Power corrections in deep inelastic structure functions at large Bjorken $x^*$

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Abstract: We study power corrections to the spin-averaged structure functions $F_2$ and $F_L$ in the semi-inclusive region. The dramatic breakdown of the operator product expansion at large Bjorken $x$ due to the formation of a narrow jet with invariant mass $W$ ($W^2 = Q^2(1 - x)/x$) calls for an alternative approach. Our main conjecture is that the dominant contribution at each twist is the one which mixes under renormalization with the leading twist (ultraviolet dominance). At twist two the dominant perturbative corrections at large $x$ due to soft and collinear radiation can be factorized into a jet function. From the ultraviolet dominance conjecture it follows that the twist two parton distribution as well as the jet function actually factorize to any order in the twist expansion. This factorization suggests that non-perturbative corrections $\sim 1/W^{2n}$ exponentiate together with the leading twist. We verify explicitly, at the level of a single dressed gluon, the cancellation between ambiguities due to infrared renormalons in the twist-two coefficient functions and those due to the ultraviolet divergence of twist-four matrix elements. Independently of the renormalon analysis, we show that the dominant contribution to twist four at large $x$ is associated with a twist-two like configuration: the final states are identical to those of twist two, whereas the initial states differ just by additional partons carrying small momentum fractions. This picture is consistent with the ultraviolet dominance conjecture.

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1 Introduction

The phenomenological study of the structure functions of deep inelastic scattering (DIS) is well developed [1] and its role in future colliders can hardly be overestimated [2]. One of the gaps in the state-of-the-art QCD description of the structure functions is the region of large Bjorken $x$, the so-called semi-inclusive region [1, 3]. The two limitations are insufficient data and lack of pure theoretical understanding of this region. The QCD analysis shows large perturbative and non-perturbative corrections which make the parton model inapplicable at large $x$.

In general, the Operator Product Expansion (OPE) [4] provides a well-established framework for analyzing the structure functions in QCD. It allows one to expand the moments of the structure functions $F_a$, where $a = L, 2$, in inverse powers of the hard scale $1/Q^2$,

$$
\int_0^1 dx \ x^{N-2} F_a(x, Q^2) = \sum_{i=q,g} \tilde{C}_{i,a}^i(N, \alpha_s) \langle O_i^{(2)}(N) \rangle + \sum_j \tilde{C}_{4,a}^j(N, \alpha_s) \langle O_j^{(4)}(N) \rangle + \ldots \quad (1)
$$

expressing each term in the expansion as a product of the matrix elements of certain local composite twist $-n$ operators between hadronic states, $\langle O_i^{(n)}(N) \rangle$, and the corresponding coefficient functions $\tilde{C}_{i,a}^i(N, \alpha_s)$. The latter can be systematically calculated in perturbative QCD, while the former bring a set of universal non-perturbative parameters, corresponding to generalized moments of some non-perturbative distributions.

The number of the matrix elements entering the expression for the $N$-th moment of the structure function (1) depends on the twist. The leading, twist-two flavour non-singlet, contribution to (1) depends, for arbitrary $N$, on just one non-perturbative parameter, the $N$-th moment of quark distribution within the hadron. The flavour singlet contribution depends on the $N$-th moment of both the quark and the gluon distributions. In any case, twist-two depends strictly on single particle distributions. The asymptotic behaviour of the structure function at large $x$ corresponds, in moment space, to large $N \sim 1/(1-x)$. An additional simplification occurs in this limit at twist two: the contribution of gluons to the structure functions becomes suppressed as compared with that of the quarks [5]–[7].

Going over to higher twists, one encounters a completely different situation: the number of matrix elements entering the twist-$n$ ($n \geq 4$) contribution depends on the moment $N$ and it grows rapidly with $N$. The reason is that the high-twist contribution depends on novel multi-particle distributions, which describe the correlations between partons in the nucleon and which do not have a simple partonic interpretation [8]–[11]. For different moments $N$, the high-twist matrix elements measure specific projections of the multi-parton distributions. One consequence is that the $Q^2$ dependence of the high-twist contribution to the structure functions is not described by the DGLAP evolution equations.

Examining the large-$x$ limit one finds that the twist expansion breaks down. The physical reason for this is that for large-$x$ kinematics the final state in the current
fragmentation region is bound to be a narrow jet with a square invariant mass $W^2 = Q^2 (1 - x)/x$ such that $W^2 \ll Q^2$. In such a situation, the long-distance effects of the jet formation dominate, in spite of the momentum transfer $Q^2$ being large: the twist expansion goes effectively in inverse powers of the smallest scale, $1/W^2$. The problem becomes acute when $W^2 \lesssim \text{few GeV}^2$.

The breakdown of the OPE at large $x$ is three-fold: the perturbative expansion breaks down, the power expansion breaks down, and the number of non-perturbative parameters increases. Perturbative corrections to the twist-two coefficient functions $\bar{C}^i_{2,a}(N, \alpha_s)$ is enhanced at large $N$ by Sudakov logarithms

$$\bar{C}^i_{2,a}(N, \alpha_s) \sim \sum_{m \geq 0} \alpha_s^m \left( a_m \ln^{2m} N + b_m \ln^{2m-1} N + \cdots \right).$$

(2)

These large corrections originate from an incomplete cancellation between the contribution of real and virtual soft gluons at large $x$. To obtain a reliable perturbative description of the coefficient functions, they need to be resummed to all orders in $\alpha_s$. Thanks to the factorization property of soft radiation such resummation can be performed [12]–[15] and it takes the form of exponentiation. One expects to find similar Sudakov corrections in the coefficient functions of higher twist, $\bar{C}^i_{n,a}(N, \alpha_s)$ with $n > 2$. Since higher-twist corrections scale as powers of $1/W^2 \sim N/Q^2$ rather than as $1/Q^2$, terms of arbitrarily high twist in (1) become relevant at large $N$. Consequently, the number of matrix elements involved increases sharply. This means that, without additional insight into the properties of the high twists, the OPE parameterization of non-perturbative corrections to the structure functions in the semi-inclusive region becomes impractical and their scale dependence is obscure.

In the absence of a practical OPE-based parametrization of power corrections to structure functions, various models were developed and applied [16]–[26]. In particular, the renormalon-based model became popular. Through the resummation of the factorially divergent perturbative series for the twist-two coefficient functions, infrared renormalons generate an ambiguous higher-twist contribution. The renormalon approach to power corrections [16] is based on the consistency of the theory as well as that of the OPE: all ambiguities in the resummed perturbative calculation must eventually cancel out. Therefore, the presence of ambiguities $\sim \mu^{2n}/Q^{2n}$ in the twist-two coefficient functions implies the existence of genuine power corrections of the same form, that is of the twist $-(2n + 2)$. In this way infrared renormalons in the twist-two coefficient functions detect some part of the higher twist contribution.

The main assumption made by the renormalon model (e.g. [21]–[23]) is that the power corrections accessible to renormalons dominate. A physical argument supporting this assumption was never provided. A further assumption which, in our view, makes this model inapplicable at large $x$, is that multi-gluon contributions are discarded by considering the large-$\beta_0$ limit [16]: in this approximation one resums a set of diagrams corresponding to a single dressed gluon. Since at large $x$ multiple gluon emission is crucial at the perturbative level (2), it is unlikely that power corrections would be entirely
associated with a single emission. Therefore, in contrast with the above mentioned
renormalon model, we shall not base the parametrization of power corrections at large $x$
on the single dressed gluon. In our approach multiple gluon emission plays a major role
also at the non-perturbative level.

The first part of this paper is devoted to show how the renormalon ambiguity cancels
out within the OPE, leading to unambiguous predictions to the structure functions. The
conjectured cancellation of infrared renormalons in physical observables is fundamental
to our understanding of power corrections in QCD. Nevertheless, not much has been done
to test it. In this context the example of deep inelastic scattering plays an important
role: since the DIS structure functions admit an OPE, the mechanism of cancellation
can be checked by explicit calculations. The case of the longitudinal structure function
$F_L(x) = F_1(x) - F_2(x)$ was analyzed in this spirit in Ref. [16, 17]. It was shown that the
renormalon ambiguity in the twist-two coefficient function indeed cancels against that of
the quadratic ultraviolet divergence of the twist-four matrix elements. The cancellation
of renormalon ambiguities for the $F_2(x)$ structure function by the same mechanism will
be demonstrated here.

The OPE is based on the separation of short and long-distance effects into the co-
efficient functions and hadronic matrix elements, respectively, by introducing the fac-
torization scale $\mu$. This scale provides an infrared cut-off for momenta of particles in
$\tilde{C}_{n,a}(N, \alpha_s)$ and, at the same time, it sets up the ultraviolet renormalization scale for
$\langle O^{(n)}_i(N) \rangle$. Although both the coefficient functions and the matrix elements have a
nontrivial dependence on $\mu$, this dependence cancels in the expression for the structure
functions (1). In particular, varying $\mu$ one finds that the twist-four operators mix among
themselves at the level of logarithmic ultraviolet divergences, $\sim \ln \mu^2$, as well as with
the operators of twist two at the level of quadratic ultraviolet divergences, $\sim \mu^2$. The
former divergences give rise to DGLAP evolution equations, while the latter introduce
an ambiguity in separating the contribution of twist two and twist four. This ambiguity
is compensated in (1) by $\sim \mu^2/Q^2$ contribution to the twist-two coefficient functions in-
duced by infrared renormalons. Thus, the cancellation of renormalon ambiguities follows
from the independence of the DIS structure functions of the factorization scale.

Let us now return to the renormalon model and address the meaning of the underlying
assumption, that is the dominance of those power corrections which are associated with
infrared renormalons. The predictive power of the model relies on the assertion that the
dependence of the renormalon ambiguity on the Bjorken variable $x$, or equivalently on
the moment $N$, represents that of the full higher-twist contribution. This is realized if the
contribution of the higher-twist matrix elements is dominated by the part that has the
highest possible ultraviolet divergence, and thus mixes under renormalization with the
leading twist. When viewed in the context of the OPE, the “renormalon dominance”
assumption is therefore understood as [27, 28] “ultraviolet dominance”. It was used
in [21]–[26], [18] to develop the phenomenology of power correction to the DIS structure
functions.

To leading order in the large-$\beta_0$ limit, i.e. at the level of a single dressed gluon,
the ambiguities in the twist-two coefficient functions of both $F_L(x)$ and $F_2(x)$ appear as power corrections $\sim \mu^2/Q^2$ and $\sim \mu^4/Q^4$ only. In the OPE, they are compensated by the contribution of the operators of twists four and six, respectively, that are given by the convolution of a specific function of $x$ with the twist two parton distributions. Then, the “renormalon dominance” assumption implies that

- Contributions to the structure functions of twist higher than six are suppressed, and can be ignored.

- The dominant twist-four and twist-six contributions are proportional to the twist-two distribution. Therefore, these contributions can be parameterized by a single non-perturbative parameter at each twist which controls the overall normalization.

Because of the small number of parameters, the renormalon-based model for power corrections seems to be very appealing phenomenologically. It is important, however, to keep in mind the strong assumptions involved. A priori, from the OPE point-of-view, ultraviolet dominance seems hard to justify. Higher-twist matrix elements measure multi-parton correlations within the hadronic target. They encode information on the detailed structure of the hadron, which is inaccessible at twist two. The renormalon model reduces this rich structure to an overall normalization of the twist-two matrix elements.

The main shortcoming of the renormalon model, which is particularly relevant at large $x$, can be understood through perturbation theory. The renormalon ambiguity described above emerges from a specific set of diagrams in which a single dressed gluon is emitted. These diagrams dominate the large-order behaviour of the series. However, it is well understood that at large $x$ multiple gluon emission is important: due to the phase-space constraint, i.e. the small invariant mass of the hadronic system, $W^2 \ll Q^2$, the emission of soft and collinear gluons is enhanced. In perturbation theory this results in the appearance of the Sudakov logarithms in the coefficient functions (2). These logarithms exponentiate in the Mellin transform (moment) space owing to the factorization property of soft and collinear gluons [12]–[15]. This implies that the large-$N$ asymptotic behaviour of the perturbative coefficients is controlled by multiple emission from the primary hard parton(s). An entirely different set of diagrams than the single dressed gluon.

