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Supersymmetric Nonlinear Sigma Models as Gauge Theories

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Abstract

Supersymmetric nonlinear sigma models are obtained from linear sigma models by imposing supersymmetric constraints. If we introduce auxiliary chiral and vector superfields, these constraints are expressed by D-terms and F-terms depending on the target manifolds. Auxiliary vector superfields appears as gauge fields without kinetic terms. If there is no D-term constraints, the target manifolds are always non-compact manifolds. When all the degrees of freedom in these non-compact directions are eliminated by gauge symmetries, the target manifold becomes compact. All supersymmetric nonlinear sigma models, whose target manifolds are the hermitian symmetric spaces, are successfully formulated as gauge theories.

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1 Introduction

When global symmetry G is spontaneously broken down to its subgroup H , there appears massless Nambu-Goldstone(NG) bosons corresponding to broken generators of the coset manifold G/H . At low energies, interactions among these massless particles are described by the so-called nonlinear sigma models, whose lagrangians are completely determined by the geometry of the target manifold G/H parameterized by NG-bosons [1].

In supersymmetric theories, there appear massless fermions as supersymmetric partners of NG-bosons [2]. These massless fermions together with NG-bosons are described by chiral superfields in four dimensional theories with $N = 1$ supersymmetry. Since chiral superfields are complex, the supersymmetric nonlinear sigma models are closely related to the complex geometry; their target manifolds, where fields variables take their values, must be Kähler manifolds [3]. If the coset manifold G/H itself happens to be a Kähler manifold, both real and imaginary parts of the scalar components of chiral superfields are NG-bosons. If G/H is not a Kähler manifold, on the other hand, there is at least one chiral superfield whose real or imaginary part is not a NG-boson. This additional massless boson is called the Quasi-Nambu-Goldstone(QNG) boson [2, 4].

The general method to construct supersymmetric nonlinear sigma models has been discussed by Bando, Kuramoto, Maskawa and Uehara (BKMU) [5]. When QNG bosons are present, their effective lagrangians includes arbitrary functions. This is always the case when the target manifold of the nonlinear sigma model is larger than the coset manifold G/H where NG-bosons reside, since the geometry of the target manifold cannot be fixed by the metric of its subspace G/H [6]–[10]. The arbitrariness reflects the ambiguity of the metric in the direction of QNG bosons. When the coset manifold G/H is itself Kähler, the effective lagrangian is uniquely determined by the geometry of G/H , as has been shown in a beautiful paper by Itoh, Kugo and Kunitomo [11] (See Appendix A for a review). Kähler potentials in this case were discussed by many authors [5, 11]–[18] (see references in Ref. [18]), and were used to construct the coset unification models where fermionic partners of

NG bosons are considered as quarks [19].

Nonlinear sigma models are considered as low energy effective theories for massless particles after integrating out the massive particles in the corresponding linear sigma models. In this context, Lerche and Shore have shown that nonlinear sigma models whose target manifolds are Kähler G/H cannot be obtained from linear sigma models [6] (See also Refs. [7] and Appendix B for a review). According to this theorem, there must exist at least one QNG bosons in effective field theories obtained from linear sigma models.

On the other hand, it is known that sigma models on some Kähler G/H manifolds, namely on CP^N or on Grassmann manifold $G_{N,M}(\mathbf{C})$, are obtained by the introduction of gauge symmetry [12, 20, 21]. Implicit assumption of Lerche and Shore is the absence of gauge interactions in the linear sigma models. It seems possible to eliminate unnecessary QNG bosons if we introduce appropriate gauge symmetry.

In this paper, we show that supersymmetric nonlinear sigma models on certain class of Kähler G/H manifolds are obtained from linear sigma models with gauge symmetry. We define nonlinear sigma models by imposing supersymmetric constraints to linear sigma models. We introduce two kinds of constraints: D-term and F-term constraints. If we introduce auxiliary fields, these correspond to vector and chiral superfields. Vector auxiliary superfields appear as gauge fields. We will successfully formulate nonlinear sigma models on (irreducible, compact) hermitian symmetric spaces¹ classified by Cartan as in Table 1 [23].²

This paper is organized as follows. In Sec. 2, we review simple cases without F-term constraints, namely the projective space CP^{N-1} and the Grassmann manifold $G_{N,M}(\mathbf{C})$. Although these cases are known, it is instructive to discuss them with emphasis on the interpretation in terms of NG and QNG boson. In Sec. 3, we

¹Symmetric spaces are homogeneous spaces G/H with an involutive automorphism. Since it can be shown that any differential form ω in a symmetric space is closed, $d\omega = 0$, a fundamental two form of a hermitian symmetric space is also closed and then is Kählerian. Hence, a term “Kähler symmetric space” has the same meaning.

²We use ‘ $\dim_{\mathbf{C}}$ ’ for complex dimensions and ‘ \dim ’ for real dimensions.

Table 1: **Hermitian Symmetric Spaces**

Type	G/H	$\dim_{\mathbf{C}}(G/H)$
AIII ₁	$\mathbf{C}P^{N-1} = SU(N)/S(U(N-1) \times U(1))$	$N-1$
AIII ₂	$G_{N,M}(\mathbf{C}) = U(N)/U(N-M) \times U(M)$	$M(N-M)$
BDI	$Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$	$N-2$
CI	$Sp(N)/U(N)$	$\frac{1}{2}N(N+1)$
DIII	$SO(2N)/U(N)$	$\frac{1}{2}N(N-1)$
EIII	$E_6/SO(10) \times U(1)$	16
EVII	$E_7/E_6 \times U(1)$	27

First three manifolds, $\mathbf{C}P^{N-1}$, $G_{N,M}(\mathbf{C})$ and $Q^{N-2}(\mathbf{C})$ are called a projective space, a Grassmann manifold and a quadratic surface respectively. The projective space $\mathbf{C}P^{N-1}$ and the Grassmann manifold $G_{N,M}(\mathbf{C})$ are a set of complex lines and M dimensional complex planes in \mathbf{C}^N respectively. BI(DI) corresponds to odd(even) N . In the mathematical literature, EIII is written as $E_6/Spin(10) \times U(1)$, since coset generators belong to the $SO(10)$ Weyl spinor.

generalize to other hermitian symmetric spaces, by introducing F-term constraints in addition to D-term constraints. Results in this section are new. As a by-product, we find explicit expressions of holomorphic constraints to embed G/H to $\mathbf{C}P^N$ or $G_{N,M}(\mathbf{C})$. Sec. 4 is devoted to conclusions and discussions. We discuss how the results can be generalized to an arbitrary Kähler G/H manifold. In Appendix A, we review the construction of the Kähler potentials for Kähler G/H by BKMU and IKK methods, in the case of the hermitian symmetric spaces. In Appendix B, we review the theorem of Lerche and Shore. Appendix C, D and E are devoted to summaries of SO , E_6 and E_7 algebras.

In the rest of this section, we summarize notations and terminology used in this paper.

The linear description of nonlinear sigma model without a gauge symmetry is given by

$$\mathcal{L} = \int d^4\theta \phi^\dagger \phi + \left(\int d^2\theta \phi_0 g(\phi) + \text{c.c.} \right), \quad (1.1)$$

where chiral superfield ϕ belongs to an irreducible representation of the global symmetry group G and ϕ_0 is an auxiliary chiral superfield. The absence of ϕ_0 corresponds to the strong coupling limit of the Yukawa theory. Although the superpotential $W = \phi_0 g(\phi)$ is G -invariant, ϕ_0 and $g(\phi)$ need not be G -invariant separately. Instead, they may have indices transforming as a non-trivial representation of G , such as $W = \phi_{0_i} g(\phi)^i$. If we integrate over the auxiliary field ϕ_0 , we obtain F-term constraints, $g(\phi) = 0$, which are holomorphic functions. Therefore, the F-term constraints are invariant under the larger group $G^{\mathbf{C}}$, the complex extension of G .

Let the number of F-term constraints be N_{F} . If it is large enough, the target manifold M' becomes a $G^{\mathbf{C}}$ -orbit of the vacuum $v = \langle \phi \rangle$. Let the complex isotropy group of the vacuum be \hat{H} ($\hat{H}v = v$). Then, the target manifold of the nonlinear sigma model is parameterized by the chiral superfields corresponding to complex broken generators in $\mathcal{G}^{\mathbf{C}} - \hat{\mathcal{H}}$.³ Therefore M' is a complex coset space, $M' \simeq G^{\mathbf{C}}/\hat{H}$ generated by these broken generators. As an example, let us consider a doublet $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ of $G = SU(2)$, and suppose that they acquire vacuum expectation values $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since the raising operator $\tau_+ = \frac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $\tau_+ v = 0$, it is the complex unbroken generator in $\hat{\mathcal{H}}$. On the other hand, τ_3 and the lowering operator $\tau_- (= \tau_+^\dagger)$ are the elements of the broken generators in $\mathcal{G}^{\mathbf{C}} - \hat{\mathcal{H}}$.

There are two kinds of broken generators: hermitian broken generator X and non-hermitian broken generator E .⁴ Superfield corresponding to non-hermitian and hermitian generators are called the pure-type and mixed-type superfields, respectively [5, 6]. In the previous example where the representative of $G^{\mathbf{C}}/\hat{H}$ is given by $\phi = \exp i(\varphi_3 \tau_3 + \varphi \tau_-) \cdot v$, φ_3 is the mixed-type and φ is the pure-type superfield. The scalar components of the mixed-type multiplets consist of a QNG boson in addition to a NG boson, whereas those of the pure-type multiplets consist of two genuine NG bosons. Since the vacuum is invariant under \hat{H} , we can multiply arbitrary element of \hat{H} to the representative of the coset manifold from the right. In our previous

³We use the calligraphy for a Lie algebra corresponding to a Lie group.

⁴In general, \hat{H} is larger than $H^{\mathbf{C}}$ by the existence of non-hermitian generators \bar{E} . \bar{E} is hermitian conjugate of E . They constitute so-called Borel subalgebra \mathcal{B} in \hat{H} [5].

example, we can rewrite it as $\exp i(\varphi_3\tau_3 + \Re\varphi\tau_1 + \Im\varphi\tau_2) \cdot v$ by multiplying appropriate factor generated by τ_+ for sufficiently small $|\varphi_3|$ and $|\varphi|$. The NG-bosons parameterizing $S^3 \simeq SU(2)$ are $\Re\varphi_3, \Re\varphi, \Im\varphi$, whereas $\Im\varphi_3$ is the QNG-boson parameterizing the radius of S^3 . The number of chiral superfields parameterizing the target manifold is

$$N_\Phi = \dim_{\mathbf{C}} V - N_F = N_M + N_P, \quad (1.2)$$

where V is the representation space. Numbers of the mixed-type and pure-type multiplets are denoted by N_M and N_P respectively.

The directions parameterized by QNG bosons are non-compact, whereas those of NG bosons are compact.⁵ From the theorem of Lerche and Shore (see Appendix B), there exist at least one mixed-type multiplet, and therefore the target manifold M' becomes non-compact. Since any two points in the non-compact direction cannot be connected by the compact isometry group G , M' is also non-homogeneous.

