Renormalisation group flows for gauge theories in axial gauges

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Abstract
Gauge theories in axial gauges are studied using Exact Renormalisation Group flows. We introduce a background field in the infrared regulator, but not in the gauge fixing, in contrast to the usual background field gauge. It is shown how heat-kernel methods can be used to obtain approximate solutions to the flow and the corresponding Ward identities. Expansion schemes are discussed, which are not applicable in covariant gauges. As an application, we derive the one-loop effective action for covariantly constant field strength, and the one-loop β-function for arbitrary regulator.

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The perturbative sector of QCD is very well understood due to the weak coupling of gluons in the ultraviolet (UV) limit, known as asymptotic freedom. In the infrared (IR) region, however, the quarks and gluons are confined to hadronic states and the gauge coupling is expected to grow large. Thus the IR physics of QCD is only accessible with non-perturbative methods. The exact renormalisation group (ERG) provides such a tool [1,2]. It is based on a regularised version of the path integral for QCD, which is solved by successively integrating-out momentum modes.

ERG flows for gauge theories have been formulated in different ways (for a review, see [3]). Within covariant gauges, ERG flows have been studied in [4–6], while general axial gauges have been employed in [7,8]. In these approaches, gauge invariance of physical Greens functions is controlled with the help of modified Ward or Slavnov-Taylor identities [5–10]. A different line has been followed in [11] based on gauge invariant variables, e.g. Wilson loops. Applications of these methods to gauge theories include the physics of superconductors [12], the computation of instanton-induced effects [13], the heavy quark effective potential [14,15], effective gluon condensation [16], Chern-Simons theory [17], monopole condensation [18], chiral gauge theories [19], supersymmetric Yang-Mills theories [20], and the derivation of the universal two-loop beta function [21].

In the present paper, we use flow equations to study Yang-Mills theories within a background field method. In contrast to the usual background field formalism [22], we use a general axial gauge, and not the covariant background field gauge. The background field enters only through the regularisation, and not via the gauge fixing. Furthermore, in axial gauges no ghost degrees of freedom are present and Gribov copies are absent. Perturbation theory in axial gauges is plagued by spurious singularities of the propagator due to an incomplete gauge fixing, which have to be regularised separately. Within an exact renormalisation group approach, and as a direct consequence of the Wilsonian cutoff, these spurious singularities are absent [7]. The resulting flow equation can be used for applications even beyond the perturbative level. This formalism has been used for a study of the propagator [23], for a formulation of Callan-Symanzik flows in axial gauges [24], and for a study of Wilson loops [25].

Here, we continue the analysis of [7,8] and provide tools for the study of Yang-Mills theories within axial gauges. We discuss a framework for the evaluation of the path integral for covariantly constant fields. We use an auxiliary background field which allows us to define a gauge invariant effective action. The background field is introduced only in the regulator, in contrast to the usual background field formalism. This way it is guaranteed that all background field dependence vanishes in the infrared limit, where the cutoff is removed. We employ heat kernel techniques for the evaluation of the ERG flow. The heat kernel is used solely as a technical devise, and not as a regularisation. The flow equation itself is by
construction infra-red and ultra-violet finite and no further regularisation is required. As an explicit application, we compute the full one-loop effective action for non-Abelian gauge theories. This includes the universal \(\beta\)-function at one loop for arbitrary regulator. We also discuss new expansions of the flow, which are not applicable for covariant gauges.

The article is organised as follows. We begin with a brief review of the Wilsonian approach for gauge theories. This includes a derivation of the flow equation, the discussion of Ward-Takahashi identities and the background field dependence, as well as the definition of a gauge invariant effective action (Section II). Next, we derive the propagator for covariantly constant fields, and explain how expansions in the fields and heat kernel techniques can be applied in the present framework (Section III). We compute the full one loop effective action using heat kernel techniques. We also show in some detail how the universal beta function follows for arbitrary regulator functions (Section IV). We close with a discussion of the main results (Section V) and leave some more technical details to the Appendices.

II. WILSONIAN APPROACH FOR GAUGE THEORIES

In this section we review the basic ingredients and assumptions necessary for the construction of an exact renormalisation group equation for non-Abelian gauge theories in general axial gauges. The issue of gauge invariance of physical Greens functions, controlled by modified Ward-Takahashi identities, is discussed. This part is based on earlier work [7,8]. New material is contained in the discussion of the background field, which, in contrast to the usual background field method [22,8], will only be introduced for the Wilsonian regulator term. Finally, we define a gauge-invariant effective action as it follows from the present formalism.

A. Derivation of the flow

The starting point for the derivation of an exact renormalisation group equation are the classical action \(S_A\) for a Yang-Mills theory, an appropriate gauge fixing term \(S_{gf}\) and a regulator term \(\Delta S_k\), which introduces an infra-red cut-off scale \(k\) (momentum cut-off). This leads to a \(k\)-dependent effective action \(\Gamma_k\). Its infinitesimal variation w.r.t. \(k\) is described by the flow equation, which interpolates between the gauge-fixed classical action and the quantum effective action, if \(\Delta S_k\) and \(\Gamma_k\) satisfies certain boundary conditions at the initial scale \(\Lambda\). The classical action of a non-Abelian gauge theory is given by

\[
S_A[A] = \frac{1}{4} \int d^4x \, F^a_{\mu\nu}(A) F^{a\mu\nu}(A) \tag{2.1}
\]

with the field strength tensor

\[
F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{a}_{\ bc} A^b_\mu A^c_\nu \tag{2.2}
\]
and the covariant derivative

\[ D_{\mu}^{ab}(A) = \delta^{ab} \partial_{\mu} + gf^{abc} A_{\mu}^c, \quad [t^b, t^c] = f^{abc}_a t^a. \]  

(2.3)

A general axial gauge fixing is given by

\[ S_{gf}[A] = \frac{1}{2} \int d^4x n_{\mu} A_{\mu}^a \frac{1}{\xi n^2} n_{\nu} A_{\nu}^a. \]  

(2.4)

The gauge fixing parameter \( \xi \) has the mass dimension \(-2\) and may as well be operator-valued [7]. The particular examples \( \xi = 0 \) and \( \xi p^2 = -1 \) are known as the axial and the planar gauge, respectively. The axial gauge is a fixed point of the flow [7].

The scale-dependent regulator term is

\[ \Delta S_k[A, \bar{A}] = \frac{1}{2} \int d^4x A_{\mu}^a R_k^{ab}[\bar{A}] A_{\nu}^b. \]  

(2.5)

It is quadratic in the gauge field and leads to a modification of the propagator. We have introduced a background field \( \bar{A} \) in the regulator function. Both the classical action and the gauge fixing depend only on \( A \). The background field serves as an auxiliary field which can be interpreted as an index for a family of different regulators \( R_{k,\bar{A}} \). Its use will become clear below.

The scale dependent Schwinger functional \( W_k[J, \bar{A}] \), given by

\[ \exp W_k[J, \bar{A}] = \int \mathcal{D} A \exp \left\{ -S_k[A, \bar{A}] + \int d^4x A_{\mu}^a J_a^\mu \right\}, \]  

(2.6)

where

\[ S_k[A, \bar{A}] = S_A[A] + S_{gf}[A] + \Delta S_k[A, \bar{A}]. \]  

(2.7)

We introduce the scale dependent effective action \( \Gamma_k[A, \bar{A}] \) as the Legendre transform of \( W_k \)

\[ \Gamma_k[A, \bar{A}] = \int d^4x J_a^\mu A_{\mu}^a - W_k[J, \bar{A}] - \Delta S_k[A, \bar{A}], \quad A_{\mu}^a = \frac{\delta W_k[J, \bar{A}]}{\delta J_a^\mu}. \]  

(2.8)

For later convenience, we have subtracted \( \Delta S_k \) from the Legendre transform of \( W_k \). Thus \( \Gamma_k[A, \bar{A}] \) is given by the integro-differential equation

\[ \exp -\Gamma_k[A, \bar{A}] = \int \mathcal{D} a \exp \left\{ -S_A[a] - S_{gf}[a] - \Delta S_k[a, \bar{A}] + \frac{\delta}{\delta A} \Gamma_k[A, \bar{A}](a - A) \right\}. \]  

(2.9)

The corresponding flow equation for the effective action

\[ \partial_t \Gamma_k[A, \bar{A}] = \frac{1}{2} \text{Tr} \left\{ G_k[A, \bar{A}] \partial_t R_k[\bar{A}] \right\} \]  

(2.10)
follows from (2.9) by using $\langle a - A \rangle = 0$. The trace sums over all momenta and indices, $t = \ln k$. $G_k$ is the full propagator of the field $A$, whereas $\bar{A}$ is not propagating. Its inverse is given by

$$
\left( G_k[A, \bar{A}] \right)^{-1} = \frac{\delta^2 \Gamma_k[A, \bar{A}]}{\delta A^a_\mu(x) \delta A^b_\mu(x')} + R_k[\bar{A}]_{ab} \mu \nu (x, x').
$$

There are no ghost terms present in (2.10) due to the axial gauge fixing.