Analyzing the properties of the resummed perturbative expressions, one finds that multiple soft gluon emission also affects the power corrections [29]–[31]. The systematic way see this is by considering the large-order behaviour of the Sudakov exponent [32]–[34]. It turns out that the coefficients of sub-leading Sudakov logarithms are enhanced factorially due to infrared renormalons. This implies that any Sudakov resummation with a fixed logarithmic accuracy is insufficient unless $W^2 \gg \Lambda^2$. In the large $x$ limit one is obliged to resum all the logarithms that are factorially enhanced. Here infrared renormalons hit again: the resummed exponent is defined only up to a power-suppressed ambiguity. The ambiguity must be compensated by genuine power corrections, which exponentiate together with resummed perturbative expansion.
The emerging structure of power corrections becomes quite different from the renormalon model: due to multiple emission all powers of $1/W^2 \sim N/Q^2$ are present:

$$\int_0^1 dx x^{N-2} F_a(x, Q^2) = \tilde{C}_{2,a}^q(N, \alpha_s) \langle O^{(2)}_q(N) \rangle \left[ 1 + \sum_{n \geq 1} \kappa_n^{(a)} \left( \frac{N\Lambda^2}{Q^2} \right)^n \right], \quad (3)$$

where $a = 2, L$ and $Q^2/N$ is fixed as $N \to \infty$, and terms which are suppressed by $1/N$ are neglected. Here, $\langle O^{(2)}_q(N) \rangle$ and $\tilde{C}_{2,a}^q(N, \alpha_s)$ are the twist-two quark-operator matrix elements (the gluon contribution is negligible) and the corresponding Sudakov- and renormalon-resummed coefficient functions, respectively, and $\kappa_n^{(a)} \Lambda^{2n}$ are $N$-independent non-perturbative scales related to hadronic matrix elements of twist-2($n + 1$) operators producing the dominant contribution as $x \to 1$.

Eventually, in order to describe the behaviour of the structure function (3) at large $x$, one has to resum an infinite set of power corrections on the scale $Q^2/N$. In the shape function approach [30, 31] the infinite set of scales $\kappa_n^{(a)} \Lambda^2$, with $n = 1, 2, \ldots$, is replaced by a single non-perturbative function. The series on the r.h.s. of (3) defines the Laplace transform of the shape function (see Eq. (57) below), which is a new non-perturbative distribution which governs the shape of the structure function at large $x$. The shape function approach has already proven useful in several processes, including Drell-Yan production [29, 34], and event-shape distributions [30]–[35].

Independently of the renormalon analysis, we examine in this paper the structure and the physical origin of the dominant power corrections at large $x$ within the framework of the OPE. At first sight, the increasing number of local matrix elements of higher-twist operators in the large-$N$ limit seems to be an obstacle to any use of the OPE to study this limit. Nevertheless, analyzing the twist-four contribution at leading order using its representation in terms of non-local light-cone operators [8]–[11] we find that, as at the leading twist, a significant simplification occurs in this limit.

Physically, this simplification occurs owing to the dominance of particular configurations in the final and initial states. The dominating final state turns out to be the same as at twist two. The initial state differs, of course: the quark–antiquark light-cone operator of twist two is replaced by a quark–gluon–antiquark operator at twist four. However, in the dominating configuration the gluon has a small longitudinal momentum fraction. Consequently, the corresponding twist-four matrix elements essentially depend only on the light-cone separation between the quark and antiquark, just as the twist-two matrix element.

We find that the relevant correlation measured by twist four at large $x$ corresponds to very specific configurations, which resemble twist two. This leads us to finally to the our main conjecture, that the simplification of higher twist at large $x$ amounts to ultraviolet dominance, namely the dominance of these operator matrix elements that mix under renormalization with the leading twist. Note that, in contrast with the renormalon model, we do not assume here the dominance of single gluon contributions; multi-gluon contributions are relevant. The ultraviolet dominance assumption implies that the twist
two matrix element factorizes out of the twist expansion at large $x$. Moreover, for the twist four contribution to be totally free of factorization-scale dependence, the resummed coefficient function must have the same logarithmic dependence on the factorization scale as the twist two coefficient function. Thus both the parton distribution function and the resummed coefficient function factorize, as summarized in (3).

This factorization implies, in particular, that the large $N$ limit has a remarkable feature: a subset of matrix elements which dominate the twist four contribution decouples from the rest and satisfies a closed set of evolution equations. In fact, such an approximation is known for the twist-three contribution to the spin structure functions of deep inelastic scattering [36], and we plan to extend it to twist four in a forthcoming publication.

We proceed as follows: in Section 2 we recall the standard formulation of the OPE, and then present the known results concerning renormalon ambiguities in the twist-two coefficient functions. In Section 3 we follow [11] describing twist four in terms of non-local light-cone operators and calculate their coefficient functions in moment space. We also examine the large-$x$ limit and show that the asymptotic behaviour of the twist four contribution in this limit is associated with specific final states. In Section 4 we verify explicitly the cancellation [16, 17] between $\sim 1/Q^2$ infrared renormalon ambiguities at twist two and the ambiguity in the definition of twist four matrix elements for both $F_L$ and $F_2$, to leading order in the flavour expansion. In Section 5 we return to the large $x$ limit and show that the physical picture emerging from general considerations at twist four is consistent with the ultraviolet dominance assumption. Our conclusions are summarised in Section 6.

2 Twist-two analysis and infrared renormalons

The spin-averaged structure functions, $F_L(x)$ and $F_2(x)$, of deep inelastic scattering of a virtual photon with momentum $q$ from a hadron with momentum $p$ are defined in terms of the hadronic tensor

$$W_{\mu\nu}(x,q^2) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{F_L(x)}{2x} - \left(g_{\mu\nu} - \frac{p_\mu q_\nu + p_\nu q_\mu}{(pq)} + \frac{q^2}{(pq)^2}p_\mu p_\nu\right)\frac{F_2(x)}{2x}, \quad (4)$$

with $x \equiv Q^2/2pq$ and $Q^2 = -q^2 > 0$. The latter is expressed through the imaginary part of the Fourier transform of the time-ordered ($T$) product of two electromagnetic currents, symmetrized over the Lorentz indices:

$$W_{\mu\nu}(x,q^2) = \frac{8}{\pi} \text{Im} \left\{ \int d^4y e^{iyq/p} |T [j_{\mu}(y)j_{\nu}\gamma_5(-y)] |p\rangle \langle p| \right\} \quad (5)$$

with $j_{\mu}(x) = \sum_{q=1}^{N_f} e_q \bar{\Psi}_q(x)\gamma_\mu \Psi_q(x)$ and $N_f$ being the number of quark flavours\(^1\). At large $Q^2$ the dominant contribution to (5) comes from the vicinity of the light-cone $y^2 \sim 1/Q^2$.

\(^1\)In what follows we shall assume for simplicity that quarks of different flavour have identical electric charges $e_q$. 

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This allows one to expand the $T$-product into the series of local composite operators with increasing twist (≡ dimension − spin), and, therefore, decreasing singularity for small $y^2$. A more efficient (although equivalent) way of writing the OPE is based on expanding the $T$-product in terms of non-local light-cone operators of increasing twist [11]. These operators serve as generating functionals for the infinite tower of high-twist local operators. This formulation of the OPE proves to be more convenient for our purpose.

To leading order in the coupling constant, the twist-two contribution to the OPE of the symmetric part of the $T$-product in (5) is given for $y^2 \to 0$ by [11]

$$iT \left[ j_{\mu}(y)j_{\nu}(-y) \right]^{tw-2} = -\frac{1}{16\pi^2 y^4} \left( y_\mu \partial_\nu + y_\nu \partial_\mu - g_{\mu\nu}(y_\alpha \partial_\alpha) \right)$$

$$\times \int_0^1 du \left[ 1 + \frac{1}{4} \ln u y^2 \partial_\beta^2 + \mathcal{O}(y^4) \right] \overline{\Psi}(uy)\hat{q}\Psi(-uy), \quad (6)$$

where $\partial \equiv \partial / \partial y$ and the sum over $N_f$ massless flavours is assumed. A path-ordered exponential between $-y$ and $y$ is required to make this operator gauge invariant. We do not write it explicitly here.

Going over to hadronic matrix elements, we introduce the standard twist-two quark distribution $q(\xi)$ by the relation

$$\langle p | \overline{\Psi}(y)\hat{q}\Psi(-y)|p \rangle_{\mu} = 2(py) \int_{-1}^1 d\xi \, e^{2\xi(py)} q(\xi, \mu^2), \quad (7)$$

where $y^2 = 0$ and $\mu$ defines the normalization scale of the non-local light-cone operator. Inserting (6) into (5) and calculating the hadronic tensor, we arrive at the well-known expressions for the Born approximation to the twist-two part of the structure functions

$$F_{L(0)}(x) = 0, \quad F_{2(0)}(x) = x \sum_q e_q^2 \left[ q(x) + \bar{q}(x) \right], \quad (8)$$

where $\bar{q}(x) = q(-x)$ stands for the antiquark distribution. Going beyond the leading order in (6), one finds that the coefficient function of the non-local quark operator is modified by $\mathcal{O}(\alpha_s)$ corrections and, moreover, the $T$-product receives contribution from non-local light-cone gluon operators. The general expression for the twist-two contribution to the structure functions is then given by

$$\frac{1}{x} F^{tw-2}_a(x, Q^2) = \int_x^1 \frac{d\xi}{\xi} C_{2,a}^q(\xi, \alpha_s(Q^2), Q^2 / \mu^2) q(x/\xi, \mu) + \ldots \quad (9)$$

where $a = L, 2$ stand for $F_L$ and $F_2$, respectively, and $C_{2,a}^q(\xi, \alpha_s(Q^2), Q^2 / \mu^2)$ are the corresponding quark coefficient functions. The ellipses stand for the antiquark and the gluon contributions.

For our purpose, the gluon contribution to (9) can be safely ignored for the following reasons. Our task in Section 4 is to verify the cancellation of infrared renormalons at twist two and ultraviolet contribution to the twist-four matrix elements, to the leading order of
the flavour expansion. However, as was argued in [16], renormalons appear in the gluon matrix elements only at the next-to-leading order in the flavour expansion. Secondly, when examining the asymptotics of the structure function $F_2(x)$ in (9) for $x \to 1$, one finds [5, 6, 7, 37] that the gluon coefficient function is subleading as compared with that of the quark. For example, to order $\alpha_s$, the $F_2$ coefficient functions are

$$C_{2,2}^{g,NS}(\xi, \alpha_s(Q^2), 1) = \delta(1-\xi) + C_F \frac{\alpha_s}{\pi} \left[ \left( \frac{\ln(1-\xi)}{1-\xi} \right) + \frac{3}{4} \left( \frac{1}{1-\xi} \right) + \right. - \frac{1}{2}(1+\xi) \ln(1-\xi) - \frac{11+\xi^2}{2(1-\xi)} \ln \xi + \frac{3}{2} + \xi - \delta(1-\xi) \left( \frac{\pi^2}{6} + \frac{9}{4} \right) \right]$$

$$C_{2,2}^g(\xi, \alpha_s(Q^2), 1) = N_f \frac{\alpha_s}{\pi} \left[ \frac{1}{2}(1-2\xi + 2\xi^2) \left[ \ln(1-\xi) - \ln(\xi) \right] - \frac{1}{2} + 4\xi(1-\xi) \right],$$

and the $F_L$ coefficient functions are

$$C_{2,L}^{g,NS}(\xi, \alpha_s(Q^2), 1) = C_F \frac{\alpha_s}{\pi} \xi$$

$$C_{2,L}^g(\xi, \alpha_s(Q^2), 1) = N_f \frac{\alpha_s}{\pi} 2\xi(1-\xi),$$

where the factorization scale was set as $\mu^2 = Q^2$. In both cases there is an additional suppression factor of $(1-\xi)$ for the gluon coefficient function compared with that of the quark, and therefore the gluon contribution can be ignored provided that the gluon density is not anomalously large compared with the quark density for $x \to 1$. We shall assume that this is indeed the case. Phenomenological studies [1] shows that the gluon density decreases faster than that of the quarks, consistently with this assumption.