As a notation, we rewrite the groups G and H defined above as G' and H' , and therefore $M' \simeq G'^{\mathbf{C}}/\hat{H}'$. In order to eliminate the degree of freedom of QNG bosons, we elevate the subgroup of G' to a local gauge symmetry. We assume G' is a direct product of a global symmetry and the gauge symmetry G_{gauge} , namely, $G' = G \times G_{\text{gauge}}$, where $G_{\text{gauge}} = U(1)$ or $U(N)$. The gauged linear Lagrangian can be written as

$$\mathcal{L} = \int d^4\theta \left(e^V \phi^\dagger \phi - cV \right) + \left(\int d^2\theta \phi_0 g(\phi) + \text{c.c.} \right), \quad (1.3)$$

where ϕ_0 and V are auxiliary chiral and vector superfields. Absence of the kinetic term of gauge field corresponds to the strong coupling limit where gauge coupling constant tends to infinity. Here, for simplicity, the gauge group is assumed to be $U(1)$. See Sec. 2.2 for a non-Abelian case. Integration over ϕ_0 gives F-term constraint to define the non-compact manifold M' as discussed before. The integration over V gives a D-term constraint that restrict to the compact manifold $M = M'/G_{\text{gauge}}^{\mathbf{C}}$ [20], whose dimension is

$$\dim_{\mathbf{C}} M = N_\Phi - \dim G_{\text{gauge}}. \quad (1.4)$$

⁵We use the word compactness in the sense of topology. The kinetic terms of QNG bosons have the same sign with those of NG bosons.

Since we introduce gauge fields to absorb all mixed type multiplets,⁶ the dimension of the gauge group and the compact manifold M are

$$\dim G_{\text{gauge}} = N_{\text{M}}, \quad \dim_{\mathbb{C}} M = N_{\text{P}}. \quad (1.5)$$

The compact manifold M is parameterized by only pure-type multiplets.

2 Nonlinear Sigma Models without F-term constraint

Although examples in this section are well known[20], we describe them in detail, since the interpretation in terms of NG and QNG bosons is useful to find the nonlinear sigma models on other compact manifolds.

2.1 Projective space: $\mathbf{CP}^{N-1} = SU(N)/S(U(N-1) \times U(1))$

We consider a global symmetry $G' = U(N) = SU(N) \times U(1)_{\text{D}} \stackrel{\text{def}}{=} G \times U(1)_{\text{D}}$. Later, the phase symmetry $U(1)_{\text{D}}$ will be gauged while $G = SU(N)$ remains to be global. We prepare the fundamental fields $\vec{\phi} \in \mathbf{N}$, which will acquire a vacuum expectation value. First of all, we consider a canonical Kähler potential,

$$K(\vec{\phi}, \vec{\phi}^\dagger) = \vec{\phi}^\dagger \vec{\phi}. \quad (2.1)$$

For later purposes, we decompose $G = SU(N)$ under a subgroup $SU(N-1) \times U(1)$. A fundamental representation \mathbf{N} is decomposed as $\mathbf{N} = (\mathbf{N}-1, 1) \oplus (1, -N+1)$, where the second factors are $U(1)$ charges. Hence, we decompose the fields as $\vec{\phi} = \begin{pmatrix} x \\ y^i \end{pmatrix}$ ($i = 1, \dots, N-1$). Generators of $SU(N)$ can also be decomposed to $SU(N-1)$ generators T_A ($A = 1, \dots, N^2 - 2N$), $U(1)$ generator T , $N-1$ raising operators E^i represented by upper triangle matrices and lowering operators

⁶Supersymmetric Higgs mechanism occurs as follows: A vector superfield absorbs one mixed-type multiplet to constitute a massive vector multiplet. If it absorbs a pure-type multiplet, one NG boson remains massless, they can not constitute massive vector multiplet, and supersymmetry is spontaneously broken [30].

represented by lower triangle matrices $\bar{E}_i = (E^i)^\dagger$. The transformation law of $\vec{\phi}$ under the complexified group $SU(N)^{\mathbf{C}}$ is

$$\begin{aligned}\delta\vec{\phi} &= \left(i\theta T + i\theta^A T_A + \bar{\epsilon}_i E^i + \epsilon^i \bar{E}_i \right) \vec{\phi} \\ &= \begin{pmatrix} i\sqrt{\frac{2(N-1)}{N}}\theta & & \bar{\epsilon}_j \\ \epsilon^i & -i\theta^A \rho(T_A)^i_j - i\sqrt{\frac{2}{N(N-1)}}\theta\delta^i_j & \end{pmatrix} \begin{pmatrix} x \\ y^j \end{pmatrix},\end{aligned}\quad (2.2)$$

where $\rho(T_A)$ is a $N-1$ by $N-1$ matrix for fundamental representation of $SU(N-1)$. We normalized these generators as $\text{tr } T_A^2 = \text{tr } T^2 = \text{tr } \bar{E}_i E^i = 2$ (no sum). When $\epsilon = \bar{\epsilon}$ and $\theta, \theta^A \in \mathbf{R}$, this transformation law reduces to that of the real group $SU(N)$. The $U(1)_D$ transformation is generated by $T_D = \mathbf{1}_N$.

When $\vec{\phi}$ acquire a vacuum expectation value, it can be transformed by $G'^{\mathbf{C}}$ to the standard form,

$$\vec{v} = \langle \vec{\phi} \rangle = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}.\quad (2.3)$$

By this vacuum, the global symmetry is spontaneously broken down as $U(N) \rightarrow U(N-1) = SU(N-1) \times U(1)' \stackrel{\text{def}}{=} H'$. Here, $U(1)'$ is generated by $T' \sim \text{diag}(0, 1, \dots, 1)$, which is a linear combination of T_D and T . The complex isotropy group \hat{H}' , which leaves \vec{v} invariant, is larger than $H'^{\mathbf{C}}$ since upper triangle generators E^i annihilate the vacuum \vec{v} . Here, E^i generators constitute a Borel subalgebra \mathcal{B} in $\hat{\mathcal{H}}'$. On the other hand, the complex broken generators are lower triangle generators \bar{E}_i and a diagonal generator $X = (1, 0, \dots, 0)$, which is also a linear combination of T and T_D . The non-hermitian generators \bar{E}_i are pure-type generators and the hermitian generator X is a mixed-type generator. The target manifold M' of nonlinear sigma model is a complex coset manifold $M' \simeq G'^{\mathbf{C}}/\hat{H}'$ generated by these complex broken generators. Since, by using its representative $\xi' = \exp(\varphi^i \bar{E}_i + i\psi X)$, the fields can be written as $\vec{\phi} = \xi' \vec{v}$, its form near the vacuum is

$$\delta\vec{v} = (i\psi X + \varphi^i \bar{E}_i) \vec{v} = \begin{pmatrix} i\psi \\ \varphi^i \end{pmatrix}.\quad (2.4)$$

We, thus, find that ψ is a mixed-type chiral superfield, whose scalar components are NG and QNG bosons, while φ^i are pure-type chiral superfields, whose scalar components are both NG bosons. Then the numbers of mixed-type and pure-type

chiral superfields are $N_M = 1$ and $N_P = N - 1$ respectively. This Kähler manifold is non-compact and non-homogeneous due to the existence of the QNG boson.

To construct a compact homogeneous manifold, we want to eliminate the QNG boson (the mixed-type multiplet). Hence, we gauge $U(1)_D$ symmetry by introducing a vector superfield V , which will absorb the mixed-type multiplet. The gauged Kähler potential is [30]

$$K(\vec{\phi}, \vec{\phi}^\dagger, V) = e^V \vec{\phi}^\dagger \vec{\phi} - cV, \quad (2.5)$$

where cV is a Fayet-Iliopoulos(FI) D-term [20, 30]. Since the transformation law of V is

$$e^V \rightarrow e^{V'} = e^V e^{i(\theta^\dagger - \theta)}, \quad e^{i\Re\theta} \in U(1)_D, \quad (2.6)$$

where θ is a chiral superfield, the Kähler potential (2.5) is invariant under the complex extension of the gauge symmetry, $U(1)_D^{\mathbf{C}}$. Note that the global symmetry $G = SU(N)$ cannot be complexified. The equation of motion of V is

$$\delta K / \delta V = e^V \vec{\phi}^\dagger \vec{\phi} - c = 0. \quad (2.7)$$

From this equation, V can be solved as

$$V(\vec{\phi}, \vec{\phi}^\dagger) = -\log\left(\frac{\vec{\phi}^\dagger \vec{\phi}}{c}\right). \quad (2.8)$$

To eliminate the gauge field, we substitute $V(\vec{\phi}, \vec{\phi}^\dagger)$ back to Eq. (2.5):

$$K(\vec{\phi}, \vec{\phi}^\dagger, V(\vec{\phi}, \vec{\phi}^\dagger)) = c \log(\vec{\phi}^\dagger \vec{\phi}), \quad (2.9)$$

where we have omitted constant terms.⁷ Since we have the gauge symmetry $U(1)_D^{\mathbf{C}}$, we can fix the gauge as

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \end{pmatrix}. \quad (2.10)$$

⁷Their contributions to the lagrangian vanishes because of $d^4\theta$ integration.

By comparing Eq. (2.4) and Eq. (2.10), we find that the mixed-type chiral superfield has been eliminated by this gauge fixing. The gauge fixed field (2.10) can be rewritten as

$$\vec{\phi} = \xi \vec{v}, \quad \xi = e^{\varphi \cdot \bar{E}} = \begin{pmatrix} 1 & \mathbf{0} \\ \varphi^i & \mathbf{1}_{N-1} \end{pmatrix} \quad (2.11)$$

where ξ can be considered as a representative of a complex coset manifold $G^{\mathbf{C}}/\hat{H} \simeq G/H = SU(N)/S(U(N-1) \times U(1))$. Since this is a compact homogeneous Kähler manifold, we have obtained a desired result. To obtain a compact manifold, gauge fields were necessary. By substituting Eq. (2.10) to Eq. (2.9), we obtain

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log(1 + |\varphi|^2). \quad (2.12)$$

This is the well known Kähler potential of the Fubini-Study metric for $\mathbf{C}P^{N-1} = SU(N)/S(U(N-1) \times U(1))$.

2.2 Grassmann manifold: $G_{N,M}(\mathbf{C}) = U(N)/U(N-M) \times U(M)$

This subsection is a generalization of the last subsection. The picture of NG and QNG bosons was discussed in Ref. [21]. We consider a global symmetry $G' = G_L \times G_R = U(N)_L \times U(M)_R$ ($N > M$). The basic fields are $\Phi \in (\mathbf{N}, \bar{\mathbf{M}})$, which are $N \times M$ matrix valued chiral superfields. The transformation law of Φ under $G'^{\mathbf{C}}$ is⁸

$$\Phi \rightarrow \Phi' = g \cdot \Phi \stackrel{\text{def}}{=} g_L \Phi g_R^{-1}, \quad g = (g_L, g_R) \in G'^{\mathbf{C}} \quad (2.13)$$

where g_L and g_R are $N \times N$ and $M \times M$ matrices, respectively.

