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The $U(1)$s in the Finite N Limit of Orbifold Field Theories

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Abstract

We study theories generated by orbifolding the $\mathcal{N} = 4$ super conformal $U(N)$ Yang Mills theory with finite N, focusing on the rôle of the remnant $U(1)$ gauge symmetries of the orbifold process. It is well known that the one loop beta functions of the non abelian $SU(N)$ gauge couplings vanish in these theories. It is also known that in the large N limit the beta functions vanish to all order in perturbation theory. We show that the beta functions of the non abelian $SU(N)$ gauge couplings vanish to two and three loop order even for finite N. This is the result of taking the abelian $U(1)$ of $U(N) = SU(N) \otimes U(1)$ into account. However, the abelian $U(1)$ gauge couplings have a non vanishing beta function. Hence, those theories are not conformal for finite N. We analyze the renormalization group flow of the orbifold theories, discuss the suppression of the cosmological constant and tackle the hierarchy problem in the non supersymmetric models.

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Contents

1 Introduction

Supersymmetric Conformal Field Theories (SCFT) in the large N limit have been extensively studied and are very well understood. Both the hierarchy problem and the cosmological constant problem are solved in SCFT. Unfortunately, we live in a non-supersymmetric non-conformal finite N world. Orbifolds of SCFT give us an opportunity to study non-SCFT using our knowledge of SCFT and, hopefully, without loosing all the properties of SCFT. In this paper, we analyze orbifolds of SCFT with finite N, focusing on the rôle of the $U(1)$ gauge symmetries that the orbifold process leaves us with.

The large N limit was first introduced by 't Hooft [?] who taught us that planar diagrams dominate the amplitudes of $U(N)$ gauge theories in the large N limit. He also noticed the analogy between the topologies of Feynman diagrams and the topologies of strings of the dual string model.

More recently, Maldacena conjectured [?] that there is a correspondence between type IIB string theory on $AdS_5 \times S^5$ and four dimensional $\mathcal{N} = 4$ $U(N)$ SCFT. In the large N limit it is a correspondence between IIB supergravity and $\mathcal{N} = 4$ SCFT. (For a review and references see [?].)

In "The Wall of the Cave" [?] Polyakov suggested that non supersymmetric non conformal field theories should be described by type 0 string theory. The problem with type 0 string theory is that it has a tachyon in the closed string sector. Klebanov and Tseytlin showed in [?] that the coupling of the tachyon to the R-R fields shifts the effective mass of the tachyon and can cure its instability.

In type 0B there is a doubling of the R-R sector. Specifically, the five-form field strength F_5 is unconstrained, giving rise to electric and magnetic D3 branes. In [?] the field theory living on N electric and N magnetic D3 branes was first analyzed. It is an $SU(N) \otimes SU(N)$ non supersymmetric theory. The gauge coupling one loop β function is zero and the two loops β function vanishes in the large N limit, suggesting that in the large N limit this is a non-supersymmetric conformal field theory.

We noticed that the two loop β function also vanishes for finite N if a diagonal $U(1)$ gauge field with a matching $U(1)$ scalar is included in the model. This observation was the trigger to this paper.

The inclusion of the $U(1)$ fields makes the $SU(N)$ two loop β function vanish. However, the β function of the $U(1)$ gauge is non vanishing already at one loop. Hence, the theory is not conformal for finite N.

The $U(N) \otimes U(N)$ model is a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ $U(2N)$ super Yang Mills where $\mathbb{Z}_2$ is in the center of the $SU(4)_R$ symmetry [?, ?]. This leads to the possibility that including the $U(1)$ fields in general $\mathcal{N} = 4$ orbifolds makes the two loop β function vanish.

Orbifolds in the AdS/CFT correspondence where first considered in [?]. In string theory the orbifold acts on the $SO(6) \sim SU(4)$ isometry group of S^5. In field theory the orbifold acts on the $SU(4)_R$ symmetry.

Naïvely, one expects that the orbifolds will have no effect on the other symmetries of the theory. On the string theory side, this means that the isometry group of AdS_5, i.e. $SO(4,2) \sim SU(2,2)$, remains intact. On the field theory side, this means that the conformal group $SO(4,2)$ is not broken, leading to a conformal field theory.

However, the naïve expectation is not realized. The one loop β function of the gauge coupling does vanish [?], But the higher loop corrections vanish only in the large N limit [?, ?]. The source of the large N requirement in orbifolds is not obvious from the field theory point of view since the original $\mathcal{N} = 4$ theory is conformal also for finite N.

We claim that taking "The $U(1)$s in the finite N limit of Orbifold Field Theories" into account is required for the understanding of the orbifolded theories. The $U(1)$s can be ignored by setting their couplings to zero, yet the vanishing of the two loop β functions when the $U(1)$s are taken into account signifies their rôle in the orbifold theories.

Orbifold theories with finite N were already analyzed in the literature but without taking the $U(1)$ factors into account. In [?], the conditions for the canceling of the two loop β functions were considered. In [?] it was suggested that softly broken conformal symmetry could solve the hierarchy problem. In [?] the RG flow of the orbifold theories was analyzed.

We start our analysis of finite N theories by presenting in section 2 the double line notation for finite N. We find a subset of diagrams that have no subleading corrections in N, and entitle them as "calculable".

In section 3 we present the double line notation for orbifold theories. We claim that it is natural to choose all the coupling in the orbifold theory equal and introduce the concept of a natural line on which all the couplings are equal and related to the original $\mathcal{N} = 4$ coupling. We prove the vanishing of the β function up to three loops for finite N on the natural line. Our proof is based on the proof for the large N limit [?, ?] combined with the fact that all diagrams up to three loops are "calculable".

In subsection 3.4 we discuss the scalar mass corrections that vanish up to three loops for most of the scalars. This helps to solve the hierarchy problem in orbifolds that do not have $U(1)$ scalars. In subsection 3.5 we discuss the vacuum bubble diagrams that vanish up to four loops. This could have solved the cosmological constant problem if it were not for the running of the $U(1)$ couplings. However, it still leads to a suppressed cosmological constant relative to general non supersymmetric field theories.

Since the orbifold theories are not conformal for finite N, an analysis of the renormalization group flow is in order. This is done in section 4. We start with the easiest case when the orbifold projection leaves us with an $\mathcal{N} = 2$ supersymmetry. In this case the $U(1)$ fields are decoupled from the $SU(N)$ fields leaving the $SU(N)$ theory conformal, but strictly speaking, those theories are not conformal because of the running of the $U(1)$ couplings.

For the $\mathcal{N} = 1$ orbifolds we use the arguments of [?] to analyze the manifold of fixed point. We find only the fixed line found in [?] when the $U(1)$ fields decouple and show that the natural line flows to the fixed line in the IR.

For the non-supersymmetric orbifolds, the lack of any non renormalization theorems limits our results to what we can directly calculate. We calculate the effective scalar potential to one loop order in an attempt to check the validity of the orbifold theory. We also calculate the β functions to determine the RG flow of the model.

In section 5 we summarize our results and discuss the prospects of generalizing the proof of the vanishing of the β function to all orders in perturbation theory. We also point out some open issues and related topics not pursued in this paper.

2 The Unitary Group in Double Line Notation

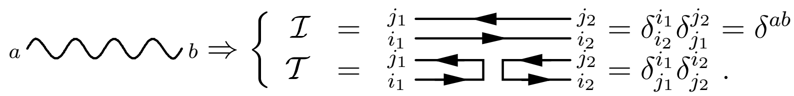
2.1 Elementary Group Theory

We start with a short presentation of the unitary group in order to introduce the double line notation. The double line notation introduced by 't Hooft [?] gives the leading order behavior in N. We present a notation that gives exact results including subleading terms in N. Our notation closely resembles Cvitanović's birdtracking notation [?, ?].

The unitary group $U(N)$ is the group of unitary transformations on a vector (quark) q with N complex components, leaving $\bar{q}q = \delta^i_j q_i q^j$ invariant. The Kronecker delta is the projection operator (propagator) of the defining (fundamental) representation

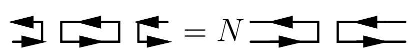
$$= \delta^i_j \ . \tag{2.1}$$

All invariant tensors can be constructed by products of Kronecker deltas. We are mainly interested in the adjoint representation, since all matter in the $\mathcal{N} = 4$ SYM model is in this representation. The adjoint representation is constructed from a quark-antiquark state. There are two invariant tensors for the quark-antiquark state, the identity $\mathcal{I}$ and the trace $\mathcal{T}$



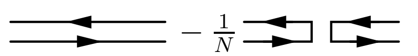
$$\begin{cases} \mathcal{I} = \delta^{i_1}_{i_2}\delta^{j_2}_{j_1} = \delta^{ab} \\ \mathcal{T} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} \ . \end{cases} \tag{2.2}$$

Where $a, b = 1 \ldots N^2$ or in other words $a = \binom{i}{j}$. The eigenvalues of the trace matrix can be calculated using the trace tensor equation

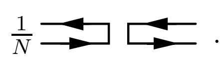
$\mathcal{T}^2 =$  $= N\mathcal{T}$.

The roots of the equation are $\lambda_1 = N$ and $\lambda_2 = 0$. With each root we can associate a projection operator $P_i = \frac{\mathcal{T} - \lambda_j \mathcal{I}}{\lambda_i - \lambda_j}$

$$P_{SU(N)} = \frac{\mathcal{T} - N\mathcal{I}}{0 - N} \tag{2.3}$$



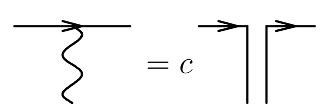
$$P_{U(1)} = \frac{\mathcal{T} - 0\mathcal{I}}{N - 0} \tag{2.4}$$



Those projection operators are orthonormal, $P_i P_j = \delta_{ij} P_j$, and complete, $\sum P_i = \mathcal{I}$, giving us the $SU(N)$ and the $U(1)$ propagators.