To see intuitively why the hierarchy between the quark and gluon coefficient function at large $x$ holds, one has to compare the final states described by the two contributions. For the quark contribution to $F_2(x)$ the final state consists of the outgoing energetic quark jet of small invariant mass $Q^2(1-x)$ surrounded by soft gluons. For the gluon contribution, coming necessarily through the box diagram, the cloud of soft particles is bound to contain a soft quark. Invoking the standard power-counting arguments, one finds that the contribution of a soft quark is suppressed by a power of the energy $\sim (1-x)$ compared with that of a soft gluon.

In addition one finds that owing to the different quantum numbers, the $F_L$ coefficient functions are suppressed at large $x$ by an overall factor of $(1-x)$, compared with $F_2$. Taking the moments of (9)

$$\tilde{F}_a(N, Q^2) \equiv \int_0^1 F_a(x, Q^2) x^{N-2} dx = \tilde{C}_2^{a,N}(N, \alpha_s) q_N$$

with $q_N \equiv \int_0^1 q(x) x^{N-1} dx$ and $\tilde{C}_2(N) \equiv \int_0^1 C_2(x) x^{N-1} dx$, one concludes that

$$\frac{\tilde{F}_L^{tw-2}(N)}{\tilde{F}_2^{tw-2}(N)} = \frac{\tilde{C}_2(N) / \tilde{C}_2^{2,2}(N)}{O(1/N)},$$

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The validity of the hierarchy (13) proves to be independent of the twist. In particular, as we shall show below, the same hierarchy (13) holds for the twist-four contribution, and it becomes a general property of the structure functions in the semi-inclusive region.

The twist-two coefficient functions $C_{2a}(\xi, \alpha_s(Q^2), Q^2/\mu^2)$ have a well-defined perturbative expansion. To any order in the coupling constant, their perturbative expressions contain terms involving logarithmic dependence on the factorization scale $\mu$ as well as the terms regular as $\mu \to 0$. The former are controlled by the DGLAP evolution equations, while the latter become sensitive to the behaviour of the coupling constant at low scale and increase factorially at high orders in $\alpha_s$ (renormalons). Infrared renormalons make the perturbative series non-summable and thus induce ambiguous contributions to the coefficient functions suppressed by powers of $1/Q^2$.

The logarithmic dependence of the twist-two parton distributions $q(x, \mu)$ on the factorization scale $\mu$ reflects the $\ln \mu$ renormalization scale dependence of the corresponding operators (7). Through renormalization, different operators mix with each other. Similarly, higher twist $n$ (with $n = 4, 6, \ldots$) distribution amplitudes admit power-like dependence on the factorization scale, $\sim \mu^{2n}$. This reflects the fact that the higher-twist operators mix under renormalization with the leading twist. As a consequence, the separation of higher-twist matrix elements from those of the leading twist depends on the factorization scale. Obviously, the structure functions $F_a(x, Q^2)$ should not depend on this scale. Within the OPE (1), the logarithmic $\mu$ dependence cancels out between the coefficient functions and the matrix elements of the same twist, whereas the power-like $\mu^{2n}$ dependence cancels between the contributions of different twists. From this we draw two important conclusions. The first is that in order to verify the cancellation of infrared renormalon ambiguities in the twist-two coefficient functions, one has to calculate the most divergent ultraviolet contribution to the matrix elements of higher-twist operators – the quadratic divergence for twist four, the quartic divergence for twist six, etc. The second conclusion has considerable implications on phenomenological applications: in general, higher-twist contributions cannot be consistently included without resumming the perturbative expansion (which is, by itself, ambiguous). The ambiguity cancels only when resummation is performed and power corrections are included.

Renormalon contributions to the twist-two coefficient functions, $\sim \alpha_s^{n+1}\beta_0^n n!$, can be extracted from the (unphysical, but technically useful) large $N_f$ limit, since $\beta_0 = (11C_A - 2N_f)/12$ is linear in $N_f$. The leading order in the flavour expansion corresponds to the lowest-order diagrams in figure 1, where the internal gluon line is dressed by any number of fermion loops.

To regularize a divergent renormalon contribution, we shall apply the Borel regularization [16]. It amounts to representing the large-$N_f$ dressed-gluon propagator by

$$\frac{1}{k^2} \frac{1}{\beta_0} \int_0^\infty du \exp \left(-u \ln \frac{k^2}{\Lambda^2}\right)$$

and identifying singularities of the resulting momentum integral in the $u$-plane located.

\footnote{A similar conclusion was reached in [19].}
Figure 1: Diagrams that contribute to the twist-two coefficient function at leading order.

at real positive \( u \). The main advantage of the Borel method is that it allows one to regularize simultaneously the renormalon contribution to the coefficient functions and power-like ultraviolet divergences to the higher-twist matrix elements.

The renormalon contribution to the twist-two coefficient functions, \( C_{q_{2,2}} \) and \( C_{q_{2,L}} \), has been calculated in the single dressed gluon approximation in Refs. [21]–[23] (see also [24] and [18]). It generates power-suppressed corrections to the coefficient functions of the form \( \sim 1/Q^2 \) and \( \sim 1/Q^4 \) only and, therefore, produce ambiguous contribution to the structure functions of twist four and twist six, respectively. In particular, the twist-four renormalon contribution has the following form

\[
\int_{0}^{1} dx x^{N-1} \Delta_{\text{IR}} \left[ \frac{F_{2}}{2x} \right]_{\text{tw}-4} = \frac{C_{F}}{4\beta_{0}} \left[ 4\psi(N + 1) + 4\gamma + \frac{2}{1 + N} + \frac{12}{2 + N} - 8 - N \right] q_{N}\frac{\Lambda^2\delta}{Q^2},
\]

\[
\int_{0}^{1} dx x^{N-1} \Delta_{\text{IR}} \left[ \frac{F_{L}}{2x} \right]_{\text{tw}-4} = \frac{C_{F}}{4\beta_{0}} \left[ -\frac{4N}{N + 2} \right] q_{N}\frac{\Lambda^2\delta}{Q^2},
\]

(14)

with \( \psi(x) = d\ln \Gamma(x)/dx \) and \( \gamma \) being the Euler constant. Here, the notation was introduced for the dimensionless constant \( \delta \), defined by the integral \( \delta \equiv \int_{0}^{1} du/(1 - u) \), which represents the difference between different deformations of the original Borel-integration contour \( (0 < u < \infty) \) in the vicinity of the pole at \( u = 1 \).

The expressions (14) simplify significantly at large \( N \),

\[
\int_{0}^{1} dx x^{N-1} \Delta_{\text{IR}} \left[ \frac{F_{2}}{2x} \right]_{\text{tw}-4} = -\frac{C_{F}}{4\beta_{0}} q_{N}\frac{N\Lambda^2\delta}{Q^2},
\]

\[
\int_{0}^{1} dx x^{N-1} \Delta_{\text{IR}} \left[ \frac{F_{L}}{2x} \right]_{\text{tw}-4} = -\frac{C_{F}}{\beta_{0}} q_{N}\frac{\Lambda^2\delta}{Q^2},
\]

(15)

so that the renormalon contribution to \( F_{L}(x, Q^2) \) is suppressed at large \( N \) by a power of \( N \) as compared with \( F_{2}(x, Q^2) \). We recall that a similar property holds for the twist-two contribution, Eq. (13). In addition, Eq. (15) suggests that, in agreement with our expectations (3), the effective parameter of the twist expansion in the limit \( x \to 1 \) is given by \( N/Q^2 \) rather than \( 1/Q^2 \). An expression similar to (14) is available for the
twist-six renormalon ambiguity [22]. In the Borel plane, this ambiguity appears as an additional pole, at \( u = 2 \). At large \( N \), this contribution contains an additional factor of \( N/Q^2 \) with respect to twist four. Again, this holds for both \( F_2(x, Q^2) \) and \( F_L(x, Q^2) \).

Consistency of the OPE of the structure functions requires that renormalon ambiguities (14) will cancel in (1) against the ultraviolet divergent contribution to the twist-four matrix elements. For the longitudinal structure function, \( F_L(x, Q^2) \), this cancellation was demonstrated in [16, 17]. Below we will conduct a similar analysis for case of the structure function \( F_2(x, Q^2) \), which according to (15) is expected to have a more singular behaviour for \( x \to 1 \). The cancellation of the twist-six renormalon ambiguities with ultraviolet divergent contribution to the twist-six matrix elements requires a complete analysis of twist-six operators and their coefficient functions. It is therefore more involved and will not be discussed here.

### 3 Twist-four analysis

To demonstrate the cancellation of the renormalon ambiguities (14), we will need the expression for the twist-four part of the structure functions. This relies on the twist-four contribution to the symmetric part of the time-ordered product (6). To leading order in the coupling constant it is given by [8]–[11],

\[
iT \left[ j_\mu(y) j_\nu(-y) \right]^{tw-4} = \frac{1}{128\pi^2} (T_{2, \mu\nu} + T_{L, \mu\nu} + \cdots)
\]

(16)

with two terms \( T_{L, \mu\nu} \) and \( T_{2, \mu\nu} \) representing the separate projections of the longitudinal and the transverse components, respectively, and the ellipses denoting the operators containing total derivatives which therefore having vanishing forward matrix elements. The operators \( T_{a, \mu\nu} \) (with \( a = 2, L \)) are defined for \( y^2 \to 0 \) as

\[
T_{L, \mu\nu} = \int_0^1 du \left[ \frac{4g_{\mu\nu}}{y^2} u(1 + \ln u) Q_1(uy) + u \ln u \partial_\mu \partial_\nu \left( \ln y^2 Q_1(uy) \right) \right]
\]

(17)

and

\[
T_{2, \mu\nu} = \int_0^1 du \left\{ -\ln y^2 \partial_\mu \partial_\nu \left[ u \ln u Q_1(uy) + \frac{1-u}{u^2} Q_2(uy) \right] + \frac{1}{y^2} (y_\mu \partial_\nu + y_\nu \partial_\mu - g_{\mu\nu}(y \cdot \partial)) \left[ -u(1+3 \ln u) Q_1(uy) + \frac{1}{u} Q_2(uy) \right] \right\}.
\]

(18)

Here, \( Q_1(y) \) and \( Q_2(y) \) are two specific combinations of non-local twist-four light-cone operators, which will be given later on.

Inserting (17), (18) and (16) into the expression for the hadronic tensor, (5), we parameterize, in analogy with (7), the matrix elements of the operators \( Q_1(y) \) and \( Q_2(y) \) in terms of the twist-four distribution amplitudes \( D_1(\xi) \) and \( D_2(\xi) \), respectively,

\[
\langle p | Q_1(y) | p \rangle = i \int_{-1}^1 d\xi \, e^{2i\xi p_+ y} D_1(\xi),
\]

(19)
with $y^2 = 0$. The symmetry of the $T$-product, Eq. (16), under $y \to -y$ implies that $Q_i(-y) = Q_i(y)$, and, as a consequence, $D_i(\xi) = D_i(-\xi)$ (with $i = 1, 2$). The functions $D_i(\xi)$ defined in this way describe the momentum fraction distribution inside the hadron and vanish outside the interval $-1 < \xi < 1$. However, in contrast to the twist-two distributions, they do not have probabilistic interpretation and, in general, are not positive definite. As we will show below, they can be expanded into a sum over multi-parton configurations with definite weights, given by the twist-four coefficient functions.