The Kähler potential is the canonical one:

$$K(\Phi, \Phi^\dagger) = \text{tr}(\Phi^\dagger \Phi). \quad (2.14)$$

Any vacuum can be transformed under $G'^{\mathbf{C}}$ to

$$V = \langle \Phi \rangle = \begin{pmatrix} \mathbf{1}_M \\ \mathbf{0} \end{pmatrix}, \quad (2.15)$$

⁸The conjugate representation $\vec{\phi} \in \bar{\mathbf{N}}$ is defined to transform $\vec{\phi} \rightarrow (g^{-1})^T \vec{\phi}$, since the group is extended to its complexification and we should preserve the chirality.

where $\mathbf{1}_M$ is the $M \times M$ identity matrix and $\mathbf{0}$ is a $(N - M) \times M$ zero matrix. The global symmetry is spontaneously broken as $U(N)_L \times U(M)_R \rightarrow U(N - M)_L \times U(M)_V$. Here, $U(N - M)_L$ is the group generated by $\left(\begin{pmatrix} \mathbf{0}_M & \mathbf{0} \\ \mathbf{0} & T \end{pmatrix}, \mathbf{0}_M \right) \in (\mathcal{G}_L, \mathcal{G}_R)$, where T are $(N - M) \times (N - M)$ matrices, and $U(M)_V$ is by $\left(\begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{N-M} \end{pmatrix}, T \right) \in (\mathcal{G}_L, \mathcal{G}_R)$ where T are $M \times M$ matrices. The complex isotropy $\hat{\mathcal{H}}'$ that leaves $\langle \Phi \rangle$ invariant is larger than $\mathcal{H}'^{\mathbf{C}}$ by $E \stackrel{\text{def}}{=} \left(\begin{pmatrix} \mathbf{0}_M & T \\ \mathbf{0} & \mathbf{0}_{N-M} \end{pmatrix}, \mathbf{0}_M \right)$, where T are $M \times (N - M)$ matrices. Here, E constitute a Borel subalgebra \mathcal{B} of $\hat{\mathcal{H}}'$ and its dimension is $\dim_{\mathbf{C}} \mathcal{B} = M(N - M)$. On the other hand, the complex broken generators consist of non-hermitian generators, $\bar{E} \stackrel{\text{def}}{=} \left(\begin{pmatrix} \mathbf{0}_M & 0 \\ T & \mathbf{0}_{N-M} \end{pmatrix}, \mathbf{0}_M \right)$, which are hermitian conjugates of E , and hermitian generators, $X \stackrel{\text{def}}{=} \left(\begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{N-M} \end{pmatrix}, -T \right)$, which are elements of an axial symmetry $U(M)_A$. The target manifold is a complex coset manifold $M' \simeq G'^{\mathbf{C}}/\hat{H}'$ and its representative is $\xi' = \exp(\varphi \cdot \bar{E} + i\psi \cdot X) \stackrel{\text{def}}{=} (\xi'_L, \xi'_R)$. The field can be written as $\vec{\phi} = \xi' \cdot V = \xi'_L V \xi'^R{}^{-1}$. Its form near the vacuum is

$$\delta V = \begin{pmatrix} 2i\psi \\ \varphi \end{pmatrix}. \quad (2.16)$$

Here, ψ is $M \times M$ matrix chiral superfield considered as mixed types and φ is $(N - M) \times M$ matrix chiral superfield considered as pure types. Hence, the number of the mixed-type and pure-type chiral superfields are $N_M = M^2$ and $N_P = M(N - M)(= \dim_{\mathbf{C}} \mathcal{B})$ respectively.

To absorb M^2 mixed-type chiral superfields, we gauge $U(M)_R$ by introducing M^2 vector superfields $V = V^A T_A$, where T_A are generators of $U(M)_R$. The gauged Kähler potential is

$$K(\Phi, \Phi^\dagger, V) = \text{tr}(\Phi^\dagger \Phi e^V) - c \text{tr} V, \quad (2.17)$$

where $c \text{tr} V$ is a Fayet-Iliopoulos D-term. Since the vector superfields are transformed as

$$e^V \rightarrow e^{V'} = g_R e^V g_R^\dagger, \quad (2.18)$$

the gauged Kähler potential is invariant under the complexified gauge symmetry $G_{\mathbb{R}}^{\mathbb{C}}$. To eliminate vector superfields, we use the equation of motion of V ,⁹

$$\delta K/\delta V = \Phi^\dagger \Phi e^V - c \mathbf{1}_M = 0. \quad (2.19)$$

Then V can be solved as

$$V(\Phi, \Phi^\dagger) = -\log\left(\frac{\Phi^\dagger \Phi}{c}\right). \quad (2.20)$$

By substituting this to Eq. (2.17), we obtain

$$K(\Phi, \Phi^\dagger, V(\Phi, \Phi^\dagger)) = c \operatorname{tr} \log(\Phi^\dagger \Phi) = c \log \det(\Phi^\dagger \Phi), \quad (2.21)$$

where we have omitted constant terms. We choose a gauge fixing as

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} \quad (2.22)$$

where φ is a $(N-M) \times M$ matrix valued chiral superfield. By comparing Eq. (2.16) and Eq. (2.22), we find that all mixed-type multiplets ψ have disappeared by this gauge fixing condition. When ξ is a representative of $G^{\mathbb{C}}/\hat{H} = U(N)/U(N-M) \times U(M)$, Φ can be rewritten as

$$\Phi = \xi \cdot V = \xi_L V \xi_R^{-1}, \quad \xi = e^{\varphi \cdot \bar{E}} = \left(\begin{pmatrix} \mathbf{1}_M & 0 \\ \varphi & \mathbf{1}_{N-M} \end{pmatrix}, \mathbf{1}_M \right) = (\xi_L, \xi_R). \quad (2.23)$$

Since the target space M is parameterized solely by pure-type multiplets, it is a compact homogeneous Kähler manifold. By substituting Eq. (2.22) to Eq. (2.21), we obtain the Kähler potential of M

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log \det(\mathbf{1}_M + \varphi^\dagger \varphi). \quad (2.24)$$

This is the Kähler potential of Grassmann manifold $G_{N,M} = U(N)/U(N-M) \times U(M)$ [3].

⁹We treat $e^{-V} \delta e^V$ as an infinitesimal parameter, since $\delta \operatorname{tr}(\Phi^\dagger \Phi e^V) = \operatorname{tr}(\Phi^\dagger \Phi e^V (e^{-V} \delta e^V))$. The second term is obtained from $\operatorname{tr}(\delta \log X) = \operatorname{tr}(X^{-1} \delta X)$, where $X = e^V$.

3 Nonlinear Sigma Models with F-term Constraints

Only D-term constraints appeared in the last two examples. In this section we also introduce appropriate F-term constraints to define other Kählerian G/H manifolds.

3.1 $SO(N)/SO(N-2) \times U(1)$

We consider a global symmetry, $G' = SO(N) \times U(1)_D = G \times U(1)_D$. We will gauge $U(1)_D$ symmetry later. The fields, which will develop a vacuum expectation value, are $\vec{\phi}$ in the defining representation \mathbf{N} of $SO(N)$. The $U(1)_D$ charge of $\vec{\phi}$ is defined to be 1. The fundamental representation is decomposed under its subgroup $SO(N-2) \times U(1)$ as $\mathbf{N} = (\mathbf{N}-\mathbf{2}, 0) \oplus (\mathbf{1}, 1) \oplus (\mathbf{1}, -1)$. Here, the second factor is the $U(1)$ charge. The fields can be written as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ z \end{pmatrix}, \quad (3.1)$$

where x , y^i ($i = 1, \dots, N-2$) and z are a scalar, a vector and a scalar of $SO(N-2)$ respectively, and their $U(1)$ charge are defined above. $SO(N)$ is defined as the group that leaves the following quadratic form

$$I_2 \stackrel{\text{def}}{=} \vec{\phi}^2 \stackrel{\text{def}}{=} \vec{\phi}^T J \vec{\phi} = 2xz + y^2, \quad (3.2)$$

where we have written the invariant tensor of rank 2 in a rather unconventional way (see Appendix C)

$$J = \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{1}_{N-2} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}. \quad (3.3)$$

The generators of $SO(N)$ consist of $SO(N-2)$ generator T_{ij} ($i, j = 1, \dots, N-2$), $U(1)$ generator T , and upper triangle matrices E^i ($i = 1, \dots, N-2$) which transform as $(\mathbf{N}-\mathbf{2}, 1)$, and their complex conjugates $\bar{E}_i = (E^i)^\dagger$ in $(\mathbf{N}-\mathbf{2}, -1)$. The $SO(N)^{\mathbb{C}}$ acts on the fundamental representation in our basis as

$$\delta \vec{\phi} = \left(i\theta T + \frac{i}{2} \theta_{ij} T_{ij} + \bar{\epsilon}_i E^i + \epsilon^i \bar{E}_i \right) \vec{\phi}$$

$$= \begin{pmatrix} i\theta & \bar{\epsilon}_j & 0 \\ \epsilon^i & \theta_{ij} & -\bar{\epsilon}_i \\ 0 & -\epsilon^i & -i\theta \end{pmatrix} \begin{pmatrix} x \\ y^j \\ z \end{pmatrix}, \quad (3.4)$$

where $\frac{i}{2}\theta_{kl}(T_{kl})^i{}_j = \theta_{ij}$. Here, these coefficients are normalized by $\text{tr} T_{ij}^2 = \text{tr} T^2 = \text{tr} E^i \bar{E}_i = 2$ (no sum). All parameters are complex when we consider $SO(N)^{\mathbf{C}}$ and real when $SO(N)$.

In order to impose the global symmetry $SO(N) \times U(1)_{\text{D}}$, we introduce a superpotential

$$W(\phi_0, \vec{\phi}) = \phi_0 \vec{\phi}^2, \quad (3.5)$$

with a lagrange multiplier field ϕ_0 , which is $SO(N)$ singlet and its $U(1)_{\text{D}}$ charge is defined to be -2 so that W is invariant under G' . Since the superpotential is a holomorphic function of ϕ and ϕ_0 , the symmetry is enhanced to its complexification $G'^{\mathbf{C}} = SO(N)^{\mathbf{C}} \times U(1)_{\text{D}}^{\mathbf{C}}$. We can eliminate the auxiliary field by using its equation of motion¹⁰

$$\partial W / \partial \phi_0 = I_2 = 2xz + y^2 = 0. \quad (3.6)$$

We, thus, obtain a F-term constraint ($N_F = 1$). This equation is immediately solved as

$$z = -\frac{y^2}{2x}. \quad (3.7)$$

Then, the field $\vec{\phi}$ constrained by F-term is written as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ -\frac{y^2}{2x} \end{pmatrix}. \quad (3.8)$$

When this develop a vacuum expectation value, any vacuum can be transformed by $G'^{\mathbf{C}}$ to the standard form,

$$\vec{v} = \langle \vec{\phi} \rangle = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \quad (3.9)$$

¹⁰There is another way to obtain the F-term constraint. If we take $K = \lambda \phi_0^\dagger \phi_0 + \vec{\phi}^\dagger \vec{\phi}$, and $W = \phi_0 \vec{\phi}^2$, then the potential reads $V = \frac{1}{\lambda} |\vec{\phi}^2|^2 + |\phi_0|^2 |\vec{\phi}|^2$. We obtain the F-term constraint in the limit $\lambda \rightarrow 0$.

By this vacuum expectation value, the global symmetry is spontaneously broken as $SO(N) \times U(1)_D \rightarrow SO(N-2) \times U(1)'$, where unbroken $U(1)'$ is generated by a linear combination of $U(1)$ subgroup and $U(1)_D$.¹¹ The complex broken generators consist of X , which is hermitian and generates a mixed-type multiplet, and E^i , which are non-hermitian and generate pure-type multiplets. Then, the number of the mixed- and pure-type multiplets are $N_M = 1$ and $N_P = N - 2$ respectively. The target manifold M' generated by these generators is a non-compact and non-homogeneous due to presence of the QNG boson. The field near \vec{v} is

$$\delta\vec{v} = (i\psi X + \varphi^i \bar{E}_i)\vec{v} = \begin{pmatrix} i\psi \\ \varphi^i \\ 0 \end{pmatrix}, \quad (3.10)$$

where ψ is a mixed-type multiplet and φ^i are pure-type multiplets.

