Stringy World Branes and Exponential Hierarchies

P. Mayr

Abstract
We describe heterotic string and M-theory realizations of the Randall-Sundrum (RS) scenario with $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry in the bulk. Supersymmetry can be broken only on the world brane, a scenario that has been proposed to account for the smallness of the cosmological constant. An interesting prediction from string duality is the generation of a warp factor for conventional type II Calabi–Yau 3-fold compactifications. On the other hand we argue that an assumption that is needed in the RS explanation of the hierarchy is hard to satisfy in the string theory context.
1. Introduction

An appealing interpretation of the hierarchy problem is in terms of a world brane scenario with large extra dimensions that dilute gravity [1][2][3]. It was further argued in [4], referred to as RS in the following, that the remaining problem of the unnatural large extra dimension can be eased if the back-reaction of gravity to the brane produces a total space-time with suitable non-product structure. Specifically the metric considered in these papers is of the warped form

\[ ds^2 = e^{A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \] (1.1)

where \( \mu, \nu = 1, \ldots, 4 \), \( A(y) = -k|y| \) and \( y \) takes values in a finite interval. Due to the exponential dependence of the warp factor on the transverse dimension, a moderate distance in the \( y \) direction

\[ \delta y \sim 2 \ln m/M \] (1.2)

may account for the large hierarchy \( m_{EW}/M_{pl} \) between the scales of gravity and electro-weak interactions.

Note that there is an important assumption in this argument, namely that the length scale \( \delta y \) is the input which assumes a natural value whereas the ratio of mass scales \( M/m \) is the derived quantity. In the framework of ref.[4], which is supposed to be the effective description of a desired setup, the validity of this assumption can not be decided. If in contrary physics tells us that \( M/m \) is the input quantity, naturally of order one, and \( \delta y \) is a derived quantity, the logarithmic relation (1.2) is largely irrelevant for the hierarchy problem. We will argue that this is indeed the generic answer one finds in a string theory realization of this geometry. Nevertheless there are also indications that a sufficiently large interval \( \delta y \) may be stabilized in the case of heterotic and M-theory world branes.

An universal warp factor present in any string theory is the dilaton factor that relates the string sigma model metric \( g^{(s)} \) and the canonical Einstein metric

\[ g_{MN}^{(E)} = e^{-2\alpha \phi} g_{MN}^{(s)}, \] (1.3)

where \( \alpha \) is a positive number. It follows that a compactification on a manifold \( X \) with a dilaton that depends on the internal dimensions gives rise to a warp factor in
space-time. If there are degrees of freedom localized on a “world brane”, they may probe this warp factor and the relation that links the masses as measured in the brane and bulk metric is \( m_{\text{brane}} = e^{-\alpha \phi} m_{\text{bulk}} \).

In this note we argue that the world brane geometries of the RS type appear naturally in heterotic string and M-theory vacua\(^2\) and study some of their properties. Indeed the generic four-dimensional heterotic string vacua, the so-called (0,2) vacua, come with warp factors \([8][9][10]\). Moreover, on a subset of the moduli space there are non-abelian gauge symmetries localized on a 5-brane and the dilaton warp factor near the brane realizes the RS geometry. So in some sense the condition for having a world brane of RS type is not (much) more special than having non-abelian gauge symmetries, which is granted by standard model physics! A similar situation arises for 5-branes in M-theory, a subclass of which can in fact be considered as the strong coupling description of the heterotic 5-branes via M-theory/heterotic duality [11].

The heterotic and M-theory brane worlds are interesting examples of theories that provide equivalent dual descriptions in terms of either a gravity free world volume theory or on the other hand a string theory in a geometric background [12][13][14][15]. The latter describes the higher-dimensional theory in which the brane is embedded as a soliton. This dual perspectives have led to an intriguing proposal [16] that could explain the observed smallness of the cosmological constant [17], if supersymmetry is broken only on the brane. We find that the heterotic brane worlds realize this situation in a very natural way.

2. Heterotic \( SO(32) \) small instantons

2.1. The warp factor

Let us start with the small instanton in the heterotic \( SO(32) \) string [18], which has a representation as a 5-brane soliton in the low energy supergravity theory [19]\(^3\). On its world volume lives an \( \mathcal{N} = 1 \) supersymmetric \( SU(2) \) gauge theory with sixteen matter multiplets in the fundamental representation. In the following we take this gauge theory on the brane as a candidate to describe part of the standard model; more general and realistic world volume theories will be mentioned later on.