Calculating the twist-four contribution to the hadronic tensor according to (5), we perform a Fourier transform of the forward matrix element of (16) and take its imaginary part. Matching the resulting expression into the definition of the structure functions, Eq. (4), we obtain after some algebra the following expressions for the twist-four contributions to the structure functions

$$\frac{F_L(x)}{2x} \bigg|_{tw-4} = \frac{1}{4q^2} \int_x^1 \frac{d\xi}{\xi} \xi^2 (1 + \ln \xi) D_1(x/\xi),$$

$$\frac{F_2(x)}{2x} \bigg|_{tw-4} = \frac{1}{16q^2} \left[ x(D_2(x) - D_1(x)) + \int_x^1 \frac{d\xi}{\xi} \xi^2 (5 + 6 \ln \xi) D_1(x/\xi) \right]. \quad (20)$$

The expressions obtained for the twist-four contribution to the structure functions have a striking similarity to the twist-two contribution given by (9). Indeed, one can rewrite (20) in the same form as (9) by replacing twist-two parton densities by their twist-four counterparts $D_i(\xi)$ and defining the corresponding coefficient functions $C_{4,a}(\xi)$. This analogy proves to be very useful as it reveals the important difference between the two terms entering the second relation in (20): the first term where the distribution amplitude is evaluated at $\xi = x$, and the second, which involves an integral over the partonic momentum fraction $\xi$. In the case of twist two, these two terms are associated with the contribution to the coefficient functions due to virtual and real perturbative corrections to the partonic cross section, respectively. The former corresponds to a single-parton final state, $C_{2,a}(\xi) \sim \delta(1 - \xi)$, and it is singular at $x = 1$. In general, to high orders in the perturbation expansion, the singular behaviour of the coefficient functions for $x \to 1$ is associated with particular partonic (short-distance) final states in which the recoiling quark creates a narrow jet. Generalization of this analysis to the twist-four contribution is straightforward.

Following [10], we can rewrite the twist-four contribution to the structure function as the sum over all possible final states. Then, the two terms in the above-mentioned expression for $F_2(x)$ correspond to two different short-distance final states: a single recoiled quark state and a state in which this quark is accompanied by an additional gluon carrying the momentum fraction $1 - \xi$. We also notice that the coefficient functions entering the expressions for $F_L(x)$ and the second term in the expression for $F_2(x)$ in (20) are not singular as $\xi \to 1$. As in the case of twist two, this suggests that the emitted gluon cannot be soft and, as a consequence, the corresponding contribution to the structure functions is subleading at large $x$ compared with the first term in the second relation in (20).
Indeed, let us examine the twist-four contributions (20) in the large \( x \) limit. It is convenient to consider the expressions (20) in moment space

\[
\int_0^1 dx x^{N-1} \left. \frac{F_L(x)}{2x} \right|_{\text{tw-4}} = \frac{1}{4q^2 (N+2)^2} \widetilde{D}_1(N+1),
\]

\[
\int_0^1 dx x^{N-1} \left. \frac{F_2(x)}{2x} \right|_{\text{tw-4}} = \frac{1}{16q^2} \left[ \widetilde{D}_2(N+1) - \frac{N(N-1)}{(N+2)^2} \widetilde{D}_1(N+1) \right],
\]

where \( \widetilde{D}_i(N) \equiv \int_0^1 d\xi \xi^{N-1} D_i(\xi) \). Assuming that the moments \( \widetilde{D}_1(N+1) \) and \( \widetilde{D}_2(N+1) \) have a similar behaviour at large \( N \), we conclude from (21) that the twist-four contribution to \( F_L(x) \) is suppressed by a power of \( N \) compared with that of \( F_2(x) \). We recall that the same property holds for the twist-two contribution to the structure functions, Eq. (13), including the renormalon contribution (15).

We also find that for large \( x \), the twist-four contribution to \( F_2(x) \) (this is not so for \( F_L(x) \)) has a natural approximation,

\[
\left. \frac{F_2(x)}{2x} \right|_{\text{tw-4}} \simeq \frac{1}{16q^2} x(D_2(x) - D_1(x)).
\]

Remarkably enough, this is similar to the expression for the twist-two structure function to the lowest order in \( \alpha_s \), Eq. (8). As explained above, the reason for this is that in both cases the partonic \textit{final} states which determine the large-\( x \) asymptotics of the coefficient functions look alike: they correspond to the propagation of a single recoiled quark. Calculating high order corrections to the twist-four coefficient functions, we expect that, as in the twist-two situation, the large-\( x \) asymptotics will correspond to the propagation in the final state of narrow jet initiated by the recoiled quark. The difference between twists resides, at large \( x \), in the different \textit{initial} states. For the twist two, the initial state consists of a single parton and the coefficient function is given by the squared modulus of the diagonal transition amplitude, or, equivalently, the partonic cross section. For twist four, there can be one or more partons in the initial state, so that the coefficient function is given by the sum of diagonal transition amplitudes as well as their interference. This implies that twist-four coefficient functions do not have an interpretation of the partonic cross section but rather of correlations between different multi-parton states within the hadron [10].

Let us examine the properties of twist-four distributions \( D_i(x) \) in more detail. According to Eq. (19), these distributions are defined through hadronic matrix elements of non-local light-cone twist-four operators \( Q_i(y) \). Their expressions have been worked out in [8]–[11] and are given by\(^3\)

\[
Q_1(y) = \int_{-1}^1 dv \left[ 4 \mathcal{O}_3(v; y) - 2i(1 - v^2) \mathcal{O}_7(v; y) \right]
\]

\(^3\)These expressions were taken from Ref. [11] and a few misprints were corrected.
+ 4 \int_{-1}^{v} dt (1-vt) \mathcal{O}_5(v,t;y) + (v-t) \mathcal{O}_6(v,t;y) \right) + (y \leftrightarrow -y)

Q_2(y) = \int_{-1}^{1} dv \left[ -4i(1-v^2) \mathcal{O}_7(v;y) + 4 \int_{-1}^{v} dt \left\{ (2-v+t-2vt) \mathcal{O}_5(v,t;y) + (v-t) \mathcal{O}_6(v,t;y) + \mathcal{O}_1^{\text{sym}}(v,t;y) + \mathcal{O}_2^{\text{sym}}(v,t;y) \right\} \right] + (y \leftrightarrow -y).

Here, the notations for the “canonical” basis of light-cone operators \( \mathcal{O}_i \) are according to \([8]\). They can be classified according to their partonic content and are defined as follows. The quark-antiquark-gluon operators are

\[
\mathcal{O}_3(v;y) = \frac{1}{2} g_{\mu\nu\rho\eta} \bar{\Psi}(y) y^\mu \gamma^\nu \gamma_5 G^{\rho\eta}(vy) \Psi(-y)
\]

\[
\mathcal{O}_7(v;y) = g \bar{\Psi}(y) \gamma_a D_a G^{\mu\nu}(vy) \Psi(-y),
\]

where \( y^2 = 0 \) and \( G_{\mu\nu} \equiv t^a G_{\mu\nu}^a \) with \( t^a \) being the color matrices in the quark representation. Other operators include a quark, an antiquark and two gluons,

\[
\mathcal{O}_5(v,t;y) = g^2 \bar{\Psi}(y) y_a G^{\alpha\eta}(vy) y_\beta G_{\beta\eta}(ty) \bar{\Psi}(-y)
\]

\[
\mathcal{O}_6(v,t;y) = ig^2 \epsilon_{\mu\nu\rho\eta} \bar{\Psi}(y) y^\rho y_\alpha G^{\alpha\mu}(vy) y_\beta G_{\beta\nu}(ty) \gamma^\eta \gamma_5 \Psi(-y),
\]

and \( C \)-even combination (sym) of two quark-antiquark pairs,

\[
\mathcal{O}_1^{\text{sym}}(v,t;y) = \frac{ig^2}{4} \left[ \bar{\Psi}(y) t^a \gamma_5 \Psi(vy) - \bar{\Psi}(vy) t^a \gamma_5 \Psi(y) \right] \times \left[ \bar{\Psi}(ty) t^a \gamma_5 \Psi(-y) - \bar{\Psi}(-y) t^a \gamma_5 \Psi(ty) \right]
\]

\[
\mathcal{O}_2^{\text{sym}}(v,t;y) = \frac{ig^2}{4} \left[ \bar{\Psi}(y) t^a \gamma_5 \Psi(vy) + \bar{\Psi}(vy) t^a \gamma_5 \Psi(y) \right] \times \left[ \bar{\Psi}(ty) t^a \gamma_5 \Psi(-y) + \bar{\Psi}(-y) t^a \gamma_5 \Psi(ty) \right].
\]

Inserting (23) into (19), we can relate the distribution amplitudes \( D_i(\xi) \) to the matrix elements of the basis twist-four operators \( \mathcal{O}_i(v;y) \), which, in turn, can be parameterized by introducing the set of twist-four multi-parton distributions (see figure 2),

\[
\langle p | \mathcal{O}_3(v;y) | p \rangle = 2(py) \int d\xi_1 d\xi_2 e^{i(py)[\xi_1(1-v)+e_2(1+v)]} T_3(\xi_1,\xi_2)
\]

\[
\langle p | \mathcal{O}_7(v;y) | p \rangle = 2(py)^2 \int d\xi_1 d\xi_2 e^{i(py)[\xi_1(1-v)+e_2(1+v)]} T_7(\xi_1,\xi_2).
\]

The distribution functions corresponding to the remaining operators from the canonical basis are defined similarly. The distributions \( T_{3,7}(\xi_1,\xi_2) \) are the analogue of the twist-two distribution \( q(\xi) \) introduced in (7). They describe the correlations inside the hadron between a state containing a quark and a gluon carrying the momentum fractions \( \xi_1, \)
Figure 2: Forward matrix elements of the non-local light-cone operators $O_{3,7}(v; y)$.

$x_2 - x_1$, respectively, and a state containing quark carrying momentum fraction $x_2$. These distributions vanish outside the “physical” region $-1 < x_1, x_2, x_2 - x_1 < 1$. Using the hermiticity properties of the operators (24), $[O_3(v; y)]^\dagger = -O_3(-v; -y)$ and $[O_7(v; y)]^\dagger = O_7(-v; -y)$ we deduce from (27) that

$$T_3^*(x_1, x_2) = T_3(x_2, x_1), \quad T_7^*(x_1, x_2) = T_7(x_2, x_1). \quad (28)$$

In general, the distributions $T_\alpha(x_1, x_2)$ are complex valued functions, but, as we will show below, the twist-four structure function receives nonzero contribution only from their real part, which is symmetric under replacement of quarks $x_1 \leftrightarrow x_2$. Expanding both sides of (27) in powers of $y$ we find that, as in the case of twist two, the moments of the distribution amplitudes $T_i(x_1, x_2)$ are related to the matrix elements of local twist-four operators.