We elevate $U(1)_D$ to a local gauge symmetry to obtain a compact manifold by eliminating the mixed-type multiplet as in the case of $\mathbf{C}P^{N-1}$. The gauged Kähler potential is the same as Eq. (2.5). By integrating out the auxiliary superfields, we obtain Eq. (2.9), with a constraint $\vec{\phi}^2 = 0$. By using a gauge symmetry $U(1)_D^{\mathbf{C}}$, we can choose the gauge fixing as $x = 1$:

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \\ -\frac{1}{2}\varphi^2 \end{pmatrix}, \quad (3.11)$$

where we rewrite y^i as φ^i . This can be rewritten by using the representative ξ of

¹¹Note that the condition $I_2 = 0$ is essential to introduce the gauge symmetry. To impose $I_2 = f^2 \neq 0$, we have to use $W = g\phi_0(I_2 - f^2)$, then there is no $U(1)_D$ symmetry, and there is a supersymmetric vacuum alignment [9, 10]. Namely, the unbroken global symmetry H can be different depending on the choice of the vacuum expectation value: $H = SO(N - 1)$ at the symmetric points, where $\phi^\dagger\phi = f^2$, and $H = SO(N - 2)$ at the non-symmetric points, where $\phi^\dagger\phi > f^2$. Whereas $I_2 = 0$ corresponds to an open orbit, $I_2 \neq 0$ correspond to closed orbits. In general, in closed orbits, there is a supersymmetric vacuum alignment. See e.g. Subsec. 3.3 for the E_6 case. In this paper, we do not discuss closed orbits, since we can not gauge $U(1)_D$ symmetry.

the complex coset manifold $G^{\mathbf{C}}/\hat{H} = SO(N)/SO(N-2) \times U(1)$ as

$$\vec{\phi} = \xi \vec{v}, \quad \xi = e^{\varphi \cdot \vec{E}} = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ \varphi^i & \mathbf{1}_{N-2} & \mathbf{0} \\ -\frac{1}{2}\varphi^2 & -\varphi^i & 1 \end{pmatrix}. \quad (3.12)$$

We, thus, obtain a Kähler potential of $G^{\mathbf{C}}/\hat{H}$,

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log \left(1 + |\varphi|^2 + \frac{1}{4} \varphi^{\dagger 2} \varphi^2 \right). \quad (3.13)$$

This is exactly the Kähler potential of $SO(N)/SO(N-2) \times U(1)$ [13, 22, 10].

In our derivation of the Kähler potential, we used the D-term constraint after solving F-term constraint first. Instead, we could impose the D-term constraint first. If we do so, we obtain the previous $\mathbf{C}P^{N-1}$ model. F-term constraints is used as the holomorphic embedding condition of $Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$ to $\mathbf{C}P^{N-1}$. It is a well-known method to obtain $Q^{N-2}(\mathbf{C})$ in the mathematical literature [23]. (See also p278 of [24].)

3.2 $SO(2N)/U(N)$ and $Sp(N)/U(N)$

In this subsection, we consider the global symmetry $G' = G_{\mathbf{L}} \times G_{\mathbf{R}}$, where $G_{\mathbf{L}}$ is either $SO(2N)$ or $Sp(N)$ and $G_{\mathbf{R}} = U(N)_{\mathbf{R}}$, which will be gauged later. To embed $G_{\mathbf{L}}$ to $2N \times 2N$ matrix of $U(2N)$, we write its elements by using four $N \times N$ matrices,

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2N). \quad (3.14)$$

This is an element of $SO(2N)$ or $Sp(N)$, if it satisfies

$$g^T J' g = J', \quad (3.15)$$

where J' is the invariant tensor of $SO(2N)$ or $Sp(N)$,

$$J' = \begin{pmatrix} \mathbf{0} & \mathbf{1}_N \\ \epsilon \mathbf{1}_N & \mathbf{0} \end{pmatrix}. \quad (3.16)$$

Here $\epsilon = +1$ corresponds to $SO(2N)$ and $\epsilon = -1$ to $Sp(N)$. Eq. (3.15) can be written explicitly as

$$\begin{pmatrix} A^T C + \epsilon C^T A & A^T D + \epsilon C^T B \\ B^T C + \epsilon D^T A & B^T D + \epsilon D^T C \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_N \\ \epsilon \mathbf{1}_N & \mathbf{0} \end{pmatrix}. \quad (3.17)$$

We consider the global symmetry either $G' = SO(2N)_L \times U(N)_R$ for $\epsilon = +1$ or $G' = Sp(N)_L \times U(N)_R$ for $\epsilon = -1$. The field contents are $\Phi \in (\mathbf{2N}, \bar{\mathbf{N}})$, which will acquire a vacuum expectation value. Its transformation law under G' is

$$\Phi \rightarrow \Phi' = g \cdot \Phi = g_L \Phi g_R^{-1}, \quad g = (g_L, g_R) \in G_L \times G_R \quad (3.18)$$

The G'^C invariant superpotential is

$$W(\Phi_0, \Phi) = \text{tr}(\Phi_0 \Phi^T J' \Phi), \quad (3.19)$$

where Φ_0 is an $N \times N$ auxiliary matrix chiral superfield, whose transformation law is

$$\Phi_0 \rightarrow g_R \Phi_0 g_R^T. \quad (3.20)$$

Since $I_2' \stackrel{\text{def}}{=} \Phi^T J' \Phi$ is symmetric (anti-symmetric) for $\epsilon = 1(\epsilon = -1)$, Φ_0 satisfies

$$\Phi_0^T = \epsilon \Phi_0. \quad (3.21)$$

Namely, Φ_0 belongs to a symmetric (anti-symmetric) rank-2 tensor representation of $SU(N)_R$ for $\epsilon = 1(\epsilon = -1)$, and its $U(1)_D(\in U(N)_R)$ charge is defined to be -2 to cancel with the Φ charge. Note that $I_2' = \Phi^T J' \Phi$ is invariant under G_L , but not invariant under G_R .

To eliminate the auxiliary field Φ_0 , we calculate its equation of motion:

$$\delta W / \delta \Phi_0 = \Phi^T J' \Phi = 0. \quad (3.22)$$

We, thus, obtained F-term constraints for the fields Φ . Their number is $N_F = \frac{1}{2}N(N+1)$ for $\epsilon = 1$ and $N_F = \frac{1}{2}N(N-1)$ for $\epsilon = -1$. Then the dimension of the resulting manifold M' constrained by the F-term is $N_\Phi = 2N^2 - \frac{1}{2}N(N+1) = \frac{3}{2}N^2 - \frac{1}{2}N$ for $\epsilon = 1$ and $N_\Phi = 2N^2 - \frac{1}{2}N(N-1) = \frac{3}{2}N^2 + \frac{1}{2}N$ for $\epsilon = -1$. When the field Φ acquires a vacuum expectation value, any vacuum can be transformed by G'^C to the standard form,

$$V = \langle \Phi \rangle = \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_N \end{pmatrix}. \quad (3.23)$$

Hence, the F-term constrained manifold is a $G^{\mathbf{C}}$ -orbit of V . The breaking pattern of the global symmetry is either $SO(2N)_{\mathbf{L}} \times U(N)_{\mathbf{R}} \rightarrow U(N)_{\mathbf{V}}$ for $\epsilon = 1$ or $Sp(N)_{\mathbf{L}} \times U(N)_{\mathbf{R}} \rightarrow U(N)_{\mathbf{V}}$ for $\epsilon = -1$. Here, in both cases, the element of $U(N)_{\mathbf{V}}$ can be written as

$$\left(\left(\begin{pmatrix} h & 0 \\ 0 & h^{-1T} \end{pmatrix}, h \right) \right) \in U(N)_{\mathbf{V}}, \quad (3.24)$$

where we have used Eq. (3.17). The complex isotropy group \hat{H}' consists of complex extension of these elements and elements of the type

$$\left(\left(\begin{pmatrix} \mathbf{1}_N & B \\ \mathbf{0}_N & \mathbf{1}_N \end{pmatrix}, \mathbf{1}_N \right) \stackrel{\text{def}}{=} e^E, \quad E = \left(\left(\begin{pmatrix} \mathbf{0}_N & B \\ \mathbf{0}_N & \mathbf{0}_N \end{pmatrix}, \mathbf{0}_N \right) \right), \quad (3.25)$$

with constraints from Eq. (3.17),

$$B + \epsilon B^T = 0. \quad (3.26)$$

These E constitute a Borel subalgebra \mathcal{B} of $\hat{\mathcal{H}}'$. Its dimensionality is $\dim_{\mathbf{C}} \mathcal{B} = \frac{1}{2}N(N-1)$ for $\epsilon = +1$ and $\dim_{\mathbf{C}} \mathcal{B} = \frac{1}{2}N(N+1)$ for $\epsilon = -1$. The pure-type broken generators are the complex conjugation of $E \in \mathcal{B}$: $\bar{E} = (E)^\dagger$.

To obtain a compact coset manifold, we gauge $U(N)_{\mathbf{R}}$ symmetry by introducing vector superfields as in the Grassmann manifold. The gauged Kähler potential is the same as Eq. (2.17), but with F-term constraints. Since the procedure of integrating out the gauge fields are also the same as the Grassmann manifold, we obtain Eq. (2.21). We can choose the gauge fixing as

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix}, \quad (3.27)$$

where φ satisfies F-term constraints Eq. (3.22),

$$\Phi^T J' \Phi = \varphi + \epsilon \varphi^T = 0. \quad (3.28)$$

The fields φ are all pure-type chiral superfields, since Φ is generated by the pure-type broken generators \bar{E} from the vacuum V :

$$\Phi = \xi \cdot V, \quad \xi = e^{\varphi \cdot \bar{E}} = \left(\left(\begin{pmatrix} \mathbf{1}_N & \mathbf{0}_N \\ \varphi & \mathbf{1}_N \end{pmatrix}, \mathbf{1}_N \right) \right). \quad (3.29)$$

Here, from Eq. (3.17), φ satisfies $\varphi + \epsilon\varphi^T = 0$, which is consistent with (3.28). By substituting Eq. (3.27) to Eq. (2.21), we obtain the Kähler potential

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log \det(\mathbf{1}_N + \varphi^\dagger \varphi), \quad \varphi + \epsilon\varphi^T = 0. \quad (3.30)$$

They are anti-symmetric (symmetric) parts of the matrix chiral superfield of the Grassmann manifold $G_{2N,N}$ for $\epsilon = +1$ (-1). Their dimensions are $\dim_{\mathbf{C}} M = \frac{1}{2}N(N-1)$ for $\epsilon = +1$ and $\dim_{\mathbf{C}} M = \frac{1}{2}N(N+1)$ for $\epsilon = -1$. Again, it is a well-known that these manifold are submanifolds of the Grassmann manifold $G_{2N,N}$ in the mathematical literature [23].

3.3 $E_6/SO(10) \times U(1)$

This and the next subsections are devoted to the gauge theory construction of the exceptional-type hermitian symmetric spaces. The situation is slightly different from the classical group cases. Namely, a F-term constrained manifold M' is characterized by the derivative of a G -invariant, $\partial I = 0$, but not the G -invariant itself, $I = 0$, as in the case of classical-types.