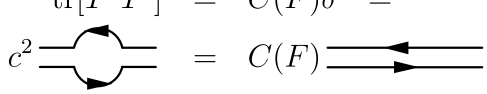
The generator of the defining representation (the quark-antiquark gluon vertex) is

$$(T^a)^i_j = \tag{2.5}$$



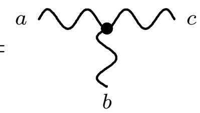
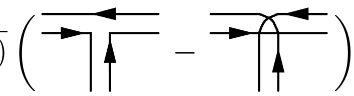
where c is an overall normalization set by the Dynkin index of the fundamental representation

$$\mathrm{tr}[T^a T^b] = C(F)\delta^{ab} =$$



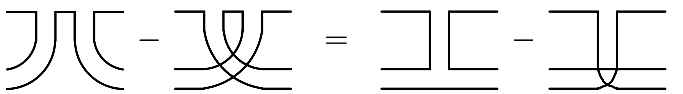
$$\Rightarrow c^2 = C(F) \ . \tag{2.6}$$

The adjoint generator is

$$(G^a)^{bc} = if^{abc} =$$  $$= \sqrt{C(F)}$$  $$, \quad (2.7)$$

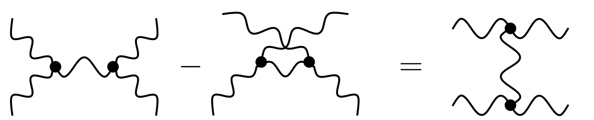
where we choose the convention that indices are marked counterclockwise. The antisymmetric form of the adjoint generator is required in order to satisfy the Lie algebra for the fundamental generators,

$$\left[T^a, T^b\right] = if^{abc}T^c \ ,$$

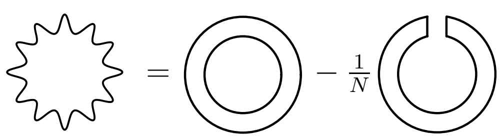
 (2.8)

It is easy to check that (2.7) also satisfies the Lie algebra for the adjoint generators,

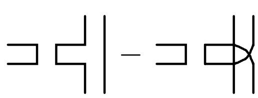
$$[G^a, G^b] = if^{abc}G^c \ ,$$

 (2.9)

In order to calculate Feynman diagrams with gauge invariant external sources we also need the one loop vacuum bubble diagram

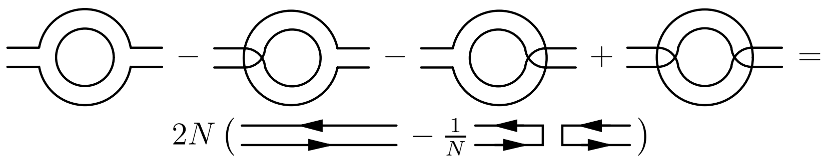
 $$= N^2 - 1 = d(G) \ . \quad (2.10)$$

Now we have the tools to calculate the group factor of any Feynman diagram. Just replace each vertex with the two vertices in (2.7), each $SU(N)$ propagator with (2.3) and each $U(1)$ propagator with (2.4), and sum all the diagrams. For a Feynman diagram with V vertices and P $SU(N)$ propagators one needs to sum 2^{V+P} diagrams. The number of diagrams one needs to sum can be reduced by using the fact that the $U(1)$ propagator decouples from the adjoint vertex (2.7)

 $$= 0 \ . \quad (2.11)$$

Consequently, we can replace each $SU(N)$ propagator with the identity propagator $\mathcal{I}$.[1]

The calculation of the Dynkin index for the adjoint representation comes from the group factor of the one loop correction to the two point function

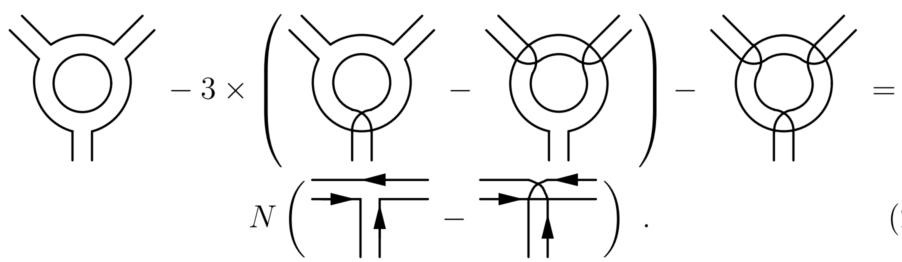


(2.13)

$$\Rightarrow C(G) = 2NC(F)\ .$$

We see that although we used $U(N)$ propagators, we actually calculated the Dynkin index for the $SU(N)$ propagator. This is a result of the $U(1)$ decoupling (2.11). The Dynkin index for the $U(1)$ propagator is zero.

It is useful (and easy) to calculate the group factor of the one loop correction to the three point function



(2.14)

The "3×" stands for the three possible permutation of each diagram. Each permutation results in a different diagram, so "×" can not be treated as the multiplicity of the diagram, but this does not matter since the term in the brackets is zero anyway. Equation (2.14) tells us that the group factor of the one loop correction to the vertex (2.7) is $NC(F) = \frac{1}{2}C(G)$.

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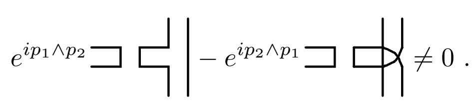
[1] Digressing to non commutative geometry, we point out that in the double line notation it is manifest that the $U(1)$ in non commutative geometry does not decouple from the adjoint vertex because we need to add different phases to each diagram[**?**]



(2.12)

2.2 't Hooft Large N Limit

Before proceeding to higher loop diagrams, we wish to recall 't Hooft results for the large N limit [?]. Using the double line notation we can get the N dependence of a Feynman diagram with adjoint fields from topological considerations. A connected diagram with $V = V_3 + V_4$ vertices, $E = \frac{1}{2}(3V_3 + 4V_4)$ edges (propagators) and F faces (closed lines in the double line notation) has a group coefficient proportional to

$$g_{YM}^{V_3+2V_4} N^F = \lambda^{E-V} N^\chi , \qquad (2.15)$$

where $\chi \equiv V - E + F = 2 - 2g$ is the Euler characteristic and g is the genus of the surface defined by the double line diagram with all the faces shrunk to a point. Each Feynman diagram is translated into a number of diagrams in the double line notation which can have different genera. The leading N contribution comes from the double line diagrams with the minimal genus. The 't Hooft limit is defined by taking N to infinity while leaving $\lambda = g^2 N$ fixed.

2.3 "Calculable" Diagrams

For some diagrams we can calculate not only the leading N contribution, but the exact N dependence using (2.9), (2.13) and (2.14). We will refer to diagrams that can be thus calculated as "calculable" (all diagrams are calculable but the "calculable" ones are easily so). We now show generally that *any "calculable" L-loop Feynman diagram of adjoint fields has a group factor proportional to*

$$g_{YM}^{V_3+2V_4} N^{L-1}(N^2 - 1) = \lambda^{E-V}(N^2 - 1) . \qquad (2.16)$$

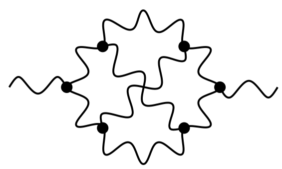
with no subleading corrections in N. Here L is the number of loops in the Feynman diagram. It can be defined as the number of momentum loops needed to be integrated over in the vacuum bubble diagram. The factor of $(N^2 - 1)$ comes from the one loop vacuum bubble diagram (2.10) that has $L = 1$. For every two 3-point vertices and for every 4-point vertex we add to the diagram we get one more loop, therefore $L = 1 + \frac{1}{2}V_3 + V_4$.

When calculating the group factor of the diagram, each time (2.13) or (2.14) is used, a factor of N is added and a loop is removed. (2.9) does not change neither the power of N nor the number of loops. Hence, after using

(2.13) or (2.14) $L-1$ times, we get the one loop vacuum bubble diagram (2.10) with a factor of N^{L-1}.

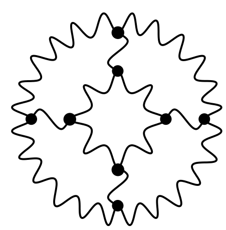
We assumed that each four point vertex has the group structure of two 3-point vertices. This is not true in general, but it is true for the $\mathcal{N}=4$ model.

Equation (2.15) tells us that the leading order contribution in N comes from the diagram with the minimal genus. Equation (2.16) tells us that for "calculable" diagrams the exact N dependence is (N^2-1). The two equations can match only for diagrams with genus zero. Hence we conclude that *non-planar "calculable" diagrams have a vanishing group factor.* For example the non planar two loop correction to the photon is "calculable" which means that



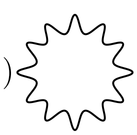
$$= 0 \ . \tag{2.17}$$

The first "incalculable" vacuum diagram is the five loop vacuum bubble diagram



$$= \tag{2.18}$$

$$2C(F)^4(N^4+12N^2)$$



$$= 2C(F)^4(N^4+12N^2)d(G) \ .$$

It is "incalculable" because in each loop there are four vertices and we can not change this with the use of (2.9). We calculated the group factor of this "incalculable" diagram by summing up all the $2^V = 256$ double line diagrams.

The first "incalculable" propagator diagram is the four loop diagram obtained by cutting out a propagator from (2.18). The first "incalculable" three vertex diagram is the three loop diagram obtained by cutting out a vertex from (2.18).

3 Orbifolds in the Double Line Notation

3.1 The Orbifold Process

The orbifold of $\mathcal{N}=4$ SYM is defined by a discrete subgroup Γ of the global R symmetry group $SU(4)_R$. The action of the orbifold on the gauge group $U(|\Gamma|N)$ is defined by the γ matrices

$$g \in \Gamma \quad : \quad g \to \gamma_g \ ,$$

where γ_g are $(|\Gamma| \times |\Gamma|) \otimes \mathbf{1}_{N\times N}$ matrices with $\gamma_1 = \mathbf{1}_{|\Gamma|\times|\Gamma|} \otimes \mathbf{1}_{N\times N} = \mathbf{1}_{|\Gamma|N\times|\Gamma|N}$. The orbifold breaks the gauge group into $U(N)^{|\Gamma|}$. For simplicity we assumed that all irreducible representations of Γ are one dimensional, namely that Γ is abelian. For a general discrete group Γ with irreducible representations labeled r_i, the orbifold breaks the gauge group $U(\sum_i d_i N)$ into the $\bigotimes_i U(d_i N)$ gauge group where $d_i = \dim r_i$.