Taking into account (27), it becomes straightforward to express the distributions $D_i(x)$ defined in (19) in terms of the canonical twist-four distributions $T_i(x_i)$. To this end one substitutes (27) into (23) and matches the result of $v$ integration into the r.h.s. of (19). Expanding the both sides of the resulting relation in powers of $(py)$, we obtain the following expression for the odd moments $N$ of the distribution amplitudes

$$\int_0^1 d\xi \xi^{N-1} D_1(x) = -\bar{T}_3(N) + \bar{T}_7(N) + \ldots$$

$$\int_0^1 d\xi \xi^{N-1} D_2(x) = 0 \cdot \bar{T}_3(N) + 2 \cdot \bar{T}_7(N) + \ldots, \quad (29)$$

where the ellipses stand for the contribution of the remaining distributions from the canonical basis, which we demonstrate in the next Section to be irrelevant for the cancellation of renormalon ambiguities to leading order in the flavour expansion. Here we have introduced the notation for the generalized moments:

$$\bar{T}_\alpha(N) = 16 \int d\xi_1 d\xi_2 T_\alpha(x_1, x_2) \Phi_{\alpha,N-2}(x_1, x_2), \quad (30)$$

with $\alpha = 3, 7$. $\Phi_{3,N}(x_i)$ and $\Phi_{7,N}(x_i)$ are homogenous polynomials of degree $N$ in the momentum fractions $x_i$. Inserting the expressions (23) for the non-local twist-four operators in terms of the canonical operators $O_\alpha$ into (19) and (29) uniquely determines the
form of the polynomials $\Phi_{\alpha,N}$:

$$
\Phi_{3,N}(\xi_1, \xi_2) = \frac{\xi_1^{N+1} - \xi_2^{N+1}}{\xi_1 - \xi_2} = \sum_{k=0}^{N} \xi_1^k \xi_2^{N-k}
$$

$$
\Phi_{7,N}(\xi_1, \xi_2) = \frac{1}{N+2} \partial_{\xi_1} \partial_{\xi_2} \Phi_{3,N+1}(\xi_1, \xi_2) = \sum_{k=0}^{N-1} \frac{(k+1)(N-k)}{N+2} \xi_1^k \xi_2^{N-k-1}.
$$

Since the functions $\Phi_{3,N}(\xi_i)$ and $\Phi_{7,N}(\xi_i)$ are symmetric under interchange of $\xi_1$ and $\xi_2$, the moments (30) only receive nonzero contribution from the real part of the distribution amplitudes $T_i(\xi_i)$, in virtue of (28).

Repeating a similar calculation, one can obtain the expressions for the coefficient functions $\Phi_{\alpha,N}(\xi_i)$ (with $\alpha = 1, 2, 5, 6$), defining the contribution of the remaining twist-four canonical operators to the r.h.s. of (29).

Using (21) and (29) we get

$$
\int_0^1 dx x^{-1} \frac{F_L(x)}{2x} \bigg|_{\text{tw}-4} = \frac{1}{4q^2 (N+2)^2} \left[-\tilde{T}_3(N+1) + \tilde{T}_7(N+1) + \ldots \right],
$$

$$
\int_0^1 dx x^{-1} \frac{F_2(x)}{2x} \bigg|_{\text{tw}-4} = \frac{1}{16q^2 (N+2)^2} \times \left[ N(N-1) \tilde{T}_3(N+1) + (N+1)(N+8) \tilde{T}_7(N+1) + \ldots \right].
$$

These are the general expressions for the contribution of the operators $O_3(v; y)$ and $O_7(v; y)$ to the moments of the structure functions. The moments $\tilde{T}_\alpha(N)$, Eq. (30), are given by the sum over matrix elements of local twist-four operators generated by the non-local light-cone operators, $O_\alpha(v; y)$. Each local operator enters into the sum with the weight given by the coefficients of the polynomials $\Phi_{\alpha,N}(\xi_i)$, expanded in powers of momentum fractions.

4 Cancellation of ambiguities

The generalized moments entering the r.h.s. of (32) depend on the factorization scale $\mu$ that sets up an ultraviolet cut-off for the matrix elements of the twist-four operators $O_\alpha(v; y)$. We now proceed to calculate the quadratically divergent, $\sim \mu^2$, contribution to the distribution amplitudes $T_\alpha(\xi_i)$. According to (32), this induces an ambiguity in the definition of the twist-four structure functions, which should cancel against the infrared renormalon contribution, Eq. (15).

To match the renormalon calculation of twist two, we must calculate the ultraviolet contribution to the twist-four distribution amplitudes in the same approximation, that is to leading order in the flavour expansion. Since this contribution does not depend on the particular choice of the matrix element, it suffices to consider the matrix elements of the operators $\langle p | O_\alpha(v; y) | p \rangle$ between quark states, with the quark being off-shell, $p^2 < 0$, and
$y^2 = 0$. To calculate the quadratic divergences we have to introduce a regularization. To be consistent with the twist-two calculation of the coefficient functions, we are obliged to use the same regularization, namely the Borel representation of the large-$N_f$ dressed-gluon propagator. The Borel variable $u$ regularizes the ultraviolet divergent integrals over the gluon momenta and quadratic divergence amounts, in this regularization, to a pole at $u = 1$, leading to an ambiguity once the Borel integral is evaluated.

Examining the matrix elements of the non-local light-cone operators, $\langle p | O_\alpha(v; y) | p \rangle$, defined in equations (24)–(26), we find that, to leading order in the flavour expansion, only the quark–gluon–antiquark operators $O_5(v; y)$ and $O_7(v; y)$ (fig. 2) contribute to the quadratic divergence, and, as a consequence, mix with the twist-two operators. The relevant diagrams are shown in figures 3 and 4.

![Figure 3: Contributions to the quadratic ultraviolet divergence of twist-four matrix elements. Insertion of fermion loops into the gluon lines is assumed. The contribution (b), corresponding to non-local operators with a quark-antiquark pair and two gluons, vanishes.](image)

The irrelevance of the operators containing two gluons, $O_5(v, t; y)$ and $O_6(v, t; y)$, was pointed out in [16] for the case of $F_L(x)$. It turns out that this holds also for $F_2(x)$. Let us briefly explain why $O_{5,6}(v, t; y)$ are not relevant. These operators mix with the leading twist only if both gluons are contracted. We recall that the twist-two renormalon contribution was calculated to leading order in the flavour expansion, so we must apply the same approximation here. If both gluons are contracted to the quark or antiquark lines the contribution is subleading in $N_f$ and therefore irrelevant. There remains the possible contribution where the gluons are contracted one to the other, as in figure 3 (b). However, as noted in [16], this diagram vanishes for $y^2 = 0$, since there is no external scale.

The vanishing of the contribution of the four-quark operators $O_{1/2}^{\text{sym}}(v, t; y)$ of Eq. (26) is less obvious. The four quark operators (see figure 4 (a)) have the form

$$O_1(s_1, s_2; s_3, s_4) = ig^2 \left[ \bar{\Psi}(s_1) t^b \gamma_5 \Psi(s_2) \right] \left[ \bar{\Psi}(s_3) t^b \gamma_5 \Psi(s_4) \right]$$

$$O_2(s_1, s_2; s_3, s_4) = ig^2 \left[ \bar{\Psi}(s_1) t^b \gamma_5 \Psi(s_2) \right] \left[ \bar{\Psi}(s_3) t^b \gamma_5 \Psi(s_4) \right].$$

(33)
In order for these operators to mix with the twist two operator (6), one of the quarks must be contracted to one of the antiquarks. In addition, for the contribution to be relevant in the large $N_f$ limit a dressed gluon must be included. Contracting between a quark and an antiquark of different pairs, e.g. $\Psi(s_2)$ and $\Psi(s_3)$, and adding a dressed gluon with $n$ fermion loops, the contribution is $O(g^4(g^2N_f)^n)$. Being sub-leading in $N_f$, this contribution is irrelevant. On the other hand, contracting between a quark and an antiquark within a pair, e.g. $\Psi(s_1)$ and $\Psi(s_2)$ as shown in Fig. 4 (b), an additional factor of $N_f$ is gained so the contribution is $O(g^4N_f(g^2N_f)^n)$, which is relevant in the large $N_f$ limit. Calculation of this diagram shows that $\langle p | O_{1,2}(s_1, s_2; s_3, s_4) | p \rangle$ is indeed quadratically divergent. Nevertheless, because of the symmetry property

$$\langle p | O_{1,2}(s_1, s_2; s_3, s_4) | p \rangle = \theta_{1,2} \langle p | O_{1,2}(s_2, s_1; s_3, s_4) | p \rangle$$

with $\theta_1 = -\theta_2 = 1$, the symmetrized matrix element $\langle p | O_{1,2}^{\text{sym}}(v, t; y) | p \rangle$ is free of any quadratic divergence. As a consequence the four-quark operators do not contribute to the quadratic divergence in (32).

To understand this property, it is useful to return to the construction of the OPE of eqs. (16)–(17) as an expansion of local Feynman diagrams in powers of $1/Q^2$. The four-quark operators $O_{1,2}^{\text{sym}}(v, t; y)$ emerge from the diagram in figure 5 (a).

The possibly relevant contributions of the non-local four-quark operators of figure 4 (b) correspond, before expansion, to the diagram in figure 5 (b). The latter vanishes since it contains a quark loop attached to one photon and two gluons, which is odd under (electric) charge conjugation. The symmetrized form of (26) which nullifies its contribution is thus a manifestation of charge-conjugation invariance.

We conclude that to leading order in the flavour expansion, the structure functions (32) receive a quadratically divergent contribution only from the generalized moments $\tilde{T}_{3,7}(N + 1)$, or equivalently from the distribution amplitudes $T_{3,7}(\xi_1, \xi_2)$. The quadratically divergent contribution to these amplitudes, $\Delta_{uv}T_{3,7}(\xi_1, \xi_2)$, was calculated in [16, 17].
Figure 5: (a) Local Feynman diagram from which four-quark operators emerge by taking the operator product expansion; (b) The origin of figure 4 (b).

Evaluating the diagram shown in Fig. 3 (a) with a dressed gluon and an off-shell external quark $p^2 < 0$ at $y^2 = 0$, where the Feynman rules for the non-local operators are defined by (24), one gets

$$
\Delta_{\text{UV}} T_3(\xi_1, \xi_2) = -\frac{C_F}{4\beta_0} \Lambda^2 \delta \left\{ \frac{1}{\xi_1} \left( 1 + \frac{\xi_2}{\xi_1} \right) q(\xi_1) \theta(\xi_1 - \xi_2) + (\xi_1 \leftrightarrow \xi_2) \right\} \theta(\xi_1) \theta(\xi_2)
$$

$$
\Delta_{\text{UV}} T_7(\xi_1, \xi_2) = 2\frac{C_F}{4\beta_0} \Lambda^2 \delta \left\{ \frac{\xi_2}{\xi_1} q(\xi_1) \theta(\xi_1 - \xi_2) + (\xi_1 \leftrightarrow \xi_2) \right\} \theta(\xi_1) \theta(\xi_2),
$$

with the parameter of Borel regularization $\delta$ defined in (14) and $q(\xi)$ being the twist-two quark distribution function. Next, the ultraviolet ambiguity of $\tilde{T}_{3,7}(N+1)$ can be evaluated by substituting (35) into (30) and performing the $\xi_i$ integrals with the coefficient functions $\Phi_{3,7}(\xi_i)$ defined in (31). The result is

$$
\Delta_{\text{UV}} \tilde{T}_3(N+1) = \frac{C_F q_N}{\beta_0} \Lambda^2 \delta \left[ 2\psi(N+1) + 2\gamma - 1 + \frac{1}{N+1} \right]
$$

$$
\Delta_{\text{UV}} \tilde{T}_7(N+1) = \frac{C_F q_N}{\beta_0} \Lambda^2 \delta \left[ N - 2\psi(N+1) - 2\gamma + 2 - \frac{2}{N+1} \right].
$$

Finally, substituting (36) into (32) we calculate the ultraviolet ambiguity of the structure functions

$$
\int_0^1 dx x^{N-1} \Delta_{\text{UV}} \left[ \frac{F_2}{2x} \right]_{\text{tw}-4} = -\frac{C_F}{4\beta_0} \left[ 4\psi(N+1) + 4\gamma + \frac{2}{N+1} + \frac{12}{N+2} - 8 - N \right] q_N \frac{\Lambda^2 \delta}{Q^2}
$$

$$
\int_0^1 dx x^{N-1} \Delta_{\text{UV}} \left[ \frac{F_L}{2x} \right]_{\text{tw}-4} = \frac{C_F}{4\beta_0} \frac{4N}{N+2} q_N \frac{\Lambda^2 \delta}{Q^2}.
$$

As expected, this ambiguity cancels exactly with the corresponding infrared renormalon ambiguity in the twist-two coefficient functions in Eq. (14). We see that when the
twist-two coefficient functions are resummed and the (ultraviolet divergent part of) the twist four is included there remains no ambiguity at the level of \(1/Q^2\) corrections in the OPE (1).