As in the $Q^{N-2}(\mathbf{C})$ case, we consider a global symmetry $G' = E_6 \times U(1)_{\mathbf{D}} = G \times U(1)_{\mathbf{D}}$. The field belongs to the fundamental representation of E_6 : $\vec{\phi} \in \mathbf{27}$, which will acquire a vacuum expectation value. We decompose E_6 under its maximal subgroup $SO(10) \times U(1)$. Since the fundamental representation can be decomposed as $\mathbf{27} = (\mathbf{1}, 4) \oplus (\mathbf{16}, 1) \oplus (\mathbf{10}, -2)$ [26] where the second factors are $U(1)$ charge, the basic field $\vec{\phi}$ can be written as

$$\vec{\phi} = \begin{pmatrix} x \\ y_\alpha \\ z^A \end{pmatrix}. \quad (3.31)$$

Here, x , y_α ($\alpha = 1, \dots, 16$) and z^A ($A = 1, \dots, 10$) are a $SO(10)$ scalar, a Weyl spinor and a vector respectively. The decomposition of the adjoint representation, $\mathbf{78} = (\mathbf{45}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{16}, 1) \oplus (\bar{\mathbf{16}}, -1)$ [26], implies that E_6 algebra can be constructed by $SO(10)$ generators T_{AB} ($A, B = 1, \dots, 10$), a $U(1)$ generator T , upper half generators E_α , which belong to a Weyl spinor of $SO(10)$, and their conjugates \bar{E}^α . See Appendix D for a detail.

The transformation law of $\vec{\phi}$ under the complex extension of E_6 is [15, 27]

$$\begin{aligned}\delta\vec{\phi} &= \left(i\theta T + \frac{i}{2}\theta_{AB}T_{AB} + \bar{\epsilon}^\alpha E_\alpha + \epsilon_\alpha \bar{E}^\alpha \right) \vec{\phi} \\ &= \begin{pmatrix} \frac{2i}{\sqrt{3}}\theta & \bar{\epsilon}^\beta & \mathbf{0} \\ \epsilon_\alpha & \frac{i}{2}\theta_{AB}(\sigma_{AB})_\alpha{}^\beta + \frac{i}{2\sqrt{3}}\theta\delta_\alpha^\beta & -\frac{1}{\sqrt{2}}(\bar{\epsilon}\sigma_B C)_\alpha \\ \mathbf{0} & -\frac{1}{\sqrt{2}}(C\sigma_A^\dagger\epsilon)^\beta & \theta_{AB} - \frac{i}{\sqrt{3}}\theta\delta_{AB} \end{pmatrix} \begin{pmatrix} x \\ y_\beta \\ z^B \end{pmatrix}, \quad (3.32)\end{aligned}$$

where $\frac{i}{2}\theta_{CD}\rho(T_{CD})^A{}_B = \theta_{AB}$ and $\rho(T_{AB})$ is the vector representation matrices of $SO(10)$. The 16×16 matrices σ_A , σ_{AB} and C are (off-diagonal blocks of) $SO(10)$ gamma matrices, spinor rotation matrices and the charge conjugation matrix, respectively. Normalizations are fixed by $\text{tr } T^2 = \text{tr } T_{AB}^2 = \text{tr } E_\alpha \bar{E}^\alpha = 6$ (no sum).¹²

The decomposition of tensor product, $\mathbf{27} \otimes \mathbf{27} = \mathbf{27}_s \oplus \dots$, implies that there exist rank-3 symmetric invariant tensor Γ_{ijk} and its complex conjugate Γ^{ijk} [27]. By using this invariant tensor, a cubic invariant I_3 of E_6 is defined as

$$I_3 \stackrel{\text{def}}{=} \Gamma_{ijk}\phi^i\phi^j\phi^k = xz^2 + \frac{1}{\sqrt{2}}z^A(yC\sigma_A^\dagger y). \quad (3.33)$$

Note that this is not invariant under $U(1)_D$.

We construct a superpotential,

$$W(\vec{\phi}_0, \vec{\phi}) = \Gamma_{ijk}\phi_0^i\phi^j\phi^k. \quad (3.34)$$

where $\vec{\phi}_0$ are auxiliary fields whose $U(1)_D$ charge should be chosen so as to make the superpotential invariant. If we assign the $U(1)_D$ charge 1 to $\vec{\phi}$, $\vec{\phi}_0$ must have charge -2 so that they belong to $(\mathbf{27}, -2)$. The equations of motion for the auxiliary fields ϕ_0^i , $\delta W/\delta\phi_0 = \Gamma_{ijk}\phi^j\phi^k = 0$, are calculated as

$$\partial W/\partial z_0^A = \Gamma_{Ajk}\phi^j\phi^k = 2z_A x + \frac{1}{\sqrt{2}}y_\alpha(C\sigma_A^\dagger)^{\alpha\beta}y_\beta = 0, \quad (3.35)$$

$$\partial W/\partial y_{0\alpha} = \Gamma_{\alpha jk}\phi^j\phi^k = \sqrt{2}(C\sigma_A^\dagger)^{\alpha\beta}y_\beta z^A = 0, \quad (3.36)$$

$$\partial W/\partial x_0 = \Gamma_{0jk}\phi^j\phi^k = z^2 = 0. \quad (3.37)$$

¹² $\text{tr } T_{AB}^2 = 6$ has been calculated from $\text{tr } \rho(T_{AB})^2 = 2$ and $\text{tr } (\sigma_{AB})^2 = 4$ and others have been fixed to this.

In the second equation, we have used the fact that $(C\sigma_A^\dagger)^{\alpha\beta}$ is symmetric. Note that these equations can be written also as

$$\partial I_3 = 0, \quad (3.38)$$

where the differentiation is with respect to ϕ^i . In these 27 equations, only the first 10 equations are independent: the first equation can be solved

$$z_A = -\frac{1}{2\sqrt{2}x}y(C\sigma_A^\dagger)y. \quad (3.39)$$

Then, the last two equations are not independent, since they are automatically satisfied as

$$\sqrt{2}(C\sigma_A^\dagger)^{\alpha\beta}y_\beta z^A = -\frac{1}{2x}(C\sigma_A^\dagger)^{\alpha\beta}y_\beta (y(C\sigma_A^\dagger)y) = 0, \quad (3.40)$$

$$z^2 = \frac{1}{8x^2} (y(C\sigma_A^\dagger)y)^2 = 0, \quad (3.41)$$

with the help of the identity,

$$(\varepsilon C\sigma_A^\dagger\psi)(\psi C\sigma_A^\dagger\eta) = -\frac{1}{2}(\varepsilon C\sigma_A^\dagger\eta)(\psi C\sigma_A^\dagger\psi). \quad (3.42)$$

Hence, the number of F-term condition is $N_F = 10$, and the dimension of M' is $N_\Phi = 27 - 10 = 17$. The manifold M' satisfying these F-term constraints can be written as

$$\vec{\phi} = \begin{pmatrix} x \\ y_\alpha \\ -\frac{1}{2\sqrt{2}x}(yC\sigma_A^\dagger y) \end{pmatrix}. \quad (3.43)$$

On M' , the value of E_6 invariant is

$$I_3 \sim (yC\sigma_A^\dagger y)^2 = 0, \quad (3.44)$$

by the identity (3.42). Note that I_3 must vanish since it is not invariant under $U(1)_D$.

When the fields $\vec{\phi}$ develop a vacuum expectation value, any vacuum can be transformed under $G'^{\mathbf{C}}$ to the standard form,

$$\vec{v} = \langle \vec{\phi} \rangle = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \quad (3.45)$$

The global symmetry is spontaneously broken as $E_6 \times U(1)_D \rightarrow SO(10) \times U(1)' = H'$.¹³ The unbroken $U(1)'$ is generated by $T' = T - \frac{2}{\sqrt{3}}\mathbf{1}_{27} = \text{diag.}(0, -\frac{\sqrt{3}}{2}\delta_\alpha^\beta, -\sqrt{3}\delta_{AB})$ and $SO(10)$ by T_{AB} . The complex isotropy $\hat{\mathcal{H}}'$ is larger than the complexification of \mathcal{H}' by E_α . These 16 E_α constitute a Borel subalgebra \mathcal{B} in $\hat{\mathcal{H}}'$. The complex broken generators are composed of pure-type generators \bar{E}^α and another combination of $U(1)$ generators of a mixed-type $X \sim (1, \dots)$. Their numbers are $N_P = 16$ and $N_M = 1$ respectively. The target manifold M' generated by these broken generators has dimension $\dim M' = N_\Phi = 17$. Since it coincide with the dimension of the manifold constrained by 10 independent F-term conditions, any vacuum which satisfies F-term constraints can be transformed to the form of Eq. (3.45) by a G'^C transformation.

To remove the mixed-type multiplet and to obtain a compact manifold, we gauge $U(1)_D$ symmetry as the case of the CP^{N-1} . The gauged Kähler potential is the same as Eq. (2.5). Since the procedure to eliminate the vector superfield is also the same as the CP^{N-1} case, we obtain Eq. (2.9). We can choose a gauge fixing as

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi_\alpha \\ -\frac{1}{2\sqrt{2}}(\varphi C \sigma_A^\dagger \varphi) \end{pmatrix}, \quad (3.46)$$

where we write φ_α for y_α . By using the representative ξ of the complex coset manifold $M = G^C/\hat{H} \simeq E_6/SO(10) \times U(1)$, it can be rewritten as

$$\vec{\phi} = \xi \vec{v}, \quad \xi = e^{\varphi \cdot \bar{E}} = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \varphi_\alpha & \mathbf{1}_{16} & \mathbf{0} \\ -\frac{1}{2\sqrt{2}}(\varphi C \sigma_A^\dagger \varphi) & -\frac{1}{\sqrt{2}}(C \sigma_A^\dagger \varphi)^\beta & \mathbf{1}_0 \end{pmatrix}. \quad (3.47)$$

By substituting Eq. (3.46) to Eq. (2.9), we obtain the Kähler potential

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log \left(1 + |\varphi|^2 + \frac{1}{8}(\varphi^\dagger \sigma^A \varphi^\dagger)(\varphi \sigma_A^\dagger \varphi) \right), \quad (3.48)$$

¹³As in the case of $SO(N)$ discussed in Subsec. 3.1, there is no $U(1)_D$ symmetry if $I_3 \neq 0$. In this case, the E_6^C -orbit is closed, and, by a supersymmetric vacuum alignment, there exist two regions with different unbroken global symmetries [9]: symmetric points and non-symmetric points. The breaking patterns of E_6 are $E_6 \rightarrow F_4$ at the symmetric points and $E_6 \rightarrow SO(8)$ at generic points [27].

where we have used the basis where $C = 1$ [15]. This coincides with the Kähler potential of $E_6/SO(10) \times U(1)$ constructed in Refs. [23, 15, 22]. (It is also equivalent to [14].) Its dimension is $\dim_{\mathbf{C}} M = 27 - 10 - 1 = 16$. If we do not introduce the superpotential, the manifold is $\mathbf{C}P^{26}$. Hence, $E_6/SO(10) \times U(1)$ is embedded in $\mathbf{C}P^{26}$ by 10 F-term constraints, $\partial I_3 = 0$. In fact, Yasui constructed $E_6/SO(10) \times U(1)$ as a submanifold of $\mathbf{C}P^{26}$ by using the Jordan algebra[16].