The cancellation of tadpoles in string theory imposes that the representation of Γ has to be regular [**?**], meaning $\mathrm{tr}[\gamma_g] = 0 \ \forall g \neq 1$. The regularity of Γ guarantees the cancellation of gauge anomalies, though it is not a necessary condition.

The spectrum of the orbifold theory is defined by the projection operator [**?**]

$$P = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} r_g \otimes \gamma_g^\dagger \otimes \gamma_g \ , \tag{3.1}$$

where r_g is the representation of the projected field under the $SU(4)_R$ symmetry group. The projection leaves only fields that are invariant under the orbifold Γ, i.e.

$$\begin{aligned} A_\mu &= \gamma_g^\dagger A_\mu \gamma_g \qquad \forall g \in \Gamma \ , \\ \phi_I &= \gamma_g^\dagger (r_g^6)_I^J \phi_J \gamma_g \qquad {\scriptstyle I,J=1\ldots6} \ , \\ \psi_I &= \gamma_g^\dagger (r_g^4)_I^J \psi_J \gamma_g \qquad {\scriptstyle I,J=1\ldots4} \ . \end{aligned}$$

For the double line notation we split the fundamental propagator (2.1) into $|\Gamma|$ parts

$$\begin{aligned} \delta^i_j &= \delta^k_l \delta^{i_k}_{j_l} \qquad {\scriptstyle k,l=0\ldots|\Gamma|-1} \ , \\ i \longrightarrow j &= \mathrm{diag}(i_0 \,\text{———}\, j_0, i_1 \,\text{- - - -}\, j_1, i_2 \cdots\cdots j_2, \ldots) \ . \end{aligned} \tag{3.2}$$

The action of the projection operator (3.1) on the adjoint propagator depends on the group element g under which the propagator transforms,

$$
\begin{aligned}
g_0 = 1:\ \delta^l_k(\mathcal{I} - \tfrac{1}{|\Gamma|N}\mathcal{T}) \;&=\; \mathrm{diag}(\,\overline{\quad\quad}\; - \tfrac{1}{N}\sqsupset\ \sqsubset\,,\\
&\qquad\quad \text{- - - -} - \tfrac{1}{N}\,\text{-}\sqsupset\ \sqsubset\text{-},\dots)\ , \qquad (3.3)\\
\delta^l_k(\tfrac{1}{|\Gamma|N}\mathcal{T}) \;&=\; \mathrm{diag}(\tfrac{1}{N}\sqsupset\ \sqsubset\,, \tfrac{1}{N}\,\text{-}\sqsupset\ \sqsubset\text{-},\dots)\ , \qquad (3.4)\\
g_{k'} = e^{\frac{2\pi i}{|\Gamma|}k'}:\ \delta^l_{k+k'}(\mathcal{I} - \tfrac{1}{|\Gamma|N}\mathcal{T}) \;&=\; \mathrm{offdiag}_{k'}(\,\overline{\text{- - - -}}\,, \overline{\text{- - - -}}\,,\dots)\ , \qquad (3.5)
\end{aligned}
$$

where $k' = 1 \dots |\Gamma| - 1$. The $\mathcal{T}$ propagator is multiplied by a factor of $|\Gamma|$ because $\delta^l_k \delta^k_l \delta^{i_k}_{j_l} = |\Gamma| \delta^{i_k}_{j_l}$. We see that fields that are invariant under $SU(4)_R$ are in adjoint representations of $\big(SU(N) \otimes U(1)\big)^{|\Gamma|}$, while non-invariant fields are in bifundamental representations. $\mathrm{offdiag}_{k'}$ means that the nonzero elements are shifted k' places off the diagonal.

In (3.5) we assumed that the orbifold group is $\Gamma = \mathbb{Z}_{|\Gamma|}$. For other groups we would have a different representation for $g_{k'}$ and would get the bifundamental fields in different permutations.

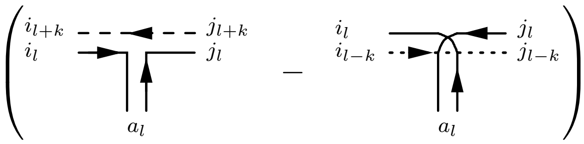
The embedding of $\mathbb{Z}_{|\Gamma|}$ in $SU(4)_R$ can be specified using four integer weights (k_1, k_2, k_3, k_4) describing how $\mathbb{Z}_{|\Gamma|}$ operates on the fundamental representation **4** of $SU(4)_R$. Because we are interested in subgroups of $SU(4)$ and not of $U(4)$, we require $k_1 + k_2 + k_3 + k_4 = 0 \mod |\Gamma|$. Consequently, the embedding of $\mathbb{Z}_{|\Gamma|}$ is parameterized by three integer numbers. The transformation of the antisymmetric representation **6** is described by the six integers $k_{(i,j)} = k_i + k_j$, where (i, j) are $\binom{4}{2}$ unordered pairs.

To describe a general orbifold group, not necessarily abelian, we can use quiver diagrams [**?**] with a node for each irreducible representation of Γ ($|\Gamma|$ nodes for abelian groups). The diagrams have four directed links from each node describing the four fermions (ψ), four directed links going into each node ($\bar{\psi}$), three undirected (double directed) links from each node describing the six real scalars (three complex scalars) and one undirected link going from each node to itself describing the gauge field in the adjoint representation.

The effect of the orbifold on the three vertex (2.7) is described by three integers that must satisfy $k_1 + k_2 + k_3 = 0 \mod |\Gamma|$ (a closed loop in the quiver language), since the vertex is $SU(4)_R$ invariant. If we choose $k_1 = -k_2 \equiv k$,

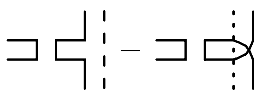
$k_3 = 0$, we get the bifundamental generator

$$(T^{a_l})^{i_l}_{j_l}\delta^{j_{l+k}}_{i_{l+k}} \quad - \quad (T^{a_l})^{j_l}_{i_l}\delta^{i_{l-k}}_{j_{l-k}} =$$

$\sqrt{C(F)}$  . (3.6)

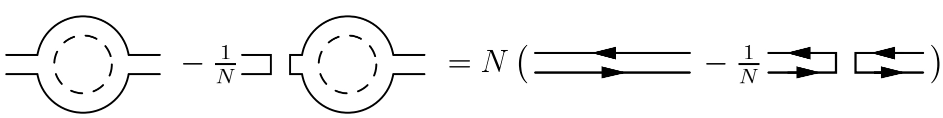
We can read from (3.6) that an adjoint in $SU(N)_l$ couples to a couple of bifundamentals, $(N_l, \overline{N}_{l+k})$ and $(N_{l-k}, \overline{N}_l)$. For $k_1, k_2, k_3 \neq 0$ we get vertices of three bifundamental. Those vertices exist only for some orbifolds, those that have a triangle with all vertices on different nodes of the quiver diagram.

It is obvious that the $U(1)$ factors do not decouple any more,

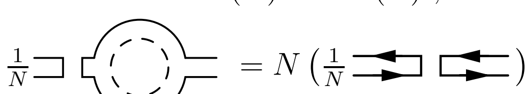
 $\neq 0$. (3.7)

The $|\Gamma|$ $U(1)$ factors of the orbifold theory are not independent, as the sum over the $U(1)$ charges for each field is zero. This is a consequence of the decoupling of the $U(1)$ from the original $U(|\Gamma|N)$ theory (2.11).

The Dynkin index of the bifundamental representation is



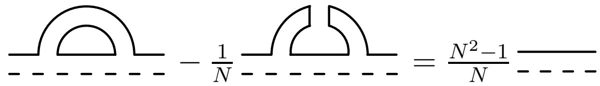
$$\Rightarrow C^{SU(N)}(B) = NC(F) \ , \tag{3.8}$$



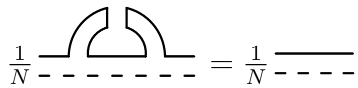
$$\Rightarrow C^{U(1)}(B) = NC(F) \ . \tag{3.9}$$

The N dependence of the $U(1)$ Dynkin index comes from the multiplicity of the bifundamental representation. Diagrams with one twisted vertex are projected out in the orbifold process (3.1). Diagrams with two twisted vertices will give the Dynkin index of the anti-bifundamental representation $C(\bar{B}) = NC(F)$. One can see that we no longer have the luxury of ignoring the trace propagator $\mathcal{T}$.

The second Casimir $(T^aT^a)^i_j = C_2(F)\delta^i_j$ of the bifundamental representation is calculated from the one loop correction to the bifundamental propagator,



$$\Rightarrow C_2^{SU(N)}(B) = \frac{N^2-1}{N} C(F) \ , \tag{3.10}$$



$$\Rightarrow C_2^{U(1)}(B) = \frac{1}{N} C(F) \ . \tag{3.11}$$

The gauge group is $(SU(N) \otimes U(1))^{|\Gamma|}$ meaning that for every $SU(N)$ second Casimir there is a $U(1)$ second Casimir. The contribution of the one loop bifundamental propagator to the Feynman diagram will alway be of the form

$$\left(g_N^2 \frac{N^2-1}{N} + g_1^2 \frac{1}{N} \right) C(F) = \left(g_N^2 N + \frac{g_1^2 - g_N^2}{N} \right) C(F) \ , \tag{3.12}$$

where g_N is the $SU(N)$ gauge coupling and g_1 is the $U(1)$ gauge coupling. From (3.12) we see that the $U(1)$ factor can be neglected in the large N limit, in the 't Hooft limit it is suppressed by a factor of $\frac{1}{N^2}$. We also see that if we choose $g_N = g_1$, the subleading corrections in N are canceled. This is the natural choice since we originally had a $U(N)$ symmetry that was split by the RG flow to $SU(N) \otimes U(1)$.

