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Supersymmetric M3-branes and G_2 Manifolds

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ABSTRACT

We obtain a generalisation of the original complete Ricci-flat metric of G_2 holonomy on $\mathbb{R}^4 \times S^3$ to a family with a non-trivial parameter λ. For generic λ the solution is singular, but it is regular when $\lambda = \{-1, 0, +1\}$. The case $\lambda = 0$ corresponds to the original G_2 metric, and $\lambda = \{-1, 1\}$ are related to this by an S_3 automorphism of the $SU(2)^3$ isometry group that acts on the $S^3 \times S^3$ principal orbits. We then construct explicit supersymmetric M3-brane solutions in $D = 11$ supergravity, where the transverse space is a deformation of this class of G_2 metrics. These are solutions of a system of first-order differential equations coming from a superpotential. We also find M3-branes in the deformed backgrounds of new G_2-holonomy metrics that include one found by A. Brandhuber, J. Gomis, S. Gubser and S. Gukov, and show that they also are supersymmetric.

1 Introduction

Recently the study of M-theory on spacs of G_2 holonomy has attracted considerable attention. In particular, it has been proposed that M-theory compactified on a certain singular seven-dimensional space with G_2 holonomy might be related to an $\mathcal{N} = 1$, $D = 4$ gauge theory [1, 2, 3, 4, 5] that has no conformal symmetry. (See also the recent papers [6, 7, 8, 9, 10].) The quantum aspects of M-theory dynamics on spaces of G_2 holonomy can provide insights into non-perturbative aspects of four-dimensional $\mathcal{N} = 1$ field theories, such as the preservation of global symmetries and phase transitions. For example, Ref. [5] provides an elegant exposition and study of these phenomena for the three manifolds of G_2 holonomy that were obtained in [11, 12].

Studying the classical geometry of eleven-dimensional supergravity on spaces with G_2 holonomy provides a starting point for investigations of $D = 4$ vacua in M-theory. Recently, in [13], new configurations in M-theory were found, which describe M3-branes with a (3+1)-dimensional Poincaré invariance on the world-volume. These configurations arise as solutions of $D = 11$ supergravity in which the 4-form field is non-vanishing in the seven-dimensional transverse space, which is deformation of the a Ricci-flat metric of G_2 holonomy.

The transverse spaces for M3-branes obtained in [13] were deformations of the three explicitly known metrics of G_2 holonomy that were found in [11, 12]. The original metrics are all of cohomogeneity one: two have principal orbits that are $\mathbb{CP}^3$ or $SU(3)/(U(1)\times U(1))$, viewed as an S^2 bundle over S^4 or $\mathbb{CP}^2$ respectively, and the third has principal orbits that are topologically $S^3 \times S^3$. The M3-branes on these deformed G_2 holonomy spaces arise from first-order differential equations derivable from a superpotential, which in fact are the integrability conditions for the existence of a Killing spinor. Thus the M3-branes obtained in [13] are supersymmetric.

A number of issues related to the properties of G_2 holonomy spaces invite further study. For example, it is of interest to find generalisations of the original three complete metrics of G_2 holonomy that were obtained in [11, 12]. Of particular interest is to find families of G_2 metrics on these manifolds that have non-trivial parameters (i.e. not merely the scale parameter that any Ricci-flat metric has). Such families of metric, when reduced on a circle could, for example, provide a connection between the six-dimensional metrics on the deformed and resolved six-dimensional conifolds [17], thus providing insights into the relation [2] between M-theory on these classes of G_2 holonomy metrics, and Type IIA string theory on the corresponding six-dimensional special holonomy spaces.[1] Recently, in Ref.

[1] For 8-dimensional spaces of $Spin(7)$ holonomy an explicit family of non-singular generalisations has been

[16], a new class of G_2 holonomy metrics with such properties was constructed.

In section 2 we construct a new two-parameter family of metrics of G_2 holonomy, on the manifold of $\mathbb{R}^4 \times S^3$ topology. We do this by starting from a rather general ansatz for metrics of cohomogeneity one on manifolds whose principal orbits are S^3 bundles over S^3, which is a generalisation of the ansatz for the original G_2 metrics of this topology in [11, 12]. The general ansatz involves nine functions of the radial coordinate, which parameterise homogeneous squashings of the base and fibre 3-spheres. We have not found a superpotential formulation for the Einstein equations in this most general case, but by making a specialisation to a 3-function ansatz that is spherically symmetric in the base and the fibres, we have found a superpotential and hence first-order equations. We show that these equations are in fact the integrability conditions for the existence of a single covariantly-constant spinor, and hence for G_2 holonomy. The first-order equations can be solved explicitly, yielding Ricci-flat metrics with two parameters, one of which is the (trivial) scale size, while the other, which we call λ, is non-trivial and characterises genuinely inequivalent metrics.[2] For a generic value of the parameter λ these metrics are singular, but they are regular for three discrete values, namely $\lambda = \{-1, 0, +1\}$. The case $\lambda = 0$ corresponds to the original metric of G_2 in [11, 12], while the cases $\lambda = \{-1, 1\}$ are related to $\lambda = 0$ by an S_3 (permutation) automorphism of the $SU(2)^3$ isometry group.[3] The S_3 action restricts the parameter space of these metrics to a "fundamental domain" $0 \le \lambda \le \frac{1}{3}$. However, when viewed as a reduction on a specific circle, M-theory on the three regular solutions is related to the type IIA theory on a deformed conifold and two types of resolved conifold respectively.

In section 3 we consider a different specialisation of the nine-function metric ansatz introduced in section 2, in which six metric functions remain. For this particular truncation a superpotential can be found, and we show that the resulting first-order equations are the integrability conditions for a covariantly-constant spinor, and hence G_2 holonomy. A specialisation in which two of the three directions on the base and fibre 3-spheres are treated on an equivalent footing yields the ansatz and first-order system introduced in [16]. We discuss

found recently [14, 15], which has such a non-trivial parameter. They provide a connection between $\mathcal{N} = 1$ $D = 3$ vacua of M-theory, and Type IIA string theory on G_2 holonomy manifolds with $\mathbb{R}^3 \times S^4$ topology.

[2]There is also a third completely trivial parameter corresponding to a constant shift of the radial coordinate.

[3]The discrete S_3 coordinate transformations (for details see section 2.3) were found in [5] as a triality symmetry among the three regular G_2 holonomy metrics with important consequences for the phase diagrams of the field theory associated with these three branches.

how this is related by dimensional reduction to the metric on the six-dimensional Stenzel manifold. A further specialisation of the six-function ansatz to a spherically-symmetric one with just two remaining functions yields a first-order system whose solution is the original metric of G_2 holonomy on the $\mathbb{R}^4$ bundle over S^3, expressed now in a manifestly $\mathbb{Z}_2$-symmetric fashion.

Another aim of the paper is to construct further examples of supersymmetric M3-branes. In section 4 we extend the methods introduced in [13] to obtain the equations of motion for M3-branes in the background of deformations of the new G_2 metrics that we constructed in section 2. We show that these equations admit a superpotential formulation, and we obtain the general M3-brane solution for these configurations. We also obtain the analogous system of equations for M3-branes in the background of deformations of the new G_2 ansatz introduced in [16], which we discussed in section 3. Again, we find a superpotential and associated first-order equations. In section 5 we show that these first-order equations are the integrability conditions for the existence of a Killing spinor. Thus the new M3-brane solutions are supersymmetric.

2 A class of metrics of G_2 holonomy

2.1 The ansatz and Einstein equations

An approach to looking for metrics of G_2 holonomy is to make a generalisation of the original G_2 metric found in [11, 12] on $\mathbb{R}^4 \times S^3$. We take as our starting point the metric

$$ds_7^2 = dt^2 + a_i^2\,(\Sigma_i + g_i\,\sigma_i)^2 + b_i^2\,\sigma_i^2\,, \tag{1}$$

where σ_i and Σ_i are left-invariant 1-forms on two $SU(2)$ group manifolds, S^3_σ and S^3_Σ, and a_i, b_i and g_i are functions of the radial coordinate t. The principal orbits are therefore S^3 bundles over S^3, and since the bundle is topologically trivial, they have the topology $S^3 \times S^3$. The isometry group in general is $SU(2)^\sigma_L \times SU(2)^\Sigma_L$, where the two factors denote left-acting $SU(2)$ transformations on S^3_σ and S^3_Σ respectively.

The original metric ansatz used in obtaining the solution in [11, 12] is obtained by taking $a_i = a$, $b_i = b$, and $g_i = -\frac{1}{2}$. Note that in this "round" case the isometry group is enhanced to $SU(2)^\sigma_L \times SU(2)^\Sigma_L \times SU(2)^D_R$, where the third factor denotes the diagonal $SU(2)$ subgroup of the product of right-acting $SU(2)^\sigma_R$ and $SU(2)^\Sigma_R$ transformations. There is a natural triality permutation symmetry S_3, generated by the $\mathbb{Z}_2$ "flop" described in [2] that interchanges the $SU(2)^\sigma_L$ and $SU(2)^\Sigma_L$ factors, and another $\mathbb{Z}_2$ "flip" operation under

which $SU(2)$ group elements G_σ and G_Σ on the two 3-spheres are transformed according to $(G_\sigma, G_\Sigma) \longrightarrow (G_\sigma^{-1}, G_\Sigma\, G_\sigma^{-1})$. These will be discussed in more detail in section 2.3.

It is straightforward to obtain the conditions for Ricci-flatness of the metric (1). We find that they can be derived from the Lagrangian $L = T - V$, together with the constraint $T + V = 0$, where $T = \frac{1}{2} g_{ij}\, (d\alpha^i/d\eta)\, (d\alpha^j/d\eta)$, and the 9×9 metric on the "non-linear sigma model" for the functions in the ansatz is given by

$$g_{ij} = \text{block-diag}(2 - 2\delta_{ab}, -a_1^2\, b_1^{-2}, -a_2^2\, b_2^{-2}, -a_3^2\, b_3^{-2})\,, \tag{2}$$

and $V = (\prod_i a_i\, b_i)^2\, (U_1 + U_2 + U_3)$, with

$$\begin{aligned} U_1 \quad &= \quad -\frac{1}{a_1^2} + \frac{a_1^2}{2a_2^2\, a_3^2} - \frac{1}{b_1^2} + \frac{b_1^2}{2b_2^2\, b_3^2} \\ &\quad +g_1^2 \Big(\frac{a_1^2}{2b_2^2\, b_3^2} + \frac{a_2^2}{2b_1^2\, a_3^2} + \frac{a_3^2}{2b_1^2\, a_2^2}\Big) + \frac{a_1^2}{2b_2^2\, b_3^2}\, (g_2^2\, g_3^2 + 2 g_1\, g_2\, g_3)\,. \end{aligned} \tag{3}$$

The functions U_2 and U_3 are related to U_1 by cyclic permutation of the indices 1, 2 and 3. The 9 functions α^i are defined by

$$\alpha^i = \log a_i \qquad \alpha^{i+3} = \log b_i\,, \qquad \alpha^{i+6} = g_i\,, \qquad i = 1, 2, 3\,, \tag{4}$$

and the indices a and b in (2) range over the 6×6 block with $1 \le a \le 6$ and $1 \le b \le 6$. The radial coordinate η is related to t by $dt = (\prod_i a_i\, b_i)\, d\eta$.