3.4 $E_7/E_6 \times U(1)$

In this subsection, we consider another exceptional group E_7 . The global symmetry is $G' = E_7 \times U(1)_{\mathbf{D}} = G \times U(1)_{\mathbf{D}}$. The basic fields $\vec{\phi}$ belongs to the fundamental representation **56**. Under a maximal subgroup $E_6 \times U(1)$, it can be decomposed as **56** = $(\mathbf{27}, -\frac{1}{3}) \oplus (\bar{\mathbf{27}}, \frac{1}{3}) \oplus (\mathbf{1}, -1) \oplus (\mathbf{1}, 1)$ [26]. Therefore, we write $\vec{\phi}$ as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ z_i \\ w \end{pmatrix}, \quad (3.49)$$

where y^i and z_i are **27** and $\bar{\mathbf{27}}$ respectively, and x and w are scalars. By a decomposition of the adjoint representation under $E_6 \times U(1)$ [26], **133** = $(\mathbf{78}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{27}, 1) \oplus (\bar{\mathbf{27}}, -1)$, we can construct the E_7 algebra from the E_6 algebra $T_A (A = 1, \dots, 78)$, $U(1)$ generator T , upper half generators $E^i (i = 1, \dots, 27)$, belonging to **27**, and their conjugates $\bar{E}_i = (E^i)^\dagger$, belonging to $\bar{\mathbf{27}}$. Their commutation relations are discussed in Appendix E. The action of the E_7 algebra on the fundamental representation is

$$\begin{aligned} \delta \vec{\phi} &= (i\theta T + i\theta_A T_A + \bar{\epsilon}_i E^i + \epsilon^i \bar{E}_i) \vec{\phi} \\ &= \begin{pmatrix} i\sqrt{\frac{3}{2}}\theta & \bar{\epsilon}_j & \mathbf{0} & 0 \\ \epsilon^i & i\theta_{A\rho}(T_A)^i_j + i\sqrt{\frac{1}{6}}\theta\delta^{ij} & \Gamma^{ijk}\bar{\epsilon}_k & \mathbf{0} \\ \mathbf{0} & \Gamma_{ijk}\epsilon^k & -i\theta_{A\rho}(T_A)^T_i^j - i\sqrt{\frac{1}{6}}\theta\delta_i^j & \bar{\epsilon}_i \\ 0 & \mathbf{0} & \epsilon^j & -i\sqrt{\frac{3}{2}}\theta \end{pmatrix} \begin{pmatrix} x \\ y^j \\ z_j \\ w \end{pmatrix}, \end{aligned} \quad (3.50)$$

where $\rho(T_A)$ is the 27×27 representation matrix for the fundamental representation, Γ_{ijk} is the E_6 invariant tensor, defined in the last subsection, and Γ^{ijk} is its conjugate. Here normalizations have been determined by $\text{tr } T^2 = \text{tr } T_A^2 = \text{tr } E^i \bar{E}_i = 12$ (no sum).¹⁴

In the tensor products [26], $\mathbf{56} \otimes \mathbf{56} = \mathbf{1}_a \oplus \dots$ and $\mathbf{56} \otimes \mathbf{56} \otimes \mathbf{56} \otimes \mathbf{56} = \mathbf{1}_s \oplus \dots$, there exist the rank-2 anti-symmetric invariant tensor $f_{\alpha\beta}$ and the rank-4 symmetric invariant tensor $d_{\alpha\beta\gamma\delta}$ respectively. Their components are calculated in Appendix E. By using this invariant tensor, we can construct the quartic invariant of E_7 as

$$\begin{aligned} I_4 &\stackrel{\text{def}}{=} d_{\alpha\beta\gamma\delta} \phi^\alpha \phi^\beta \phi^\gamma \phi^\delta \\ &= -\frac{1}{2}(xw - y^i z_i)^2 - \frac{1}{3}w \Gamma_{ijk} y^i y^j y^k - \frac{1}{3}x \Gamma^{ijk} z_i z_j z_k \\ &\quad + \frac{1}{2} \Gamma^{ijk} \Gamma_{ilm} z_j z_k y^l y^m. \end{aligned} \quad (3.51)$$

Again, note that this is not invariant under $U(1)_D$.

The superpotential invariant under $E_7 \times U(1)_D$ is

$$W(\vec{\phi}_0, \vec{\phi}) = d_{\alpha\beta\gamma\delta} \phi_0^\alpha \phi^\beta \phi^\gamma \phi^\delta, \quad (3.52)$$

where ϕ_0^α are auxiliary fields, belonging to $(\mathbf{56}, -3)$. Here the second component is the $U(1)_D$ charge assigned to cancel with $U(1)_D$ charge of ϕ^α . (The term with rank-2 tensor $f_{\alpha\beta}$ is forbidden by $U(1)_D$ symmetry.) To eliminate the auxiliary fields ϕ_0 , we consider F-term constraints obtained by their equation of motions:

$$\begin{aligned} \partial W / \partial y_0^i &= w(xz_i - \Gamma_{ijk} y^j y^k) - z_i y^j z_j + \Gamma^{jkl} \Gamma_{jim} z_k z_l y^m = 0, \\ \partial W / \partial w_0 &= x y^i z_i - w x^2 - \frac{1}{3} \Gamma_{ijk} y^i y^j y^k = 0, \\ \partial W / \partial z_{0i} &= x(w y^i - \Gamma^{ijk} z_j z_k) - y^i y^j z_j + \Gamma^{jik} \Gamma_{jlm} z_k y^l y^m = 0, \\ \partial W / \partial x_0 &= w y^i z_i - x w^2 - \frac{1}{3} \Gamma^{ijk} z_i z_j z_k = 0. \end{aligned} \quad (3.53)$$

Note that these equations can be written as

$$\partial I_4 = 0, \quad (3.54)$$

¹⁴ $\text{tr } T_A^2 = 12$ have been calculated in the normalization $\text{tr } (\rho(T_A)^2) = 6$ for E_6 fundamental representation as in the previous subsection. Other normalizations have been fixed relative to this. In the calculation of $\text{tr } E^i \bar{E}_i = 12$, we have used the identity, Eq. (D.5).

where the differentiations are with respect to ϕ^α . We show that only half of these 58 equations are independent. To solve these equations, we put an ansatz

$$z_i = \frac{c}{x} \Gamma_{ijk} y^j y^k, \quad (3.55)$$

where c is a constant. By substituting this ansatz to the first and second equations, we obtain

$$w(c-1) \Gamma_{ijk} y^j y^k + \frac{c^2}{3x^2} \Gamma_{ijk} \Gamma_{lmn} y^j y^k y^l y^m y^n = 0, \quad (3.56)$$

$$w = \frac{c - \frac{1}{3}}{x^2} \Gamma_{ijk} y^i y^j y^k. \quad (3.57)$$

From these equations we obtain the equation,

$$\frac{4(c - \frac{1}{2})^2}{3x^2} \Gamma_{ijk} y^j y^k \Gamma_{lmn} y^l y^m y^n = 0, \quad (3.58)$$

which gives $c = \frac{1}{2}$. By substituting $c = \frac{1}{2}$ back to Eqs. (3.55) and (3.57), we obtain the results,

$$z_i = \frac{1}{2x} \Gamma_{ijk} y^j y^k, \quad w = \frac{1}{6x^2} \Gamma_{ijk} y^i y^j y^k. \quad (3.59)$$

In the same way, the third and the fourth equations in Eqs. (3.61) can be solved as

$$y^i = \frac{1}{2w} \Gamma^{ijk} z_j z_k, \quad x = \frac{1}{6w^2} \Gamma^{ijk} z_i z_j z_k. \quad (3.60)$$

We can show that these equations are not independent of Eqs. (3.59) by the help of the Springer relation, Eq. (D.6). Then the number of F-term constraints is $N_F = 28$, and the dimension of M' is $\dim_{\mathbb{C}} M' = 56 - 28 = 28$. Thus, the F-term constraints can be solved as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ \frac{1}{2x} \Gamma_{ijk} y^j y^k \\ \frac{1}{6x^2} \Gamma_{ijk} y^i y^j y^k \end{pmatrix}. \quad (3.61)$$

On these points, the value of the E_7 invariant is

$$I_4 = 0, \quad (3.62)$$

where we have used the Springer relation (D.6). Note that $U(1)_D$ invariance requires $I_4 = 0$.

By using $G'^{\mathbf{C}}$, any vacuum expectation value of $\vec{\phi}$ can be transformed to

$$\vec{v} = \langle \vec{\phi} \rangle = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \quad (3.63)$$

On this vacuum, global symmetry is spontaneously broken as $E_7 \times U(1)_D \rightarrow E_6 \times U(1)' \stackrel{\text{def}}{=} H'$. Here $U(1)'$ is generated by a linear combination of $U(1)$ generator T and $U(1)_D$ generator $T_D = \mathbf{1}_{56}$. From Eq. (3.50), the complex isotropy $\hat{\mathcal{H}}'$ is larger than $\mathcal{H}'^{\mathbf{C}}$ by E^i , which constitutes a Borel subalgebra. The complex broken generators are an hermitian generator X , which is a linear combination of T_D and T , and non-hermitian generators \bar{E}_i . Hence, the numbers of mixed- and pure-type multiplets are $N_M = 1$ and $N_P = 27$ respectively. The target manifold M' is generated by these broken generators and its dimension is $\dim_{\mathbf{C}} M' = 28$, which coincides with the dimension of the manifold constrained by the F-term conditions in Eq. (3.61).

The target manifold M' obtained above is non-compact due to the QNG boson. We gauge $U(1)_D$ symmetry to remove the mixed-type multiplet and to obtain a compact manifold. Since the situation is the same as the $\mathbf{C}P^{N-1}$, $Q_{N-2}(\mathbf{C})$ and $E_6/SO(10) \times U(1)$ cases, by integrating out the vector superfield, we obtain Eq. (2.9). We can choose a gauge fixing as

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \\ \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k \\ \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k \end{pmatrix}, \quad (3.64)$$

where we rewrite y^i as φ^i . As in the previous subsections, this can be written as

$$\vec{\phi} = \xi \vec{v}, \quad \xi = e^{\varphi \cdot \bar{E}} = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} & 0 \\ \varphi^i & \mathbf{1}_{27} & \mathbf{0}_{27} & \mathbf{0} \\ \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k & \Gamma_{ijk}\varphi^j & \mathbf{1}_{27} & \mathbf{0} \\ \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k & \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k & \varphi^i & 1 \end{pmatrix}. \quad (3.65)$$

Hence the target manifold M , obtained by integrating out the vector superfield, is the coset manifold generated by \bar{E}_i , which is $M \simeq E_7/E_6 \times U(1)$. Then, by

substituting (3.64) to Eq. (2.9), we obtain the Kähler potential

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log \left(1 + |\varphi|^2 + \frac{1}{4} |\Gamma_{ijk} \varphi^j \varphi^k|^2 + \frac{1}{36} |\Gamma_{ijk} \varphi^i \varphi^j \varphi^k|^2 \right). \quad (3.66)$$

This form coincides with Ref. [13]. Its dimension is $\dim_{\mathbf{C}} M = 56 - 28 - 1 = 27$. It can be embedded to $\mathbf{C}P^{55}$ by holomorphic constraints $\partial I_4 = 0$.

4 Conclusions and Discussions

We have obtained nonlinear sigma models, whose target manifolds are the hermitian symmetric spaces G/H , which are compact and homogeneous, from linear models. For this purpose, we introduced appropriate superpotentials for $G = SO, SU, Sp, E_6, E_7$ to impose F-term constraints. By solving these F-term constraint equations, we have obtained constrained manifolds M' , which are non-compact and non-homogeneous due to the existence of QNG bosons. When there is no gauge symmetry, there must be at least one QNG boson by the theorem of Lerche and Shore [6], and the manifold inevitably becomes non-compact and non-homogeneous (See Appendix B). In order to get rid of these unwanted QNG-bosons, we further introduced suitable local gauge symmetry. By choosing suitable gauge conditions, we obtained the Kähler potentials of all the hermitian symmetric spaces, where decay constants (overall constants of Kähler potentials) are originated from FI-terms of gauge fields.