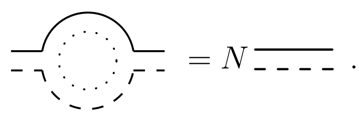
3.2 The Natural Line

The orbifold theory has $2|\Gamma|$ gauge couplings. In the space of gauge couplings we choose the two dimensional manifold parametrized by (g_N, g_1) for which all the $SU(N)$ couplings are the same and all the $U(1)$ couplings are the same. It is the natural choice since we originally had a $U(|\Gamma| N)$ symmetry. For the $\mathbb{Z}_{|\Gamma|}$ orbifold this manifold has a $\mathbb{Z}_{|\Gamma|}$ symmetry and all the RG equations have a $\mathbb{Z}_{|\Gamma|}$ symmetry. Accordingly, if we start in this manifold, we will stay in it. We also choose the Yukawa couplings to be equal to the gauge coupling and the quartic couplings to be equal to the gauge coupling squared. The RG flow can make those couplings different.

In the space of couplings, we choose to start the RG flow from a point on a one dimensional manifold (line) parametrized by the coupling g to which all the couplings are equal at some renormalization scale μ_N. g can be related to the coupling of the original $\mathcal{N}=4$ theory. This is the natural submanifold to choose because of the $\mathcal{N}=4$ origin of the orbifold theory and we will refer to this submanifold as the *natural line*.

In view of the AdS/CFT correspondence, g can be related to the string coupling $g^2 \sim g_s$. There are two scales in the field theory orbifold, the regularization scale Λ and the renormalization scale μ_N. In the AdS/CFT correspondence the regularization scale is related to the string scale $\Lambda \sim \frac{1}{\sqrt{\alpha'}}$, and the renormalization scale is related to the AdS_5 fifth coordinate $\mu \sim U = \frac{r}{\alpha'}$. Our model is not conformal, therefore we do not expect an AdS geometry. The renormalization scale μ_N where all the couplings are equal, should be related to some unique U_N in the new geometry. From dimensional consideration it should also be somehow related to the string scale. The regularization scale in field theory is not physical, but we have to take it into consideration when we discuss the hierarchy problem and the cosmological constant problem.

In (3.8-3.11) we calculated diagrams obtained by orbifold projections of the one loop diagram (2.13). We found out that their group factor is proportional to g^2N with no subleading corrections on the natural line. The only other non trivial orbifold projection of (2.13) is



(3.13)

The vertices in this diagram can only be Yukawa vertices and since the Yukawa couplings are equal to the gauge couplings on the natural line, this diagram also has a group factor of g^2N.

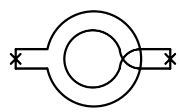
The orbifold projections of the one loop correction to the three point vertex (2.14) also have a group factor proportional to g^2N with no subleading corrections in N, when all the couplings are equal to g. "Calculable" diagrams were defined as diagrams that can be calculated using (2.13), (2.14) together with (2.9). Hence, we conclude that *"calculable" diagrams of orbifold theories have no subleading corrections in N on the natural line.* The group factor of those diagrams is the same as in (2.16).

3.3 Vanishing of the β Functions

In [?, ?] it was shown that in the large N limit the correlation functions of the orbifold theories coincide with those of $\mathcal{N}=4$. This leads to the vanishing of the β function of the orbifold theories to all orders in perturbation theory in the large N limit. We want to generalize the proof for finite N, but not to all orders, only to orders for which all diagrams in that order are "calculable".

The proof in [?, ?] was for planar diagrams with all external legs attached to the same boundary. The fact that "calculable" diagrams have no subleading corrections on the natural line leads to the conclusion that non planar "calculable" diagrams have a vanishing contribution. For example, all orbifold projections of (2.17) will vanish on the natural line. If the external legs are attached to different boundaries there are several possibilities [?]

- For two point functions, each leg is attached to a boundary of itself. The color indices of the leg are traced, meaning that they are $U(1)$ legs. This is the source of the running of the $U(1)$ coupling constant


.

- For three point functions, one of the external legs must be attached to a boundary of itself. This is the source of the running of the Yukawa coupling of the $U(1)$ fields.

- For four point functions, the previous argument does not apply and we have $SU(N)$ diagrams. Those diagrams have a different group structure from that of the original $\mathcal{N}=4$ quartic couplings. In orbifolds with at least $\mathcal{N}=1$ supersymmetry, the perturbative non renormalization of the superpotential guarantees that new quartic couplings will not be generated. In $\mathcal{N}=0$ orbifolds, new quartic couplings are generated as we shall see in the next section.

- Five or more point functions can not be generated because we are dealing with renormalizable field theories.

Two point functions are "calculable" up to three loops, and three point functions are "calculable" up to two loops, hence there are no subleading corrections in N on the natural line and the corresponding diagrams coincide with the $\mathcal{N}=4$ ones. We conclude that *on the natural line the two point functions of the non abelian fields have a zero β function up to three loops and the three point functions of the non abelian fields have a zero β function up to two loops.*

The orbifold theories on the natural line are not finite because the β functions of the $U(1)$ couplings are non-zero already at one loop, which may cause the orbifold theories to flow away from the natural line. To see the flow of the $SU(N)$ couplings one has to calculate the second derivative of the coupling, since the first derivative (the β function) is zero on the natural line. Since the first derivative is a function of the $U(1)$ coupling, the second derivative will depend on the $U(1)$ β function.

3.4 The Hierarchy Problem

Generally in field theory, the scalar two point functions diverge quadratically leading to scalar masses of the order of Λ^2, where Λ is some cutoff scale. To keep the scalars light, mass counterterms must be very fine tuned. This is called the hierarchy problem.

In supersymmetric theories, the mass of the scalars is protected by the non renormalization of the superpotential, solving the hierarchy problem. In non supersymmetric orbifolds, a mass counterterm is not needed for most of the scalar fields, at least up to three loops, because of the vanishing of the scalar two point function. This is again a result of the matching between the $\mathcal{N} = 4$ diagrams and orbifold diagrams for the scalar two point functions. This could have helped to solve the hierarchy problem for non supersymmetric theories, but this matching does not work for the $U(1)$ scalars which are diverging already at one loop.

The problem of the diverging $U(1)$ scalar mass is not general to all $\mathcal{N} = 0$ orbifolds. There are $\mathcal{N} = 0$ orbifolds with no scalars in the adjoint representation and hence no $U(1)$ scalars, e.g., the $\mathbb{Z}_4$ orbifold with weights $(1, 1, 1, 1)$.

The divergence of the $U(1)$ scalar mass and the cancellation of the other scalar masses are demonstrated in subsection 4.4 for the Z_2 non supersymmetric orbifold, using the effective potential formalism.

The mass of the $U(1)$ scalar depends on the regularization scheme. If we claim that field theory is related to string theory by the AdS/CFT correspondence, the scheme we should choose is adding massive fields corresponding to the massive open strings between D3 branes in the orbifolded Type IIB string theory. Those massive fields will act as a cutoff. We can hope that in this scheme the $U(1)$ scalars will be massless.

In [**?**] it was already suggested that non supersymmetric conformal field theories with a softly broken conformal symmetry may solve the hierarchy problem. We suggest that the $U(1)$ couplings behave as naturally occurring soft symmetry breaking terms of the conformal symmetry in the sense that the flow of the $U(1)$ couplings induces the flow of the other couplings. The "soft breaking" parameter is $\frac{1}{N}$.

In [**?**] the $U(1)$ factors were not taken into account resulting in a mass term for all scalar fields suppressed by a factor of $\frac{1}{N}$. There it was suggested to solve the hierarchy problem by choosing a *very large* but finite N. Notice that when the $U(1)$ factors are taken into account the scalar mass vanish, but only at the renormalization scale μ_N. At other scales we will get Λ^2 contributions again.

3.5 The Cosmological Constant

The vacuum energy in field theories is generally of the order of Λ^4, where Λ is some cutoff scale. In field theory the constant shift of the vacuum energy is unobservable. In general relativity the vacuum energy plays the rôle of the cosmological constant. The cosmological constant is expected to be very small while Λ^4 is very large. This is called the cosmological constant problem.

The contributions to the cosmological constant come from the zero point functions (vacuum bubble diagrams). To one loop order, bosons and fermions of the same mass have equal and opposite contributions to the cosmological constant. In orbifold theories the cosmological constant vanishes to one loop because the number of bosons and fermions is the same. In non supersymmetric orbifolds this would not be true without taking the $U(1)$ factors into account.

We can go further than that in the loop expansion. We can use the vanishing of the vacuum energy in the $\mathcal{N} = 4$ theory to conclude that the vacuum bubble diagrams of orbifold theories vanish on the natural line up to four loops. (In five loops we have the "incalculable" bubble diagram (2.18).) This leads to a suppression of the cosmological constant by a factor of g^8.

In [**?**] it was suggested that the cosmological constant in orbifold theories vanishes in cases where one has a non supersymmetric fixed line. In our finite N non supersymmetric models we have found no fixed line, therefore we do not expect that the cosmological constant would vanish.

4 The Renormalization Group Flow of Orbifold Theories

4.1 General β Functions

Now we want to analyze the RG flow from the natural line on which the couplings reside at the renormalization scale μ_N. We have the freedom to choose the coupling we start with, so we can rely on perturbation theory by choosing small g. We are also taking all the masses to zero at the renormalization scale μ_N.

For our analysis we calculate the gauge coupling β function up to two loop order because the first loop correction vanishes. The Yukawa coupling β function is calculated up to one loop. We do not need to calculate the quartic coupling β function because the quartic coupling does not participate in the evolution of the gauge and Yukawa couplings at the orders we are looking at. The quartic coupling β function will be calculated for the $\mathcal{N} = 0$ orbifold using the effective action.