2.2 Specialisation to the "round" metric, and its general solution

2.2.1 Superpotential and first-order equations

There does not appear to exist a superpotential and associated first-order equations for the 9-function ansatz (1). However, we have found a superpotential for the case where we take all three directions equal, so that just three functions remain. It is convenient to write these as

$$a_i = a\,, \qquad b_i = b\,, \qquad g_i = g - \tfrac{1}{2}\,, \tag{5}$$

and so the ansatz (1) now reduces to

$$ds_7^2 = dt^2 + a^2\, [\Sigma_i + (g - \tfrac{1}{2})\, \sigma_i]^2 + b^2\, \sigma_i^2\,. \tag{6}$$

Since in this case both the base and the fibre are round three spheres, giving th enhanced $SU(2)^3$ symmetry discussed in section 2.1, we shall refer to this as the "round G_2 manifold."

The Lagrangian $L = T - V$ describing the Ricci-flat equations for this is given by $T = \frac{1}{2} g_{ij}\, (d\alpha^i/d\eta)\, (d\alpha^j/d\eta)$, where $\alpha^1 \equiv \log a$, $\alpha^2 \equiv \log b$ and $\alpha^3 \equiv g$, η is defined by $dt = a^3\, c^3\, d\eta$, and

$$g_{ij} = \begin{pmatrix} 12 & 18 & 0 \\ 18 & 12 & 0 \\ 0 & 0 & -\frac{3a^2}{b^2} \end{pmatrix} . \tag{7}$$

The Lagrangian is of a "non-linear sigma model" type, where the metric g_{ij} depends on the functions in the system. We find that there is a superpotential for this system, such that $V = -\frac{1}{2} g^{ij}\, (\partial W/\partial\alpha^i)\, (\partial W/\partial\alpha^j)$, where W is given by

$$W = \frac{3}{4} a^2\, b\, K\,, \tag{8}$$

and

$$K \equiv \sqrt{[a^2\, (1-2g)^2 + 4b^2]\, [a^2\, (1+2g)^2 + 4b^2]}\,. \tag{9}$$

The associated first-order equations, expressed in terms of the original radial variable t, are then

$$\begin{aligned} \dot a &= \frac{16b^4 - (1-4g^2)^2\, a^4}{8b^2\, K}\,, \qquad \dot b = \frac{a\, [a^2\, (1-4g^2)^2 + 4(1+4g^2)\, b^2]}{4b\, K}\,, \\ \dot g &= \frac{2g\, [a^2\, (1-4g^2) - 4b^2]}{a\, K}\,. \end{aligned} \tag{10}$$

2.2.2 Covariantly-constant spinor and G_2 holonomy

We now show that the first-order equations (10) are the integrability conditions for the existence of a covariantly-constant spinor. In turn, this demonstrates that the metrics (1) with functions given by (5) satisfying (10) have the special holonomy G_2.

From the spin connection for the metric, we find that in the obvious choice of orthonormal basis $e^0 = dt$, $e^i = b\,\sigma_i$ and $e^{\tilde i} = a\,(\Sigma_i + (g - \frac{1}{2})\,\sigma_i)$, and spin frame, a covariantly-constant spinor η satisfying $D\eta \equiv d\eta + \frac{1}{4}\omega_{ab}\,\Gamma_{ab}\,\eta = 0$ is independent of the coordinates of the $S^3 \times S^3$ principal orbits, but it does depend on the radial coordinate t. After some algebra we find that if the first-order equations (10) are satisfied there is exactly one solution, which is given by

$$\eta = f_1\, \eta_1 + f_2\, \eta_2\,, \tag{11}$$

where

$$\begin{aligned} f_1 &= \frac{(a(1-2g) - 2\mathrm{i}\, b)(a + 2\mathrm{i}\, b)}{\sqrt{(a^2(1-2g)^2 + 4b^2)(a^2 + 4b^2)}}\,, \\ f_2 &= \frac{(a(1+2g) - 2\mathrm{i}\, b)(a + 2\mathrm{i}\, b)}{\sqrt{(a^2(1+2g)^2 + 4b^2)(a^2 + 4b^2)}}\,, \end{aligned} \tag{12}$$

where η_1 and η_2 are constant spinors, satisfying the constraints

$$(\Gamma_1 - \mathrm{i}\,\Gamma_4)\,\eta_2 = 0\,, \qquad \Gamma_{56}\,\eta_1 = \mathrm{i}\,\Gamma_{01}\,\eta_2\,, \qquad (\Gamma_{26} - \Gamma_{35})\,\eta_1 = 2\Gamma_{01}\,\eta_2\,. \tag{13}$$

Here the explicit index values 1, 2 and 3 refer to the vielbein components e^i in the S^3 base manifold, while 4, 5 and 6 refer to the vielbein components $e^{\tilde{i}}$ in the S^3 fibres.

2.2.3 The general solution to the first-order equations

We can obtain the general solution to the first-order equations (10) as follows. Defining a new radial coordinate w by $dw = a\,K^{-1}\,dt$, and new variables α, β and γ by $a^2 = \alpha$, $b^2 = \beta$ and $g^2 = \frac{1}{4}\gamma$, the first-order equations become

$$\begin{aligned}
\frac{d\alpha}{dw} &= 4\beta - \frac{\alpha^2\,(1-\gamma)^2}{4\beta}\,,\\
\frac{d\beta}{dw} &= \tfrac{1}{2}\alpha\,(1-\gamma)^2 + 2\beta\,(1+\gamma)\,,\\
\frac{d\gamma}{dw} &= 4\gamma\,(1-\gamma) - \frac{16\beta\,\gamma}{\alpha}\,.
\end{aligned} \tag{14}$$

which, after the further redefinition $\gamma = \psi^{-4}$, becomes

$$\frac{d^2\psi}{dw^2} + 4\psi^{-3} - 4\psi = 0\,. \tag{15}$$

From this we obtain $(d\psi/dw)^2 = \tilde{k} + 4\psi^2 + 4\psi^{-2}$, and hence, after writing $e^{4w} = \tilde{k}\,\rho$ and then $\tilde{k} = 8/\lambda$, we arrive at the general solution

$$a^2 = F^{-1/3}\,Y\,, \qquad b^2 = \tfrac{1}{4}F^{2/3}\,Y^{-1}\,, \qquad g = m\,\lambda\,\rho\,Y^{-1}\,, \tag{16}$$

where

$$\begin{aligned}
F &\equiv 3\rho^4 - 8m\,\rho^3 + 6m^2\,(1-\lambda^2)\,\rho^2 - m^4\,(1-\lambda^2)^2\,,\\
Y &\equiv \rho^2 - 2m\,\rho + m^2\,(1-\lambda^2)\,,
\end{aligned} \tag{17}$$

(Note that $dF/d\rho = 12\rho\,Y$.) Thus the metric in this general solution is given by

$$ds_7^2 = F^{-1/3}\,d\rho^2 + F^{-1/3}\,Y\,(\Sigma_i + (g - \tfrac{1}{2})\,\sigma_i)^2 + \tfrac{1}{4}F^{2/3}\,Y^{-1}\,\sigma_i^2\,. \tag{18}$$

The metric is asymptotically conical (AC), with base the non-product Einstein metric on $S^3 \times S^3$. Note that if $m = 0$, the λ dependence of the solution drops out, and the metric becomes simply the Ricci-flat cone over the $S^3 \times S^3$ base.

When $|\rho|$ is large, the functions F and Y are both large and positive, with $F \sim 3\rho^4$, $Y \sim \rho^2$ and $g \sim 0$. Letting $\rho = r^3$, we see that at large r, after dropping an overall constant scale factor $12(3)^{1/3}$, we shall have

$$ds_7^2 \sim dr^2 + \tfrac{1}{9}r^2\,(\Sigma_i - \tfrac{1}{2}\sigma_i)^2 + \tfrac{1}{12}r^2\,\sigma_i^2\,. \tag{19}$$

Thus in this region the metric is asymptotic to the cone over $S^3 \times S^3$.

As $|\rho|$ reduces from infinity, the metric remains regular until the first singularity in the metric functions is encountered. Let us first check the function Y. This has zeros at $\rho = m\,(1 \pm \lambda)$. Substituting these values into F, we see that it is given by $F = -4(1\pm\lambda)^2\,\lambda^2\,m^4$. Thus leaving aside the special cases $\lambda = 0$ or $\lambda = \pm 1$ for now, we see from the fact that F is negative here that the first zero of F must have occurred *before* Y reached its first zero. Thus when $\lambda \ne \pm 1$ or 0, it is the first zero of F, and not Y, that defines the inner endpoint $\rho = \rho_0$ of the range of the radial variable. This occurs at $F(\rho_0) = 0$, at which point $Y(\rho_0)$ will still be positive; we shall define $Y_0 \equiv Y(\rho_0) > 0$.

Near to $\rho = \rho_0$, we shall have $F(\rho) \approx (\rho - \rho_0)\,F'(\rho_0)$, which means $F(\rho) \approx 12(\rho - \rho_0)\,Y_0$. Letting $12(\rho - \rho_0)\,Y(\rho_0) = r^{6/5}$, we see that near $\rho = \rho_0$ the metric takes the form

$$ds_7^2 \sim \frac{1}{25Y_0^2}\Big[dr^2 + 100Y_0^3\,r^{-2/5}\,[\Sigma_i + (b - \tfrac{1}{2})\,\sigma_i]^2 + 25Y_0\,r^{4/5}\,\sigma_i^2\Big]\,, \tag{20}$$

with $b = 8\rho_0/Y_0$. The metric is clearly singular at $r = 0$ (i.e. at $\rho = \rho_0$).

Having shown that the metric is singular for general values of the non-trivial parameter λ, we now consider the three special values, $\lambda = 0$ and $\lambda = \pm 1$, for which were excluded from the general analysis.

2.2.4 The non-singular solutions

There are three values of the non-trivial integration constant λ for which we find that the generically singular G_2 metric (18) is regular, namely $\lambda = 0$ and $\lambda = \pm 1$. We shall consider these one by one.