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\(^2\) World brane embeddings in orientifold/F-theory have been discussed in [5][6][7].

\(^3\) This 5-brane can be visualized as the 5+1 dimensional hypersurface transverse to a 4-plane \( H \) with \( \int_H F \wedge F = N \) and localized at the center of the instanton in \( H \) in the zero size limit.
If the 5-brane is embedded in flat ten-dimensional space-time, the six-dimensional
Planck mass on the brane is infinite. For this reason we will consider the situation
where the transverse dimensions are compactified on a 4-manifold $Y$. It is important
however that the gravity on $Y$ is not of the standard form but highly altered by the
presence of the 5-brane.

Specifically it is well-known [8][9][10] that the generic heterotic string compactifica-
tion comes with a warp factor which is entirely due to a variation of the dilaton as
in (1.3). The warp factor does in general not lead to exotic effects for the low energy
theory as long as the latter depends only on the quantities averaged over the internal
manifold. A more interesting story arises in the presence of the small instantons.
Firstly the small instantons introduce the localized degrees of freedom that probe the
warp factor. Secondly they lead to an extreme local behavior of the supergravity fields
and especially the warp factor.

In the presence of small instantons, the Bianchi identity for the anti-symmetric
tensor field strength becomes

$$dH = -tr R \wedge R + \frac{1}{30} Tr F \wedge F + \sum_{i=1}^{N} \delta_{5B},$$

(2.1)

where $\delta$ denotes a 4-form supported on the submanifold on which the $N$ instantons
localize. The first two terms on the right hand side of eq.(2.1) are forced by anomaly
cancellation in the ten-dimensional heterotic string [20]. The last term describes the
small instanton. It displays the important fact that the latter contribute to the cancel-
lation of gravitational anomalies localized on a 5-brane in space-time [21]. The integral
over the right hand side must vanish for any 4-dimensional compact submanifold $Y$ in
space-time:

$$c_2(Y) - c_2(F) - N = 0.\ 
(2.2)$$

The equation that determines the local variation of the dilaton - and thus the
warp factor - near a generic point in the transverse directions, must be of the same
form as that derived in [8] for a transverse K3 space:

$$\Delta e^{2\phi} = -tr R^2 + \frac{1}{30} Tr F^2 + \sum_{i=1}^{N} \delta_{5B}.\ 
(2.3)$$
Locally the solution for the dilaton and the metric will be approximated by the neutral 5-brane solution of refs.[19]. In the string frame one gets for $N$ small instantons located at a point $r = 0$:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + e^{2\phi}\delta_{mn}dy^m dy^n, \quad e^{2\phi} = e^{2\phi_0} \left(1 + \frac{N\alpha'}{r^2}\right),$$

(2.4)

where $x^\mu$, $\mu = 1, \ldots, 6$ and $y^m$, $m = 1, \ldots, 4$ are coordinates in the dimensions parallel and transverse to the 5-brane, respectively. To obtain a four-dimensional gauge theory on the world volume, two tangential directions of the 5-brane will be compactified on a complex curve $B$ with coordinate $z$. Introducing a new coordinate $y = \ln r$, the Einstein metric close to $r = 0$ becomes:

$$ds^2 = c^{-1/2}e^y\eta_{\mu\nu}dx^\mu dx^\nu + c dy^2 + \{g_{z\bar{z}}(B) \, dz \, d\bar{z} + c d\Omega_3\},$$

(2.5)

where $\mu, \nu = 1, \ldots, 4$, $c = e^{2\phi_0}N$ and $d\Omega_n$ is the metric on the $n$-sphere. Note that the first bracket describes a five-dimensional space with a metric of the RS form (1.1) with the world brane located at $y = -\infty$. The second bracket contains the metric for the additional transverse and compactified parallel dimensions which do not interfere with the warped five-dimensional geometry.

**Localization of gravity**

Let us compare the global structure of the warp factor as constrained by eq.(2.2) with that in the RS proposals [4]. The first of these paper assumes localization of gravity on a hidden “Planck brane” with a sink in the warp factor at the world brane, whereas the second describes localization of the graviton at a single brane. The behavior of the metric (2.5) of the small instanton is of the first type.