We conclude that to leading order in the flavour expansion, infrared renormalons “probe” the contribution to the longitudinal, \(F_L(x, Q^2)\), and transverse structure functions, \(F_2(x, Q^2)\), coming from only two twist-four operators, \(\mathcal{O}_3\) and \(\mathcal{O}_7\), defined in (24). An additional simplification occurs for \(x \to 1\): from (36) it follows that for \(N \gg 1\)

\[
\Delta_{uv} \tilde{T}_7(N + 1)/\Delta_{uv} \tilde{T}_3(N + 1) \sim N, \tag{38}
\]

and, therefore, the contribution of the operator \(\mathcal{O}_3\) to the structure functions is subdominant at large \(x\), as far as the cancellation of the renormalon ambiguity is concerned. As we will argue in the next Section, (38) is not limited the ultraviolet divergent contribution of the twist-four operators, but is a general property of these operators. Taking into account this hierarchy, we find that to leading order in the flavour expansion, the twist-four contribution to the structure functions for \(N \to \infty\) is entirely associated with the quark-antiquark-gluon operator \(\mathcal{O}_7\),

\[
\int_0^1 dx \frac{x^{N-1} F_L(x)}{2x} \bigg|_{tw-4} = \frac{1}{4q^2 N} \left[ \tilde{T}_7(N + 1) + \mathcal{O}(N^0) \right], \tag{39}
\]

\[
\int_0^1 dx \frac{x^{N-1} F_2(x)}{2x} \bigg|_{tw-4} = \frac{1}{16q^2} \left[ \tilde{T}_7(N + 1) + \mathcal{O}(N^0) \right].
\]

In addition, based on (36) we know that \(\tilde{T}_7(N + 1)\) contains a piece of the form

\[
\tilde{T}_7^{uv}(N + 1) \equiv \kappa_1^{[7]} \frac{C_F}{\beta_0} N q_N \Lambda^2, \tag{40}
\]

where we replaced the ambiguous \(\delta\) by a new \(N\)-independent dimensionless parameter, \(\kappa_1^{[7]} = \mathcal{O}(1)\), and, as before, we suppressed terms that are subleading at large \(N\). Under the ultraviolet dominance assumption the remaining piece is negligible, so

\[
\tilde{T}_7(N + 1) \simeq \tilde{T}_7^{uv}(N + 1) = \kappa_1^{[7]} \frac{C_F}{\beta_0} N q_N \Lambda^2. \tag{41}
\]

This way the normalization of the twist-four contribution (39) is fixed by a single non-perturbative parameter \(\kappa_1^{[7]}\), as anticipated in (3).

5 Large-\(x\) behaviour of twist four

We saw that the ultraviolet dominance assumption leads to a great simplification of the parameterization of the twist-four contribution at large \(x\): instead of a large number of local matrix elements, which increases \(N\), this contribution is parametrized in terms of a single parameter. But is the ultraviolet dominance assumption justified? A clear
cut answer to this question cannot be given within the framework of perturbative QCD, since it requires calculation of the matrix elements. However, by identifying the partonic configurations which become dominant at large \( x \), we can understand the physical origin of the simplification that occurs at large \( x \).

We already saw, independently of the ultraviolet dominance assumption, that at large \( x \), the twist-four contribution to \( F_2 \) has a natural approximation (22) in terms of the distribution amplitudes \( D_i(\xi) \), evaluated at \( \xi = x \), which is analogous to the Born-level twist-two result (8). At both twist two and twist four the recoiling system corresponds to a single quark. We note that for \( F_L \) at twist four, just as at twist two [37], the recoiling system includes a hard quark and a hard gluon. In each case, at higher orders in perturbation theory the recoiling system is accompanied by soft and collinear radiation. The fact that the “bare” recoiling system is the same for twist four and for twist two opens up the possibility that the jet function is the same, as we shall argue below. Clearly, this condition is not sufficient. In general, the coefficient functions at twist four differ from those of twist two, since the latter strictly consist of single-particle initial states in the amplitude and the complex-conjugate amplitude, whereas the former contain more than one parton, at least in one of them. Indeed, \( D_i(\xi) \), contrary to \( q(\xi) \), describes correlations of several partons in the hadron rather than a single-parton probability distribution.

To investigate the relevant initial states probed at large \( x \), let us apply (32) and (30) to express the structure functions through the twist-four distributions \( T_\alpha(\xi_1, \xi_2) \) defined in (27). In this way, we obtain the following expression for the longitudinal structure function in the momentum fraction representation,

\[
\left. \frac{F_L(x)}{2x} \right|_{\text{tw-4}} = \frac{4}{q^2} \int_{-1}^{1} d\xi_1 d\xi_2 \left\{ -\hat{T}_3(\xi_1, \xi_2) \frac{x^2}{(\xi_2 - \xi_1)\xi_2^2} \left( 1 + \ln \frac{x}{\xi_2} \right) \theta(\xi_2 - x) \right. \\
+ \hat{T}_7(\xi_1, \xi_2) \frac{x^2}{(\xi_2 - \xi_1)^3 \xi_2^2} \left[ \xi_2 - \xi_1 + (3\xi_2 - \xi_1) \ln \frac{x}{\xi_2} \right] \theta(\xi_2 - x) + (\xi_1 \leftrightarrow \xi_2) \left. \right\} 
\]

and a similar expression for the transverse structure function,

\[
\left. \frac{F_T(x)}{2x} \right|_{\text{tw-4}} = \frac{1}{q^2} \int_{-1}^{1} d\xi_1 d\xi_2 \left\{ \hat{T}_3(\xi_1, \xi_2) \left[ \frac{x\delta(\xi_2 - x)}{\xi_2 - \xi_1} - \frac{x^2\theta(\xi_2 - x)}{(\xi_2 - \xi_1)\xi_2^2} \right] \left[ 5 + 6 \ln \frac{x}{\xi_2} \right] \right. \\
+ \hat{T}_7(\xi_1, \xi_2) \left[ \frac{x\delta(\xi_2 - x)}{(\xi_2 - \xi_1)^2} + \frac{x^2\theta(\xi_2 - x)}{(\xi_2 - \xi_1)^3 \xi_2^2} \right] (3\xi_2 - 5\xi_1 + 6(3\xi_2 - \xi_1) \ln \frac{x}{\xi_2}) \right\} + (\xi_1 \leftrightarrow \xi_2) \right. 
\]

Here, the following notation was introduced for real-valued distribution functions,

\[
\hat{T}_3(\xi_1, \xi_2) = \frac{1}{2} \text{Re} \left[ T_3(\xi_1, \xi_2) - T_3(-\xi_1, -\xi_2) \right] , \\
\hat{T}_7(\xi_1, \xi_2) = \frac{1}{2} \text{Re} \left[ T_7(\xi_1, \xi_2) + T_7(-\xi_1, -\xi_2) \right] .
\]
The coefficient functions entering (42) and (43) have poles at \( \xi_1 = \xi_2 \). Their regularization is uniquely fixed by going from the moment space in (32) to the momentum-fraction representation as

\[
\frac{1}{(\xi_1 - \xi_2)^n} \equiv \lim_{\varepsilon \to 0} \frac{\theta(|\xi_1 - \xi_2| - \varepsilon)}{(\xi_1 - \xi_2)^n},
\]

and it amounts to creating a puncture\(^4\) on the integration axis at \( \xi_1 = \xi_2 \). We recall that \( \xi_q = \xi_1 \) and \( \xi_{\bar{q}} = -\xi_2 \) have the meaning of the longitudinal momentum fractions carried by quark and antiquark, respectively, whereas \( \xi_g = \xi_2 - \xi_1 \) defines the momentum fraction of a gluon, so that \( \xi_q + \xi_{\bar{q}} + \xi_g = 0 \). The singularities of the integrand in (42) and (43) at \( \xi_1 = \xi_2 \) can be attributed to a wee gluon in the initial state with a vanishing momentum fraction. As a check, it is straightforward to verify that the moments of (42) and (43) coincide with (32), given the regularization (45).

Obviously, Eqs. (42) and (43) have to be in agreement with (39) as \( x \to 1 \). To see this, we note first that, owing to the presence of \( \delta(\xi_i - x) \) and \( \theta(\xi_i - x) \) terms in the r.h.s. of (42) and (43), the momentum fraction of one of the quarks is restricted to \( x \leq \xi_i < 1 \). Performing integration over the momentum fraction of the other quark, we notice that the dominant contribution comes from the vicinity of the pole at \( \xi_1 = \xi_2 \). As a consequence, the asymptotic behaviour of the structure functions (42) and (43) is driven by the terms most singular at \( |\xi_1 - \xi_2| \to 0 \). For \( F_L(x) \) these terms are \( \sim \theta(\xi_2 - x)/(\xi_1 - \xi_2)^2 \) and for \( F_2(x) \) they look like \( \sim \delta(\xi_2 - x)/(\xi_1 - \xi_2)^2 \) and \( \sim \theta(\xi_2 - x)/(\xi_1 - \xi_2)^3 \). Assuming that the non-perturbative distributions \( \hat{T}_3(\xi_1, \xi_2) \) and \( \hat{T}_7(\xi_1, \xi_2) \) have the same behaviour for \( \xi_g = \xi_2 - \xi_1 \to 0 \), we find that the large \( x \) asymptotics of (42) and (43) is governed by the distribution amplitude \( T_7(\xi_1, \xi_2) \).

The asymptotic behaviour of the structure functions can be associated with the properties of the specific configuration probed at large \( x \): it describes the correlations between a state consisting of a quark carrying momentum fraction \( \xi_1 \sim x \) and a state consisting of a quark and a gluon, carrying momentum fractions \( \xi_2 \sim x \) and \( \xi_g = \xi_2 - \xi_1 \to 0 \), respectively.

One can come to the same conclusion in the moment space by examining the properties of the coefficient functions entering (30). Assuming that the functions \( \hat{T}_a(\xi_1, \xi_2) \) have a similar, non-singular behaviour, the large \( N \) behaviour of \( \hat{T}_a(N) \) is controlled by the large \( N \) behaviour of \( \Phi_{a,N}(\xi_1, \xi_2) \). One finds that at large \( N \) the functions \( \Phi_{3,N}(\xi_1, \xi_2) \) and \( \Phi_{7,N}(\xi_1, \xi_2) \), defined in the region \(-1 \leq \xi_1, \xi_2, \xi_1 - \xi_2 \leq 1 \), are peaked at the corners of the phase space \( \xi_1 = \xi_2 = \pm 1 \) and, therefore, the moments of the structure functions \( \hat{F}_2(N) \) and \( \hat{F}_L(N) \) receive dominant contribution at large \( N \) from the vicinity of these points. Comparing the contribution of the two operators in (29), we note that the peak value of \( \Phi_{7,N}(\xi_1, \xi_2) \) at \( \xi_1 = \xi_2 = \pm 1 \) is \( \Phi_7 = N(N^2 - 1)/6/(N + 2) \), which is larger than the corresponding peak of \( \Phi_{3,N} = N + 1 \). At large \( N \), the resulting hierarchy between the two is proportional to \( N \), in agreement with the conclusion we reached examining the ultraviolet divergent contribution in Eq. (38).

\(^4\)The limits \( \varepsilon \to 0 \) and \( N \to \infty \) do not commute. One has to calculate the moments first and then take the limit \( \varepsilon \to 0 \).
We conclude that the configuration which dominates the Born-level twist four contribution at large $x$ includes a gluon with a small longitudinal momentum fraction $\xi_g = \xi_2 - \xi_1 \to 0$. In this way it approaches the twist-two situation where the gluon is absent altogether. We also saw that the domination of the twist-four operator $O_7$ with respect to $O_3$ at large $x$ can be established independently of the ultraviolet dominance assumption.