The gauge coupling beta function for a product gauge group, up to two loops, depends on the gauge couplings $\{g\}$ and the Yukawa coupling matrices Y [?],

$$\beta_{g_k}(\{g\}, Y) = \frac{dg_k}{d\log\mu} = \beta^{(1)}_{g_k}(g_k) + \beta^{(2)}_{g_k}(\{g\}, Y) \quad , \tag{4.1}$$

$$\beta^{(1)}_{g_k}(g_k) = -\frac{g_k^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G_k) - \frac{2}{3}\sum_{\text{fermions}} C(F_k) - \frac{1}{6}\sum_{\text{scalars}} C(S_k)\right] , \tag{4.2}$$

$$\begin{aligned}\beta^{(2)}_{g_k}(\{g\}, Y) = -\frac{g_k^3}{(4\pi)^2}\Bigg[&\frac{34}{3}\frac{g_k^2}{(4\pi)^2}C_2(G_k)^2 \\ &- \sum_{\text{fermions}}\left(\sum_{l\in\substack{\text{gauge}\\\text{groups}}} 2\frac{g_l^2}{(4\pi)^2}C_2(F_l) + \frac{10}{3}\frac{g_k^2}{(4\pi)^2}C_2(G_k)\right)C(F_k) \\ &- \sum_{\text{scalars}}\left(\sum_{l\in\substack{\text{gauge}\\\text{groups}}} 2\frac{g_l^2}{(4\pi)^2}C_2(S_l) + \frac{1}{3}\frac{g_k^2}{(4\pi)^2}C_2(G_k)\right)C(S_k) \\ &+\frac{1}{(4\pi)^2}Y_4(F)\Bigg] \quad ,\end{aligned} \tag{4.3}$$

where G_k is the adjoint representation of the kth gauge field, F_k is the representation of the fermions under the kth gauge group, S_k is the representation of the scalars under the kth gauge group, $(Y^S)^{F1}_{F^2}$ is the Yukawa coupling matrices representing the coupling between a scalar S and two fermions F^1, F^2, and $Y_4(F)$ is the Yukawa coupling contribution defined as

$$Y_4(F)\delta^{ab} \quad = \quad \mathop{\mathrm{tr}}_{\text{fermions}} \sum_{\text{scalars}} Y^S Y^{\dagger S} T^a T^b \ . \tag{4.4}$$

The summations are over Weyl fermions and real scalars. The Dynkin index of the fundamental representation is normalized to $C(F) = \frac{1}{2}$.

The one loop β function for the Yukawa coupling is [?]

$$\beta_{Y^S} = \frac{dY^S}{d\log\mu} =$$
$$\frac{1}{(4\pi)^2}\left[\frac{1}{2}\left(Y^{\dagger S'}Y^{S'}Y^S + Y^S Y^{S'} Y^{\dagger S'}\right) + 2Y^{S'}Y^{\dagger S}Y^{S'} + Y^{S'}\mathrm{tr}\left(Y^{\dagger S'}Y^S\right)\right.$$
$$\left. - 3\sum_{k\in{\text{gauge}\atop\text{groups}}}\left(g_k^2 C_2(F_k^1)Y^S + Y^S C_2(F_k^2)\right)\right] \ . \tag{4.5}$$

The first two terms are the scalar loop corrections to the two fermion legs. The third term is the one point irreducible scalar correction. The fourth term is the fermion loop correction to the scalar leg. The last two terms are the gauge bosons loop corrections to the two fermion legs.

In the following subsections we analyze orbifold theories with $\mathcal{N} = 2, 1$ and 0 supersymmetries.

4.2 $\mathcal{N} = 2$ Orbifolds

The orbifold leaves an $\mathcal{N} = 2$ supersymmetry if $\Gamma \subset SU(2) \subset SU(4)_R$. The simplest case is the $\mathbb{Z}_2$ orbifold with weights $(1, 1, 0, 0)$ that leaves the following matter content,

	$SU(N)$ ⊗	$SU(N)$ ⊗	$U(1)$ ⊗	$U(1)$	$SU(2)$
V_{N_1}	G	1	0	0	1
V_{N_2}	1	G	0	0	1
V_{1_1}	1	1	0	0	1
V_{1_2}	1	1	0	0	1
H^1_2	N	$\overline{N}$	1	−1	2
H^2_1	$\overline{N}$	N	−1	1	2

where V and H are the vector and hyper multiplet of $\mathcal{N}=2$. The $SU(2)$ is a global symmetry that is a remnant of the original $SU(4)_R$ symmetry. There is also the usual $SU(2)_R \otimes U(1)_R$ symmetry.

The $U(1)$ charges are written up to a normalization factor. When calculating the group factor of Feynman diagrams, we use the double line notation as in (3.9) and (3.11).

The non-renormalization theorem of $\mathcal{N}=2$ guarantees that there can only be one loop corrections to the perturbative β function. The different gauge bosons can only interact through the bifundamental hypermultiplet and those interactions only occur at two loop order. Consequently, the different gauge bosons do not interact.

We choose all the $SU(N)$ couplings to be the same and all the $U(1)$ couplings to be same, and then there are only two independent couplings, with the β functions

$$\begin{aligned} \beta_{g_N} &= 0 \,, \\ \beta_{g_1} &= 2\frac{g_1^3 N}{(4\pi)^2} \,. \end{aligned} \tag{4.6}$$

If we start on the natural line, we have $g_N = g_1$ and we get a theory that is not finite because of the running of the $U(1)$ coupling constant. However, we can choose $g_1 = 0$ and get a finite theory. In other words, since the $U(1)$ gauge couplings are IR free, we can say that the $U(1)$ decouples in the IR, and the theory is IR finite.

For a general $Z_{|\Gamma|}$ orbifold we have a $|\Gamma|$ dimensional manifold of fixed points parameterized by the $|\Gamma|$ $SU(N)$ gauge couplings.

4.3 $\mathcal{N}=1$ Orbifolds

The orbifold leaves an $\mathcal{N}=1$ supersymmetry if $\Gamma \subset SU(3) \subset SU(4)_R$. The simplest case is the $\mathbb{Z}_3$ orbifold with weights $(1,1,1,0)$ [**?**] having the following matter content,

	$SU(N)$ $\otimes$	$SU(N)$ $\otimes$	$SU(N)$ $\otimes$	$U(1)$ $\otimes$	$U(1)$ $\otimes$	$U(1)$	$SU(3)$
V_{N_1}	G	1	1	0	0	0	1
V_{N_2}	1	G	1	0	0	0	1
V_{N_3}	1	1	G	0	0	0	1
V_{1_1}	1	1	1	0	0	0	1
V_{1_2}	1	1	1	0	0	0	1
V_{1_3}	1	1	1	0	0	0	1
Φ_2^1	N	$\overline{N}$	1	1	-1	0	3
Φ_3^2	1	N	$\overline{N}$	0	1	-1	3
Φ_1^3	$\overline{N}$	1	N	-1	0	1	3

where V and Φ are the vector and chiral multiplets of $\mathcal{N}=1$. The $SU(3)$ is a global symmetry remnant of the original $SU(4)_R$ symmetry. There is also the usual $U(1)_R$ symmetry. The superpotential is[2]

$$\sqrt{2}h\sum_{k=1}^{3}\mathrm{tr}\left([\Phi_{k+1}^{k},\Phi_{k+2}^{k+1}]\Phi_{k}^{k+2}\right) \ . \tag{4.7}$$

The trace here stands for taking the singlet representation under all gauge groups and the $SU(3)$ global symmetry group.

Generally we can have different h_k for the three summands. However, for the sake of simplicity and naturalness, we choose $h_k=h$, $g_N^k=g_N$ and $g_1^k=g_1$. The RG flow will not alter this choice. Before the orbifolding, $h=g$ would have yielded an $\mathcal{N}=4$ theory.

The Yukawa terms in the Lagrangian are (in components)

$$\begin{aligned}&\sqrt{2}g_N\sum_{k=1}^{3}\left(\phi_{k+1}^{k}\bar{\psi}_{k}^{k+1}\lambda^{N_k}+\phi_{k}^{k-1}\bar{\psi}_{k-1}^{k}\lambda^{N_k}\right)+ \\ &\sqrt{2}g_1\sum_{k=1}^{3}\left(\phi_{k+1}^{k}\bar{\psi}_{k}^{k+1}\lambda^{1_k}+\phi_{k}^{k-1}\bar{\psi}_{k-1}^{k}\lambda^{1_k}\right)+\frac{h}{\sqrt{2}}\sum_{k=1}^{3}\phi_{k+1}^{k}\psi_{k+2}^{k+1}\psi_{k}^{k+2}+\text{h.c.}\end{aligned} \tag{4.8}$$

where the trace over the gauge and global indices is implicit. The coupling of the first two terms has to be equal to the gauge coupling as a consequence of supersymmetry. The third term comes from the superpotential.

[2] The entire Lagrangian is normalized by a factor of $\frac{1}{C(F)}$. This is due to the use of the notation $\Phi\equiv T^a\Phi^a$ that contributes a factor of $C(F)$. For example, $\mathrm{tr}(D_\mu\Phi D^\mu\Phi)=\mathrm{tr}(T^aT^b)D_\mu\Phi^aD^\mu\Phi^b=C(F)D_\mu\Phi^aD^\mu\Phi^a$.

We can use the Leigh-Strassler arguments [?] to check whether there is a manifold of fixed points. From $\mathcal{N}=1$ SUSY arguments we know [?] that the β functions have the form (when absorbing a factor of 4π into the couplings),

$$\begin{aligned}
\beta_h &= h\left(-3+\sum\left(d(\Phi)+\tfrac{1}{2}\gamma\right)\right) = \tfrac{3}{2}h\gamma\ , \\
\beta_{g_N} &= -\frac{g_N^3}{1-g_N^2C_2(G)}\left(3C_2(G)-\sum C(R)(1-\gamma)\right) = \frac{-3g_N^3N\gamma}{1-g_N^2N}\ , \\
\beta_{g_1} &= -\frac{g_1^3}{1-g_1^2C_2(G)}\left(3C_2(G)-\sum C(R)(1-\gamma)\right) = 3g_1^3N(1-\gamma)\ .
\end{aligned} \tag{4.9}$$

The denominator of β_{g_N} is zero at $g_N^2N=1$, but for small couplings it is smooth and positive.