\section*{B. Requirements for the regulator}

For the time being, we made no specifications regarding the regulator function. However, such specifications are crucial for two reasons. First of all, we have to ensure that the flow (2.10) interpolates between the gauge-fixed classical action and the full quantum effective action. Secondly, aiming at the construction of a gauge-invariant effective action, we have to ensure that the regulator, which depends on the background field $\bar{A}$, transforms appropriately under gauge transformations of the background field.

With this in mind, we impose the following properties on $R_k$. We require that $R_k[\bar{A}]$ transforms as a tensor under gauge transformations of $\bar{A}$ given in (2.21) below. Furthermore, we restrict ourselves to the following tensor structure\footnote{A discussion of general cut-off functions $R_k$ is given in [7].}

$$
R_k[\bar{A}] = D_T r(\bar{D}_T)
$$

with the yet unspecified function $r$. We introduced $D_T$, the Laplace operator for spin 1,

$$
D_{T,\mu \nu}^a (A) := -(D_\rho D^a_\rho)_{\mu \nu} (A) \delta_{\mu \nu} - 2g F^a_{\mu \nu} (A)
$$

and $\bar{D}_T = D_T(\bar{A})$. For vanishing background field the Laplacean $D_T$ reduces to the free Laplacean $D_T(0) = p^2$. In this case we have $R_k = p^2 r(p^2)$. We require the following properties,

$$
\lim_{k^2/p^2 \to 0} R_k = 0, \quad \lim_{k^2/p^2 \to \infty} R_k \to \infty, \quad \lim_{p^2 \to \infty} R_k = 0.
$$

Written in terms of some general Laplace operator $P^2(\bar{A})$, a typical example for the regulator functions $R_k(P^2)$ and $r(P^2)$ is

$$
R_k(P^2) = \frac{P^2}{\exp P^2/k^2 - 1}, \quad r(P^2) = \frac{1}{\exp P^2/k^2 - 1}.
$$

(2.15)
which meets the general properties as described in (2.14). For later use, we distinguish different regulators by their precise infra-red behaviour for $P^2 \to 0$. Consider

$$\lim_{P^2 \to 0} r(P^2) \sim \left( \frac{k^2}{P^2} \right)^\gamma. \quad (2.16)$$

Regulators with $\gamma = 1$ have a mass-like infra-red behaviour with $R_k(0) \sim k^2$. The example in (2.15) has $\gamma = 1$. In turn, regulator with $\gamma > 1$ diverge for small momenta.

It is seen by inspection of (2.9) and (2.14) that the saddle-point approximation about $A$ becomes exact for $k \to \infty$. Here, $\Gamma_k$ approaches the classical action. For $k \to 0$, in turn, the cut-off term disappears and we end up with the full quantum action. Hence, we confirmed that the functional $\Gamma_k$ indeed interpolates between the gauge-fixed classical and the full quantum effective action:

$$\lim_{k \to \infty} \Gamma_k[A, \bar{A}] \equiv S[A] + S_{gf}[A], \quad (2.17a)$$

$$\lim_{k \to 0} \Gamma_k[A, \bar{A}] \equiv \Gamma[A]. \quad (2.17b)$$

Notice that both limits are independent of $\bar{A}$ supporting the interpretation of $\bar{A}$ as an index for a class of flows. It is worth emphasising that both the infrared and ultraviolet finiteness of (2.10) are ensured by the conditions (2.14) on $R_k$.

### C. Modified Ward-Takahashi Identities

We now address the issue of gauge invariance for physical Greens functions. The problem to face is that the presence of a regulator term quadratic in the gauge fields is, a priori, in conflict with the requirements of a (non-linear) gauge symmetry. This question has been addressed earlier for Wilsonian flows within covariant gauges [4–7,9]. The resolution to the problem is that modified Ward-Takahashi identities (as opposed to the usual ones) control the flow such that physical Greens functions, obtained from $\Gamma_k$ at $k = 0$, satisfy the usual Ward-Takahashi identities.

The same line of reasoning applies in the present case even though in the presence of the background field $\bar{A}$ some refinement is required [8]. In this particular point it is quite similar to the symmetry properties of the full background field formalism as discussed in [10]. The background field makes it necessary to deal with two kinds of modified Ward-Takahashi Identities. The first one is related to the requirement of gauge invariance for physical Green functions, and is known as modified Ward Identity (mWI). The second one has to do with the presence of a background field $\bar{A}$ in the regulator term $R_k$, and will be denoted as the background field Ward-Takahashi Identity (bWI).

To simplify the following expressions let us introduce the abbreviation $\delta_\omega$ and $\delta_\omega$ for the generator of gauge transformations on the fields $A$ and $\bar{A}$ respectively:
\[ \delta_\omega A = D(A)\omega \quad \bar{\delta}_\omega \bar{A} = 0 \]  \hspace{1cm} (2.18a)
\[ \bar{\delta}_\omega A = 0 \quad \delta_\omega \bar{A} = D(\bar{A})\omega. \]  \hspace{1cm} (2.18b)

The action of the gauge transformations \( \delta_\omega \) and \( \bar{\delta}_\omega \) on the effective action \( \Gamma_k \) can be computed straightforwardly. It is convenient to define

\[ W_k[A, \bar{A}; \omega] \equiv \delta_\omega \Gamma_k[A, \bar{A}] - \text{Tr} \left( n_\mu \partial_\mu \omega \right) \frac{1}{n^2} n_\nu A_\nu + \frac{1}{2} \text{Tr} \omega \left[ G_k[A, \bar{A}], R_k[\bar{A}] \right] \]  \hspace{1cm} (2.19a)
\[ \bar{W}_k[A, \bar{A}; \omega] \equiv \bar{\delta}_\omega \Gamma_k[A, \bar{A}] - \frac{1}{2} \text{Tr} \omega \left[ G_k[A, \bar{A}], R_k[\bar{A}] \right]. \]  \hspace{1cm} (2.19b)

In terms of (2.19), the behaviour of \( \Gamma_k[A, \bar{A}] \) under the transformations \( \delta_\omega \) and \( \bar{\delta}_\omega \), respectively, is given by

\[ W_k[A, \bar{A}; \omega] = 0 \]  \hspace{1cm} (2.20a)
\[ \bar{W}_k[A, \bar{A}; \omega] = 0 \]  \hspace{1cm} (2.20b)

In (2.20b) we have used that the regulator function transforms as a tensor under \( \delta_\omega \),

\[ \bar{\delta}_\omega R_k[\bar{A}] = \left[ R_k[\bar{A}], \omega \right]. \]  \hspace{1cm} (2.21)

Eq. (2.20a) is referred to as the modified Ward-Takahashi identity, and (2.20b) as the background field Ward-Takahashi identity.