The gauging procedures to eliminate QNG bosons can be summarized as follows.¹⁵

$$\begin{aligned} \mathbf{R}^+ \times \frac{SU(N) \times U(1)_D}{SU(N-1) \times U(1)'} &\xrightarrow{U(1)_D} \frac{SU(N)}{S(U(N-1) \times U(1))} \\ (\mathbf{R}^+)^{M^2} \times \frac{U(N)_L \times U(M)_R}{U(N-M)_L \times U(M)_V} &\xrightarrow{U(M)_R} \frac{U(N)_L}{U(N-M)_L \times U(M)_L} \end{aligned}$$

¹⁵By using the result in Ref. [9], in all cases considered in this paper, there exists no supersymmetric vacuum alignment, since there is no non-singlet broken generators under the real unbroken subgroup H . Hence, the F-term constrained manifolds $M' \simeq G'^{\mathbf{C}}/\hat{H}'$ are topologically isomorphic to direct products of a QNG boson factor $\mathbf{R}^+ = \{\theta | \theta \in \mathbf{R}, \theta > 0\}$, which is non-compact, and a NG bosons factor G'/H' , which is compact. For example, in the case of \mathbf{C}^N without F-term constraint, $M' \simeq G'^{\mathbf{C}}/\hat{H}' = \frac{(SU(N) \times U(1)_D)^{\mathbf{C}}}{(SU(N-1) \times U(1))^{e \wedge B}} \simeq \mathbf{R}^+ \times \frac{SU(N) \times U(1)_D}{SU(N-1) \times U(1)'} = \mathbf{R}^+ \times \frac{G'}{H'}$. Then, by gauging $U(1)$, we obtained $G^{\mathbf{C}}/\hat{H} \simeq G/H = \mathbf{C}P^{N-1}$.

$$\begin{array}{ccc}
\mathbf{R}^+ \times \frac{SO(N) \times U(1)_D}{SO(N-2) \times U(1)'} & \xrightarrow{U(1)_D} & \frac{SO(N)}{SO(N-2) \times U(1)} \\
(\mathbf{R}^+)^{N^2} \times \frac{SO(2N)_L \times U(N)_R}{U(N)_V} & \xrightarrow{U(N)_R} & \frac{SO(2N)_L}{U(N)_L} \\
(\mathbf{R}^+)^{N^2} \times \frac{Sp(N)_L \times U(N)_R}{U(N)_V} & \xrightarrow{U(N)_R} & \frac{Sp(N)_L}{U(N)_L} \\
\mathbf{R}^+ \times \frac{E_6 \times U(1)_D}{SO(10) \times U(1)'} & \xrightarrow{U(1)_D} & \frac{E_6}{SO(10) \times U(1)} \\
\mathbf{R}^+ \times \frac{E_7 \times U(1)_D}{E_6 \times U(1)'} & \xrightarrow{U(1)_D} & \frac{E_7}{E_6 \times U(1)}
\end{array}$$

The left hand sides denote the F-term constrained manifolds M' (if there are the superpotential). All M' are non-compact and non-homogeneous due to the existence of QNG bosons represented by \mathbf{R}^+ indicating that they are scale factors. The arrows represent the gauging and the right hand sides denote the manifold M obtained by integrating out the vector superfields. The relation of M and M' is a Kähler quotient, $M = M'/G_{\text{gauge}}^{\mathbf{C}}$. All M are compact and homogeneous, since they are parameterized by only NG bosons. In the cases of $\mathbf{C}P^{N-1}$ and $G_{N,M}$, there are no F-term constraints. Other cases have $G^{\mathbf{C}}$ -invariants, superpotentials and F-term constraints as summarized in Table 2.

The F-term constraints can be classified into two types:

- $G = SO, Sp$: $I = 0$. They are $G^{\mathbf{C}}$ -invariants.
- $G = E_6, E_7$: $\partial I = 0$. Although ∂I are not $G^{\mathbf{C}}$ -invariant, those constraints themselves are $G^{\mathbf{C}}$ -invariant.

In each case, the value of $G^{\mathbf{C}}$ -invariant vanishes on the constrained manifolds, since, even in the case of exceptional cases, the constraints $\partial I = 0$ lead to $I = 0$. This remarkable fact can be understood as follows: *Note that, in each case, $G^{\mathbf{C}}$ -invariant I is not invariant under a gauge group. Hence, it must vanish to be consistent with a gauge symmetry. We call it the consistency condition with a gauge symmetry.*¹⁶

¹⁶By combining the result in Ref. [10], this condition can be understood as the condition that the the manifold before gauging must be an open orbit but not a closed orbit. In Ref. [10], it was shown that the open orbit includes a compact and homogeneous manifold as a submanifold, on the other hand, the closed orbit does not have it.

Table 2: **F-term constraints and embedding**

G/H	$G^{\mathbf{C}}$ -invariants	superpotentials	constraints	embedding
$\frac{SO(N)}{SO(N-2) \times U(1)}$	$I_2 = \vec{\phi}^T J \vec{\phi}$	$\phi_0 I_2$	$I_2 = 0$	$\mathbf{C}P^{N-1}$
$\frac{SO(2N)}{U(N)}, \frac{Sp(N)}{U(N)}$	$I_2' = \Phi^T J' \Phi$	$\text{tr}(\Phi_0 I_2')$	$I_2' = 0$	$G_{2N,N}$
$\frac{E_6}{SO(10) \times U(1)}$	$I_3 = \Gamma_{ijk} \phi^i \phi^j \phi^k$	$\Gamma_{ijk} \phi_0^i \phi^j \phi^k$	$\partial I_3 = 0$	$\mathbf{C}P^{26}$
$\frac{E_7}{E_6 \times U(1)}$	$I_4 = d_{\alpha\beta\gamma\delta} \phi^\alpha \phi^\beta \phi^\gamma \phi^\delta$	$d_{\alpha\beta\gamma\delta} \phi_0^\alpha \phi^\beta \phi^\gamma \phi^\delta$	$\partial I_4 = 0$	$\mathbf{C}P^{55}$

Here, J , J' , Γ and d are rank-2, rank-2, rank-3 and rank-4 invariant symmetric tensors of $SO(N)$, $SO(2N)$ or $Sp(N)$, E_6 and E_7 respectively, and I_2 , I_2' , I_3 and I_4 are $G^{\mathbf{C}}$ -invariants composed of them. Each superpotential gives a F-term constraint, which is $I = 0$ in the case of classical groups or $\partial I = 0$ in the case of exceptional groups. Only 10 equations in the 27 equations are independent in the E_6 case, and only 28 equations among 56 equations are independent in the E_7 case. The last column denotes the projective or Grassmann manifold, to which each hermitian symmetric space is embedded by the F-term constraint.

If we forget the F-term constraints and impose only the D-term constraints, the manifolds become $\mathbf{C}P^{N-1}$ or $G_{2N,N}$. This means that all of the hermitian symmetric spaces are holomorphically embedded to $\mathbf{C}P^{N-1}$ or $G_{N,M}$ by F-term constraints, as is shown in the last column of Table 2. Although, some constraints are known in the mathematical literature, the explicit form of constraints in the E_6 and E_7 cases are new results: $E_6/SO(10) \times U(1)$ is holomorphically embedded to $\mathbf{C}P^{26}$ by 16 quadratic homogeneous constraints, and $E_7/E_6 \times U(1)$ to $\mathbf{C}P^{55}$ by 28 triplectic homogeneous constraints. The consistency condition with a gauge symmetry can be understood if we interpret the F-term constraints as the embedding conditions. Since the $G_{N,M}$ can be embedded to $\mathbf{C}P^N$, all hermitian symmetric spaces are embedded to $\mathbf{C}P^N$. If we want to embed M to $\mathbf{C}P^N$, the constraint must be homogeneous, when it is written in terms of homogeneous coordinates.¹⁷

In this paper, we used the equation of motion for the vector auxiliary field. In path integral formalism, this procedure corresponds to integrating over the vector

¹⁷The manifold, which can be embedded to $\mathbf{C}P^N$, is a (projective) algebraic variety and can be understood as a Hodge manifold.

field. In a separate paper [25], we show that the path integration can be performed exactly.

Now we discuss possible generalizations of our result to wider class of Kählerian G/H . In this paper, we treated the hermitian symmetric spaces, which are a special class of homogeneous Kähler manifolds. We confined ourselves to the gauge groups of $U(1)$ or $U(N)$.

1. Even within this limitation, it is possible to generalize our construction to wider class of homogeneous Kähler manifolds. Let us consider Kähler G/H where H have only one $U(1)$ factor, $H = H_{\text{ss}} \times U(1)$ with H_{ss} being semisimple subgroup of H . To be specific, let us generalize $SO(2N)/U(N)$. By generalizing Φ to $2N \times M$ matrix ($N \geq M$) transforming under $SO(2N) \times U(M)$ as $\Phi \rightarrow g_L \Phi g_R^{-1}$, with the same superpotential (3.19), where J is the same as Eq. (3.16), we obtain

$$(\mathbf{R}^+)^{M^2} \times \frac{SO(2N)_L \times U(M)_R}{SO(2N-2M)_L \times U(M)_V} \xrightarrow{U(M)_R} \frac{SO(2N)_L}{SO(2N-2M)_L \times U(M)_L}.$$

This reduces to $SO(2N)/SO(2N-2) \times U(1)$ when $M = 1$ and to $SO(2N)/U(N)$ when $N = M$. Similarly, $Sp(N)/U(N)$ can also be generalized. By generalizing Φ to $2N \times M$ matrix ($N \geq M$), we obtain

$$(\mathbf{R}^+)^{M^2} \times \frac{Sp(N)_L \times U(M)_R}{Sp(N-M)_L \times U(M)_V} \xrightarrow{U(M)_R} \frac{Sp(N)_L}{Sp(N-M)_L \times U(M)_L}.$$

2. Generalization to many $U(1)$ factors. Remember that the FI parameter c becomes a decay constant, which represents the size of G/H , after integrating out the vector superfield. Then, we can consider that there is a one to one correspondence: the decay constants \leftrightarrow FI-parameters. Hence, to obtain G/H with $H = H_{\text{ss}} \times U(1)^n$ we must prepare n FI-parameters. We, thus, consider a global symmetry, $G' = G \times G_1 \times \cdots \times G_n$, where each G_i includes a $U(1)$ factor. If we gauge all G_i , the gauged Kähler potential has n FI terms. After integrating out vector superfields, we will obtain $G/H' \times G_1 \times \cdots \times G_n = G/H_{\text{ss}} \times U(1)^n$, where H' is the remaining part after embedding all G_i to G . Here we have put $H_{\text{ss}} = H' \times G_{1\text{ss}} \times \cdots \times G_{n\text{ss}}$. In the case of the hermitian symmetric spaces,

we introduced an irreducible representation of G as basic field. It seems that we have to introduce more irreducible representations in these generalizations. Then we have to impose orthogonality relations of these fields by D-term or F-term constraints. At moment, we are unable to find consistent constraints in these cases.

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A BKMU-IKK construction of Kähler potentials of compact homogeneous Kähler manifolds

Bando et.al. gave the general method to construct G invariant Kähler potential of $G^{\mathbb{C}}/\hat{H}$ [5]. However, there remained an ambiguity to choose the projection operators η_i introduced below, Eq. (A.1). Itoh et.al. constructed them explicitly when the target is compact, namely $G^{\mathbb{C}}/\hat{H} \simeq G/H$ [11]. Note that their method does not ensure that such models can be obtained from linear models. In this appendix, we review their method to compare with our method, which, on the other hand, gives a linear origin.