Moreover the non-trivial structure of gravity in the transverse dimensions, namely peaks and sinks for the warp factor corresponding to localization or dilution of the graviton wave functions, is obviously characterized by the non-vanishing of the term $c_2(Y)$. Since it is the integral of the ten-dimensional counter term for gravitational anomalies, it depends on the transverse manifold $Y$. Specifically, if $c_2(Y) = 0$, then either there is a world-brane and the space has to be non-compact to be consistent with (2.2) - which makes gravity on the six-dimensional brane trivial. Or $Y$ is a flat torus $T^n$ and the six-dimensional Planck mass is finite, but the exponential sink at the world brane is prohibited by the the absence of appropriate compensating sources on the compact manifold.

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On the other hand, if $Y$ is a compact space with $c_2(Y) \neq 0$, the global condition (2.2) allows for the presence of 5-branes. The metric is locally determined by (2.5), with an exponential sink at the world brane, and gives a perfect string theory embedding of the RS scenario. The Planck brane is replaced by nearby peaks in the graviton wave-function which are related to the non-vanishing value of the anomaly counter term evaluated on $Y$. A qualitative picture is shown in Fig. 1.

**Fig. 1:** A sketch of the shape of gravity on the manifold $Y$. The local structure near the small instanton is identical to the RS model, while globally the Planck brane is replaced by sources of the graviton wave function arising from a non-zero value of the “anomaly counter term” $c_2(Y)$.

**Scales and couplings**

The simplest case of a transverse compact space $Y$ with $c_2 \neq 0$ is a compact K3 manifold with $c_2 = 24$. To obtain a four-dimensional gauge theory the six-dimensional world volume will be further compactified on a complex curve $B$. For the moment we assume that $B$ is a $T^2$ which gives the total manifold a product structure $K3 \times T^2$. The following discussion of scales and couplings generalizes straightforwardly to the more general compactifications discussed in sect. 4.

The six-dimensional gravitational coupling constant is given by

$$M^4_{pl} = M^8_{str} \int_Y \sqrt{g(Y)} e^{-2\phi} = e^{-2\phi_0} M^4_{str}, \quad e^{-2\phi_0} = \frac{M^4_{str} V_Y}{\lambda^2_H}, \quad (2.6)$$

where $\lambda_H$ is the ten-dimensional dilaton and $V_Y$ is the volume of K3 without the warp factor in string units. In the second relation we have used the fact that we can calculate the value of the integral for a configuration with constant dilaton. Moreover,
in the string frame, the value of the six-dimensional gauge couplings in the perturbative
gauge sector and on the brane are [18]:

\[ \frac{1}{g_{6,p}^2} \sim M_{str}^2 e^{-2\phi_0}, \quad \frac{1}{g_{6,b}^2} \sim M_{str}^2. \]  

(2.7)

Note that, contrary to what happens for the perturbative gauge groups where one has
the relation \( g_6^2 M_{str}^2 \sim (M_{str}/M_{pl})^4 \), the string scale can be arbitrarily small for a fixed
gauge coupling and fixed Planck mass by making the volume \( V_Y \) large or the coupling
\( \lambda_H \) small. This has led to the idea of weak scale superstrings [2][1]. They realize a
world localized on a brane with gravity diluted by extra large dimensions, an idea
which is also central to refs.[3]. The four-dimensional couplings and scales follow from
a trivial multiplication of the six-dimensional quantities with the volume of \( B \), as the
world volume fills the entire six-dimensional space time.

However, the above discussion neglects the warp factor. Even if \( M_{str} \sim M_{pl} \),
that is there is no large dimension, the hierarchy can be generated by an exponential
warp factor \( \epsilon \equiv e^{-\delta y/2} \sim m_{EW}/M_{pl} \) in a finite interval \( \delta y \). From eq.(2.4) it follows
that the upper end of this interval, which we may choose to correspond to \( y = 0 \), is
set by a multiple of the string scale \( M_{str} \). The hierarchy \( \epsilon \) is then determined by the
lower end, or IR cutoff, \( y = -\delta y \). We will comment on the question to which extent
a large interval \( \delta y/2 \sim 20 \) is natural and stable in the present situation in sect. 6.
Of course, more generally the hierarchy can also arise from a combined effect of the
two mechanisms, with a moderately large dimension leading to an intermediate scale
\( M_{str} < M_{pl} \) and the remaining hierarchy produced by a warp factor.