Let us show that (2.20) is consistent with the basic flow equation (2.10). With consistency, we mean the following. Assume, that a functional \( \Gamma_k \) is given at some scale \( k \) which is a solution to both the mWI and the bWI. We then perform a small integration step from \( k \) to \( k' = k - \Delta k \), using the flow equation, and ask whether the functional \( \Gamma_{k'} \) again fulfills the required Ward identities (2.20). That this is indeed the case is encoded in the following flow equations for (2.20), namely

\[ \partial_t W_k[A, \bar{A}; \omega] = -\frac{1}{2} \text{Tr} \left( G_k \frac{\partial R_k}{\partial t} G_k \frac{\delta}{\delta A} \otimes \frac{\delta}{\delta \bar{A}} \right) W_k[A, \bar{A}; \omega] \]  \hspace{1cm} (2.22a)
\[ \partial_t \bar{W}_k[A, \bar{A}; \omega] = \frac{1}{2} \text{Tr} \left( G_k \frac{\partial R_k}{\partial t} G_k \frac{\delta}{\delta \bar{A}} \otimes \frac{\delta}{\delta A} \right) \bar{W}_k[A, \bar{A}; \omega], \]  \hspace{1cm} (2.22b)

where \( \left( \frac{\delta}{\delta A} \otimes \frac{\delta}{\delta \bar{A}} \right)_{\mu\nu}^{ab} (x, y) = \frac{\delta}{\delta A(x)} \frac{\delta}{\delta \bar{A}(y)} \). Eq. (2.22) states that the flow of mWI is zero if the mWI is satisfied for the initial scale. The required consistency follows from the fact that the flow is proportional to the mWI itself (2.22a), which guarantees that (2.20a) is a fixed point of (2.22a). The same follows for the bWI by using (2.22b). There is no fine-tuning involved in lifting a solution to (2.20a) to a solution to (2.20b). It also straightforwardly follows from (2.22a) and (2.22b).

We close with a brief comment on the use of mass term regulators. Such a regulator corresponds simply to \( R_k = k^2 \) and leads to a Callan-Symanzik flow. The regulator is
momentum-independent, which implies that the loop term in (2.19a) vanishes identically. Hence one concludes that the modified Ward identity reduces to the usual one for all scales $k$. This happens only for an axial gauge fixing [7].

D. Gauge invariant effective action

Returning to our main line of reasoning and taking advantage of the results obtained in the previous section, we define a gauge invariant effective action only dependent on $A$ by identifying $\bar{A} = A$. It is obtained for a particular choice of the background field, and provides the starting point for our formalism.

It is a straightforward consequence of the mWI (2.20a) and the bWI (2.20b) that the effective action $\Gamma_k[A, \bar{A}]$ is gauge invariant – up to the gauge fixing term – under the combined transformation

$$ (\delta_\omega + \bar{\delta}_\omega) \Gamma_k[A, \bar{A}] = \text{Tr} n_\mu (\partial_\mu \omega) \frac{1}{n^2 \xi} n_\nu A_\nu. \quad (2.23) $$

We define the effective action $\hat{\Gamma}_k[A]$ as

$$ \hat{\Gamma}_k[A] = \Gamma_k[A, \bar{A} = A]. \quad (2.24) $$

The action $\hat{\Gamma}_k[A]$ is gauge invariant up to the gauge fixing term, to wit

$$ \delta_\omega \hat{\Gamma}_k[A] = \text{Tr} \left\{ n_\mu (\partial_\mu \omega) \frac{1}{n^2 \xi} n_\nu A_\nu \right\}. \quad (2.25) $$

This follows from (2.23). Because of (2.17b), the effective action $\hat{\Gamma}_k=0[A]$ is the full effective action. The flow equation for $\hat{\Gamma}_k[A]$ can be read off from the basic flow equation (2.10),

$$ \partial_t \hat{\Gamma}_k[A] = \frac{1}{2} \text{Tr} \left\{ G_k[A, A] \partial_t R_k[A] \right\}, \quad (2.26) $$

Notice that the right-hand side of (2.26) is not a functional of $\hat{\Gamma}_k[A]$. The flow depends on the full propagator $G_k[A, A]$, which is the propagator of $A$ in the background of $\bar{A}$ taken at $\bar{A} = A$. Thus for the flow of $\hat{\Gamma}_k[A]$ one needs to know the flow (of a subset) of vertices of $\delta^2 \Gamma_k[A, \bar{A}] / (\delta A)^2$ at $\bar{A} = A$. Still, approximations, where this difference is neglected are of some interest.

We argue that (2.25) has far reaching consequences for the renormalisation procedure of $\hat{\Gamma}_k[A]$ as is well-known for axial gauges and the background field formalism. $\Gamma_k[A]$ is gauge invariant up to the breaking due to the gauge fixing term. We define its gauge invariant part as

$$ \Gamma_{k,\text{inv}}[A] = \Gamma_k[A] - S_{gf}[A] \quad (2.27a) $$

$$ \delta_\omega \Gamma_{k,\text{inv}}[A] = 0. \quad (2.27b) $$
Eq. (2.27) implies that the combination $gA$ is invariant under renormalisation, $\partial_t(gA) = 0$. If one considers wave function renormalisation and coupling constant renormalisation for $A$ and $g$ respectively

\[
A \rightarrow Z_{F}^{1/2} A \tag{2.28a}
\]
\[
g \rightarrow Z_g g \tag{2.28b}
\]
we conclude that
\[
Z_g = Z_{F}^{-1/2}. \tag{2.29}
\]

E. Background field dependence

By construction, the effective action $\Gamma_k[A, \bar{A}]$ at some finite scale $k \neq 0$ will depend on the background field $\bar{A}$. This dependence disappears for $k = 0$. The effective action $\hat{\Gamma}_k[A]$ is the simpler object to deal with as it is gauge invariant and only depends on one field. As we have already mentioned below (2.26), its flow depends on the the propagator $\delta^2 \Gamma_k[A, \bar{A}]$ at $A = \bar{A}$. Eventually we are interesting in approximations where we substitute this propagator by $\delta^2 \hat{\Gamma}_k$. The validity of such an approximation has to be controlled by an equation for the background field dependence of $\Gamma_k[A, \bar{A}]$. The flow of the background field dependence of $\Gamma_k[A, \bar{A}]$ can be derived in two ways. $\delta_\bar{A} \partial_t \Gamma_k$ can be derived from the flow equation (2.10),

\[
\frac{\delta}{\delta \bar{A}} \partial_t \Gamma_k[A, \bar{A}] = \frac{1}{2} \frac{\delta}{\delta \bar{A}} \text{Tr} \left\{ G_k[A, \bar{A}] \partial_t \bar{A} R_k[\bar{A}] \right\}. \tag{2.30}
\]

The flow $\partial_t \delta_\bar{A} \Gamma_k$ follows the observation that the only background field dependence of $\Gamma_k$ originates in the regulator. Thus, $\delta_\bar{A} \Gamma_k$ is derived along the same lines as the flow itself and we get

\[
\partial_t \frac{\delta}{\delta \bar{A}} \Gamma_k[A, \bar{A}] = \frac{1}{2} \text{Tr} \partial_t \left\{ G_k[A, \bar{A}] \delta_\bar{A} R_k[\bar{A}] \right\}, \tag{2.31}
\]

which turns out to be important also for the derivation of the universal one loop $\beta$-function in Sect. IVB. The difference of (2.30) and (2.31) has to vanish

\[
\left[ \frac{\delta}{\delta \bar{A}}, \partial_t \right] \Gamma_k[A, \bar{A}] = 0. \tag{2.32}
\]

Eq. (2.32) combines the flow of the intrinsic $\bar{A}$-dependence of $\Gamma_k[A, \bar{A}]$ (2.31) with the $\bar{A}$-dependence of the flow equation itself (2.30). It provides a check for the validity of a given approximation. Using the right hand sides of (2.30) and (2.31) the consistency condition (2.32) can be turned into
\[
\text{Tr} \left\{ G_k \frac{\delta \Gamma_k^{(2)}}{\delta A} G_k \partial_t G_k \right\} = \text{Tr} \left\{ G_k \frac{\delta R_k}{\delta A} G_k \partial_t \Gamma_k^{(2)} \right\},
\]

(2.33)

where

\[
\Gamma_k^{(2)}[A, \bar{A}]_{\mu \nu} (x, x') = \frac{\delta^2 \Gamma_k[A, \bar{A}]}{\delta A_\mu^a (x) \delta A_\nu^b (x')},
\]

(2.34)

With (2.33), we control the approximation

\[
\left. \frac{\delta^2 \Gamma_k[A, \bar{A}]}{\delta A \delta A} \right|_{\bar{A} = A} = \frac{\delta^2 \hat{\Gamma}_k[A]}{\delta A \delta A} + \text{sub-leading terms}
\]

(2.35)

For this approximation the flow (2.26) is closed and can be calculated without the knowledge of \(\Gamma_k^{(2)}\), but with \(\hat{\Gamma}_k^{(2)}\). Amongst others, the approximation (2.35) is implicitly made within proper-time flows, where the use of heat-kernel methods is even more natural [26]. This is discussed in [27]. Let us finally comment on the domain of validity for the approximation (2.35). In the infrared \(k \to 0\), the dependence of the effective action \(\Gamma_k[A, \bar{A}]\) on the background field \(A\) becomes irrelevant, because the regulator \(R_k[A]\) tends to zero. Therefore we can expect that (2.35) is reliable in the infrared, which is the region of interest.