- $\underline{\lambda = 0}$: In this case, we have

$$F = (3\rho + m)(\rho - m)^3\,, \qquad Y = (\rho - m)^2\,, \qquad g = 0\,. \tag{21}$$

If we define a new radial coordinate $r = \sqrt{3}\,(3\rho + m)^{1/3}$ and scale parameter $\ell^3 = 4\sqrt{3}\,m$, the metric (18) then becomes

$$ds_7^2 = \frac{dr^2}{\Big(1 - \frac{\ell^3}{r^3}\Big)} + \tfrac{1}{9}\,r^2\Big(1 - \frac{\ell^3}{r^3}\Big)\,(\Sigma_i - \tfrac{1}{2}\sigma_i)^2 + \tfrac{1}{12}\,r^2\,\sigma_i^2\,. \tag{22}$$

This is the previously-known complete metric of G_2 holonomy on the $\mathbb{R}^4$ bundle over S^3, as found in [11, 12].

- $\underline{\lambda = -1}$: Now we have

$$F = \rho^3\,(3\rho - 8m)\,, \qquad Y = \rho\,(\rho - 2m)\,, \qquad g = -\frac{m}{(\rho - 2m)}\,. \tag{23}$$

After making the redefinitions $r = \sqrt{3}\,(3\rho - 8m)^{1/3}$ and $\ell^3 = -24\sqrt{3}\,m$, the metric (18) becomes

$$ds_7^2 = \frac{dr^2}{\Big(1 - \frac{\ell^3}{r^3}\Big)} + \tfrac{1}{9} r^2 \Big(1 - \frac{\ell^3}{4r^3}\Big)\,[\Sigma_i + (g - \tfrac{1}{2})\,\sigma_i]^2 + \tfrac{1}{12} r^2\, \frac{\Big(1 - \frac{\ell^3}{r^3}\Big)}{\Big(1 - \frac{\ell^3}{4r^3}\Big)}\,\sigma_i^2\,, \tag{24}$$

with

$$g = \frac{3\ell^3}{8\Big(1 - \frac{\ell^3}{4r^3}\Big)}\,. \tag{25}$$

This metric is clearly regular, and indeed it can easily be seen to be equivalent to the original metric of [11, 12]. This can be done by first "recompleting the square" so that the terms in the metric are instead organised with the structure $x_1\,(\sigma_i + y\,\Sigma_i)^2 + x_2\,\Sigma_i^2$, and then making the transformation

$$\sigma_i \longrightarrow \Sigma_i\,, \qquad \Sigma_i \longrightarrow \sigma_i\,. \tag{26}$$

After doing this, the metric becomes precisely (22). Note that this $\mathbb{Z}_2$ transformation is precisely the "flop" described in [2].

- $\underline{\lambda = +1}$: In this case we have

$$F = \rho^3\,(3\rho - 8m)\,, \qquad Y = \rho\,(\rho - 2m)\,, \qquad g = \frac{m}{(\rho - 2m)}\,. \tag{27}$$

Under the same transformations $r = \sqrt{3}\,(3\rho - 8m)^{1/3}$ and $\ell^3 = -24\sqrt{3}\,m$ as for the case $\lambda = -1$, the metric (18) becomes

$$ds_7^2 = \frac{dr^2}{\Big(1 - \frac{\ell^3}{r^3}\Big)} + \tfrac{1}{9} r^2 \Big(1 - \frac{\ell^3}{4r^3}\Big)\,[\Sigma_i + (g - \tfrac{1}{2})\,\sigma_i]^2 + \tfrac{1}{12} r^2\, \frac{\Big(1 - \frac{\ell^3}{r^3}\Big)}{\Big(1 - \frac{\ell^3}{4r^3}\Big)}\,\sigma_i^2\,, \tag{28}$$

with

$$g = -\frac{3\ell^3}{8\Big(1 - \frac{\ell^3}{4r^3}\Big)}\,. \tag{29}$$

This differs from $\lambda = -1$ case only in the sign of g, and the metric is again regular. However, if one repeats the previous procedure of recompleting the square and interchanging σ_i and Σ_i, one now finds that (28) simply maps into itself. It is, nevertheless, equivalent to the original metric (22), but the required coordinate transformation that explicitly implements this is more complicated. This will be discussed in the next subsection.

2.3 S_3 transformations of the round G_2 metrics

In this subsection, we shall show that there is an S_3 triality transformation of the metric (18) that maps it back into its original isotropic form, with transformed values of its parameters.[4] The three permutation transformations can be generated by two types of operation. The first is a generalisation of the one we already used in order to transform $\lambda = -1$ regular metric into the $\lambda = 0$ metric, which amounts to an exchange of the base and the fibre 3-spheres. The second is a more complicated one, involving transformations of the coordinates on the 3-spheres. We shall discuss these in succession.

- The interchange of the 3-spheres

  This transformation follows from the "recompletion of squares" identity

$$x_1\,(\Sigma_i + y\,\sigma_i)^2 + x_2\,\sigma_i^2 = \hat{x}_1\,(\sigma_i + \hat{y}\,\Sigma_i)^2 + \hat{x}_2\,\Sigma_i^2\,, \tag{30}$$

where

$$\hat{x}_1 = x_1\,y^2 + x_2\,, \qquad \hat{x}_2 = \frac{x_1\,x_2}{x_1\,y^2 + x_2}\,, \qquad \hat{y} = \frac{x_1\,y}{x_1\,y^2 + x_2}\,. \tag{31}$$

Since after exchanging σ_i and Σ_i this leaves the metric (18) in the same class of "round" metrics as (1), (5), and since (18) is the general solution of the first-order equations, it follows that the transformed metric must be contained within the general solutions (18), after appropriate transformations of the parameters.[5]

It is easily seen from (31) that the effect of the $\sigma_i \leftrightarrow \Sigma_i$ flop automorphism is to map the solutions as follows:

$$\begin{aligned} m &\longrightarrow \hat{m} = \tfrac{1}{2}m\,(3\lambda - 1)\,,\\ \lambda &\longrightarrow \hat{\lambda} = \frac{\lambda + 1}{3\lambda - 1}\,,\\ r &\longrightarrow \hat{r} = r + m\,(\lambda - 1)\,. \end{aligned} \tag{32}$$

Clearly this is an involution, giving a Z_2 transformation on the solutions.

A special case of the above is to take $\lambda = -1$, which maps to $\hat{\lambda} = 0$. This is the transformation that we used in the previous subsection to show that (24) is equivalent to the original G_2 metric (22). On the other hand if we apply it to the case $\lambda = 1$, we see that

[4] This triality symmetry among the regular G_2 holonomy metrics was discussed in detail in [5]. M.C. would like to thank E. Witten for sharing with her the draft of [5] prior to the publication.

[5] There are really three parameters in the general solutions, namely the non-trivial parameter λ, the scale parameter m, and a trivial constant shift of the radial coordinate that we have not made explicit in (18). Our statement about the mapping of solutions into solutions must be understood to include the need to perform such a constant shift.

this is a fixed point of the transformation, which was already observed in our discussion of the metric (28).

- $SU(2)$ transformations

Let τ_i be the Pauli matrices, and let U and V be 2×2 matrices for two $SU(2)$ groups. Then we can define the left-invariant 1-forms σ_i and Σ_i as follows:

$$U^{-1}\,dU = \tfrac{\rm i}{2}\,\tau_i\,\sigma_i\,,\qquad V^{-1}\,dV = \tfrac{\rm i}{2}\,\tau_i\,\Sigma_i\,. \tag{33}$$

Now, make a coordinate transformation to new $SU(2)$ matrices $\widetilde U$ and $\widetilde V$, related to U and V as follows:

$$U = \widetilde U^{-1}\,,\qquad V = \widetilde V\,\widetilde U^{-1}\,. \tag{34}$$

(This is the $(G_\sigma, G_\Sigma)\longrightarrow (G_\sigma^{-1}, G_\Sigma\, G_\sigma^{-1})$ automorphism of the round metrics that we mentioned in section 2.1.) It follows that we shall have

$$U^{-1}\,dU = -\widetilde U\,(\widetilde U^{-1}\,d\widetilde U)\,\widetilde U^{-1}\,,\quad V^{-1}\,dV = \widetilde U\,(\widetilde V^{-1}\,d\widetilde V - \widetilde U^{-1}\,d\widetilde U)\,\widetilde U^{-1}\,. \tag{35}$$

We also define "tilded" left-invariant 1-forms

$$\widetilde U^{-1}\,d\widetilde U = \tfrac{\rm i}{2}\,\tau_i\,\tilde\sigma_i\,,\qquad \widetilde V^{-1}\,d\widetilde V = \tfrac{\rm i}{2}\,\tau_i\,\widetilde\Sigma_i\,. \tag{36}$$

Since we can write

$$\sigma_i^2 = -2{\rm tr}(\tfrac{\rm i}{2}\,\tau_i\,\sigma_i)^2 = -2{\rm tr}(U^{-1}\,dU)^2\,, \tag{37}$$

and so on, it now follows that

$$x_1\,[\Sigma_i + (g-\tfrac12)\,\sigma_i]^2 + x_2\,\sigma_i^2 = x_1\,[\widetilde\Sigma_i + (-g-\tfrac12)\,\tilde\sigma_i]^2 + x_2\,\tilde\sigma_i^2\,, \tag{38}$$

In other words, the net effect of the transformation (34) is simply to send $g\longrightarrow -g$. Expressed as a transformation of the parameters in the general solution (18), this therefore just amounts to

$$\lambda\longrightarrow\tilde\lambda = -\lambda\,, \tag{39}$$

while leaving the other parameters unchanged. Thus we see that the S_3 permutation group acting on the parameter λ is generated by $\lambda\rightarrow(\lambda+1)/(3\lambda-1)$ and $\lambda\rightarrow-\lambda$. The fundamental domain of λ is therefore given by

$$0\le\lambda\le\tfrac13\,. \tag{40}$$

For a generic value of m, the solutions form triplets under the S_3 group. When $m=0$, corresponding to the Ricci-flat cone over $S^3\times S^3$, the triplet degenerates to a singlet.

A particular application of this transformation is to the metric (28), which corresponds to $\lambda = +1$. It is therefore mapped into the previous example (24), which corresponds to $\lambda = -1$. Indeed, we may note that as solutions of (18) these metrics differ only in the sign of g. This, therefore, provides the additional transformation that is needed in order to show that (28) is again equivalent to the original G_2 metric (22).

As another application of the general result (34), we may note that

$$(\sigma_i - \Sigma_i)^2 = \widetilde{\Sigma}_i^2\,, \qquad (\sigma_i + \Sigma_i)^2 = 4(\tilde{\sigma}_i - \tfrac{1}{2}\widetilde{\Sigma}_i)^2\,. \tag{41}$$

This enables us to rewrite the original G_2 metric (22) in a form that is manifestly invariant under the interchange of σ_i and Σ_i, namely

$$ds_7^2 = \frac{dr^2}{\Big(1 - \frac{\ell^3}{r^3}\Big)} + \tfrac{1}{12} r^2\,(\sigma_i - \Sigma_i)^2 + \tfrac{1}{36}\,r^2\,\Big(1 - \frac{\ell^3}{r^3}\Big)\,(\sigma_i + \Sigma_i)^2\,. \tag{42}$$

As we shall see below, this basis is closely related to the one used in [19] for constructing the metric on the cotangent bundle of S^3.