3. The cosmological constant and supersymmetry breaking

Irrespectively of the hierarchy problem, there are quite interesting features of the
identification of the (heterotic) world brane theories with the observable world. An
intriguing property is that a change in the vacuum energy of the world brane fields does
not change the effective four-dimensional cosmological constant, differently then what
happens, at last naively, for the perturbative, non-localized heterotic gauge sector.
Indeed the effective cosmological constant \( \Lambda_4 \) gets contribution from both the brane
vacuum energy and the variation of the metric transverse to the brane and its total
value is given by an integration constant \( c \):

\[ \Lambda_4 = \lambda_{brane} + \lambda_{bulk} + (\partial_y \phi)^2 = c, \]  

(3.1)
where $\lambda_{brane}$ and $\lambda_{bulk}$ are the contributions from the brane and bulk fields other than the metric, respectively. For any chosen value of $c$, a change in the vacuum energy of the brane is canceled by the appropriate back-reaction of the transverse metric such that $\Lambda_4$ is kept fixed. This is the mechanism of Rubakov and Shaposhnikov [22] which was rediscovered in the context of AdS/CFT dualities in refs. [23].

As a consequence, the cosmological constant depends only on the initial value of $c$, but not on phase transitions or quantum corrections in the gauge theory on the brane. In the present context, a supersymmetric string compactification to a maximal symmetric non-compact space-time $M$ requires $M$ to be Minkowskian [24][8] and so $c = 0$. In other words, requiring supersymmetry of the bulk theory, selects the appropriate boundary condition such that $\Lambda_4$ is zero, irrespectively of the vacuum energy of the world brane fields.

The interesting question however is what determines the size of $\Lambda_4$ when supersymmetry is broken. If supersymmetry is, in first approximation, only broken on the brane, then the above mechanism still works to cancel the effective cosmological constant from a brane vacuum energy by an adjustment of the transverse metric. A non-zero cosmological constant arises therefore only from the sub-leading effect that the cancellation mechanism will be disturbed by the propagation of supersymmetry breaking into the bulk. Thus the leading order contribution to $\Lambda_4$ is determined by the scale $m_{susy}^{bulk}$, rather then $m_{susy}^{brane}$, which characterizes the mass splittings in the observable sector.

Now morally the hierarchy between the mass scales of the bulk and the brane arises from the fact that the interactions between the two type of fields are suppressed by the warp factor $\epsilon = m_{EW}/M_{pl} \sim m_{susy}^{brane}/M_{pl} \ll 1$. The idea of ref.[16] is that if a similar factor suppresses the propagation of supersymmetry breaking back into the bulk, this may lead to a sufficiently small value of $\Lambda_4$. In fact it has been argued in ref.[17] that the 1-loop contribution to the cosmological constant induced by the supersymmetry breaking in the bulk leads to values close to the observed data.

To introduce a mechanism that breaks supersymmetry exclusively on the brane is not trivial, however. In the holographic correspondence the mass operator on the brane is dual to the vev of a supergravity bulk field. Generically, an operator that leads to a mass $m$ on the brane corresponds to a mass scale $M$ in the bulk that is larger, $m^2 = e^y M^2$. It is therefore unclear whether it is even in principle possible to obtain an appropriate warped geometry that describes a supersymmetry breaking sufficiently confined to a neighborhood of the brane in the pure supergravity sector.
An extra sector that comes for free in the case of the heterotic world branes and seems to be tailor-made to realize the above situation is the perturbative gauge sector of the heterotic string with gauge group $G_0 = SO(32)$. In an $\mathcal{N} = 1$ supersymmetric vacuum, strong gauge interactions in this string sector lead to a fermion condensate at a scale $\Lambda_F \sim e^{-8\pi^2/\beta_0 g_4^2}$ that may break supersymmetry non-perturbatively [25]. Let us assume that this scale is very small in the bulk, as a consequence of a small asymptotic value of the dilaton away from the world branes or since $G_0$ is broken to a subgroup with small $b$, or a combination of these effects. No matter how small the mass scale $\Lambda_F$ is away from the five-branes it will grow with decreasing distance and induce a large scale $\Lambda_F$ of the order of the string scale in the $G_0$ gauge theory near the brane

$$\Lambda_F(y) \sim (\Lambda_F)^{\exp(-2\phi)} \frac{y - \delta y}{\delta y} (\Lambda_F)^4 \sim 1, \quad (3.2)$$

in units of $M_{\text{str}}$. Supersymmetry breaking is therefore restricted to the immediate local neighborhood of the world branes.