There is a linear relation between the first two equations, so setting the three β functions to zero gives us two conditions on the three couplings. This yields a fixed line at $\gamma=0$, $g_1=0$.

This fixed line, however, is not the natural line $g_N=g_1=h$ from which we want to start the RG flow. We have already shown that on the natural line β_h vanishes up to two loop order and β_{g_N} vanishes up to three loop order. The form of the β functions (4.10) tells us that if one β function vanishes then so does the other. It is tempting to speculate that on the natural line both β_h and β_{g_N} vanish to all orders in perturbation theory.

By an explicit calculation of the β functions we can parameterize the fixed line $\gamma(g_N,g_1,h)|_{g1=0}=0$ and check whether the natural line flows to the fixed line in the IR. The β functions of the gauge couplings up to 2-loop order and the Yukawa coupling up to 1-loop order are

$$\begin{aligned}
\beta_h &= \frac{6}{N}\frac{h}{(4\pi)^2}\left(N^2h^2-N^2g_N^2-(g_1^2-g_N^2)\right)\ , \\
\beta_{g_N} &= -12\frac{g_N^3}{(4\pi)^4}\left(N^2h^2-N^2g_N^2-(g_1^2-g_N^2)\right)\ , \\
\beta_{g_1} &= 3\frac{g_1^3N}{(4\pi)^2}-12\frac{g_1^3}{(4\pi)^4}\left(N^2h^2-N^2g_N^2-(g_1^2-g_N^2)\right)\ .
\end{aligned} \tag{4.10}$$

The β functions are of the expected form (4.10), giving a consistency check of our calculations. The first two β functions vanish on the two dimensional manifold defined by the relation

$$h^2 = g_N^2+\frac{g_1^2-g_N^2}{N^2}\ . \tag{4.11}$$

The natural line $g_1 = g_N = h$ is obviously on this manifold. The relation between h, g_N on the fixed line ($g_1 = 0$) is

$$h^2 = (1 - \frac{1}{N^2}) g_N^2 \ . \tag{4.12}$$

In the large N limit the fixed line coincides with the natural line.

In figure 1 we plot the numerical solution of the β functions (4.10). We plot g_1^2, g_N^2, h^2 as a function of g_1^2 and not as a function of the energy scale μ. This is permissible because g_1 is a monotonic increasing function of μ (β_{g_1} is positive definite).

The solution demonstrates how the natural line flows to the fixed line in the IR. In the IR ($g_1 = 0$) we get the expected ratio (4.12) between h and g_N.

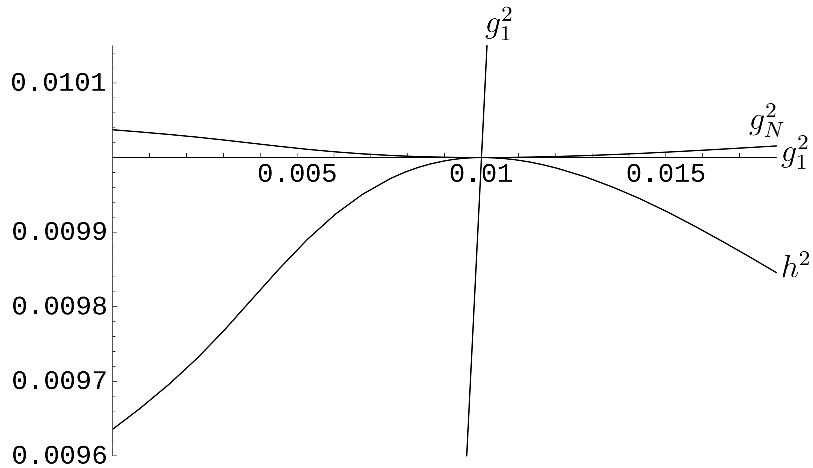


Figure 1: The RG flow of the $\mathcal{N} = 1$ orbifold theory from the natural line $g_1^2 = g_N^2 = h^2 = 0.01$ to the fixed line in the IR. g_1^2, g_N^2, h^2 are plotted as a function of g_1^2. The graph was plotted for $N = 5$ and gives the expected h/g_N ratio at $g_1 = 0$.

4.4 $\mathcal{N} = 0$ Orbifolds

For the non supersymmetric orbifolds we will focus on the $\mathbb{Z}_2$ orbifold with weights $(1, 1, 1, 1)$. The $\mathbb{Z}_2$ is in the center of $SU(4)_R$, as can be seen from the weights that do not break the $SU(4)$ symmetry. We will specify which of the essential results are general and which are specific to this case. Applying the $\mathbb{Z}_2$ orbifold to a Type II string theory reproduces the Type 0 string theory. The field theory we are describing lives in Type 0B string theory on N dyonic (electric-magnetic) D3 branes [?]. The theory has the following matter content,

	$SU(N)$ ⊗	$SU(N)$ ⊗	$U(1)$ ⊗	$U(1)$	$SU(4)$
$A^\mu_{N_1}$	G	1	0	0	1
$A^\mu_{N_2}$	1	G	0	0	1
$A^\mu_{1_1}$	1	1	0	0	1
$A^\mu_{1_2}$	1	1	0	0	1
$(\psi^I)^1_2$	N	$\overline{N}$	1	−1	4
$(\psi^I)^2_1$	$\overline{N}$	N	−1	1	4
$\phi^{IJ}_{N_1}$	G	1	0	0	6
$\phi^{IJ}_{N_2}$	1	G	0	0	6
$\phi^{IJ}_{1_1}$	1	1	0	0	6
$\phi^{IJ}_{1_2}$	1	1	0	0	6

where $I, J = 1 \ldots 4$ are the $SU(4)$ fundamental indices. The fermions are Weyl spinors and the scalars are real.

Because the scalars are in the adjoint representation there can only a be Coulomb branch and the gauge group can maximally break into its Cartan subalgebra $U(1)^{2N}$. Therefore, there is no spontaneous symmetry breaking of the $U(1)$ gauge fields specified in the table above. This is not true for all $\mathcal{N} = 0$ orbifolds because in general, we can have scalars in the bifundamental representation, for example, $\mathbb{Z}_5$ with weights $(1, 2, 3, 4)$ [?]. On the other hand there are $\mathcal{N} = 0$ orbifolds with no scalars at all in the adjoint representation, for example, $\mathbb{Z}_5$ with weights $(1, 1, 1, 2)$.

The Yukawa terms in the Lagrangian are

$$\frac{Y_N}{2} \sum_{k=1}^{2} \varepsilon_{IJKL} \mathrm{tr} \left(\phi^{IJ}_{N_k} (\psi^K)^1_2 (\psi^L)^2_1 \right) + \frac{Y_1}{2} \sum_{k=1}^{2} \varepsilon_{IJKL} \mathrm{tr} \left(\phi^{IJ}_{1_k} (\psi^K)^1_2 (\psi^L)^2_1 \right) . \quad (4.13)$$

The $U(1)$ scalars couple to the theory only through the Y_1 Yukawa coupling.

The quartic terms in the Lagrangian are

$$\frac{\lambda}{4} \mathrm{tr} \left(\left[\phi^{IJ}_{N_1}, \phi^{KL}_{N_1} \right]^2 + \left[\phi^{IJ}_{N_2}, \phi^{KL}_{N_2} \right]^2 \right) . \quad (4.14)$$

This is the classical scalar potential. The $U(1)$ scalars do not participate in this potential because they are abelian.

We have a classical moduli space that can be parametrized by the diagonalized scalar *vev* matrices

$$\left.\begin{aligned}(\phi^{\rm c}_{N_k})^{IJ} &= \mathrm{diag}\left((y_k^{IJ})^1, \ldots, (y_k^{IJ})^N\right) - \frac{1}{N}\sum_{i=1}^{N}(y_k^{IJ})^i \cdot \mathbf{1}_{N\times N} \ , \\ (\phi^{\rm c}_{1_k})^{IJ} &= \frac{1}{N}\sum_{i=1}^{N}(y_k^{IJ})^i \equiv \frac{1}{N}\mathrm{tr}(y_k^{IJ}) \ .\end{aligned}\right\}_{k=1,2} \tag{4.15}$$

Since there is no supersymmetry, nothing protects the fields from acquiring a mass by quantum corrections. There are also new quartic scalar coupling terms that can appear in the renormalization process [**?**], for example,

$$\lambda' \mathrm{tr}\left(\varepsilon_{IJKL}\phi^{IJ}_{N_1}\phi^{KL}_{N_1}\right)\mathrm{tr}\left(\varepsilon_{IJKL}\phi^{IJ}_{N_2}\phi^{KL}_{N_2}\right) \ . \tag{4.16}$$

Those quantum corrections will in general lift the classical moduli.

In order to analyze the behavior of the theory we use the Coleman-Weinberg effective potential [**?**] as was done in [**?**, **?**] for this model without the $U(1)$ factors. The "zero loop" (tree level) effective potential comes from expanding the classical potential around the classical *vev*s, i.e. setting $\phi = \phi^{\rm c} + \phi'$. There are no masses in the original Lagrangian, but in the effective Lagrangian the fields acquire masses from the scalar *vev*s. The eigenvalues of the mass matrices are

$$\begin{aligned}\mu^{ij}_{\rm gauge}(\phi^{\rm c})^2 &= g_N^2|y^i - y^j|^2 \ , \\ \mu^{ij}_{\rm scalar}(\phi^{\rm c})^2 &= \lambda|y^i - y^j|^2 \ , \\ \mu^{ij}_{\rm fermion}(\phi^{\rm c}_1, \phi^{\rm c}_2)^2 &= \Big|Y_N\left(\left(y_1^i - \tfrac{1}{N}\mathrm{tr}(y_1)\right) - \left(y_2^j - \tfrac{1}{N}\mathrm{tr}(y_2)\right)\right) \\ &\quad + Y_1\left(\tfrac{1}{N}\mathrm{tr}(y_1) - \tfrac{1}{N}\mathrm{tr}(y_2)\right)\Big|^2 \ .\end{aligned} \tag{4.17}$$

The $SU(4)$ indices are implicit, where $|y|^2 = \frac{1}{2}\varepsilon_{IJKL}y^{IJ}y^{KL}$ and $\mu_{\rm fermion}$ has two $SU(4)$ indices related to the $SU(4)$ indices of the fermion to which the mass couples. Therefore, $\mu_{\rm fermion}$ has also spinor indices.