2.4 S^1 reduction to type IIA

Since the G_2 manifold is Ricci-flat with special holonomy, it provides a natural compactification (reduction) space for M-theory. The vacuum solution is just a metric product of four-dimensional Minkowski space-time and that of the G_2 manifold. Because of the discrete triality automorphism of the round G_2 manifold, it follows that there are three different ways of reducing it on S^1, so that the principal orbits of the lower dimensional manifold are the $T^{1,1} = SO(4)/SO(2)$ coset space.[6] In order to do so, we note that the Σ_i can be written in terms of Euler angles as

$$\Sigma_1 = \cos\psi\,d\theta + \sin\psi\,\sin\theta\,d\varphi\,, \quad \Sigma_2 = -\sin\psi\,d\theta + \cos\psi\,\sin\theta\,d\varphi\,, \quad \Sigma_3 = d\psi + \cos\theta\,d\varphi\,. \tag{43}$$

One can then verify that [20]

$$\sum_i [\Sigma_i + (g - \tfrac{1}{2})\,\sigma_i]^2 = [\cos\theta\,d\psi + (g - \tfrac{1}{2})\,\mu^i\,\sigma_i]^2 + \sum_i (D_g \mu^i)^2\,, \tag{44}$$

where

$$\mu_1 = \sin\theta\,\sin\psi\,, \qquad \mu_2 = \sin\theta\,\cos\psi\,, \qquad \mu_3 = \cos\theta\,, \tag{45}$$

[6] Some further details of the relation to $T^{1,1}$ and the six-dimensional Stenzel manifold T^*S^3 are discussed in section 3.

and $D_g\mu_i = d\mu_i + \epsilon_{ijk}\,(g-\frac{1}{2})\,\sigma_j\,\mu_k$. The $\lambda = 0$ solution therefore reduces to become a wrapped D6-brane in type IIA:

$$\begin{aligned} ds_{10}^2 &= e^{-\frac{1}{6}\phi}\Big[dx^\mu\,dx_\mu + \frac{dr^2}{1-\frac{\ell^3}{r^3}} + \tfrac{1}{9}r^2(1-\frac{\ell^3}{r^3})\,(D_{g=0}\mu^i)^2 + \tfrac{1}{12}r^2\,\sigma_i^2\Big]\,, \\ e^{\frac{4}{3}\phi} &= \tfrac{1}{9}r^2(1-\frac{\ell^3}{r^3})\,, \qquad \mathcal{A}_{(1)} = \cos\theta\,d\psi + \tfrac{1}{2}\mu_i\,\sigma_i\,. \end{aligned} \tag{46}$$

The six-dimensional transverse space here has a similar structure to the deformed conoifold. The $\lambda = -1$ solution becomes

$$\begin{aligned} ds_{10}^2 &= e^{-\frac{1}{6}\phi}\Big[dx^\mu\,dx_\mu + \frac{dr^2}{1-\frac{\ell^3}{r^3}} + \frac{1}{9}r^2\,(1-\frac{\ell^3}{4r^3})(D_g\mu^i)^2 + \tfrac{1}{12}r^2\,\frac{1-\frac{\ell^3}{r^3}}{1-\frac{\ell^3}{4r^3}}\,\sigma_i^2\Big]\,, \\ e^{\frac{4}{3}\phi} &= \tfrac{1}{9}r^2\,(1-\frac{\ell^3}{4r^3})\,, \qquad \mathcal{A}_{(1)} = \cos\theta\,d\psi - (g-\tfrac{1}{2})\,\mu_i\,\sigma_i\,, \end{aligned} \tag{47}$$

where g is given by (25). The six-dimensional transverse space has a structure similar to the resolved conifold. These two reductions were previously obtained in [2]. The $\lambda = 0$ reduction is singular from the ten-dimensional point of view, but the $\lambda = -1$ reduction is regular.

Employing the S_3 triality transformations, discussed in the previous subsections, we can obtain the third case, namely $\lambda = 1$. The form of the metric is the same as (47) but with g replaced by $-g$. It should be emphasised that from the type IIA point of view, the choice of sign for g has a non-trivial effect. The $\lambda = 0$ solution gives the same topology as the deformed conifold, whilst the two solutions $\lambda = \pm 1$ give the topology of the resolved conifold. These two resolutions arise from the fact that there can be a sign choice for the Fayet-Illiopoulos term for the field theory arising from type IIA string theory on a resolved conifold.[7]

3 Squashed manifolds with G_2 holonomy

We can make a different truncation of the 9-function ansatz (1), in which six functions remain, which again allows us to contruct a superpotential whose first-order equations yield first integrals of the Ricci-flat Einstein equations. A further truncation in this example, to an ansatz with four remaining functions, gives the system considered first in [16].[8] We shall

[7]We thank E. Witten for making this observation.

[8]We are grateful to A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov for informing us of their four-function ansatz prior to publication. The truncation of the nine-function ansatz of section 2 to the six-function ansatz that we consider here was motivated by their results, together with results in [19] on the construction of Stenzel metrics.

take a somewhat different approach here, and motivate the six-function and four-function ansätze from a six-dimensional viewpoint, based on earlier results in [19] on the Stenzel construction of Ricci-flat Kähler metrics on the cotangent bundle of S^{n+1}.

We take as our starting point the construction of the Stenzel metrics on T^*S^{n+1}, using the formalism and notation of [19]. The metrics are of cohomogeneity one, with principal orbits that are $SO(n+2)/SO(n)$. Denoting the left-invariant 1-forms of $SO(n+2)$ by L_{AB}, satisfying

$$dL_{AB} = L_{AC} \wedge L_{CB}\,, \tag{48}$$

and decomposing under the $SO(n)$ subgroup according to $A = (1, 2, i)$, the following ansatz for Stenzel metrics was made in [19]:

$$ds^2 = dt^2 + a^2\,(L_{1i})^2 + b^2\,(L_{2i})^2 + c^2\,(L_{12})^2\,. \tag{49}$$

Note that the 1-forms L_{1i}, L_{2i} and L_{12} are precisely the ones that live in the coset $SO(n+2)/SO(n)$.

Let us specialise now to $n = 1$, which gives a construction of the 6-dimensional Stenzel metric, with principal orbits $SO(4)/SO(2)$. In this case, we can now give a natural extension of the ansatz to one for 7-dimensional metrics of cohomogeneity one, where the principal orbits are $SO(4)$, by appending an additional term $f^2\,(L_{34})^2$:

$$ds_7^2 = dt^2 + 4a^2\,(L_{1i})^2 + 4b^2\,(L_{2i})^2 + 4c^2\,(L_{12})^2 + 4f^2\,(L_{34})^2\,. \tag{50}$$

(The factors of 4 are introduced for later convenience.) It is helpful now to exploit the local isomorphism between $SO(4)$ and $SU(2)\times SU(2)$, which is made manifest by forming self-dual and anti-self-dual combinations of the L_{AB}. A convenient choice for these combinations is

$$\begin{aligned}
\sigma_1 &= L_{42} + L_{31}\,, \qquad &\sigma_2 &= L_{23} + L_{41}\,, \qquad &\sigma_3 &= L_{34} + L_{21}\,,\\
\Sigma_1 &= L_{42} - L_{31}\,, \qquad &\Sigma_2 &= L_{23} - L_{41}\,, \qquad &\Sigma_3 &= L_{34} - L_{21}\,.
\end{aligned} \tag{51}$$

It now follows from (48) that the σ_i and Σ_i are left-invariant 1-forms for the two $SU(2)$ factors, satisfying

$$d\sigma_i = -\tfrac12 \epsilon_{ijk}\,\sigma_j\wedge\sigma_k\,, \qquad d\Sigma_i = -\tfrac12 \epsilon_{ijk}\,\Sigma_j\wedge\Sigma_k\,. \tag{52}$$

In terms of these, the metric ansatz (50) becomes

$$ds_7^2 = dt^2 + a^2\,[(\sigma_1-\Sigma_1)^2+(\sigma_2-\Sigma_2)^2] + b^2\,[(\sigma_1+\Sigma_1)^2+(\sigma_2+\Sigma_2)^2] + c^2\,(\sigma_3-\Sigma_3)^2 + f^2\,(\sigma_3+\Sigma_3)^2\,. \tag{53}$$

The principal orbits are $SO(4)$ (or its double cover Spin(4)), but the isometry group may be larger in special cases. The form (53) is the same as the ansatz previously introduced in [16]. Indeed the above derivation of the ansatz (53) shows how the ansatz of [16] is related to the construction of the six-dimensional Stenzel metric [17, 18, 19], which is the deformed conifold. In particular, since $L_{34} = \frac{1}{2}(\sigma_3 + \Sigma_3)$, the reduction to six dimensions is achieved by omitting the last term in (53).

A further generalisation of the ansatz clearly suggests itself, namely to write

$$ds_7^2 = dt^2 + \tfrac{1}{2} a_{AB}^2 \, (L_{AB})^2 \,, \tag{54}$$

where the functions a_{AB} depend only on t. In the $SU(2) \times SU(2)$ basis for $SO(4)$, this becomes

$$ds_7^2 = dt^2 + a_i^2 \, (\sigma_i - \Sigma_i)^2 + b_i^2 \, (\sigma_i + \Sigma_i)^2 \,. \tag{55}$$

By completing the square, it is easily seen that this ansatz is a special case of our original 9-function ansatz (1).

Note that this ansatz is subsumed in the (1) with the specialisation

$$g_i^2 = 1 - \frac{b_i^2}{a_i^2} \,. \tag{56}$$

Calculating the curvature, we find that the conditions for Ricci-flatness can be derived from the Lagrangian $L = T - V$ for the functions a_i and b_i, together with the constraint $T + V = 0$. Defining $\alpha^i \equiv \log a_i$ and $\alpha^{i+3} \equiv \log b_i$, the kinetic energy is given by $T = \frac{1}{2} g_{ab} \, (d\alpha^a/d\eta) \, (d\alpha^b/d\eta)$ with

$$g_{ab} = 2 - 2\delta_{ab} \,, \tag{57}$$

where the new radial variable η is defined by $dt = \prod_i (a_i \, b_i) \, d\eta$. We find that the potential energy V can be derived from a superpotential W, so that $V = -\frac{1}{2} g^{ab} \, (\partial W/\partial\alpha^a) \, (\partial W/\partial\alpha^b)$, with

$$\begin{aligned} 2W &= -a_1 \, a_2 \, a_3 \, (b_1^2 + b_2^2 + b_3^2) + b_1 \, b_2 \, b_3 \, (a_1 \, b_1 + a_2 \, b_2 + a_3 \, b_3) \\ &\quad + b_1 \, b_2 \, a_3 \, (a_1^2 + a_2^2) + b_1 \, b_3 \, a_2 \, (a_1^2 + a_3^2) + b_2 \, b_3 \, a_1 \, (a_2^2 + a_3^2) \,. \end{aligned} \tag{58}$$

(Other sign choices can also arise, but these just correspond to relabellings of indices, and so they give equivalent results.)