Although the process of supersymmetry breaking is non-perturbative in the heterotic theory and therefore difficult to access, one can, in favorable cases, use heterotic/type II duality to determine these effects by mirror symmetry [26]. We will discuss the heterotic/type II duality in presence of the world branes in more detail in the next section.

**Fig. 2:** The behavior of the warp factor $e^\phi$ and the mass scale $\Lambda_F(y)$ of the fermion condensate in the perturbative gauge sector as a function of the distance from the brane $y$. The warp factor decays exponentially towards the brane located to the left. In contrary the gaugino condensate increases to the value $\Lambda_F(-\delta y) \sim M_{\text{str}}$ in the neighborhood of the brane.
4. Brane compactifications and string duality

We proceed with a short description of four-dimensional compactifications with 5-branes with $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry and discuss some of their properties. An interesting implication arises from heterotic/type II duality.

4.1. Four-dimensional compactifications with $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry

For simplicity we will mention in the following mainly the $SU(2)$ world volume theory of one small instanton at a generic point in the transverse space-time. Obtaining a phenomenological viable spectrum is, as usual, a question of model building. However it is important to note that the available spectra are sufficiently general to embed the standard model in many ways. Specifically more general world volume theories arise for $N$ coinciding instantons and instantons at non-generic singular points in the transverse space-time. A brief summary of, and references on, these spectra are included in the appendix.

Brane worlds with $\mathcal{N} = 2$ supersymmetry

The simplest case of a four-dimensional compactification with 5-branes is the manifold $X$ conformal to K3 times an extra $T^2$. In addition to the $SU(2)$ world volume gauge theory with matter there is a single hypermultiplet for each 5-brane that parametrizes its position on K3. Moreover, in four-dimensions the gauge theory is $\mathcal{N} = 2$ supersymmetric with the extra scalars in the vector multiplets representing the Wilson lines on $T^2$. There are also Wilson lines in the perturbative $SO(32)$ gauge factor. They parametrize the masses of the fundamental hypermultiplets in the world volume theory.

The compactification on $X$ with non-trivial dilaton in the absence of 5-branes has been studied in [8] and it is entirely straightforward to adapt the discussion to include the 5-branes. As $c_2(X) = 24$, the global constraint (2.2) can be satisfied by taking a trivial gauge background $F$ and adding 24 small instantons transverse to K3. The precise warp factor is determined by (2.3) with the Laplacian understood to act only in the four directions of K3. An explicit expression for the curvature terms can be given in the orbifold limit. Including the small instantons, the equation for the warp factor becomes

$$\Delta e^{2\phi} = -\frac{3}{2} \sum_{k=1}^{16} \delta^4(x - x_k) + \sum_{i=1}^{24} \delta(x - \tilde{x}_i).$$

(4.1)
Here $x_k$ are the positions of the 16 orbifold fixed points at which the curvature of K3 has been localized. Moreover the $\tilde{x}_i$ are the locations of the the 24 5-branes.

**Brane worlds with perturbative $\mathcal{N} = 1$ supersymmetry**

Four-dimensional theories with $\mathcal{N} = 1$ supersymmetry are obtained by compactifying the heterotic string on a Calabi–Yau 3-fold $X_3$ [24]. Similar as before, evaluating $c_2(S)$ on a holomorphic four-dimensional submanifold $S$ in $X_3$ determines the possible number of sinks due to world branes transverse to $S$. The cycle $C$ dual to $S$ is holomorphic, and thus a supersymmetric cycle, on which the 5-brane of the small instanton may be wrapped [1]. For a generic position of the 5-brane the behavior of the local - or low energy - description of the warp factor is still described by (2.4).