### III. ANALYTIC METHODS

In this section we develop analytical methods to study flow equations for gauge theories in general axial gauges. The flow equation is a one-loop equation which makes it possible to use heat kernel techniques for its solution. The main obstacles, technically speaking, are the constraint imposed by the modified Ward identity and the necessity to come up with a closed form for the full propagator. We first derive such an expression for the case of covariantly constant fields within general axial gauges. In addition a generic expansion procedure in powers of the fields is discussed. Finally, we give the basic heat kernels to be employed in the next section.

#### A. Propagator for covariantly constant fields

We derivate an explicit expression for the full propagator for specific field configurations. This is a prerequisite for the evaluation of the flow equation (2.10). To that end we restrict ourselves to field configurations with covariantly constant field strength (see e.g. [28]), namely \(D_\mu F_{\nu \rho} = 0\). This is a common procedure within the algebraic heat kernel approach. We also use the existence of the additional Lorentz vector to demand \(n_\mu A^\mu = n_\mu F_{\mu \nu} = 0\). That this can be achieved is proven by the explicit example of \(n_\mu = \delta_{\mu 0}\) and \((A_\mu) = (A_0 = 0, A_i(\vec{x}))\). These constraints lead to
\[ [D_\mu, F_{\nu\rho}] = 0, \quad n_\mu A_\mu = 0 \]  
\[ n_\mu F_{\mu\nu} = 0. \]  
(3.1a)

To keep finiteness of the action of such configurations we have to go to a theory on a finite volume. However, the volume dependence will drop out in the final expressions and we smoothly can take the limit of infinite volume. For the configurations satisfying (3.1) we derive the following properties

\[ [D^2, D_\mu] = -2gF_{\mu\rho}D_\rho, \]  
(3.2a)

\[ D_{T,\mu\rho}D_\rho = -D_\mu D^2, \]  
(3.2b)

\[ [n_\rho D_\rho, D_\mu] = 0. \]  
(3.2c)

Defining the projectors \( P_n \) and \( P_D \) with

\[ P_{n,\mu\nu} = \frac{n_\mu n_\nu}{n^2}, \]  
(3.3a)

\[ P_{D,\mu\nu} = D_\mu \frac{1}{D^2} D_\nu, \]  
(3.3b)

we establish that

\[ P_D D_T = -P_D D^2 P_D, \quad P_n D_T = -P_n D^2 \]  
(3.4)

holds true. After these preliminary considerations we consider the gauge-fixed classical action given in (2.1). We need the propagator on tree level to obtain the traces on one-loop level. The initial action reads

\[ \Gamma_A[A] = S_A + S_{gf}. \]  
(3.5)

From (3.5) we derive the full inverse propagator as

\[ \Gamma^{(2)ab}_{k,\mu\nu}[A, A] = \left( D^{ab}_{T,\mu\nu} + (D_\mu D_\nu)^{ab} + \frac{1}{\xi n^2} n_\mu n_\nu \delta^{ab} \right) + O(g^2; D_T, D_\mu D_\nu). \]  
(3.6)

The inverse propagator (3.6) is an operator in the adjoint representation of the gauge group. We now turn to the computation of the propagator (2.11) for covariantly constant fields. Using (3.6), (3.1) and (3.2), we find

\[ G_k[A, A]^{ab}_{\mu\nu} = -\left( \left( \frac{a_1}{D_T} \right)_{\mu\nu} + D_\mu \frac{a_2}{D^2} D_\nu + n_\mu \frac{a_3}{D^2(nD)} D_\nu + D_\mu \frac{a_3}{D^2(nD)} n_\nu + n_\mu \frac{a_4}{n^2 D^2} n_\nu \right), \]  
(3.7)

with the dimensionless coefficient functions.
They imply that the propagator (3.6) corresponds to the particular case where we take the full (covariant) momentum dependence of the propagator into account. The inverse expression for the full propagator, similar to (3.7), follows from (3.12). Such approximations (3.7) simplifies considerably. With (3.6) and (3.9) we get

$$a_1 = \frac{1}{1 + r_T},$$  \hspace{2cm} (3.8a) \\
$$a_2 = \frac{1 - \xi D^2(1 + r_D)}{(1 + r_D)} \left( s^2 + r_D[1 - D^2\xi(1 + r_D)] \right)^{-1},$$  \hspace{2cm} (3.8b) \\
$$a_3 = -\frac{s^2}{(1 + r_D)} \left( s^2 + r_D[1 - D^2\xi(1 + r_D)] \right)^{-1},$$  \hspace{2cm} (3.8c) \\
$$a_4 = -\frac{r_D}{(1 + r_D)} \left( s^2 + r_D[1 - D^2\xi(1 + r_D)] \right)^{-1}. $$  \hspace{2cm} (3.8d)

Notice that $a_1$ is a function of $D_T$ while $a_2$, $a_3$ and $a_4$ are functions of both $D^2$ and $(nD)^2$. We also introduced the convenient short-hands

$$r_T \equiv r_k(D_T), \quad r_D \equiv r_k(-D^2), \quad s^2 \equiv \frac{(nD)^2}{(n^2D^2)}. $$  \hspace{2cm} (3.9)

The regulator function, as introduced in (2.12), depends on $D_T$. The dependence on $D^2$, as apparent in the terms $a_2$, $a_3$ and $a_4$, comes into game due to the conditions (3.1) and (3.2). They imply

$$r_k(D_T)D_\mu = D_\mu r_k(-D^2), \quad r_k(D_T)n_\mu = n_\mu r_k(-D^2),$$  \hspace{2cm} (3.10)

which can be shown term by term for a Taylor expansion of $r_k$ about vanishing argument. For vanishing field $A = 0$ the propagator (3.7) reduces to the one already discussed in [7]. There, it has been shown that the regularised propagator (3.7) (for $r \neq 0$) is not plagued by the spurious propagator singularities as encountered within standard perturbation theory, and in the absence of a regulator term ($r = 0$). For the axial gauge limit $\xi = 0$ the expression (3.7) simplifies considerably. With (3.6) and (3.9) we get

$$G_{k,\mu\nu}[A] = \left( \frac{1}{D_T(1 + r_T)} \right)_{\mu\nu} - D_\mu D_\nu \frac{1}{D^4(1 + r_D)(s^2 + r_D)} + \frac{n_\mu}{n^2} \frac{nD}{D^4(1 + r_D)(s^2 + r_D)} D_\nu$$

$$+ D_\mu \frac{nD}{D^4(1 + r_D)(s^2 + r_D)} n_\nu \frac{r_D}{D^2(1 + r_D)(s^2 + r_D)} P_{n,\mu\nu}. $$  \hspace{2cm} (3.11)

The propagators (3.7) and (3.11) are at the basis for the following computations. Notice that this analysis straightforwardly extends to approximations for $\Gamma_k[A, \bar{A}]$ beyond the one-loop level. Indeed, it applies for any $\Gamma_k[A, \bar{A}]$ such that $\Gamma_{k,\mu\nu}^{(2)}[A, A]$ is of the form

$$\Gamma_{k,\mu\nu}^{(2)}[A, A] = f_k^{D_T} D_T \mu\nu + D_\mu f_k^{DD} D_\nu + n_\mu f_k^{nD} n_\nu + n_\mu f_k^{nn} n_\nu. $$  \hspace{2cm} (3.12)

Here, the scale-dependent functions $f_k^{D_T}$ and $f_k^{DD}$ can depend on $D_T$, $D^2$ and $nD$. In turn, the functions $f_k^{nD}$ and $f_k^{nn}$ can depend only on $D^2$ and $nD$. An explicit analytical expression for the full propagator, similar to (3.7), follows from (3.12). Such approximations take the full (covariant) momentum dependence of the propagator into account. The inverse propagator (3.6) corresponds to the particular case $f_k^{D_T} = f_k^{DD} = 1$, $f_k^{nD} = 0$, and $f_k^{nn} = 1/\xi$.  