From the superpotential we can derive the first-order equations $d\alpha^a/d\eta = g^{ab} \, \partial W/\partial\alpha^b$. Expressed back in terms of the original radial variable t, these are

$$\dot{a}_1 = \frac{a_1^2}{4a_3 \, b_2} + \frac{a_1^2}{4a_2 \, b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2} \,,$$

$$
\begin{aligned}
\dot{a}_2 &= \frac{a_2^2}{4a_3\, b_1} + \frac{a_2^2}{4a_1\, b_3} - \frac{a_1}{4b_3} - \frac{a_3}{4b_1} - \frac{b_1}{4a_3} - \frac{b_3}{4a_1}\,, \\
\dot{a}_3 &= \frac{a_3^2}{4a_2\, b_1} + \frac{a_3^2}{4a_1\, b_2} - \frac{a_1}{4b_2} - \frac{a_2}{4b_1} - \frac{b_1}{4a_2} - \frac{b_2}{4a_1}\,, \\
\dot{b}_1 &= \frac{b_1^2}{4a_2\, a_3} - \frac{b_1^2}{4b_2\, b_3} - \frac{a_2}{4a_3} - \frac{a_3}{4a_2} + \frac{b_2}{4b_3} + \frac{b_3}{4b_2}\,, \\
\dot{b}_2 &= \frac{b_2^2}{4a_3\, a_1} - \frac{b_2^2}{4b_3\, b_1} - \frac{a_1}{4a_3} - \frac{a_3}{4a_1} + \frac{b_1}{4b_3} + \frac{b_3}{4b_1}\,, \\
\dot{b}_3 &= \frac{b_3^2}{4a_1\, a_2} - \frac{b_3^2}{4b_1\, b_2} - \frac{a_1}{4a_2} - \frac{a_2}{4a_1} + \frac{b_1}{4b_2} + \frac{b_2}{4b_1}\,,
\end{aligned} \tag{59}
$$

We may note that a consistent truncation of these equations is to set $a_2 = a_1$ and $b_2 = b_1$, leading to the simpler system of equations

$$
\begin{aligned}
\dot{a}_1 &= \frac{a_1^2}{4a_3\, b_1} - \frac{a_3}{4b_1} - \frac{b_1}{4a_3} - \frac{b_3}{4a_1}\,, \\
\dot{a}_3 &= \frac{a_3^2}{2a_1\, b_1} - \frac{a_1}{2b_1} - \frac{b_1}{2a_1}\,, \\
\dot{b}_1 &= \frac{b_1^2}{4a_1\, a_3} - \frac{a_1}{4a_3} - \frac{a_3}{4a_1} + \frac{b_3}{4b_1}\,, \\
\dot{b}_3 &= \frac{b_3^2}{4a_1^2} - \frac{b_3^2}{4b_1^2}\,.
\end{aligned} \tag{60}
$$

These are the first-order equations found in [16].[9] The solutions give rise to metrics with a Taub-NUT like squashing of the S^3 fibres (as well as a squashing of the S^3 base), with the circle corresponding to $(\sigma_3 + \Sigma_3)$ tending to a constant radius at infinity. Thus the metrics are asymptotically locally conical (ALC).

We can make a further consistent truncation of the first-order equations (60), by setting $a_1 = -a_3 = a$, $b_1 = b_3 = b$. They then become

$$
\dot{a} = \frac{b}{2a}\,, \qquad \dot{b} = \frac{1}{4} - \frac{b^2}{4a^2}\,. \tag{61}
$$

By the standard methods, we can solve these by defining a new radial coordinate ρ such that $dt = d\rho/b$, leading to $a^2 = \rho$ and $b^2 = \frac{1}{3}\rho + k\,\rho^{-1/2}$. After setting $\rho = r^2$, and $k = -\ell^3/3$, we get (dropping an overall factor of 12 from the front)

$$
ds_7^2 = F^{-1}\, dr^2 + \tfrac{1}{12} r^2\, (\sigma_i - \Sigma_i)^2 + \tfrac{1}{36}\, r^2 F\, (\sigma_i + \Sigma_i)^2\,, \tag{62}
$$

where

$$
F \equiv 1 - \frac{\ell^3}{r^3}\,. \tag{63}
$$

[9] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov have obtained an explicit solution of these equations, which yields a new complete non-compact metric of G_2 holonomy [16]. We are grateful to them for telling us about their results prior to publication.

This is in fact the original G_2 metric on $\mathbb{R}^4 \times S^3$, written in the Z_2-symmetric form that we presented in (42).

3.1 Parallel spinor and G_2 holonomy

It is a straightforward matter to solve the equations $D\eta \equiv d\eta + \frac{1}{2}\omega_{ab}\,\Gamma^{ab}\,\eta = 0$ for covariantly-constant spinors, in the metrics (55). We find that there is precisely one such spinor, if and only is the six metric functions a_i and b_i satisfy the first-order equations (59) (or the discrete set of other possibilities corresponding to sign-convention choices). Thus the first-order equations (59) coming from the superpotential have the interpretation of being the integrability conditions for the metrics (55) to admit a covariantly-constant spinor, and hence for them to have G_2 holonomy.

In the orthonormal frame with $e^0 = dt$, $e^i = a_i\,(\sigma_i - \Sigma_i)$ and $e^{\tilde{i}} = b_i\,(\sigma_i + \Sigma_i)$, with the obvious choice for the spin frame, we find that the covariantly-constant spinor η has constant components, and it satisfies the constraints

$$(\Gamma_{16} + \Gamma_{34})\,\eta = 0\,, \qquad (\Gamma_{35} + \Gamma_{26})\,\eta = 0\,, \qquad (\Gamma_{01} - \Gamma_{26})\,\eta = 0\,. \tag{64}$$

Here the indices 1, 2 and 3 refer to the vielbein components e^i, while 4, 5 and 6 refer to $e^{\tilde{i}}$. It is easy to verify that, up to overall scale, the conditions (64) specify the spinor uniquely. Note that η can also be expressed as $\eta = \eta_1 + \eta_2$, where η_1 and η_2 satisfy the constraint (13).

4 Massless M3-branes

4.1 M3-branes on round G_2 manifolds

In [13], M3-brane were constructed on the background of the deformed G_2 manifolds, where the 4-form field strength of the eleven-dimensional supergravity was turned on. When the 4-form field strength is set to zero, the solution becomes the product of the Minkowski space-time and the regular G_2 manifold. In particular, we shall begin by considering the example studied in [13] that is based on the G_2 manifold of the $\mathbb{R}^4$ bundle over S^3. The metric for the associated massless M3-brane is [13]

$$ds_{11}^2 = H^2\,dx^\mu\,dx_\mu + 12 H^4\,Y^{-1}\,dr^2 + \tfrac{4}{3} r^2\,H^{-2}\,Y\,(\sigma_i - \tfrac{1}{2}\Sigma_i)^2 + r^2\,H^{-2}\,\Sigma_i^2\,, \tag{65}$$

where

$$Y \equiv 1 - \frac{\ell^3}{r^3}\,, \qquad H = \Big(1 - \frac{c^2}{r^{12}\,Y^3}\Big)^{-1/6}\,. \tag{66}$$

We can now perform the two Z_2 transformations given in section 2.3, to obtain the previous M3-brane solution (65) written in the other two forms. Firstly, if we apply the $\sigma_i \leftrightarrow \Sigma_i$ transformation as in (30), then we find that (65) becomes

$$ds_{11}^2 = H^2\, dx^\mu\, dx_\mu + 12H^4\, Y^{-1}\, dr^2 + \tfrac{4}{3} r^2\, H^{-2}\, Z\, (\Sigma_i + v\, \sigma_i)^2 + r^2\, H^{-2}\, \frac{Y}{Z}\, \sigma_i^2\,, \tag{67}$$

where H and Y are given in (66) and

$$Z \equiv 1 - \frac{\ell^3}{4r^3}\,, \qquad v = -\frac{Y}{2Z}\,. \tag{68}$$

Instead, we can apply to (65) the $SU(2)$ transformation as in (34). This gives the M3-brane in the form

$$ds_{11}^2 = H^2\, dx^\mu\, dx_\mu + 12H^4\, Y^{-1}\, dr^2 + \tfrac{1}{3} r^2\, H^{-2}\, Y\, (\sigma_i + \Sigma_i)^2 + r^2\, H^{-2}\, (\sigma_i - \Sigma_i)^2\,. \tag{69}$$

Indeed, (67) and (69) are equivalent, however they lead to very different solutions after reduction to type IIA theory on the circle corresponding to $(\sigma_3 + \Sigma_3)$.