The spectrum of the world volume theory of a small instanton wrapped on a general curve $C$ is determined by the normal bundle of the curve\(^4\). In particular the number of supersymmetries will depend on the dimension of the moduli space of $C$; an isolated (non movable) cycle $C$ is related to $\mathcal{N} = 1$ supersymmetry while wrapping on curves of genus $g = 1$ may lead to $\mathcal{N} = 2$ supersymmetric theories [28].

**4.2. A puzzle from Heterotic type/IIA duality**

String dualities map the gauge sector of one theory to a gauge sector of in general completely different origin in another theory. It is far from being obvious that they will map a gauge theory on a world brane with exponential warp factor to a localized gauge theory with a similar warp factor in the dual theory.

E.g., the heterotic string has on one hand the perturbative, non-localized gauge theory with gauge group $G_0$ and on the other hand the gauge theories on the world branes with a warp factor discussed in this paper. For a $K3 \times T^2$ compactification this string is dual to a type IIA compactification on a Calabi–Yau 3-fold $X$ [29], where both gauge symmetries arise from localized singularities [30] and thus live on a brane. No warp factor has been found so far in this type IIA compactification. How can two such theories be dual? The answer to this question is bound to be interesting:

**Resolution 1: No warp factor**: The first possibility is that the localized (heterotic) world brane theory with a warp factor maps to a theory without a warp factor in the dual (type IIA) theory. In fact it has been speculated for a while what might be the proper generalization of the AdS/CFT duality to describe a four-dimensional world

\(^4\text{See refs. [27] for a discussion of various aspects of such 5-brane wrappings.}\)
with dynamical gravity, similar to ours. A proper description of the field theory side coupled to gravity must involve string theory, so one expects that AdS/CFT duality is replaced by a suitable string/string duality. In an appropriate scaling limit one must recover string theory on 'AdS' on the one side and the 'CFT' on the other side. If the type IIA string metric remains its product form including $\alpha'$ and quantum corrections, it is likely to represent a string duality embedding of a 'AdS/CFT' correspondence. In the appropriate limit, the type IIA string represents the decoupled four-dimensional 'CFT' while the heterotic side describes its 'AdS' dual.

Resolution 2: A “quantum” warp factor. The second possibility is that the localized theories with warp factor match 1-1 to each other under the duality. In fact we want to argue that this is what happens in the present case of the heterotic/type IIA duality.

To see which effect in the type IIA theory might result in a so far neglected warp factor, note that the terms on the right hand side of eqs. (2.1) and (2.3) are higher order effects in $\alpha'$ in the heterotic string. In fact precisely the same terms have played a crucial role in the equivalence of the hypermultiplet moduli spaces of the heterotic string on K3 singularities of type $G$ and three-dimensional $G$ gauge theories with $\mathcal{N} = 4$ supersymmetry [31]. In particular the $\alpha'$ correction to the heterotic moduli maps to a 1-loop term in the 3d gauge theory. The isomorphicity of these moduli spaces for general $G$ has been established in [32] by showing that the gauge theory comes to life in a type II compactification which is the string dual of the heterotic string on the K3 singularity times a $T^3$.

This suggests that a similar effect as the one that corrects the hypermultiplet moduli of the type IIA theory in the above equivalence may also create a warp factor. An important role must be played by the $R^4$ terms in the type II action which contribute to the hypermultiplet metric and the Einstein term in four dimensions [34]. Note that to reproduce the structure of warp factors in the heterotic string, the relevant corrections in the type II compactification should arise only from those singularities that describe the non-perturbative world volume theories, but not from the others that support the gauge theories dual to the perturbative gauge group of the heterotic string.

An interesting consequence of such a warp factor for the type IIA gauge theory would be that we could extend the geometric engineering approach of refs.[28][35] to describe AdS/CFT duality. In lowest order, the type IIA vacuum is of the form

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5 See also ref. [33] for a M-theory derivation of the equivalence of these moduli spaces.
\(X_3 \times M_4\) and describes the (possibly conformal) field theory on flat \(M_4\). Adding the warp factor should deform this geometry locally to one of the type \(AdS \times Y_5\), where \(Y_5\) is a 5-dimensional space. Note that there are no RR backgrounds involved and therefore the determination of \(\alpha'\) corrections, which is essential for field theories with a low amount of supersymmetry may be available by conventional methods, such as mirror symmetry. More details will appear elsewhere [36].