Finally, let us discuss the reason for the operator $O_7$ to be dominant at large $x$ as compared with $O_3$. The difference between the two operators (24) manifests itself in the different power of the pre-factor $(py)$ in the r.h.s. of (27). The factor $(py)$ has the meaning of the light-cone time. For $x$ away from the end-point region $x = 1$ one has $(py) = O(1)$. On the other hand, for $x \to 1$, or equivalently $N \to \infty$, one finds that $(py) \sim N$. As a consequence, the contribution of the operator $O_7$ to the moments of the structure functions is enhanced by a factor $N$ as compared with $O_3$. To show this, it proves convenient to return to (21) and rewrite the moments $\tilde{D}_i(N+1)$ at large (even) $N$ in the following form

$$\tilde{D}_i(N+1) \equiv \int_0^1 d\xi \xi^N D_i(\xi) \approx \int_0^\infty d\xi e^{-N(1-\xi)} D_i(\xi), \quad (46)$$

where we have replaced $\xi^N = \exp(N \ln \xi) = \exp(-N [1 - \xi + \mathcal{O}((1-\xi)^2)])$, and extended the integration region by taking into account that $D_i(\xi) = 0$ for $\xi > 1$. Using (19), we express $D_i(\xi)$ as the Fourier transform of the matrix elements of non-local light-cone operators $Q_i(y)$,

$$D_i(\xi) = \int_{-\infty}^{\infty} \frac{d(py)}{\pi} e^{-2i\xi(py)} \langle p|Q_i(y)|p \rangle. \quad (47)$$

Substituting this relation into (46) and performing integration over $\xi$ one obtains,

$$\tilde{D}_i(N+1) = \int_{-\infty}^{\infty} \frac{d(py)}{2\pi i} \langle p|Q_i(y)|p \rangle \frac{\exp(-N)}{(py) + iN/2}. \quad (48)$$

To evaluate the integral over $(py)$, one uses the fact $\langle p|Q_i(y)|p \rangle$ has no singularities in the lower half plane\(^5\). Then, assuming an appropriate behaviour of the matrix element $\langle p|Q_i(y)|p \rangle$ at infinity and closing the integration contour in (48) in the lower half plane we get the following remarkable relation [15],

$$\tilde{D}_i(N+1) = -\langle p|Q_i(y)|p \rangle e^{-2i(py)} \bigg|_{2(py)=-iN}, \quad (49)$$

which is valid at large $N$. Here the factor $\exp(-2i(py))$ compensates the phases coming from the wave functions of quarks separated along on the light-cone at the large distance $\sim (py)$. Substituting (49) into (21), we find that the moments of the twist-four structure functions $F_L(x)$ and $F_2(x)$ at large $N$ are determined by the matrix elements of non-local light-cone operators $\langle p|Q_i(y)|p \rangle$ analytically continued into the complex plane and

\(^5\)This can be seen from examining Eq. (47) for $\xi > 1$, where $D_i(\xi)$ must vanish, so that closing the $(py)$ contour in the lower half plane encloses no poles.
evaluated at at large light-cone separations, \( 2(p_y) = -iN \). A similar relation can be established \[15\] for the large \( N \) asymptotics of twist two,

\[
q_N = \frac{\langle p | \Psi(y) \bar{\Psi}(-y) | p \rangle}{2(p_y)} e^{-2i(p_y)} \bigg|_{2(p_y) = -iN} .
\] (50)

Recall now that the operators \( Q_i(y) \) are given by a linear combination (23) of twist-four light-cone operators, \( O_\alpha \). Comparing the two matrix elements defined in (27), we observe that \( \langle p | O_7(v; y) | p \rangle \) is enhanced by an additional power of \( (p_y) \) as compared with \( \langle p | O_5(v; y) | p \rangle \). According to (49), the contribution of the latter operator to the moments of the twist-four structure functions \( F_L(x) \) and \( F_2(x) \) is suppressed by an additional factor \( N \) with respect to the former. To arrive at (29) one has to take into account the fact that the \( v \) integrals over the position of gluon field on the light-cone are dominated by \( v \sim 1/(p_y) \sim 1/N \). This brings the additional factor \( 1/N \) in front of \( \langle p | O_\alpha | p \rangle \) in agreement with large \( N \) behaviour of infrared renormalon contribution (36).

6 Conclusions

The breakdown of the twist expansion at large \( x \), and, in particular, the increase in the number of local matrix elements required to parametrize the contribution of each twist, makes that framework, as it stands, impractical. Nevertheless, the twist expansion can be used, as we did here at twist four, to identify the source of the dominant contributions at large \( x \), which scale as \( \sim N \Lambda^2/Q^2 \).

On very general grounds, one expects some simplification in the description of DIS structure functions at large \( x \). The leading twist simplifies in this limit since one is essentially probing only the valence parton distribution in the hadron. On the other hand, in this limit there are large perturbative and non-perturbative corrections associated with the formation of a narrow jet in the final state. Since the formation of the jet proceeds through colour exchange with the remnants of the target, the same corrections can be associated, by the OPE, to specific hadronic matrix elements. The physical picture we obtained is that the valence parton distribution remains the relevant physical quantity at large \( x \), in spite of the fact that multi-parton Fock states give important contributions, which lead to the breakdown of the twist expansion. The multi-parton correlations measured by individual higher twist terms reduce in this limit to a trivial part which is associated with the power-like evolution of the valence quark distribution. Since this evolution is a consequence of the exchange of soft gluons (with momenta \( \sim \sqrt{Q^2(1-x)} \)) the finer structure of the hadron is not resolved.

Our main conjecture in this paper is that the simplification of structure functions at large \( x \) is realised within the twist expansion through ultraviolet dominance, namely that

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6The role of the \( v \) integration differs for the case where \(|p_y|\) is large from the expansion in small \((p_y)\) used to derive Eq. (31). It should be noted that the function of \( v \) multiplying \( O_7 \) in Eq. (23) is related to the additional power of \((p_y)\) in Eq. (27) by the requirement that the matrix elements, \( \langle p | Q_i(y) | p \rangle \), should be analytic at small \((p_y)\).
each twist is dominated by the contribution which mixes under renormalization with the
leading twist. To prove this statement, it is necessary to calculate the relevant matrix
elements, a task which is definitely beyond reach of the tools we are using. Nevertheless,
examining in some detail the twist-four contribution at leading order provides significant
evidence supporting this conjecture.

After reviewing the standard twist-two and twist-four analysis we have examined
the fate of infrared renormalons. Following on from [16, 17], we have demonstrated
that infrared renormalon ambiguities appearing through the resummation of running-
coupling effects in the twist-two coefficient functions cancel within the OPE, for both
$F_2$ and $F_L$, with another ambiguity associated with the definition of twist-four matrix
elements of operators that mix under renormalization with twist two. While the former
ambiguity appears due to infrared sensitivity of twist two, the latter is directly related
to the ultraviolet renormalization properties of the twist-four operators. In fact, these
are two manifestations of the same phenomenon, namely the arbitrariness in the separa-
tion of different twists. The argument that exact cancellation of the ambiguity must
occur, is equivalent to the statement that the structure functions must be totally free of
factorization scale dependence.

Renormalon ambiguities pose no problem of consistency for the OPE. Full consis-
tency to order $1/Q^2$ is achieved upon resumming infrared renormalons at twist two and
including the parameterization of twist four within the same regularization prescription,
e.g. using Borel transforms.

The ultraviolet dominance assumption [21]–[26] amounts to a significant simplifica-
tion: instead of many parameters (a large number of local matrix elements), the entire
twist-four contribution is proportional to the twist-two matrix elements, and the only
non-perturbative parameter is the overall normalization of this contribution. Under this
assumption, the multi-parton correlation within the target reduces to the trivial part
which is proportional to the twist two matrix elements, whereas any more detailed cor-
relation is neglected. From the OPE perspective this assumption seems very strong, and
it is probably not justified for general $x$. On the other hand, the dominant contributions
at large $x$ are indeed associated with configurations similar to twist two:

- the recoiling system corresponding to the dominant twist-four contribution to $F_2$ at
leading order is that of a single hard quark (22). For $F_L$ the leading-order recoiling
system contains a hard quark and a hard gluon. These are the same configurations
that appear at twist two.

- the initial state at twist four contains a gluon in addition to the quark or anti-
quark. However, at large $x$ this gluon is constrained to have a small longitudinal
momentum fraction and, therefore, the matrix elements $\langle p | O_\alpha (v; y) | p \rangle$ are effec-
tively independent of $v$. Whereas the detailed multi-parton correlation of twist
four includes separate dependence on $v$ and $y$, its dominant large-$x$ ingredient
depends only on $py$, just like the twist-two matrix elements. Consequently the
number of relevant parameters does not increase at large $x$. 25
Having realized that the dominant contributions at large $x$ are associated with the twist-two like configurations, the assumption of ultraviolet dominance becomes natural. We conjecture that the simplification that occurs at large $x$ amounts to the dominance of this contribution which mixes under renormalization with the leading twist.

An additional simplification that occurs at large $x$, is that certain operators, whose matrix element become the largest at large light-cone separations $|p y| \sim N$, dominate. To leading order in the flavour expansion, two operators, $\mathcal{O}_7$ and $\mathcal{O}_3$, mix with the leading twist. However, at large $N$ the contribution of $\mathcal{O}_7$ alone dominates. Beyond the leading order in the flavour expansion, more operators, such as the two-gluon operators $\mathcal{O}_5$ and $\mathcal{O}_6$ or the four-quark operators $\mathcal{O}_1$ and $\mathcal{O}_2$ may contribute as well.

Our analysis of infrared renormalons was restricted to leading order in flavour expansion and, in parallel, we have applied the OPE (16) to the Born level. It is clearly of interest to extend the analysis beyond this order, particularly since we know from the standard twist-two analysis that multi-gluon emission is important at large $x$.

Higher-order corrections to the twist-two coefficient functions at large $x$ are well understood [12]–[15], [37]. They come from three different subprocesses: QCD evolution of the incoming quark state, $q_N$, hard scattering of the incoming quark off the virtual photon, $H$, and the propagation of the narrow quark jet in the final state, $J_N$. Due to different time scales involved in these three processes, they are quantum-mechanically incoherent, so their contribution is factorized in the moment space,

$$\tilde{F}_{tw}^{-2}(N, Q^2) = H_a(Q^2/\mu^2) J_a(Q^2/N\mu^2) q_N(\mu^2),$$

with $\mu^2$ being the factorization scale. Here, we have indicated explicitly the dependence of $H$ and $J$ on the relevant physical scales involved: the momentum transferred $Q^2$, and the invariant mass of the final state $W^2 \equiv Q^2(1-x)/x \sim Q^2/N$, respectively. The $\mu^2$ evolution of $q_N$ is induced by the emission of soft gluons with momentum $k \sim Q(1-x) \sim Q/N$. The $\mu$ dependence of the quark distribution $q_N$ follows from the renormalization properties of the non-local light-cone operator in the r.h.s. of (50)

$$\frac{d}{d \ln \mu^2} q_N(\mu^2) = -\Gamma_{cusp}(\alpha_s(\mu^2)) \cdot (\ln N) \cdot q_N(\mu^2),$$

with the anomalous dimension $\Gamma_{cusp}(\alpha_s) = \alpha_s C_F/\pi + \mathcal{O}(\alpha_s^2)$. This equation coincides with the DGLAP evolution equation with the matrix of anomalous dimension replaced by their large $N$ asymptotic behaviour. It is important to notice that at large $N$ the quark distribution evolves autonomously and does not mix with the twist-two gluon distribution. The latter will necessary involve the emission of a soft quark into the final state, so the corresponding contribution is suppressed\(^7\) by a power of $(1-x) \sim 1/N$. Solving (52) we can resum large Sudakov (double logarithmic) corrections associated with the infrared evolution of the quark distribution functions from the hard scale $Q^2$.

\(^7\)This can be also seen from the asymptotics of the DGLAP evolution kernel $P_{gg}(x) \sim (1-x)^0$ as $x \rightarrow 1$. 

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down to the scale $Q^2/N^2$, corresponding to the transverse momenta of soft gluons in the final state.