Since the regularity of the G_2 background in the above solutions is in any case lost when the M3-brane is introduced by turning on the 4-form field strength [13]) , it is not unreasonable to construct the more general M3-branes on deformations of the round G_2 manifolds of section 2.2. In analogy with [13], the metric ansatz will be given by

$$ds_{11}^2 = H^2\, dx^\mu\, dx_\mu + d\rho^2 + a^2\, h_i^2 + b^2\, \sigma_i^2\,, \tag{70}$$

where $h_i = \Sigma_i + (g - \frac{1}{2})\, \sigma_i$. The ansatz for the 4-form field strength, and its resulting dual, are given by

$$\begin{aligned} F_{(4)} &= f_1\, h_i \wedge h_j \wedge \sigma_i \wedge \sigma_j + f_2\, d\rho \wedge h_1 \wedge h_2 \wedge h_3 + \tfrac{1}{2} f_3\, \epsilon_{ijk}\, d\rho \wedge h_i \wedge \sigma_j \wedge \sigma_k \\ &\quad + \tfrac{1}{2} f_4 \epsilon_{ijk}\, d\rho \wedge h_i \wedge h_j \wedge \sigma_k\,, \\ H^{-4} {*F_{(4)}} &= \frac{2f_1}{a\, b}\, d^4x \wedge d\rho \wedge h_i \wedge \sigma_i + \frac{2f_2\, b^3}{a^3}\, d^4x \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &\quad + \frac{f_3\, a}{2b}\, \epsilon_{ijk}\, d^4x \wedge h_i \wedge h_j \wedge \sigma_k - \frac{f_4\, b}{2a}\, \epsilon_{ijk}\, d^4x\, h_i \wedge \sigma_j \wedge \sigma_k\,. \end{aligned} \tag{71}$$

Thus we have

$$\begin{aligned} F_{00}^2 &= 6(\frac{f_2^2}{a^6} + \frac{3f_3^2}{a^2\, b^4} + \frac{3f_4^2}{a^4\, b^2})\,, \\ F_{11}^2 &= F_{22}^2 = F_{33}^2 = 6(\frac{8f_1^2}{a^4\, b^4} + \frac{2f_3^2}{a^2\, b^4} + \frac{f_4^2}{a^4\, b^2})\,, \\ F_{44}^2 &= F_{55}^2 = F_{66}^2 = 6(\frac{8f_1^2}{a^4\, b^4} + \frac{f_2^2}{a^6} + \frac{f_3^2}{a^2\, b^4} + \frac{2f_4^2}{a^4\, b^2})\,, \\ F_{14}^2 &= F_{25}^2 = F_{36}^2 = \frac{12f_3\, f_4}{a^3\, b^3} + \frac{6f_2\, f_4}{a^5\, b}\,. \end{aligned} \tag{72}$$

The Bianchi identity and the equation motion for F_4 implies

$$\begin{aligned} dF_{(4)} = 0: &\qquad 2f_1' + (g^2 - \tfrac{1}{4})\, f_2 + f_3 + 2g\, f_4 = 0\,, \\ d{*}F_{(4)} = 0: &\qquad f_4 = 2a^2\, b^{-2}\, g\, f_3\,, \\ &\qquad b^3\, a^{-3}\, f_2 - 3(g^2 - \tfrac{1}{4})\, a\, b^{-1}\, f_3 = \lambda\, H^{-4}\,, \\ &\qquad (a\, b^{-1}\, H^4\, f_3)' + 2a^{-1}\, b^{-1}\, H^4\, f_1 = 0\,. \end{aligned} \tag{73}$$

We find that the existence of a superpotential of such a system seems to require that the constant $\lambda = 0$. We shall set this constant to zero from now on. Thus we find that the 4-form ansatz is determined by one function only, namely f_3. Thus, the total number of functions in our ansatz is 5, consisting of a, b, g, H and f_3. After some algebra, we find that the Einstein equations and that for f_3 can be obtained from the Lagrangian $L = T - V$, where the kinetic term T can be expressed as $T = \frac{1}{2} g_{ij}\, \dot\alpha^i\, \dot\alpha^j$, where $\alpha^i = (\log a, \log b, \log H, g, f_3)$, and a dot denotes a derivative with respect to η defined by $dt = a^3\, b^3\, H^4\, d\eta$. The metric g_{ij} is somewhat complicated, but its inverse is simpler, which we present here

$$g^{ij} = \frac{1}{18} \begin{pmatrix} -2 & 1 & 1 & 0 & -f_3 \\ 1 & -2 & 1 & 0 & -7f_3 \\ 1 & 1 & -\frac{4}{5} & 0 & 5f_3 \\ 0 & 0 & 0 & -\frac{6b^2}{a^2} & 0 \\ -f_3 & -7f_3 & 5f_3 & 0 & 2(3b^4 - 13f_3^2) \end{pmatrix}. \tag{74}$$

The potential can be expressed in terms of a superpotential W, namely $V = -\frac{1}{2} g^{ij}\, \partial_i W\, \partial_j W$, where W is given by

$$W = \tfrac{3}{4} a^2\, b\, H^4\, K_1\, K_2\,, \tag{75}$$

where

$$K_1 = \sqrt{[a^2(1-2g)^2 + 4b^2][a^2(1+2g)^2 + 4b^2]}\,, \qquad K_2 = \sqrt{1 - b^{-4}\, f_3^2}\,. \tag{76}$$

The first-order equations can now be straightforwardly obtained, and they are given by

$$\begin{aligned} a' &= \frac{16b^8 - a^4\,(1-4g^2)^2\,(b^4 - 2f_3^2) + 8a^2\, b^2\,(1+4g^2)\, f_3^2}{8b^6\, K_1\, K_2}\,, \\ b' &= \frac{a^4\,(1-4g^2)^2\,(2b^4 - f_3^2) + 8a^2\, b^6\,(1+4g^2) + 16b^4\, f_3^2}{8a\, b^5\, K_1\, K_2}\,, \\ g' &= -\frac{2g\,(4b^2 - a^2\,(1-4g^2))K_2}{a\, K_1}\,, \qquad H' = -\frac{f_3^2\, K_1}{8a\, b^6\, K_2}\,, \\ f_3' &= \frac{f_3\,(16b^4\,(-3b^4 + 5f_3^2) + a^4\,(1-4g^2)^2\,(b^4 + f_3^2) - 8a^2\, b^2\,(1+4g^2)(b^4 - 3f_3^2))}{8a\, b^6\, K_1\, K_2}\,. \end{aligned} \tag{77}$$

To solve them we begin by defining new hatted variables

$$\hat{a} \equiv a\,H\,, \qquad \hat{b} \equiv b\,H\,, \qquad \hat{f}_3 \equiv f_3\,H^2\,, \tag{78}$$

and a new radial coordinate x by $dx = H\,K_2\,d\rho$. In terms of these, we find that the first-order equations (77) become

$$\begin{aligned} \frac{d\hat{a}}{dx} &= \frac{16\hat{b}^4 - (1-4g^2)^2\,\hat{a}^4}{8\hat{b}^2\,\hat{K}_1}\,, \qquad \frac{d\hat{b}}{dx} = \frac{\hat{a}\,[\hat{a}^2\,(1-4g^2)^2 + 4(1+4g^2)\,\hat{b}^2]}{4\hat{b}\,\hat{K}_1}\,,\\ \frac{dg}{dx} &= \frac{2g\,[\hat{a}^2\,(1-4g^2) - 4\hat{b}^2]}{\hat{a}\,\hat{K}_1}\,, \qquad \frac{1}{H}\,\frac{dH}{dx} = -\frac{\hat{f}_3^2\,\hat{K}_1}{8\hat{a}\,\hat{b}^6\,\hat{K}_2^2}\,,\\ \frac{d\hat{f}_3}{dx} &= -\frac{\hat{f}_3[48\hat{b}^4 - \hat{a}^4\,(1-4g^2)^2 + 8\hat{a}^2\,\hat{b}^2\,(1+4g^2)]}{8\hat{a}\,\hat{b}^2\,\hat{K}_1}\,, \end{aligned} \tag{79}$$

where

$$\hat{K}_1 \equiv \sqrt{[\hat{a}^2(1-2g)^2 + 4\hat{b}^2][\hat{a}^2(1+2g)^2 + 4\hat{b}^2]}\,. \tag{80}$$

We also find that

$$\hat{a}^3\,\hat{b}\,\hat{f}_3 = \kappa\,, \tag{81}$$

where κ is a constant of integration.

The general solution of these equations can be obtained as follows. To begin with, we note that the first three equations in (79) are precisely the same as (10), in terms of the new variables. Thus the solution for $\hat{a}$, $\hat{b}$ and g as functions of x will be identical that for a, b and g as functions of t for the Ricci-flat G_2 holonomy 7-metrics (6).[10] Thus in terms of a new radial variable r, defined by $dx = F^{-1/6}\,dr$, we obtain from (16),

$$\hat{a}^2 = F^{-1/3}\,Y\,, \qquad \hat{b}^2 = \tfrac{1}{4}F^{2/3}\,Y^{-1}\,, \qquad g = m\,\lambda\,r\,Y^{-1}\,, \tag{82}$$

where

$$\begin{aligned} F &\equiv 3r^4 - 8m\,r^3 + 6m^2\,(1-\lambda^2)\,r^2 - m^4\,(1-\lambda^2)^2\,,\\ Y &\equiv r^2 - 2m\,r + m^2\,(1-\lambda^2)\,. \end{aligned} \tag{83}$$

From (81) we then obtain

$$\hat{f}_3 = 2\kappa\,F^{1/6}\,Y^{-1}\,. \tag{84}$$

Finally, solving the equation for H in (79), and fixing a constant of integration without loss of generality, we obtain

$$H = (1 - \frac{64\,\kappa^2}{F})^{-1/6}\,. \tag{85}$$

[10]This phenomenon, where functionally-rescaled metric components in the 7-dimensional space transverse to the massless M3-brane turn out to satisfy the same first-order equations as the metric components of the original Ricci-flat 7-metrics, was seen also in the original examples obtained in [13].

In general the new Ricci-flat G_2 manifolds found in section 2 are singular where $F = 0$. Here, we see that when the 4-form field strength is turned on, the associated M3-brane metric becomes singular before this point is reached, namely at the value ofr r for which $F = 64\kappa^2$ (and hence $H = 0$). Again, the function H falls off too rapidly as a function of the proper radial distance for the M3-brane to have non-zero charge and ADM mass.

It is worth noting that in this case, as in the previous M3-brane examples in [13], the integration constant κ appears in the 4-form field strength linearly, but in the metric quardratically. It follows that also in this case the metric remains real if κ is replaced by $\mathrm{i}\,\kappa$. As discussed in [13], the replacement $\kappa \longrightarrow \mathrm{i}\,\kappa$ can be viewed as the replacement of a real 4-form by a Hodge-dual 3-form that is again real, in the positive-definite 7-dimensional space obtained by Kaluza-Klein reduction on 4-dimensional world-volume of the M3-brane. In this dual description, the 3-form in $D = 7$ can be viewed as arising from the world-volume dimensional reduction of an NS-NS 2-brane configuration in ten dimensions. Thus the replacement $\kappa \longrightarrow \mathrm{i}\,\kappa$ induces a transition between the M3-brane and an NS-NS 2-brane configuration.