5. Small \(E_8\) instantons and the M-theory 5-brane

Instead of the small \(SO(32)\) instanton, we can also consider the small \(E_8\) instanton which is described by the same equations (2.1-2.5). The main difference is the spectrum on the world volume. For a six-dimensional K3 compactification, the \(SU(2)\) gauge theory is replaced by a \(\mathcal{N} = 1\) tensor multiplet together with the hypermultiplet that describes the location on K3. The \(B_{\mu\nu}\) field in the tensor multiplet has an anti-self-dual field strength and couples to a tensionless string [37]. A nice geometric description is in terms of the M-theory dual [11] on \(S^1/Z_2 \times K3\), where the small instanton corresponds to a 5-brane stuck on one of the two end of the world 9-branes. The tensionless strings on the 5-brane represent the boundary of a membrane ending on it. Upon a compactification on a circle we obtain conventional gauge theories. From the tensionless strings one obtains also matter multiplets that are charged under the perturbative gauge symmetry of the heterotic string [38]. In fact a large class of these theories will be equivalent to the theories from the small \(SO(32)\) instanton by the usual T-duality [39].

An interesting new branch in the moduli space of these theories is the one where the 5-brane leaves the 9-brane, which corresponds to a vev for the scalar in the tensor multiplet (and a Coulomb branch in the T-dual \(SO(32)\) theory). For \(N\) coinciding 5-branes in the bulk compactified further on a circle one obtains a \(SU(N)\) gauge theory on the world volume theory with one adjoint hypermultiplet [40]. Except for the interactions with the 9-branes, which are suppressed by the distance to them, this 5-brane is the same as the M-theory 5-brane.

Similar as the small heterotic instanton (2.1), the M-theory 5-brane contributes to gravitational anomalies. It also contributes to the Bianchi identity of the 4-form field strength \(dF = \sum_k \hat{\delta}_k\). The details of the anomaly cancellations are different [41][42] as the eleven-dimensional supergravity has no anomalies on a smooth manifold. For this reason the 5-branes arise in M-theory compactification with singularities, such
as the $S^1/Z_2$ compactification and the orbifolds without a heterotic dual described in [41][43]. These singularities contribute the necessary terms in the gravitational equations of motions which take over the role of the term $c_2(X)$ in the case of the perturbative heterotic string. The solution for the warp factor in a neighborhood of $N$ M-theory 5-branes is [44]

$$ds^2 = \Delta^{-1/3} \eta_{\mu\nu} dx^\mu dx^\nu + \Delta^{2/3} \delta_{mn} dy^m dy^n, \quad \Delta = 1 + \frac{N \pi l_{11}^3}{r^3}. \quad (5.1)$$

After compactification on the curve $B$ this is again of the RS form near the brane location $r = 0$

$$ds^2 = c^{-1/3} e^y \eta_{\mu\nu} dx^\mu dx^\nu + c^{2/3} dy^2 + (g_{zz}(B) dz d\bar{z} + c^{2/3} d\Omega_4), \quad (5.2)$$

with $c = \pi N$ and $\mu, \nu = 1, \ldots, 4$. The duality between the world volume theory and the supergravity or string background for this case has been discussed in detail in [12][15].

At this point the reader might wonder why the exponential warp factor did not appear in the five-dimensional supergravity description of refs.[1][45], which give a linear warp factor in the direction separating the 9-branes, usually denoted as $x^{11}$. However, neither for the heterotic 5-brane which is stuck on the 9-brane, nor the M-theory 5-brane in the bulk, there is a reason to single out the $x^{11}$ direction near the 5-brane. Locally the only natural direction is the radial distance $r$ from the brane in which the warp factor is of the exponential form described in eqs.(2.5) and (5.2).