The one loop effective potential includes the tree level scalar potential (4.14), renormalization counterterms, and the one loop correction to the effective potential,

$$
\begin{aligned}
V_{\text{eff}}^{\text{1-loop}} &= \quad 2\sum_{i,j=1}^{N}\left[V\left(\mu^{ij}_{\text{gauge}}(\phi^{\text{c}}_1)^2\right)+V\left(\mu^{ij}_{\text{gauge}}(\phi^{\text{c}}_2)^2\right)\right] \\
&\quad +6\sum_{i,j=1}^{N}\left[V\left(\mu^{ij}_{\text{scalar}}(\phi^{\text{c}}_1)^2\right)+V\left(\mu^{ij}_{\text{scalar}}(\phi^{\text{c}}_2)^2\right)\right] \qquad (4.18)\\
&\quad -8\sum_{i,j=1}^{N}\left[V\left(\mu^{ij}_{\text{fermion}}(\phi^{\text{c}}_1,\phi^{\text{c}}_2)^2\right)+V\left(\mu^{ij}_{\text{fermion}}(\phi^{\text{c}}_2,\phi^{\text{c}}_1)^2\right)\right]\ ,\\
V(\mu^2) &= \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\ln\frac{p^2+\frac{1}{2}\mu^2}{p^2}\\
&= \frac{2\pi^2}{2(2\pi)^4}\left(\frac{1}{4}\Lambda^2\mu^2+\frac{1}{16}\mu^4\left(\ln\frac{\frac{1}{2}\mu^2}{\Lambda^2}-\frac{1}{2}\right)+O\left(\frac{\mu^6}{\Lambda^2}\right)\right)\ . \qquad (4.19)
\end{aligned}
$$

The Λ^2 term provides the one loop correction to the scalar masses. As can be seen from the effective potential, the scalar mass is corrected by a gauge boson loop, a scalar loop and a fermion loop. The Λ^2 term is

$$
\begin{aligned}
(V_{\text{eff}}^{\text{1-loop}})_{\Lambda^2} = \frac{\Lambda^2}{64\pi^2}\sum_{i,j=1}^{N}\ & \left(2g_N^2+6\lambda\right)\left(|y_1^i-y_1^j|^2+|y_2^i-y_2^j|^2\right) \qquad (4.20)\\
& -16Y_N^2\left|\left(y_1^i-\tfrac{1}{N}\text{tr}(y_1)\right)-\left(y_2^j-\tfrac{1}{N}\text{tr}(y_2)\right)\right|^2\\
& -16Y_1^2\left|\tfrac{1}{N}\text{tr}(y_1)-\tfrac{1}{N}\text{tr}(y_2)\right|^2\ .
\end{aligned}
$$

On the natural line, where all the couplings are equal to g we get

$$
\begin{aligned}
(V_{\text{eff}}^{\text{1-loop}})_{\Lambda^2} &= -\frac{\Lambda^2}{4\pi^2}g^2N^2\left|\tfrac{1}{N}\text{tr}(y_1)-\tfrac{1}{N}\text{tr}(y_2)\right|^2\\
&= \frac{m^2}{2}\left(\phi^{\text{c}}_{1_1}-\phi^{\text{c}}_{1_2}\right)^2\ , \qquad (4.21)
\end{aligned}
$$

which generates a quadratically divergent tachyonic mass for the diagonal $U(1)$ scalar. The mass of the $U(1)$ scalar comes from the fermion loop and can be set to zero by choosing $Y_1=0$ as was done in [**?**]. Otherwise, we should include a renormalization counterterm for the scalar mass. Using this counterterm we can set the renormalized mass to zero. It would have been much more elegant if there were some underlying mechanism that naturally sets this scalar mass to zero as discussed in subsection 3.4.

The logarithmic term in the effective potential provides the one loop correction to the quartic coupling λ. It is sufficient to compute this term when only one component out of the **6** of $SU(4)$ is nonzero. On the natural line the logarithmic term is

$$\begin{aligned}\left(V_{\rm eff}^{\text{1-loop}}\right)_{log(\Lambda)} \quad = \quad & -\frac{8g^4}{256\pi^2}\Big(6\left(\mathrm{tr}({\phi^{\rm c}_{N_1}}^2)-\mathrm{tr}({\phi^{\rm c}_{N_2}}^2)\right)^2 \\ & - 8N\left(\mathrm{tr}({\phi^{\rm c}_{N_1}}^3)-\mathrm{tr}({\phi^{\rm c}_{N_2}}^3)\right)(\phi^{\rm c}_{1_1}-\phi^{\rm c}_{1_2}) \\ & - 12N\left(\mathrm{tr}({\phi^{\rm c}_{N_1}}^2)+\mathrm{tr}({\phi^{\rm c}_{N_2}}^2)\right)(\phi^{\rm c}_{1_1}-\phi^{\rm c}_{1_2})^2 \\ & - 2N^2(\phi^{\rm c}_{1_1}-\phi^{\rm c}_{1_2})^4\Big) \ . \end{aligned} \tag{4.22}$$

There is no $\mathrm{tr}({\phi^{\rm c}_N}^4)$ term. This is a manifestation of the fact that the one loop β function for the original quartic coupling (4.14) vanishes on the natural line. In models with scalars in the bifundamental representation this vanishing occurs only if the $U(1)$ factors are taken into account.

The logarithmic term is of the form $(\phi^{\rm c})^4\ln(\phi^{\rm c})^2$. The quartic coupling is the fourth derivative of the effective potential with respect to the scalar fields, $\lambda' = \frac{d^4V_{\rm eff}}{d\phi^{\rm c4}}$. This derivative diverges at the origin ($\phi^{\rm c}=0$), requiring a definition of a renormalized coupling away from the singularity at some arbitrary renormalization scale M,

$$\lambda' = \left.\frac{d^4V_{\rm eff}}{d\phi^{\rm c4}}\right|_{\phi^{\rm c}=M} = C_{\lambda'} + c_{\lambda'}\frac{3g^4}{4\pi^2}\left(\ln\frac{\frac{g^2}{2}M^2}{\Lambda}+\frac{11}{3}\right) \ , \tag{4.23}$$

where $C_{\lambda'}$ is the renormalization counterterm, and $c_{\lambda'}$ is a constant depending on the $\phi^{\rm c}$ we differentiate with respect to, calculated in (4.22). Since λ' did not exist in the original Lagrangian we would like to require $\lambda'=0$. We are not allowing arbitrary couplings, therefore the renormalization scale M is the renormalization scale μ_N defined for the natural line. Finally, the effective potential for the new quartic couplings is

$$V_{\rm eff} = c_{\lambda'}\frac{g^4\phi^{\rm c4}}{32\pi^2}\left(\ln\frac{\phi^{\rm c2}}{\mu_N^2}-\frac{25}{6}\right) \ . \tag{4.24}$$

The first term in (4.22) is the term found in [**?**]. This term leads to a repulsive potential between *vev*s of scalars of the same type. For example taking $y_1^1 = -y_1^2 = \rho$ leads to a repulsive potential of the form $\rho^4 \ln \rho^2$. Notice also that there are flat directions in (4.22), for example $y_1^1 = y_2^1 = -y_1^1 = -y_1^2 = \rho$.

The other terms in (4.22) give a potential to the diagonal $U(1)$ scalar. Because of the opposite sign this potential is attractive at short distances. At long distances this potential is repulsive, but at the scale it becomes so, higher loop terms should be taken into account.

Assuming that the choice of taking all the masses to zero on the natural line is consistent, we go ahead and calculate the explicit β functions for the gauge couplings to two loop order and for the Yukawa couplings to one loop order,

$$\begin{aligned}
\beta_{g_N} &= -\frac{g_N^3}{(4\pi)^4}\left(-24N^2 g_N^2 - 8(g_1^2 - g_N^2) + 24N^2 Y_N^2 + 24(Y_1^2 - Y_N^2)\right) \ , \\
\beta_{g_1} &= \frac{11}{3}\frac{g_1^3 N}{(4\pi)^2} \\
&\quad -\frac{g_1^3}{(4\pi)^4}\left(-8N^2 g_N^2 - 8(g_1^2 - g_N^2) + 24N^2 Y_N^2 + 24(Y_1^2 - Y_N^2)\right) \ , \\
\beta_{Y_N} &= \frac{Y_N}{(4\pi)^2}\left(-6N g_N^2 - \frac{6}{N}(g_1^2 - g_N^2) + 6N Y_N^2 + \frac{2}{N}(Y_1^2 - Y_N^2)\right) \ , \\
\beta_{Y_1} &= \frac{Y_1}{(4\pi)^2}\left(-6N g_N^2 - \frac{6}{N}(g_1^2 - g_N^2) + 2N Y_N^2 + 4N Y_1^2 + \frac{2}{N}(Y_1^2 - Y_N^2)\right) \ .
\end{aligned} \tag{4.25}$$

As expected, the β functions for g_N and Y_N vanish on the natural line where all the couplings are the same. Notice that the β function for Y_1 also vanishes on the natural line. Moreover, $Y_1 = Y_N \Rightarrow \beta_{Y_1} = \beta_{Y_N}$, meaning that if the Yukawa couplings start the same at the renormalization scale, they will stay the same, at least to one loop order.