4.2 M3-branes on squashed G_2 manifolds

In this section, we shall study the massless M3-brane whose transverse space is a deformation of the class of metrics of G_2 holonomy whose first-order equations, given in (60), have been constructed in [16]. These solutions will provide further examples of massless M3-brane configurations that are of the same general pattern as those constructed in [13]. We therefore begin by making the ansatz

$$
\begin{aligned}
ds_{11}^2 &= H^2\, dx^\mu\, dx_\mu + d\rho^2 + a^2\,(\tilde h_1^2 + \tilde h_2^2) + c^2\,\tilde h_3^2 + b^2\,(h_1^2 + h_2^2) + f^2\, h_3^2\,,\\
F_{(4)} &= G_{(4)}\,,
\end{aligned}
\tag{86}
$$

where

$$
h_i \equiv \sigma_i + \Sigma_i\,,\qquad \tilde h_i = \sigma_i - \Sigma_i\,,
\tag{87}
$$

and H, a, b, c and f are functions of the radial variable ρ in the 7-metric transverse to the 3-brane world-volume, whose coordinates are x^μ. The 4-form $G_{(4)}$ is constructed from isometry-invariants of the transverse 7-metric. In terms of the natural vielbein basis

$$
e^0 = h\, dr\,,\quad e^1 = a\,\tilde h_1\,,\quad e^2 = a\,\tilde h_2\,,\quad e^3 = c\,\tilde h_3\,,\quad e^4 = b\, h_1\,,\quad e^5 = b\, h_2\,,\quad e^6 = f\, h_3
\tag{88}
$$

for the metric

$$ds_7^2 = d\rho^2 + a^2\,(\tilde h_1^2 + \tilde h_2^2) + c^2\,\tilde h_3^2 + b^2\,(h_1^2 + h_2^2) + f^2\,h_3^2 \tag{89}$$

in the transverse space, the appropriate invariant ansatz for the 4-form is

$$\begin{aligned} G_{(4)} \;=\;& u_1\, e^1\wedge e^2\wedge e^4\wedge e^5 + u_2\, e^2\wedge e^3\wedge e^5\wedge e^6 + u_2\, e^3\wedge e^1\wedge e^6\wedge e^4 \\ & + u_3\, e^0\wedge e^4\wedge e^5\wedge e^6 + u_4\, e^0\wedge e^1\wedge e^2\wedge e^6 \\ & + u_5\, e^0\wedge e^2\wedge e^3\wedge e^4 + u_5\, e^0\wedge e^3\wedge e^1\wedge e^5\,. \end{aligned} \tag{90}$$

It is straightforward to calculate F^2_{ab} in the vielbein basis, leading to

$$\begin{aligned} &F^2_{00} = 6(u_3^2 + u_4^2 + 2u_5^2)\,,\quad F^2_{11} = F^2_{22} = 6(u_1^2 + u_2^2 + u_4^2 + u_5^2)\,,\quad F^2_{33} = 6(2u_2^2 + 2u_5^2)\,,\\ &F^2_{44} = 6(u_1^2 + u_2^2 + u_3^2 + u_5^2)\,,\qquad F^2_{66} = 6(2u_2^2 + u_3^2 + u_4^2)\,. \end{aligned} \tag{91}$$

We parameterise the u_i functions as

$$\begin{aligned} &u_1 = \frac{f_1}{a^2\,b^2}\,,\qquad u_2 = \frac{f_2}{a\,b\,c\,f}\,,\qquad u_3 = \frac{f_3}{b^2\,f}\,,\\ &u_4 = \frac{f_4}{a^2\,f}\,,\qquad u_5 = \frac{f_5}{a\,b\,c}\,. \end{aligned} \tag{92}$$

The Bianchi identity and equations of motion for the 4-form field strength in $D = 11$ then imply

$$\begin{aligned} dF_4 = 0:\quad & f_1' + \tfrac12 f_3 + \tfrac12 f_4 - f_5 = 0\,,\qquad f_2' + \tfrac12 f_3 - \tfrac12 f_4 = 0\,,\\ d{*}F_4 = 0:\quad & \Big(\frac{b^2\,c\,H^4\,f_4}{a^2\,f}\Big)' + \frac{c\,f\,H^4\,f_1}{2a^2\,b^2} - \frac{H^4\,f_2}{c\,f} = 0\,,\\ & \frac{a^2\,c\,H^4\,f_3}{b^2\,f} + \frac{c\,f\,H^4 f_1}{2a^2\,b^2} + \frac{H^4\,f_2}{c\,f} = 0\,,\\ & f_5 = \frac{\lambda\,c}{2f\,H^4} - \frac{a^2\,c^2\,f_3}{2b^2\,f^2} - \frac{b^2\,c^2\,f_4}{2a^2\,f^2}\,, \end{aligned} \tag{93}$$

where λ is a constant of integration. Thus we can solve for f_3, f_4 and f_5 in terms of the functions f_1 and f_2, which satisfy two second-order differential equations.

A relatively simple way to obtain the equations following from eleven-dimensional supergravity is to perform a Kaluza-Klein reduction on the 4-dimensional world-volume of the 3-brane, and hence to re-express the eleven-dimensional equations in terms of seven-dimensional ones. After doing this, it is then a routine exercise to construct a Lagrangian $L = T - V$ from which, together with the constraint $T + V = 0$, the conditions implied by the eleven-dimensional supergravity equations may be derived. We then look for a superpotential, which will enable us to construct first-order equations that are first integrals of the

original second-order equations. In order to find a superpotential for the system, it seems to be necessary to take the integation constant λ in (93) to be zero. A second constant of integration also needs to be set to zero in order to find a superpotential, implying that f_1 and f_2 become algebraically related:

$$f_1 = \frac{f_2\,(2a\,b\,c - (a^2-b^2)\,f)}{(a^2+b^2)\,f}\,. \tag{94}$$

Now our second-order differential equations involve six functions, namely a, b, c, f, H and f_2. We find that their kinetic energy $T = \frac{1}{2} g_{ij}\,(d\alpha^i/d\eta)\,(d\alpha^j/d\eta)$ has a metric whose inverse is given by $g^{ij} = g_0^{ij} + g_1^{ij}$, where

$$g_0^{ij} = \frac{1}{18}\begin{pmatrix} -\frac{7}{2} & 1 & 1 & 1 & 1 & 0 \\ 1 & -\frac{7}{2} & 1 & 1 & 1 & 0 \\ 1 & 1 & -8 & 1 & 1 & 0 \\ 1 & 1 & 1 & -8 & 1 & 0 \\ 1 & 1 & 1 & 1 & -\frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{9K_3}{2K_1} \end{pmatrix}, \tag{95}$$

and

$$g_1^{ij} = f_2 \begin{pmatrix} \frac{1}{2} f_2\,Y_2^2 & -\frac{1}{2} f_2\,Y_2^2 & -c\,f_2\,Y_2\,Y_3 & c\,f_2\,Y_2\,Y_3 & 0 & -\frac{1}{2} f^2\,Y_1\,Y_2\,Y_3 \\ -\frac{1}{2} f_2\,Y_2^2 & \frac{1}{2} f_2\,Y_2^2 & c\,f_2\,Y_2\,Y_3 & -c\,f_2\,Y_2\,Y_3 & 0 & \frac{1}{2} f^2\,Y_1\,Y_2\,Y_3 \\ -c\,f_2\,Y_2\,Y_3 & c\,f_2\,Y_2\,Y_3 & 2c^2\,f_2\,Y_3^2 & -2c^2\,f_2\,Y_3^2 & 0 & c\,f^2\,Y_1\,Y_3^2 \\ c\,f_2\,Y_2\,Y_3 & -c\,f_2\,y_2\,Y_3 & -2c^2\,f_2\,Y_3^2 & 2c^2\,f_2\,Y_3^2 & 0 & -c\,f^2\,Y_1\,Y_3^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} f^2\,Y_1\,Y_2\,Y_3 & \frac{1}{2} f^2\,Y_1\,Y_2\,Y_3 & c\,f^2\,Y_1\,Y_3^2 & -c\,f^2\,Y_1\,Y_3^2 & 0 & -2f^2\,f_2\,Y_1^2 \end{pmatrix}, \tag{96}$$

where

$$\begin{aligned} K_1 &= (3a^4+2a^2\,b^2+3b^4)\,c^2+4a\,b\,c\,f\,(a^2-b^2)+4a^2\,b^2\,f\,, \\ K_2 &= (a^2+b^2)^2\,f^2+4f_2^2\,, \qquad K_3 = f^2\,(a^2+b^2)\,((a^4+b^4)\,c^2+2a^2\,b^2\,f^2)\,, \\ Y_1 &= c\,(a^4+b^4)+a\,b\,(a^2-b^2)\,, \quad Y_2 = c\,(a^2-b^2)+2a\,b\,f\,, \quad Y_3 = a^2+b^2\,. \end{aligned} \tag{97}$$

We find that the potential V can be expressed as $V = -\frac{1}{2} g^{ij}\,(\partial W/\partial\alpha^i)\,(\partial W/\partial\alpha^j)$, where the superpotential W is given by

$$W = H^4\,(-a\,b\,f\,(a^2+b^2+c^2) + 1/2(a^2-b^2)\,c\,g)\,K\,, \tag{98}$$

and

$$K = \sqrt{1 + 4f_2^2\,(a^2+b^2)^{-2}\,g^{-2}}\,. \tag{99}$$

The first-order equations can then be straightforwardly derived, and are given by

$$
\begin{aligned}
K\,\frac{(a\,H)'}{a\,H} &= \frac{a}{4b\,c} - \frac{b}{4a\,c} - \frac{c}{4a\,b} - \frac{f}{4a^2} + (K^2-1)\,X\,,\\
K\,\frac{(b\,H)'}{b\,H} &= -\frac{a}{4b\,c} + \frac{b}{4a\,c} - \frac{c}{4a\,b} + \frac{f}{4b^2} + (K^2-1)\,X\,,\\
K\frac{(c\,H)'}{c\,H} &= -\frac{a}{2b\,c} - \frac{b}{2a\,c} + \frac{f}{2a\,b} - 2(K^2-1)\,X\,,\\
K\frac{(f\,H)'}{f\,H} &= \frac{f}{4a^2} - \frac{f}{4b^2} - 2(K^2-1)\,X\,,\\
K' &= \frac{(K^2-1)(2a\,b\,(a^2+b^2+c^2)-(a^2-b^2)\,c\,f)}{4a^2\,b^2\,c}\,,
\end{aligned}
\tag{100}
$$

together with $H = K^{1/3}$, and X is given by

$$
X = \frac{a\,b\,(a^2+b^2-2c^2)+(a^2-b^2)\,c\,f}{12a^2\,b^2\,c}\,. \tag{101}
$$

In the previous examples of M3-branes, both in [13] and in section 4.1 above, it turns out that by making appropriate redefinitions of variables, the first-order equations can be reduced to a subset that are equivalent to the original first-order equations for the undeformed Ricci-flat metric of G_2 holonomy, together with the additional equations governing the M3-brane metric function H and the functions appearing in the ansatz for the 4-form. In particular, this meant that in the M3-brane solution in section 4.1, the transverse part of the eleven-dimensional metric is of the same form as the original undeformed Ricci-flat 7-metric of G_2 holonomy, except for "warp factors" appearing in the various terms. In the situation in the present example, by contrast, it can be seen from (100) that the first-order equations for this M3-brane cannot be decomposed in such a way that the original first-order G_2 holonomy equations (60) arise as a subset. Thus in the present "squashed" example, it is not straightforward to obtain an explicit solution of the M3-brane equations.

5 Supersymmetry of massless M3-branes

One might expect that the existence of superpotentials, and hence first-order equations for the M3-brane systems, found in the previous section, would imply that these solutions would be supersymmetric. In this section, we show that this is indeed the case, for both of the new classes of M3-branes obtained above.