6. Comments

In this note we argued that world brane geometries of the Randall-Sundrum type appear naturally in generic heterotic string and M-theory compactifications with enhanced non-abelian gauge symmetries. The heterotic and M-theory 5-branes give an interesting and natural example of world branes with a holographic duality and dynamical gravity on the brane. The four-dimensional theories obtained by compactifying two of the brane dimensions are interesting candidates for a ‘standard model’ on the brane. In particular it will be interesting to determine the structure of supersymmetry breaking for $\mathcal{N} = 1$ compactifications and to compare the resulting value of the cosmological constant with the proposals in refs.[16][17].
On the other hand, although the RS geometry appears naturally in string theory, it does not necessarily lead to an explanation of the hierarchy: why should we consider the length scale $\delta y$ as more fundamental than the ratio of mass scales $m/M$? Note that a simple line of argument which would identify the distance of the branes with a modulus $\phi$ and then assign some natural vev to it applies equally well to the alternative identification of $\phi$ with the mass and so does not resolve the issue.

More concretely, the UV end, say at $y = 0$, of the interval $\delta y$ will be set by some multiple of the string scale $c M_{str}$. Smaller values of $y$ correspond to the IR flow of the world volume theory. If the world volume theory is conformal, there is an infinitely long homogeneous flow to $y = -\infty$. On the other hand if the theory has a mass scale $m$, as it should if it serves as a model for a world brane, the exponential flow of the warp factor will stop at $y = 2 \ln m/M_{str}$. E.g. this is precisely what happens in the case of D3-branes, which have an $\mathcal{N} = 4$ supersymmetric world volume theory [46]. Operators in the world volume theory that break conformal invariance and part of the supersymmetry, such as fermion masses, correspond linearly to vev’s of anti-symmetric tensors in the dual supergravity description.

Now the point is that the scale characterizing this anti-symmetric tensor backgrounds, as well as any other generic mass scale $m$ in the string theory, is naturally of the order of $M_{str}$. Thus the length $\delta y$ of the exponential flow of the warp factor is a derived quantity, proportional to $\ln m/M_{str}$. In particular the natural value of the length scale $\delta y$ is very small in string units and does not account for the required hierarchy.

It should therefore be clear that a situation in string theory, where $\delta y$ is naturally of order one must be a very peculiar one. In this respect it does not seem that the exponential warp factor leads to a substantial facilitation of the hierarchy problem in the well-defined context of string theory.

However it is worth pointing out that the situation for the heterotic world branes is at least more promising than for D-branes in this respect. The reason is that the mere existence of the world volume degrees of freedom implies that however small the asymptotic string coupling $e^{\phi_0}$ is away from the brane, the string coupling $e^{\phi}$ has to be at least of order one at the brane [18]. So the quantity $e^{\delta \phi} = e^{\phi - \phi_0} \sim e^{-2}$ which characterizes at the same time the hierarchy between the scales on the brane and in the bulk, can be large$^6$. It would be interesting to study further whether this property

$^6$ A similar conclusion has been reached in [46] for type II NS 5-brane solutions which are related by $S$-duality to the perturbed D3 brane configurations.
of the world branes considered in this note may be used to make the RS argument work in a string theory context.

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Appendix A. More references on more world volume theories

For \( N \) coinciding small \( SO(32) \) instantons the world volume theory is a \( Sp(N) \) \( \mathcal{N} = 1 \) supersymmetric gauge theory in six dimensions together with a hypermultiplets in the \((32, 2k)\) representation, and an antisymmetric representation of \( Sp(k) \) [18]. For \( N \) coinciding small \( E_8 \) instantons one obtains a world volume theory of \( N \mathcal{N} = 2 \) tensor multiplets and in addition tensionless strings [37]. Upon compactification on a circle the tensor multiplets become vector multiplets and the strings wrapped on \( S^1 \) particles. For a certain subset of the moduli space, the theory will be T-dual to the compactification of the \( SO(32) \) small instantons.

A large variation of spectra arises if the \( N \) instantons are moved to singular points in the transverse K3 [47]. In particular the \( E_8 \) instanton may lead to extra gauge degrees of freedom located at the singularity, while the \( SO(32) \) instantons have now also tensor multiplets in their spectrum.

Upon compactification on a complex curve \( B \), the spectrum on the world volume depends on the embedding of \( B \) in the space-time and in particular on the number of 1-cycles \( \gamma_i \) in \( H_1(B) \); reduction of a six-dimensional vector (tensor) on \( \gamma_i \) leads generically to a scalar (vector). The theory on the world volume is twisted [48] and the precise spectrum depends on the normal bundle to \( B \) in space-time. See e.g. [28][49] for calculations of this type in special cases.

References


[38] See first reference in [27].