It would be interesting to look for fixed points where all the β functions are zero. If we take all the β functions to the first non vanishing order, then the only fixed point is the trivial fixed point where all the couplings are zero. If we use the β_{g_1} that we calculated to two loop order we get several non trivial fixed points. There is one fixed point on the natural line,

$$\frac{g_N^2}{(4\pi)^2} = \frac{g_1^2}{(4\pi)^2} = \frac{Y_N^2}{(4\pi)^2} = \frac{Y_1^2}{(4\pi)^2} = \frac{11}{48N} \ . \tag{4.26}$$

This fixed point, however, is inconsistent, because we did not evaluate all the β functions to the same order and ignored the quartic couplings. The fixed point is also not very interesting because it is a non stable fixed point. All the same, the calculations give the order of magnitude of the range of validity of our calculations. We can trust the lowest order perturbative expansion as long as $\frac{g^2 N}{(4\pi^2)} \ll \frac{11}{48}$.

In figure 2 we plot the numerical solution of the β functions. We start the RG flow from a point on the natural line. In the IR all the couplings flow to zero. The β functions (4.26) were calculated assuming that all the fields are massless, so we can trust them only as long as the energy scale is larger than the fields masses. Still, if we get an IR free theory when ignoring the masses, we will get an IR free theory also with the masses taken into account.

Zooming on the natural point we see that the $SU(N)$ gauge coupling has a local minimum on the natural line while the Yukawa couplings have a local maximum. This can be calculated directly from the second derivative of the couplings (first derivative of the β function).

It is possible that higher loop calculations will give more interesting results than the flow to the trivial fixed point. This can be seen by analyzing the β functions around the unstable fix point (4.26). There seem to be solutions in which all or some of the couplings flow to infinity in the IR.

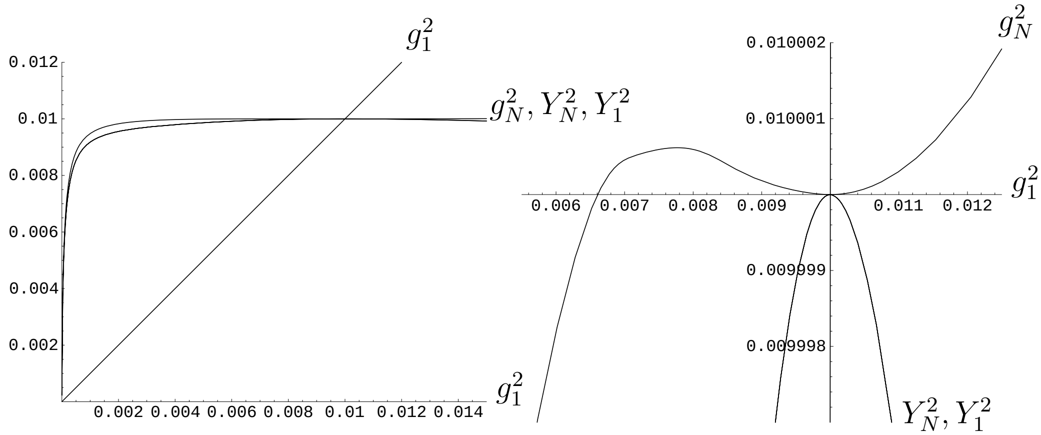


Figure 2: The RG flow of the $\mathcal{N} = 0$ theory from the natural line $g_1 = g_N = Y_1 = Y_N = 0.01$. The graph was plotted for $N = 5$. In the IR we get a free theory. The plot on the right is a zoom on the natural line.

5 Summary and Discussion

We analyzed the finite N limit of orbifold field theories. For finite N the remnant $U(1)$ factors from the orbifolding procedure should be considered. The $U(1)$ factors seem to cancel the $\frac{1}{N}$ contributions in Feynman diagrams (3.12), at least up to three loops. This encourages us to study the orbifolded theory on the natural line where all the couplings are equal.

The simplest case to analyze is of the $\mathcal{N} = 2$ orbifolds, but it turns out that it is too simple. There is no interaction between the different gauge groups due to the non renormalization theorem. The $SU(N)$ couplings are finite and the $U(1)$ couplings are IR free. The theory on the natural line is not finite, but we can easily make it finite by choosing zero $U(1)$ couplings or by taking the IR limit.

The case of $\mathcal{N} = 1$ orbifolds is more interesting because here the $U(1)$ factors do affect the rest of the gauge couplings. Still, the final conclusions are the same as in the $\mathcal{N} = 2$ case. The theory on the natural line is not conformal, but in the IR limit it flows to a point on the fixed line.

In the case of $\mathcal{N} = 0$ orbifolds all hell breaks loose. The fact that we get the $\mathcal{N} = 0$ theory from the $\mathcal{N} = 4$ SCFT leads to "miraculous" cancellation of Feynman diagrams up to three loops on the natural line, yet this does not seem to be enough. Quantum effects can generate new terms that did not exist in the original theory, like the mass term of the $U(1)$ scalar (4.21). The theory seems to be inconsistent unless we add a mass term to it, but adding a mass term is inconsistent with our attempt to look at the orbifolded theory exactly as it emerges from the orbifolding process.

The scalar mass diverges polynomially, meaning that it is scheme depended. We hope that there is a regularization scheme in which all the polynomial divergences cancel out. The alleged scheme might be defined from the AdS/CFT correspondence by adding massive fields that correspond to massive open strings between D3 branes in the orbifolded type IIB string theory.

There is also a problem of renormalization scheme dependence related to our claims on the cancellation of 3-loop Feynman diagrams. All the explicit calculations of the β functions we performed where scheme independent, but the 3-loop Feynman diagrams are scheme dependent. Again, we are not sure in what scheme our claims are valid, but we assume that such a scheme exists.

Another property of the $\mathcal{N} = 0$ theories is that we can no longer assume that the $U(1)$ factors decouple in the IR, leaving us with an $SU(N)^{|\Gamma|}$ theory.

Our analysis shows that, at least for small couplings, the entire theory is IR free. Moreover, if we start with the same Yukawa couplings for the $SU(N)$ scalars and the $U(1)$ scalars, then our calculation shows that at least to one loop order, they will remain the same.

The five loop bubble diagram (2.18) hindered us from claiming that the cancellation of the Feynman diagrams continues to all orders. There are a few hints that this cancellation might survive to all orders, based on the fact that the $\mathcal{N} = 4$ theory is finite to all orders.

The first hint is that the proof that the correlation functions of orbifold theories coincide with those of $\mathcal{N} = 4$ can be generalized to non planar diagrams. The proof is valid for abelian orbifolds, at least as long as all the external legs are on the same face. In [**?**] it was shown, that non planar diagrams in the orbifold theories are different from the $\mathcal{N} = 4$ diagrams, using as example, the non planar diagram (2.17). However, in eq. (13) there is a missing γ_3 factor. After adding it, there is a match in the non planar diagram (2.17) between the orbifold and $\mathcal{N} = 4$ theories, at least for abelian orbifolds.

The second hint comes from dividing the Feynman diagrams of each order into subsets defined by their N dependence. For example, five loop bubble diagrams can be divided into two subsets, "calculable" diagrams with a group factor of $N^4 d(G)$ and diagrams with subleading N terms (like the diagram in (2.18)).

We know that the $\mathcal{N} = 4$ theory is finite to all orders independent of the coupling g, meaning that there is a cancellation of the Feynman diagrams at every order. The finitude of the $\mathcal{N} = 4$ theory also does not depend on N meaning that there is a cancellation of the Feynman diagrams in each subgroup defined above.

We can use this cancellation to claim that the five loop diagrams in the first subset must have zero contribution to orbifold theories. However, we can not make the same claim for the second subset because in that subset there are different diagrams with different N dependence.

We have only analyzed orbifolds from the field theory point of view. It would be interesting to find a manifestation of the $U(1)$ running coupling constants in string theory orbifolds and in brane configurations. The brane configuration for the $\mathcal{N} = 2$ theory, for example, is a set of $|\Gamma|$ NS5 branes on a circle with N D4 branes stretched between them. The decoupling of the $U(1)$ factors can be seen directly from the brane configuration [**?**].

In the AdS/CFT correspondence it would be interesting to find string states corresponding to operators with $U(1)$ factors. The correspondence was done in [**?**] for orbifolds in the large N limit. We do not know how to generalize it for finite N, but it might be possible to use our knowledge in field theory to gain some insight of the string theory.

In addition to the conformal $SO(4,2)$ symmetry, the original $\mathcal{N} = 4$ theory has an $SL(2,\mathbb{Z})$ symmetry. It would be interesting to analyze the effect of the orbifold on this symmetry. The $SL(2,\mathbb{Z})$ symmetry acts on the coupling constant and it is not clear whether it has any meaning in non conformal theories. In [**?**] it was suggested that $SL(2,\mathbb{Z})$ is also a symmetry of the type 0B string. If this is true, then it would be interesting to investigate the manifestation of the $SL(2,\mathbb{Z})$ symmetry in the non supersymmetric non conformal "type 0" field theory.

To summarize, orbifolds of the $\mathcal{N} = 4$ field theory give us non supersymmetric non conformal theories with very interesting features. The exquisite cancellation of the vacuum bubble diagrams in those theories up to at least four loop order suggests that the cosmological constant is very small in those theories. Consequently, the cosmological constant problem could be solved in the orbifolded theories. We do not claim that the cosmological constant vanishes completely in those theories because the running of the $U(1)$ factors stimulate the running of the bubble diagrams.

For finite N the $U(1)$ factors also behave as soft symmetry breaking terms of the conformal symmetry in the sense that the flow of the $U(1)$ couplings induces the flow of the other couplings. The "soft breaking" parameter is $\frac{1}{N}$. The conformal symmetry is broken explicitly, but it is broken by terms that occur naturally in the orbifold process. We propose this soft symmetry breaking term for solving the hierarchy problem as suggested in [**?**].

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