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Even for analytic calculations one wishes to include more than covariantly constant gauge fields, and to expand in powers of the fields, or to make a derivative expansion. Eventually one has to employ numerical methods where it is inevitable to make some sort of approximation. Therefore it is of importance to have a formulation of the flow equation which allows for simple and systematic expansions.

In this section we are arguing in favour for a different splitting of the propagator which makes it simple to employ any sort of approximation one may think of. For this purpose we employ the regulator \( R_k[D^2(A)] \). This is an appropriate choice since it has no negative eigenvalues. We split the inverse propagator into

\[
\Gamma_{k,\mu\nu}^{(2)ab}[A] = \Delta_{\mu\nu}^{ab} - (2gF_{\mu\nu}^{ab} - (D_\mu D_\nu)^{ab})
\]

with

\[
\Delta_{\mu\nu}^{ab} = \left\{-D^2(1 + r_D)\right\}^{ab}_{\mu\nu} + \frac{1}{\xi n^2}n_\mu n_\nu \delta^{ab}.
\]

The operator \( \Delta \) can be explicitly inverted for any field configuration (and \( A = \bar{A} \)). We have

\[
\Delta^{-1} = -\frac{1}{D^2(1 + r_D)}I + \frac{1}{D^2(1 + r_D)}\frac{1}{1 + \xi D^2(1 + r_D)}P_n.
\]

With (3.13) and (3.15) we can expand the propagator as

\[
G_k[A, A] = \Delta^{-1} \sum_{n=0}^{\infty} \left[(2gF - D \otimes D) \Delta^{-1}\right]^n.
\]

where \((D \otimes D)^{ab}_{\mu\nu}(x, y) = D^{ac}_{\mu}D^{cb}_{\nu}\delta(x - y)\). For \( \xi = 0 \) (the axial gauge), \( \Delta^{-1} \) can be neatly written as

\[
\Delta^{-1}(\xi = 0) = -\frac{1}{D^2(1 + r_D)}(I - P_n),
\]

which simplifies the expansion (3.16). The most important points in (3.16) concern the fact that it is valid for arbitrary gauge field configurations and each term is convergent for arbitrary gauge fixing parameter \( \xi \). Moreover such an expansion is not possible in the case of covariant gauges. Both facts mentioned above are spoiled in this case.

C. Heat kernels

We present closed formulae for the heat-kernel of the closely related operators \( D_T \) and \(-D^2 = D_T + 2gF\). These are needed in order to evaluate the traces in (4.15). We define the heat-kernels as \( K_O(\tau) = \exp\{\tau \mathcal{O}\}(x, x) \)
\[ K_{D^2}(\tau) = \int \frac{d^4p}{(2\pi)^4} e^{\tau X_\mu X^\mu}, \]
\[ K_{-D_T}(\tau) = e^{2\tau F} K_{D^2}(\tau), \]

where \( X_\mu = ip_\mu + D_\mu \) in the corresponding representation. Here we used that \( 2gF \) commutes with \( X_\mu \) for covariantly constant fields. All kernels are tensors in the Lie algebra (\( K_{-D_T} \) is also a Lorentz tensor because of the prefactor). For the calculation of the momentum integral we just refer the reader to the literature (e.g. [28]) and quote the result for covariantly constant field strength

\[ K_{D^2}(\tau) = \frac{1}{16\pi^2 \tau^2} \det \left[ \frac{\tau gF}{\sinh \tau gF} \right]^{1/2}, \]
\[ K_{-D_T}(\tau) = \exp(2\tau gF) K_{D^2}(\tau). \]

Here, the determinant is performed only with respect to the Lorentz indices. For the computation of the one-loop beta function we need to know \( K(\tau) \) in (3.19) up to order \( F^2 \) (equivalently to order \( \tau^0 \)). Expanding \( K_{D^2} \) in \( \tau gF \) we get

\[ K_{D^2}(\tau) = \frac{1}{16\pi^2} \left( \frac{1}{\tau^2} - \frac{1}{12} g^2 (F^2)_{\rho\rho} \right) + O[\tau, (gF)^3]. \]

With (3.20) and the expansion (\( \exp 2\tau gF \)) \( \mu\nu = 1 + 2\tau gF_{\mu\nu} + 2\tau^2 g^2 (F^2)_{\mu\nu} + O[\tau, (gF)^3] \) we read off the coefficient of the \( K(\tau) \) proportional to \( F^2 \),

\[ \text{Tr} K_{D^2}|_{F^2} = -\frac{1}{16\pi^2} \frac{4}{3} Ng^2 S_A[A], \]
\[ \text{Tr} K_{-D_T}|_{F^2} = \frac{1}{16\pi^2} \frac{20}{3} Ng^2 S_A[A], \]

where the trace \( \text{Tr} \) denotes a sum over momenta and indices. We have also used that \( S_A[A] = \frac{1}{2} \int \text{tr} F^2 \) with \( \text{tr}_f t^a t^b = -\frac{1}{2} \delta^{ab} \). Since the operators \( D_T \) and \( D^2 \) carry the adjoint representation the trace \( \text{Tr} \) includes \( \text{tr}_{ad} \) with \( 2N \text{tr}_f t^a t^b = \text{tr}_{ad} t^a t^b \).

**IV. APPLICATIONS**

In order to put the methods to work we consider in this section the full one-loop effective action for \( SU(N) \) Yang-Mills theory which entails the universal one-loop beta function for arbitrary regulator function.

**A. Effective action**

For the right hand side of the flow we need
\[ \Gamma_k[A, \bar{A}] = \frac{1}{2} \int Z_f(t) \text{tr}_f F^2(A) + S_{gf}[A] + O((gA)^5, g^2 \partial A], \quad \text{tr}_f t^a t^b = -\frac{1}{2} \delta^{ab} \quad (4.1) \]

where \( \text{tr}_R \) denotes the trace in the representation \( R \), \( R = f \) stands for the fundamental representation, \( R = ad \) for the adjoint representation. Only the classical action can contribute to the flow, as \( n \)-loop terms in (4.1) lead to \( n + 1 \)-loop terms in the flow, when inserted on the right hand side of (2.26). This Ansatz leads to the propagator (3.11) which together with our choice for the regulator (2.12) is the input in the flow equation (2.26). We also use the right hand side of (2.26). This Ansatz leads to the propagator (3.11) which together with our choice for the regulator (2.12) is the input in the flow equation (2.26). We also use the following in the evaluation of the different terms in (2.26):

\[ \text{tr} D^2 = 4 \text{tr} D \otimes D \quad (4.2) \]

