha I_0(\nu d)}{K_0'(\nu d) + \alpha I_0'(\nu d)} \] (A.13)
or
\[ \frac{K_0(\nu d) + \alpha I_0(\nu d)}{K_0'(\nu d) + \alpha I_0'(\nu d)} = -\frac{j\beta\gamma Z_d/Z_0}{Z_0} \] (A.14)

From Eq. (A.14) we can obtain
\[ \frac{K_0(\nu d) + \alpha I_0(\nu d)}{K_0'(\nu d) - K_0(\nu d)(I_0'(\nu d)/I_0(\nu d))} = -\frac{j\beta\gamma Z_d/Z_0}{1 + j\beta\gamma(Z_d/Z_0)(I_0'(\nu d)/I_0(\nu d))} \] (A.15)

Using the Wronskian relation
\[ K_0(\nu d)I_0'(\nu d) - K_0'(\nu d)I_0(\nu d) = 1/\nu d \] (A.16)
we find, for $\nu d \ll 1$,
\[
G_0(\nu d) \equiv K_0(\nu d) + \alpha I_0(\nu d) \simeq \frac{j\beta \gamma(Z_d/Z_0)/\nu d}{1 + j\beta \gamma(Z_d/Z_0)(\nu d/2)}. \tag{A.17}
\]

Thus, we find, from Eqs. (A.12) and (A.15):
\[
Z^{(4)} = -\frac{(jnZ_0/\beta \gamma^2)j\beta \gamma(Z_d/Z_0)(\beta \gamma c/\omega d)}{1 + j(\omega d/2c)(Z_d/Z_0)} = \frac{(L/2\pi d)Z_d}{1 + j(\omega d/2c)Z_d/Z_0} \tag{A.18}
\]
or
\[
Z^{(4)} = \frac{1}{(2\pi d/LZ_d) + (j\omega \pi d^2/jcLZ_0)}. \tag{A.19}
\]

We thus conclude that $Z^{(4)}$ is the parallel combination of impedances
\[
\frac{LZ_d}{2\pi d} \text{ and } \frac{cLZ_0}{j\omega \pi d^2} = \frac{L}{j\pi \omega \epsilon_0 d^2}. \tag{A.20}
\]

Figure 4: Equivalent circuit for $Z^{(4)}$.

It is likely that the capacitance, whose “gap” is the circumference of the ring (!) is in effect an open circuit and that, for all intents and purposes
\[
Z^{(4)} \simeq \frac{L}{2\pi d} Z_d \tag{A.21}
\]
at all reasonable frequencies.