The calculations are best performed abstractly, by using the first-order equations for these solutions. By this means, one can establish that in these cases, as in the earlier examples in [13], these first-order equations are precisely the integrability conditions for the existence of the Killing spinor.

In fact an instructive way to study the Killing spinors is to reverse the logic, and to *derive* first-order equations by requiring the existence of such spinors. We can then show that these first-order equations are equivalent to those obtained in sections 4.1 and 4.2 via a construction of the superpotential. In the following two subsections we shall describe this procedure for the "round" and "squashed" M3-branes of sections 4.1 and 4.2 respectively.

5.1 Killing spinor in the "round" M3-brane

The gravitino transformation rule in $D = 11$ supergravity is given by

$$\delta\hat{\psi}_A = D_A\,\hat{\epsilon} - \frac{1}{288}\,F_{BCDE}\,\hat{\Gamma}_A{}^{BCDE}\,\hat{\epsilon} + \frac{1}{36}\,F_{ABCD}\,\hat{\Gamma}^{BCD}\,\hat{\epsilon}\,. \tag{102}$$

It is useful to make a $4+7$ split of the eleven-dimensional Dirac matrices:

$$\hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbb{1}\,, \qquad \hat{\Gamma}_a = \gamma_5 \otimes \Gamma_a\,, \tag{103}$$

and to seek Killing spinors of the form $\hat{\epsilon} = \epsilon\otimes\eta$. Since the field strength $F_{(4)}$ has components only in the 7-dimensional space transverse to the 3-brane, it follows from (102) that in the 3-brane world-volume directions the requirement $\delta\hat{\psi}_\mu = 0$ leads to $\partial_\mu\,\epsilon = 0$, $\gamma_5\,\epsilon = \pm\epsilon$, and the conditions

$$\tfrac{1}{2}H^{-1}\,\frac{dH}{d\rho}\,\Gamma_0\,\eta \mp \tfrac{1}{288}\,F_{abcd}\,\Gamma^{abcd}\,\eta = 0\,, \tag{104}$$

respectively, where, as usual, the "0" index denotes the radial direction in the transverse space. The matrix that acts on η here has zero eigenvalues if H satisfies one of the following conditions

$$\begin{aligned} \frac{1}{H}\,\frac{dH}{d\rho} &= \pm\tfrac{1}{6}\sqrt{9u_1^2 - 9u_4^2 - (u_2 - 3u_3)^2}\,,\\ \frac{1}{H}\,\frac{dH}{d\rho} &= \pm\tfrac{1}{6}\sqrt{u_1^2 - u_4^2 - (u_2 + u_3)^2}\,, \end{aligned} \tag{105}$$

where u_1, u_2, u_3 and u_4 are the vielbein components of the field strength $F_{(4)}$ given in (71), i.e.

$$\begin{aligned} &F_{1245} = F_{2356} = F_{3164} = u_1\,, \qquad F_{0456} = u_2\,,\\ &F_{0126} = F_{0234} = F_{0315} = u_3\,, \qquad F_{0156} = F_{0264} = F_{0345} = u_4\,, \end{aligned} \tag{106}$$

with $u_1 = 2f_1/(a^2\,b^2)$, $u_2 = f_2/a^3$, $u_3 = f_3/(a\,b^2)$ and $u_4 = f_4/(a^2\,b)$. (The $\pm$ signs in (105) are not correlated with those in (104).) Specifically, the matrix acting on η has one zero eigenvalue for each sign choice in the upper equation in (105), and three zero eigenvalues for each sign choice in the lower equation. It turns out that it is the upper equation that

is relevant for our purposes. Either choice of sign in the upper equation is allowed, since it can be reversed by a change of orientation conventions. Equally, an orientation-related sign choice was possible when we derived the first-order equations in section 4.1. The convention choices in these first-order equations and in the Killing-spinor conditions must be appropriately matched if one is to relate the two. In fact to obtain agreement with the results in section 4.1, we should choose

$$\frac{1}{H}\,\frac{dH}{d\rho} = -\tfrac{1}{6}\sqrt{9u_1^2 - 9u_4^2 - (u_2 - 3u_3)^2}\,. \tag{107}$$

Since the matrix acting on η in (104) has just one zero eigenvalue if (107) is imposed, this means that requiring $\delta\hat\psi_\mu = 0$ determines the Killing spinor η in $D = 7$ completely, up to an overall multiplicative function of the 7-dimensional coordinates on the transverse space. From the conditions $\delta\hat\psi_a = 0$ in the six directions on the S^3 bundle over S^3 of the principal orbits, we then find that η must be independent of the six coordinates on the two 3-spheres. These components of the gravitino variation also give rise to first-order equations. Since the full set of equations are somewhat involved and cumbersome to present, we shall not give them explicitly here, but simply record that they eventually turn out to be equivalent to the first-order equations for the M3-brane solutions that we obtained in section 4.1.

Finally, from the component of the gravitino variation in the radial direction, we can determine the radial dependence of the Killing spinor. The result for the case $\gamma_5\,\epsilon = +\epsilon$ is

$$\eta = g_1\,\eta_1 + g_2\,\eta_2\,, \tag{108}$$

where

$$\begin{aligned} g_1 &= H^{1/2}\,(1 + \frac{f_3}{b^2})^{1/2}\,\frac{(a(1-2g) - 2\mathrm{i}\,b)(a + 2\mathrm{i}\,b)}{\sqrt{(a^2(1-2g)^2 + 4b^2)(a^2 + 4b^2)}}\,,\\ g_2 &= H^{1/2}\,(1 - \frac{f_3}{b^2})^{1/2}\,\frac{(a(1+2g) - 2\mathrm{i}\,b)(a + 2\mathrm{i}\,b)}{\sqrt{(a^2(1+2g)^2 + 4b^2)(a^2 + 4b^2)}}\,, \end{aligned} \tag{109}$$

and η_1 and η_2 are covariantly constant spinors satisfying the constraint (13). For $\gamma_5\,\epsilon = -\epsilon$, the associated spinor η in the transverse space is again given by (108), but now with f_3 replaced by $-f_3$. The general solution for a Killing spinor can then be written as a linear combination of $\epsilon_+ \otimes \eta_+$ and $\epsilon_- \otimes \eta_-$, where the plus and minus subscripts refer to the two chirality choices under γ_5. Thus in total there will be 4 real solutions, implying $N = 1$ supersymmetry in the world-volume of the M3-brane.

5.2 Killing spinor for the "squashed" M3-brane

We can follow a similar procedure for the "squashed" M3-brane configurations obtained in section 4.2. Again, one can derive first-order equations as integrability conditions for the

existence of Killing spinors, and again it turns out that these coincide with the first-order equations for the M3-brane solutions. Thus the squashed M3-branes are also supersymmetric.

We find that the Killing spinors can again be expressed as a linear combination of terms $\epsilon_+ \otimes \eta_+$ and $\epsilon_- \otimes \eta_-$, where $\gamma_5\, \epsilon_\pm = \pm\epsilon_\pm$. The spinor η_+ in the transverse space is given by

$$\eta_+ = g_1\, \eta_1 + g_2\, \eta_2\,, \tag{110}$$

where

$$\begin{aligned} g_1 &= \sqrt{\frac{H\,(a^2+b^2)\, f}{\sqrt{4f_2^2 + (a^2+b^2)^2\, f^2)} - 2f_2}}\,, \\ g_2 &= \sqrt{\frac{H\,(a^2+b^2)\, f}{\sqrt{4f_2^2 + (a^2+b^2)\, f^2} + 2f_2}}\,, \end{aligned} \tag{111}$$

and η_1 and η_2 are covariantly constant spinors satisfying the constraint (13). The spinor η_- is given by the same expressions, but with f_2 sent to $-f_2$. Thus again we have 4 real solutions in total, and hence $N = 1$ supersymmetry in the world-volume of the M3-brane.

6 Conclusions

The two main results of the paper are (i) the construction of a new class of metrics with G_2 holonomy, and (ii) construction of new massless M3-brane solutions whose transverse spaces are deformations of the new G_2 metrics. We then showed that these M3-branes are supersymmetric.

The new class of metrics with G_2 holonomy that we found can be viewed as generalisation of the original G_2 metrics of $\mathbb{R}^4 \times S^3$ topology [11, 12], which have just a single (trivial) scale parameter, to a new family of *two-parameter* metrics of G_2 holonomy, on the manifold of the same $\mathbb{R}^4 \times S^3$ topology. This result was obtained by actually starting with a rather general ansatz for cohomogeneity one metrics whose principal orbits are S^3 bundles over S^3. For the most general ansatz that we considered, which contains nine functions of the radial coordinate, we did not find any first-order system of equations as first integrals of the Einstein equations. However, for a specialisation to a 3-function ansatz, which is spherically symmetric both in the base and in the fibre, we were able to find first-order equations derivable from the superpotential. The explicit general solution yields metrics with two parameters (aside from the completely trivial constant translation of the radial coordinate), one corresponding to the (trivial) scale size, and the second, called λ, being non-trivial

and characterising inequivalent metrics. While for a generic value of the parameter λ these metrics are singular, they become regular for $\lambda = \{-1, 0, +1\}$. The case $\lambda = 0$ corresponds to the original metric of G_2 holonomy in [11, 12]. The solutions possess an S_3 automorphism that allows one to map $\lambda = \{-1, 1\}$ to $\lambda = 0$, and that allows the continuum of values for the λ parameter to be restricted to a fundamental domain $0 \le \lambda \le \frac{1}{3}$.

The second set of results involved explicit solutions for M3-branes whose transverse spaces are deformations of the two-parameter metrics of G_2 holonomy obtained in setion 2. We obtained these M3-brane solutions by first constructing a system of first-integrals for the equations of $D = 11$ supergravity, following from a superpotential. We also constructed first-order equations for M3-branes whose transverse spaces are deformations of the new metrics of G_2 holonomy whose first-order equations were obtained in [16]. Just like the M3-branes in [13], these new M3-branes have neither mass nor charge, and have a naked singularity. Since this singularity occurs outside the radius of the innermost "endpoint" of the original undeformed metric of G_2 holonomy, this means that the singularities in the generic Ricci-flat metrics in section 2 do not materially affect the singularity structure of the associated M3-branes. We then showed that the first-order equations in both cases are precisely the integrability conditions for the existence of a Killing spinor, and hence that the M3-branes are supersymmetric.

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Note added

In an earlier version of this paper, and in [13], it was claimed that the M3-brane solutions were not supersymmetric, but instead were "pseudo-supersymmetric" with respect to a modified $D = 11$ supersymmetry transformation rule. This incorrect conclusion resulted from a systematic error in a computer program that we used for calculating the Killing spinors. We are grateful to Jim Liu for calculations that encouraged us to recheck the computer programs and discover the error.

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