With this we finally arrive at

\[ \partial_t \hat{\Gamma}_k = \frac{1}{2} \text{Tr} \left\{ \frac{\partial_t r(D_T)}{1 + r(D_T)} - \frac{1}{2} \frac{\partial_t r(-D_T)}{1 + r(-D_T)} + \frac{1}{4} \frac{\partial_t r(-D_T)}{s^2 + r(-D_T)} \right\}, \quad (4.3) \]

where the trace \( \text{Tr} \) contains also the Lorentz trace and the adjoint trace \( \text{tr}_{ad} \) in the Lie algebra. The first term on the right-hand side in (4.3) has a non-trivial Lorentz structure, while the two last terms are proportional to \( \delta_{\mu\nu} \). We notice that the flow equation (4.3) is well-defined in both the IR and the UV region. We apply the heat-kernel results of section III C to the calculation of (4.3). To that end we take advantage of the following fact: Given the existence (convergence, no poles) of the Taylor expansion of a function \( f(x) \) about \( x = 0 \) we can use the representation

\[ f(-\mathcal{O}) = f(-\partial_\tau) \exp\{\tau \mathcal{O}\}|_{\tau = 0} \quad (4.4) \]

Due to the infrared regulator the terms in the flow equation (4.3) have this property, where \( \mathcal{O} = D_T, D^2 \). Hence we can rewrite the arguments \( D_T \) and \( -D^2 \) in (4.3) as derivatives w.r.t. \( \tau \) of the corresponding heat kernels \( K_{-D_T}(\tau) \) and \( K_{D^2}(\tau) \). Applying this to the flow equation (4.3) we arrive at

\[ \partial_t \hat{\Gamma}_k = \frac{1}{2} \left[ \frac{\partial_t r(-\partial_\tau)}{1 + r(-\partial_\tau)} \text{Tr}K_{-D_T}(\tau) - \frac{1}{2} \frac{\partial_t r(\partial_\tau)}{1 + r(\partial_\tau)} \text{Tr}K_{D^2}(\tau) \right. \\
\left. + \frac{1}{4} \int dp_n \frac{(p_n^2 - \partial_\tau)\partial_t r(p_n^2 - \partial_\tau)}{p_n^2} \frac{\tau^{1/2}}{\sqrt{\pi}} \text{Tr}K_{D^2}(\tau) \right] \bigg|_{\tau = 0} \quad (4.5) \]

The two terms in the first line follow from (4.3). The last term is more involved because it depends on both \( D^2 \) and \( nD \) due to \( s^2 \equiv (nD)^2/n^2 D^2 \). We note that \( nD = (n\delta) \) holds for configurations satisfying (3.1a) and only depends on the momentum parallel to \( n_\mu \). Furthermore it is independent of the gauge field. Now we use the splitting of \( (p_\mu) = (p_n, \bar{p}) \) where \( p_n = P_n p \) and \( \bar{p} = (1 - P_n))p \). The heat kernel related to \( \bar{D}^2 \) follows from the one for \( D^2 \) via the relation \( K_{\bar{D}^2}(\tau) = \frac{\tau^{1/2}}{\sqrt{\pi}} K_{D^2}(\tau) \) as can be verified by a simple Gaussian integral in the \( p_n \)-direction.
With these prerequisites at hand, we turn to the full effective action at the scale $k$, which is given by

$$
\hat{\Gamma}_k = \hat{\Gamma}_\Lambda + \int_\Lambda^k dk' \frac{\partial \hat{\Gamma}_{k'}}{\partial k'},
$$

(4.6)

where $\Lambda$ is some large initial UV scale. We start with the classical action $\Gamma_\Lambda = S_A + S_{gf}$. Performing the $k$-integral in (4.6) we finally arrive at

$$
\hat{\Gamma}_k[A] = \left(1 + \frac{8g^2}{16\pi^2} \left(\frac{22}{3} - 7(1 - \gamma)\right) \ln k/\Lambda\right) S_A[A] + S_{gf}[A] + \sum_{m=1}^\infty C_m(k^2/\Lambda^2) \Delta \Gamma^{(m)}[g F/k^2] + \text{const.}
$$

(4.7)

The combination $S_A + S_{gf}$ on the right-hand side of (4.7) is the initial effective action. All further terms stem from the expansion of the heat kernels (3.19) in powers of $\tau$. The terms $\sim \tau^{-2}$ give field-independent contributions, while those $\sim \tau^{-1}$ are proportional to $\text{tr} F$ and vanish. The third term on the right-hand side of (4.7) stems from the $\tau^0$ coefficient of the heat kernel. This term also depends on the regulator function through the coefficient $\gamma$ (2.16). All higher order terms $\sim \tau^m, m > 0$ are proportional to the terms $C_m(k^2/\Lambda^2) \Delta \Gamma^{(m)}[g F/k^2]$. These terms have the following structure: They consist of a prefactor

$$
C_m(x) = -\frac{1}{4m} \frac{(-)^m}{m!} (1 - x^m)
$$

(4.8a)

and scheme-dependent functions of the field strength, $\Delta \Gamma^{(m)}[g F]$, each of which is of the order $2 + m$ in the field strength $g F$. They are given explicitly as

$$
\Delta \Gamma^{(m)}[g F] = B_m^{D_T} \text{Tr} K^{(m)}_{D_T}(0) + \left(B_m^{D^2} + B_m^{nD}\right) \text{Tr} K^{(m)}_{D^2}(0).
$$

(4.8b)

Here, $K^{(m)}_{D^2}(0)$ and $K^{(m)}_{D_T}(0)$ denote the expansion coefficients of the heat kernels. We use the following identity

$$
f^{(m)}(0) = f(\partial \tau) \tau^m|_{\tau=0},
$$

(4.9)

and $f^{(m)}(x) = (\partial_x)^m f(x)$. In addition, the terms in (4.8b) contain the scheme-dependent coefficients

$$
B_m^{D_T} = \left(\frac{\dot{r}_1}{1 + r_1}\right)^{(m)}(0),
$$

(4.10a)

$$
B_m^{D^2} = -\frac{1}{2} B_m^{D_T},
$$

(4.10b)

$$
B_m^{nD} = \frac{(-1)^{m+1}}{4} \int_0^\infty dx \left(\partial_x - \frac{1}{x} \alpha \partial_\alpha\right)^{m+1} \frac{\dot{r}_1(x)}{\sqrt{r_1(x)} \sqrt{r_1(x) + \alpha}}|_{\alpha=1}.
$$

(4.10c)
The coefficients $B^{D_T}$, $B^{D^2}$ and $B^{nD}$ follow from the first, second and third term in (4.3). We introduced dimensionless variables by defining $r_1(x) = r(xk^2)$ and $\dot{r}_1(x) \equiv \partial x r_1(x) = -2xk^2 r'(xk^2) = -2xr'_1(x)$, in order to simplify the expressions and to explicitly extract the $k$-dependence into (4.8a). The explicit derivation of $B^{nD}$ is tedious but straightforward and given – together with some identities useful for the evaluation of the integral and the derivatives – in appendix A. All coefficients $B^{D_T}$, $B^{D^2}$ and $B^{nD}$ are finite. In particular, we can read off the coefficients for $m = 0$ which add up to the prefactor of the classical action in (4.7):

$$B^{D_T}_0 = 2\gamma, \quad B^{D^2}_0 = -\gamma, \quad B^{nD}_0 = -\frac{1}{2}(1 - \gamma),$$

(4.11)

where we have used (A.5) in the appendix. Together with the heat kernel terms proportional to $\tau^0$ given in (3.21) this leads to (4.7).

This application can be extended to include non-perturbative truncations. The flow of the coefficients (4.8b) becomes non-trivial, and regulator-dependent due to the regulator-dependence of the coefficients (4.10). Then, optimisation conditions for the flow can be employed to improve the truncation at hand [29].

Finally, we discuss the result (4.7) in the light of the derivative expansion. Typically, the operators generated along the flow have the structure $F f_k[(D^2 + k^2)/\Lambda^2] F$, and similar to higher order in the field strength. For dimensional reasons, the coefficient function $f_k(x)$ of the operator quadratic in $F$ develops a logarithm $\sim \ln x$ in the infrared region. An additional expansion of this term in powers of momenta leads to the spurious logarithmic infrared singularity as seen in (4.7). To higher order in the field strength, the coefficient function behave as powers of $1/(D^2 + k^2)$, which also, at vanishing momenta, develop a spurious singularity in the IR, and for the very same reason. All these problems are absent for any finite external gluon momenta, and are an artifact of the derivative expansion. A second comment concerns the close similarity of (4.7) with one-loop expressions found within the heat-kernel regularisation. In the latter cases, results are given as functions of the proper-time parameter $\tau$ and a remaining integration over $d \ln \tau$. Expanding the integrand in powers of the field strength and performing the final integration leads to a structure as in (4.7), after identifying $\tau \sim k^{-2}$. In particular, these results have the same IR structure as found in the present analysis.