Another source of Sudakov corrections to the twist-two structure functions in (51) is the jet function $J$. Requiring that the l.h.s. of (51) should not depend on the factorization scale $\mu$ and taking into account that the $\mu$ dependence enters into the jet function only through $\alpha_s(\mu^2)$ and $Q^2/(N\mu^2)$, one can obtain from (51) and (52) the evolution equation on the jet function [38]. Solving this equation, one can resum large Sudakov logarithms to the twist-two structure functions (51). These corrections originate from an incomplete cancellation between virtual and real soft gluon contributions and can be attributed to QCD evolution of the outgoing quark jet carrying large energy $\sim Q$ but small invariant mass $\sim \sqrt{Q^2(1-x)}$.

The factorization of the twist-two structure functions into the product of three factors, $H$, $J$ and $q$, corresponding to three different subprocesses is a general property of QCD dynamics in DIS, which should be valid beyond the perturbative level. The twist four ingredient which mixes under renormalization with twist two has its own logarithmically-enhanced higher-order corrections, which include an evolution factor and a jet-function factor,

$$\tilde{F}^{tw-4}_a(N, Q^2)\bigg|_{UV} = H^{tw-4}_a(Q^2/\mu^2) J^{tw-4}_a(Q^2/N\mu^2) \frac{N\Lambda^2}{Q^2} q^{tw-2}_N(\mu^2).$$

The contribution (53) depends, of course, on the prescription used to regularize the renormalon sum in $J_a$ at twist two (51). We do not specify the regularization prescription here and below. The reader should keep in mind that this regularization is not associated with the conventional factorization (which is based on dimensional regularization and deals only with the logarithmic divergence), and therefore it is not necessarily associated with the factorization scale $\mu$; it can be introduced for example by the Borel transform as in Eq. (41). The relation between the principal value regularization of the Borel transform and the cutoff regularization of the corresponding momentum integrals has been investigated in [32]. The two differ just by renormalon-type power corrections of calculable magnitude.

The next observation is that independence of (53) of the factorization scale $\mu^2$ at the logarithmic level implies that the product of the factors $H$ and $J$ obeys the same evolution equation as the resummed twist-two coefficient function in (51). Thus, the coefficient functions in (53) differ from those in (51) only by the initial condition for the evolution

$$\kappa^{(a)}_1 \equiv \left[ H^{tw-4}_a(Q^2/\mu^2) J^{tw-4}_a(Q^2/N\mu^2) \right] / \left[ H_a(Q^2/\mu^2) J_a(Q^2/N\mu^2) \right],$$

which is a constant, $\kappa^{(a)}_1 = \mathcal{O}(1)$, up to corrections suppressed by powers of $\alpha_s$. We stress that $\kappa^{(a)}_1$ is strictly independent of the ratios $Q^2/\mu^2$ and $Q^2/(N\mu^2)$.

In conclusion, we find that ultraviolet dominance implies that the large $N$ asymptotic behaviour of the structure functions to twist-four accuracy takes the form,

$$\tilde{F}_a(N, Q^2) = \tilde{F}^{tw-2}_a(N, Q^2) + \tilde{F}^{tw-4}_a(N, Q^2) + \cdots$$
\[ H_a(Q^2/\mu^2) J_a(Q^2/N\mu^2) \frac{1}{N^{\frac{\Lambda^2}{Q^2}}} + \cdots, \]

where the ellipses stand for higher twist contributions, as well as for terms that are suppressed by $1/N$ at twist two and four.

Going to large $x$ one cannot consider twist four as the only relevant power correction. As mentioned above, the twist six contribution is expected to scale as $N^2/Q^4$. This is reflected in the renormalon contribution [22]. The analysis of twist six within the light-cone expansion is significantly more complicated than that of twist four. However, based on the experience gained here, we assume that the same picture emerges: amongst the large number of matrix elements needed to describe twist six contribution, the dominating contribution will be the one which mixes under renormalization with the leading twist. In particular, at Born-level for twist six, the dominant configuration to $F_2$ will be the one containing a single quark in the final state. Similarly, $F_L$ at twist six is dominated by the configuration containing a quark and a gluon in the final state, i.e. the same configuration as the lower-twist contributions.

Thus, “extrapolating” from the twist-four results, we expect that the dominant contributions to any twist is proportional to the twist-two parton distribution $q_N$. This leads to a generalization of the large-$x$ factorization formula (55) to any order in $N\Lambda^2/Q^2$. All these power corrections can be resummed into some shape function of a single variable $J_{aNP}(N\Lambda^2/Q^2)$ that multiplies $J_a(Q^2/N\mu^2)$,

\[ \tilde{F}_a(N, Q^2) = H_a(Q^2/\mu^2) J_a(Q^2/N\mu^2) J_{aNP}(N\Lambda^2/Q^2) q_N^{tw-2}(\mu^2). \]  

(56)

We stress that this formula is valid up to perturbative and non-perturbative corrections which are suppressed by $1/N$. The jet function itself now contains two factors, a perturbative factor $J_a(Q^2/N\mu^2)$ that can be calculated simply using the twist-two operators, and a non-perturbative factor $J_{aNP}(N\Lambda^2/Q^2)$ which sums up power corrections to all orders. The separation between them by regularizing the renormalon sum in $J_a(Q^2/N\mu^2)$ is, of course, arbitrary. However, the product is regularization independent: both factors together describe the hadronization of the recoiling jet. Dependence on the target is, in principle, contained in $J_{aNP}(N\Lambda^2/Q^2)$.

Let us try to understand the physical interpretation of the obtained expression for the structure functions (56). It is convenient to introduce a new non-perturbative distribution function $j_a(m^2)$ of a single argument [30, 31],

\[ J_{aNP}(N\Lambda^2/Q^2) = \int_0^\infty dm^2 \exp(-Nm^2/Q^2) j_a(m^2). \]  

(57)

By the definition, this function is an inclusive distribution describing the fragmentation of a partonic system with the invariant mass $m^2$ into the hadronic final state. $j_a(m^2)$ does not depend on the momentum transfer $Q^2$ and, it is, in this sense, a universal distribution function. Nevertheless, it does depend on the target as well as on the renormalon regularization prescription (see below). Inserting (57) into (56) and going
over from the moment space to the momentum-fraction representation\(^8\), we obtain the following expression for the structure function with the power corrections included,

\[
F_a(x, Q^2) = \int_x^1 d\xi F_a^{tw-2}(\xi, Q^2) [Q^2 j_a(Q^2(\xi - x))],
\]

where we assumed that \(j_a(m^2)\) vanishes fast for \(m^2 \gtrsim Q^2(1 - x)\). Here \(F_a^{tw-2}\) stands for the twist-two structure function in which infrared renormalon in the resummed coefficient functions have been regularized.

We conclude that the effect of the power corrections to all orders in \((N/Q^2)\) is to smear the twist-two structure functions close to the end-point, \(x \sim 1\). Whereas both \(F_a^{tw-2}\) and \(j_a\) depend on the renormalon regularization prescription, this dependence cancels out in their convolution in the r.h.s. of Eq. (58).

Let us consider the general properties of the distribution function \(j_a(m^2)\). It follows from our analysis that

\[
J_a^{NP}(N\Lambda^2/Q^2) = 1 + \kappa_1^{(a)} \frac{N\Lambda^2}{Q^2} + \kappa_2^{(a)} \left(\frac{N\Lambda^2}{Q^2}\right)^2 + \cdots,
\]

where \(\kappa_n\) are dimensionless non-perturbative parameters representing the leading contribution of terms with twist \(2(n + 1)\). \(\kappa_1^{(a)}\) was defined in (54). Note that it is not necessarily saturated by the single dressed gluon contribution, which is strictly associated with the light-cone operator \(\mathcal{O}_7\). Due to contributions of the two-gluon and the four-quark light-cone operators, which appear beyond the large \(\beta_0\) limit, \(\kappa_1^{(a)}\) can differ from \(\kappa_1^{[7]}\) of Eq. (41). In particular, at this level, the power corrections to \(F_2\) and \(F_L\) may not be controlled by the same operators.

This OPE based relation leads to a similar expansion of the function for \(j_a(m^2)\),

\[
j_a(m^2) = \delta(m^2) + \kappa_1^{(a)} \Lambda^2 \delta'(m^2) + \kappa_2^{(a)} \Lambda^4 \delta''(m^2) + \cdots.
\]

This expansion is singular as it goes over the derivatives of the delta function. In order to reconstruct the shape of the distribution \(j_a(m^2)\), one has to resum the whole series in \(N/Q^2\) in (59). Obviously, this will require a full knowledge of non-perturbative effects, which is far beyond reach. Nevertheless, based on a general properties of non-perturbative distributions [39], we expect that \(j_a(m^2)\) should vanish for \(m^2 < 0\), increase as a power at small \(m^2\), approach its maximal value at \(m^2 \sim \Lambda^2\) and, then, rapidly decrease at large \(m^2 \sim Q^2(1 - x)\).

Moreover, there is way to probe the structure of the non-perturbative distribution \(j_a(m^2)\) by Dressed Gluon Exponentiation [34], deducing an ansatz for \(J_a(m^2)\). Relying on the perturbative treatment of soft and collinear radiation one can conclude that renormalon related power-corrections exponentiate as well. This can be systematically established by studying the large-order behaviour of the Sudakov exponent: it turns out

\(^8\)At the accuracy considered, i.e. when terms suppressed by \(1/N\) are neglected \(x^{N-1} \simeq e^{-N(1-x)}\), so the Mellin transform is equivalent to a Laplace transform.
that the coefficients of sub-leading Sudakov logarithms are enhanced factorially due to infrared renormalons [32]. The enhancement of sub-leading logs implies that Sudakov resummation with a fixed logarithmic accuracy is insufficient in the limit under consideration. Instead, one is obliged to resum all the logarithms that are factorially enhanced in \( J_a \). As usual, when \( J_a \) is calculated perturbatively to power accuracy, a renormalon ambiguity shows up. This ambiguity is compensated at the non-perturbative level by \( J_a^{NP} \). The fact that the renormalon ambiguity in the Sudakov exponent is additive is fully consistent with the factorized form of (56) we obtained above.

At the level of a single dressed gluon, the exponent is equal to the logarithmically-enhanced part in the single dressed gluon result. It therefore has two Borel singularities, at \( u = 1, 2 \), suggesting the following ansatz for the non-perturbative jet function [34],

\[
J_a^{NP}(N\Lambda^2/Q^2) = \exp \left\{ -\omega_1^{(a)} \frac{C_F}{\beta_0} \frac{N\Lambda^2}{Q^2} - \frac{1}{2} \omega_2^{(a)} \frac{C_F}{\beta_0} \frac{N^2\Lambda^4}{Q^4} \right\},
\]

where \( \omega_1^{(a)} = -\kappa_1^{(a)} \) and \( \omega_2^{(a)} = -\kappa_2^{(a)} + \frac{1}{2} (\kappa_1^{(a)})^2 \) determine to the center and the width of the distribution \( j_a(m^2) \), respectively. Clearly, \( \omega_{1,2} \) depend on the regularization of the renormalon sum in the resummed coefficient function \( J_a \). It is interesting to note that the same structure of the exponent, and thus the same ansatz for power corrections in the large-\( x \) region, appears also in the case of the transverse and the total fragmentation functions in \( e^+e^- \) annihilation.

Eq. (61) should be regarded as a minimal ansatz for the parametrization of power corrections at large \( x \), since further renormalon ambiguities in \( J_a \) will probably appear once the analysis of the Sudakov exponent [34] is generalized to include more than a single dressed gluon. In this case further power corrections should be included in the exponent of Eq. (61). We must stress, that the suppression of such multi-gluon contributions to the non-perturbative exponent is not at all obvious, and it remains to be checked\(^9\). On the other hand, a non-perturbative distribution which is more involved than (61) can only be consistently used provided that renormalon resummation in \( J_a \) at the corresponding level is performed.

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References


\(^9\)Some phenomenological evidence of such suppression was found in [33], in the case of the thrust distribution in \( e^+e^- \) annihilation, barring hadron mass corrections.