**B. Running coupling**

We now turn to the computation of the beta function at one loop. We prove that the result is independent of the choice of the regulator and agrees with the standard one. However, it turns out that the actual computation depends strongly on the precise small-momentum behaviour of the regulator, which makes a detailed discussion necessary.
Naively we would read-off the $\beta$-function from the $t$-running of the term proportional to the classical action $S_A$ in (4.7). Using (2.29) leads to $\partial_t \ln Z_g = -\frac{1}{2} \partial_t \ln Z_F$. We get from (4.7) 

$$
Z_F = \left(\frac{22}{3} - 7(1 - \gamma)\right) \frac{Ng^2}{16\pi^2} t \quad \rightarrow \quad \partial_t \ln Z_g = -\left(\frac{11}{3} - \frac{7}{2}(1 - \gamma)\right) \frac{Ng^2}{16\pi^2} + O(g^4).
$$

(4.12)

We would like to identify $\beta = \partial_t \ln Z_g$. This relation, however, is based on the assumption that at one loop one can trade the IR scaling encoded in the $t$-dependence of this term directly to a renormalisation group scaling. This assumption is based on the observation that the coefficient of $S_A[A]$ is dimensionless and at one loop there is no implicit scale dependence. It is the latter assumption which in general is not valid. A more detailed analysis of this fact is given in [21]. Here, we observe that the background field dependence of the cut-off term inflicts contributions to $\partial_t Z_{F,S_{cl}}$. These terms would be regulator-dependent constants for a standard regulator without $\bar{A}$. As mentioned below (2.5), one should see the background field as an index for a family of different regulators. We write the effective action as

$$
\Gamma_k[A, \bar{A}] = \Gamma_{k,1}[A] + \Gamma_{k,2}[\bar{A}] + \Gamma_{k,3}[A, \bar{A}].
$$

(4.13)

The second term only depends on $\bar{A}$ and is solely related to the $\bar{A}$-dependence of the regulator. The last term accounts for gauge invariance of $\Gamma_k$ under the combined transformation $\delta_\omega + \bar{\delta}_\omega$. This term vanishes in the present approximation, because of the observation that our Ansatz is invariant – up to the gauge fixing term – under both $\delta_\omega$ and $\bar{\delta}_\omega$ separately. The physical running of the coupling is contained in the flow of $\Gamma_{k,1}[A]$. This leads to

$$
\beta = -\frac{1}{2} \partial_t Z_F + \frac{1}{2} \partial_t Z_{F,2}^2.
$$

(4.14)

where $Z_{F,2}$ is the scale dependence of $\Gamma_{k,2} \propto Z_{F,2} S_A[A]$. We rush to add that this procedure is only necessary because we are interested in extracting the universal one-loop $\beta$-function from the flow equation. For integrating the flow itself this is not necessary since for $k = 0$ the background field dependence disappears anyway. For calculating $\partial_t \ln Z_{F,2}$ we use (3.11) and (4.2) and get

$$
\partial_t \frac{\delta}{\delta A_\mu} \Gamma_k[A, \bar{A} = A] = \frac{1}{2} \text{Tr} \partial_t \left\{ \frac{R'_k[D_T]}{D_T + R_k[D_T]} \frac{\delta D_T}{\delta D_T} + \frac{1}{2} \frac{R'_k(-D^2)}{-D^2 + R_k[-D^2]} \frac{\delta D^2}{\delta D^2} \right. \\
- \frac{1}{4} \frac{R'_k[-D^2]}{(-nD)^2 + R_k[-D^2]} \frac{\delta D^2}{\delta D^2} \left. \right\},
$$

(4.15)

where we have introduced the abbreviation

$$
R'_k(x) = \partial_x R_k(x).
$$

(4.16)

For the derivation of (4.15) one uses the cyclicity of the trace and the relations (3.2). We notice that (4.15) is well-defined in both the IR and the UV region. The explicit calculation is done in appendix B. Collecting the results (B.2),(B.3),(B.4) we get
\[
\partial_t \delta A_k [A, \bar{A} = A]_{F^2} = -\frac{Ng^2}{16\pi^2} 7(1 - \gamma) \delta_A S_A [A] \rightarrow \partial_t Z_{F,2} = -\frac{Ng^2}{16\pi^2} 7(1 - \gamma) \quad (4.17)
\]

We insert the results (4.12) for \( \partial_t Z_F \) and (4.17) for \( \partial_t Z_{F,2} \) in (4.14) and conclude

\[
\beta = -\frac{11}{3} \frac{Ng^2}{16\pi^2} + O(g^6). \quad (4.18)
\]

which is the well-known one-loop result. For regulators with a mass-like infrared limit, \( \gamma = 1 \), there is no implicit scale dependence at one loop. It is also worth emphasising an important difference to Lorentz-type gauges within the background field approach. In the present case only the physical degrees of freedom scale implicitly with \( t = \ln k \) for \( \gamma \neq 0 \). This can be deduced from the prefactor \( 7(1 - \gamma) \) in (4.17). Within the Lorentz-type background gauge, this coefficient is \( \frac{22}{3}(1 - \gamma) \) [21]. The difference has to do with the fact that in the axial gauge one has no auxiliary fields but only the physical degrees of freedom. In a general gauge, this picture only holds true after integrating-out the ghosts. This integration leads to non-local terms. They are mirrored here in the non-local third term on the right hand side of the flow (4.5) and in the third term on the right hand side of (4.15) [see also (B.4)].

V. CONCLUSIONS

We have shown how the exact renormalisation group can be used for gauge theories in general axial gauges. We have addressed various conceptual points, in particular gauge invariance, which are at the basis for a reliable application of this approach. Our main goal was to develop methods which allow controlled and systematic analytical considerations. The formalism has the advantage that ghost fields are not required. Also, no additional regularisation – in spite of the axial gauge fixing – is needed. This is a positive side effect of the Wilsonian regulator term.

In addition, we worked in a background field formulation, which is helpful in order to construct a gauge invariant effective action. Also, it allows to expand the flow equation around relevant field configurations. Instead of relying on the standard background field gauge, we have introduced the background field only in the regulator term. The axial gauge fixing is independent on the background field. This way, it is guaranteed that the background field dependence vanishes in the IR limit. It is important to discuss how this differs from the usual background field approach to Wilsonian flows. In both cases, applications of the flow require an approximation, where derivatives w.r.t. the background field are neglected, cf. (2.35). In the present approach, this approximation improves in the infrared, finally becoming exact for \( k = 0 \) as the background field dependence disappears. For the background field gauge this does not happen, because the full effective action still depends non-trivially on the background field.
As an application, the full one-loop effective action and the universal beta-function have been computed. This enabled us to address some of the more subtle issues of the formalism like the implicit scale dependence introduced by the cutoff, which has properly to be taken into account for the computation of universal quantities, and the scheme independence of the beta-function. The equation which controls the additional background field dependence introduced by the cutoff contains the related information.

These results are an important step towards more sophisticated applications, both numerically and analytically. A natural extension concerns dynamical fermions. The present formalism is also well-adapted for QCD at finite temperature $T$, where the heat-bath singles-out a particular Lorentz vector. Here, an interesting application concerns the thermal pressure of QCD.

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A. EVALUATION OF THE ONE LOOP EFFECTIVE ACTION

The calculation of the last term in (4.7) is a bit more involved. Note that the following argument is valid for $m \geq -1$, $m > -1$ is of importance for the evaluation of (4.7), $m = -1$ will be used in Appendix B. We first convert the factor $\tau^{m+1/2}$ appearing in the expansion of the heat kernel using $\tau^{1/2+m} = (-1)^{m+1} \sqrt{\pi} \int dz \hat{g}^{m+1} e^{-\tau z^2}$. We further conclude that

$$B_m^{nD} = \frac{1}{4\pi} \int dp_n dz \frac{(p_n^2 - \partial_x) \partial_t r(p_n^2 - \partial_x)}{p_n^2 + (p_n^2 - \partial_x) r(p_n^2 - \partial_x)} \tau^{m+1} e^{-\tau z^2} \bigg|_{\tau=0}$$

$$= \frac{(-1)^{m+1}}{4\pi} \int dp_n dz \frac{\partial_t r(p_n^2 - \partial_x)}{p_n^2 + (p_n^2 - \partial_x) r(p_n^2 - \partial_x)} (p_n^2 - \partial_x) e^{-\tau z^2} \bigg|_{\tau=0}$$

$$= \frac{(-1)^{m+1}}{4\pi} \int dp_n dz \frac{\partial_t r(z^2 + p_n^2)}{z^2 + p_n^2 + r(z^2 + p_n^2)}.$$

The expression in (A.1) can be conveniently rewritten as

$$B_m^{nD} = \frac{(-1)^{m+1}}{8\pi} \int_0^\infty dx \int_0^{2\pi} d\phi \left( \partial_x - \frac{1}{x} \alpha \partial_\alpha \right)^{m+1} \frac{\partial_t r(x)}{\alpha \sin^2 \phi + r(x)} \bigg|_{\alpha=1}$$

$$= \frac{(-1)^{m+1}}{4} \int_0^\infty dx \left( \partial_x - \frac{1}{x} \alpha \partial_\alpha \right)^{m+1} \frac{\partial_t r(x)}{r(x) \sqrt{r(x) + \alpha}} \bigg|_{\alpha=1}. \quad (A.2)$$
where \( x = z^2 + p_n^2 \) and \( \sin^2 \phi = p_n^2/(z^2 + p_n^2) \). It is simple to see that \(-1/x \alpha \partial_\alpha\) is a representation of \( \partial_z \) on \( \sin^2 \phi = p_n^2/(z^2 + p_n^2) \) and \( \partial_\phi \) a representation of \( \partial_z \) on functions of \( x \) only. The expression in (A.2) is finite for all \( m \geq 0 \). Evidently it falls of for \( x \to \infty \). For the behaviour at \( x = 0 \) the following identity is helpful:

\[
\left( \partial_x - \frac{1}{x} \alpha \partial_\alpha \right)^{m+1} = \sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} \partial_x^{m+1-i} \alpha \partial_\alpha^{m+1-i}, \quad (A.3)
\]

Eq. (A.3) guarantees that the integrand in (A.2) only contains terms of the form

\[
\partial_x \left( \frac{r}{\sqrt{r+1}} (x+xr)^{i-m-1} \right) \quad (A.4)
\]

with \( i = 0, \ldots, m + 1 \). For \( x \to 0 \) one has to use that \( \partial_x r \to 2nr \) and \( r \to \frac{e^{2\gamma}}{x^n} \). The terms of integrand in (A.2) as displayed in (A.4) are finite for \( x = 0 \).

We are particularly interested in \( B_0 D \) relevant for the coefficient of \( S_A \) in the one loop effective action (4.7). With (A.2) it follows

\[
B_0 D = -\frac{1}{4} \int_0^\infty dx \left( \partial_x - \frac{1}{x} \alpha \partial_\alpha \right) \frac{\partial_x r(x)}{\sqrt{r(x)} \sqrt{r(x) + \alpha}} \bigg|_{\alpha=1}^{x=\infty} = -\frac{1}{4} \left( \frac{\partial_x r(x)}{\sqrt{r(x) + r(x)}} - 2 \frac{\sqrt{r(x)}}{\sqrt{1 + r(x)}} \right) \bigg|_{x=0} = -\frac{1}{2} (1 - \gamma), \quad (A.5)
\]

where we have used \( \partial_x r(z) = -2z \partial_x r(z) \) and the limits for \( \partial_x r(z \to 0) = 2\gamma z^{-\gamma}, r(z \to 0) = z^{1-\gamma}, r(z \to \infty) = 0 \).

**B. \( \bar{A} \)-DERIVATIVES**

For the calculation of (4.15) the following identity is useful:

\[
\text{Tr} \left( \frac{\delta}{\delta A_\mu^a} X_\mu X_\mu \right) e^{\tau X_\mu X_\mu} = \frac{1}{\tau} \text{Tr} \frac{\delta}{\delta A_\mu^a} e^{\tau X_\mu X_\mu}. \quad (B.1)
\]

Now we proceed in calculating the first term in (4.15) by using a similar line of arguments as in the calculation of (4.7) and in Appendix A. We make use of the representation of \( \tau^{-1} = \int_0^\infty dz \exp -\tau z \) and arrive at

\[
\frac{1}{2} \text{Tr} \partial_t \left( \frac{R_k[D_T]}{D_T + R_k[D_T]} \frac{\delta D_T}{\delta A_\mu^a} \right) = \frac{1}{2} \text{Tr} \partial_t \left( \frac{R_k(-\partial_T)}{-\partial_T + R_k(-\partial_T)} \frac{1}{\tau} \frac{\delta}{\delta A_\mu^a} K_{-D_T}(\tau) \right) \bigg|_{\tau=0} = \frac{1}{2} \int_0^\infty dx \partial_t \left( \frac{R_k[x]}{1 + r[x]} \right) \frac{Ng^2}{16\pi^2} \frac{20}{3} \frac{\delta}{\delta A_\mu^a} (S_A[A] + O[g]) = -\frac{Ng^2}{16\pi^2} \frac{20}{3} (1 - \gamma) \frac{\delta}{\delta A_\mu^a} (S_A[A] + O[g]). \quad (B.2)
\]
Note that $\partial_t$ acts as $-2x\partial_x$ on functions which solely depend on $x/k^2$. The term $R'/(1+r)$ is such a function. The second term can be calculated in the same way leading to

$$\frac{1}{4}\text{Tr} \partial_t \left\{ \frac{-R_k[D^2]}{-D^2 + R_k[-D^2]} \frac{\delta D^2}{\delta A^a_\mu} \right\} = \frac{1}{4} \int_0^\infty \frac{dx}{x} \partial_t \left( \frac{R'_k[x]}{1 + r[x]} \right) \frac{Ng^2 4 \delta}{16\pi^2 3 \delta A^a_\mu} (S[A] + O[g])$$

$$= -\frac{Ng^2 2}{16\pi^2 3} (1 - \gamma) \frac{\delta}{\delta A^a_\mu} (S[A] + O[g]).$$

(B.3)

The calculation of the last term in (4.15) is a bit more involved, but boils down to the same structure as for the other terms. Along the lines of Appendix A it follows that this term can be written as

$$\frac{1}{8}\text{Tr} \partial_t \left\{ \frac{-R'_k[-D^2]}{(-nD)^2 + R_k[-D^2]} \frac{\delta D^2}{\delta A^a_\mu} \right\} = \frac{1}{8}\text{Tr} \partial_t \left\{ \int dp_n \frac{R'_k[p_n^2 - \partial_r]}{p_n^2 + R_k[p_n^2 - \partial_r]} \sqrt{\frac{\tau}{\pi \delta A^a_\mu}} K_{D^2}(\tau) \right\}_{\tau=0}$$

$$= -\frac{1}{8} \int_0^\infty \frac{dx}{x} \partial_t \left( \frac{R'_k}{\sqrt{r} \sqrt{1 + r}} \frac{NG^2 4 \delta}{16\pi^2 3 \delta A^a_\mu} (S[A] + O[g]) \right),$$

$$= \frac{Ng^2}{16\pi^2} \frac{1}{3} (1 - \gamma) \frac{\delta}{\delta A^a_\mu} (S[A] + O[g]).$$

(B.4)

Note that when rewriting the left hand side of (B.4) as a total derivative w.r.t. $A$ this also includes a term which stems from $\frac{\delta}{\delta A}(nD)^2$. This, however, vanishes because it is odd in $p_n$.

REFERENCES


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