First of all, we need the projection matrices, which project a fundamental representation space onto a \hat{H} invariant subspace [5]. They satisfy the projection conditions,

$$\eta^\dagger = \eta, \quad \eta \hat{H} \eta = \hat{H} \eta, \quad \eta^2 = \eta. \quad (\text{A.1})$$

In the arbitrary Kähler G/H , there exist projection matrices as many as the $U(1)$ factors in H . Since there is only one $U(1)$ factor in the hermitian symmetric cases, there is one projection matrix. In each case, it can be written as [11]

$$\eta = \begin{pmatrix} 1 & \\ & \mathbf{0} \end{pmatrix}. \quad (\text{A.2})$$

By using this, the Kähler potentials of compact Kähler manifolds can be written as [5]

$$K = c \log \det_{\eta} \xi^{\dagger} \xi, \quad (\text{A.3})$$

where ξ is a representative of the complex coset $G^{\mathbf{C}}/\hat{H}$. Since the form of ξ can be calculated as Eqs. (2.11), (2.23), (3.12), (3.29), (3.47) and (3.65), they give the same Kähler potential obtained from linear models in this paper.

B The non-compactness theorem of Lerche and Shore

The nonlinear sigma model, whose target manifold is compact and homogeneous, has the unique Kähler potential as discussed in the last appendix [5, 11]. Although these models do not include neither any QNG boson nor an arbitrariness in the Kähler potential and they are mathematically beautiful, they can not be obtained from any linear model at least when there is no gauge symmetry: it was shown that there exist at least one QNG boson and therefore the target must be non-compact and non-homogeneous. In this appendix, we review the theorem obtained by Lerche and Shore [6]. (See also Refs. [7].)

The linear model origin means that the target manifold can be obtained from some F-term conditions (if there is no gauge symmetry). Since they are holomorphic equations, the invariance under the global symmetry G enlarges to the complexification $G^{\mathbf{C}}$, and the manifold becomes $G^{\mathbf{C}}$ -orbit of the vacuum expectation value v .¹⁸ The pure type multiplets require that the real broken generators must be divided to complex unbroken and complex broken generators, E^i and $\bar{E}_i (= (E^i)^{\dagger})$. Since \bar{E}_i is broken, we obtain

$$0 \neq |\bar{E}_i v|^2 = v^{\dagger} [E^i, \bar{E}_i] v = \alpha(i)^a v^{\dagger} H_a v, \quad (\text{B.1})$$

¹⁸If there are not enough F-term constraints, the manifold may become larger than $G^{\mathbf{C}}$ -orbit. However, the proof is valid also in such cases, since they includes at least one $G^{\mathbf{C}}$ -orbit.

where $\alpha(i)^a$ is a root vector and H_a is a Cartan generator. Therefore, at least one Cartan generator, H_a , must be broken. Since it is hermitian, there exists at least one mixed-type generator, and therefore at least one QNG boson.

C $SO(N)$ algebra

Since the basis of $SO(N)$ used in Subsec. 3.1 are not the standard form, we give a relation to the ordinary basis. The $SO(N)$ generators in ordinary basis are

$$(T_{ij})^k{}_l = \frac{1}{i}(\delta_i^k \delta_{jl} - \delta_j^k \delta_{il}). \quad (\text{C.1})$$

In the basis, the vacuum expectation value satisfying $\vec{v}^2 = 0$ can be written as

$$\vec{v} = \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \mathbf{0} \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (\text{C.2})$$

The real unbroken generators, at the center of the matrix, generate $SO(N-2)$. The complex unbroken and broken generators are

$$E^i = \begin{pmatrix} | & \dots & -\frac{i}{\sqrt{2}} & \dots & | \\ \hline \vdots & & & & \vdots \\ \frac{i}{\sqrt{2}} & & \mathbf{0}_{N-2} & & -\frac{1}{\sqrt{2}} \\ \vdots & & & & \vdots \\ \hline | & \dots & \frac{1}{\sqrt{2}} & \dots & | \end{pmatrix}, \quad \bar{E}_i = \begin{pmatrix} | & \dots & -\frac{i}{\sqrt{2}} & \dots & | \\ \hline \vdots & & & & \vdots \\ \frac{i}{\sqrt{2}} & & \mathbf{0}_{N-2} & & \frac{1}{\sqrt{2}} \\ \vdots & & & & \vdots \\ \hline | & \dots & -\frac{1}{\sqrt{2}} & \dots & | \end{pmatrix}, \quad (\text{C.3})$$

where $i = 1, \dots, N-2$ and only i -th components are nonzero. The broken $U(1)$ generator is

$$T = \begin{pmatrix} | & & -i \\ \hline & \mathbf{0}_{N-1} & \\ \hline i & & \end{pmatrix}. \quad (\text{C.4})$$

This generator will become unbroken after gauging $U(1)_D$. Here, we change basis by an unitary transformation by

$$U = \begin{pmatrix} \frac{i}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ \hline & \mathbf{1}_{N-1} & \\ \hline -\frac{i}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (\text{C.5})$$

Since U is a unitary matrix, $U^\dagger U = U U^\dagger = 1$, $\vec{\phi}^\dagger \vec{\phi}$ is invariant, and then $\log(\vec{\phi}^\dagger \vec{\phi})$. By the unitary transformation, the vacuum expectation value is transformed to the standard form,

$$U\vec{v} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \quad (\text{C.6})$$

The $SO(N-2)$ generators are not transformed and the other generators are transformed as

$$\begin{aligned}
UE^i U^\dagger &= \left(\begin{array}{c|cc|c} & \dots & 1 & \dots \\ \hline & & & \\ \vdots & & & \vdots \\ 0 & & \mathbf{0}_{N-2} & -1 \\ \vdots & & & \vdots \\ \hline & \dots & 0 & \dots \end{array} \right), & U\bar{E}_i U^\dagger &= \left(\begin{array}{c|cc|c} & \dots & 0 & \dots \\ \hline & & & \\ \vdots & & & \vdots \\ 1 & & \mathbf{0}_{N-2} & 0 \\ \vdots & & & \vdots \\ \hline & \dots & -1 & \dots \end{array} \right), \\
UTU^\dagger &= \left(\begin{array}{c|c|c} 1 & & \\ \hline & \mathbf{0}_{N-1} & \\ \hline & & -1 \end{array} \right). \quad (\text{C.7})
\end{aligned}$$

We, thus, obtain the transformation law (3.4) used in Subsec. 3.1. Moreover, the second rank invariant tensor is transformed as $\delta_{ij} \rightarrow (U\delta U^T)_{ij} = J_{ij}$, where J is defined in Eq. (3.3).

D E_6 algebra

In this appendix, we collect the E_6 algebra by referring Refs. [15, 27].

D.1 Construction of E_6 algebra

Since an adjoint representation is decomposed as $\mathbf{78} = (\mathbf{45}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{16}, 1) \oplus (\bar{\mathbf{16}}, -1)$ [26], we construct the E_6 algebra as $\mathcal{E}_6 = \mathcal{SO}(10) \oplus \mathcal{U}(1) \oplus \mathbf{16} \oplus \bar{\mathbf{16}}$: we prepare $SO(10)$ generator T_{AB} , $U(1)$ generator T , $\mathbf{16}$ as E_α and $\bar{\mathbf{16}}$ as $\bar{E}^\alpha = (E_\alpha)^\dagger$. Then their commutation relations can be calculated as follows [15, 27]:

$$[T_{AB}, T_{CD}] = -i(\delta_{BC}T_{AD} + \delta_{AD}T_{BC} - \delta_{AC}T_{BD} - \delta_{BD}T_{AC}), \quad [T, T_{AB}] = 0,$$

$$\begin{aligned}
[T_{AB}, E_\alpha] &= -(\sigma_{AB})_\alpha{}^\beta E_\beta, & [T_{AB}, \bar{E}^\alpha] &= (\sigma_{AB}^*)^\alpha{}_\beta \bar{E}^\beta, \\
[T, E_\alpha] &= \frac{\sqrt{3}}{2} E_\alpha, & [T, \bar{E}_\alpha] &= -\frac{\sqrt{3}}{2} \bar{E}_\alpha, \\
[E_\alpha, E_\beta] &= [\bar{E}^\alpha, \bar{E}^\beta] = 0, & [E_\alpha, \bar{E}^\beta] &= -\frac{1}{2}(\sigma_{AB})_\alpha{}^\beta T_{AB} + \frac{\sqrt{3}}{2} \delta_\alpha{}^\beta T. \quad (D.1)
\end{aligned}$$

The $U(1)$ charge of E_α is determined by the difference of $U(1)$ charges of x and y or y and z in Eq. (3.32): $\frac{2}{\sqrt{3}} - \frac{1}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} - (-\frac{1}{\sqrt{3}}) = \frac{\sqrt{3}}{2}$. The second coefficient of the last equation is the same value as $U(1)$ charge of E_α from the anti-symmetric property of the structure constants. The relative weight of the first and the second term is determined by using the Jacobi identity, $[\bar{E}, [E, E]] + (\text{cyclic}) = 0$, and the nontrivial identity for the spinor generators [15, 27],

$$\Sigma_{AB}(\sigma_{AB})_\alpha{}^{[\beta}(\sigma_{AB})_{\gamma}{}^{\delta]} = \frac{3}{2}\delta_\alpha{}^{[\beta}\delta_\gamma{}^{\delta]}. \quad (D.2)$$

D.2 Invariant tensor of E_6

By a tensor product [26], $\mathbf{27} \otimes \mathbf{27} = \bar{\mathbf{27}}_s \oplus \dots$, there exist rank-3 symmetric tensor invariant under E_6 . The components of Γ_{ijk} are [27]

$$\Gamma_{ijk} = \begin{cases} \Gamma_{0AB} = \delta_{AB} \\ \Gamma_{A\alpha\beta} = \frac{1}{\sqrt{2}}(C\sigma_A^\dagger)^{\alpha\beta} \\ \text{otherwise } 0 \end{cases}. \quad (D.3)$$

These components can be calculated as follows. First of all, construct the $SO(10) \times U(1)$ invariant of order three:

$$I_3 = Axz^2 + \frac{1}{\sqrt{2}}z^A y_\alpha (C\sigma_A^\dagger)^{\alpha\beta} y_\beta. \quad (D.4)$$

By the requirement of invariance of E or \bar{E} , we can conclude $A = 1$. Here we have used the identity (3.42). The components (D.3) can be read from this invariant.

It is known that there is an identity [28]¹⁹

$$\Gamma_{ijk}\Gamma^{ijl} = 10\delta_i^l. \quad (D.5)$$

¹⁹In the calculation of $\Gamma_{ijA}\Gamma^{ijB} = 10\delta_{AB}$, we have used the identity $2^{-4}\text{tr}(C\sigma_A^\dagger\sigma_B C) = \delta_{AB}$ [27].

Under the normalization in Eq. (D.5), there is the Springer relation [28]

$$\Gamma_{ijk}(\Gamma^{jl\{m\Gamma^{no\}k}) = \delta_i^{\{l\Gamma^{mno\}}, \quad (\text{D.6})$$

where we have used a notation $A^{\{ij\cdots\}} = A^{ij\cdots} + A^{ji\cdots} + \cdots$. These identities are used many times in the analysis of E_7 algebra.

E E_7 algebra

In this appendix, we construct the E_7 algebra by the same way in the last appendix.

E.1 Construction of E_7 algebra

The decomposition of the adjoint representation of E_7 under a maximal subgroup $E_6 \times U(1)$ is $\mathbf{133} = (\mathbf{78}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{27}, 1) \oplus (\bar{\mathbf{27}}, -1)$, where second components are $U(1)$ charge [26]. Hence, we can construct E_7 algebra by adding generators E^i and $\bar{E}_i (= (E^i)^\dagger)$ ($i = 1, \dots, 27$), which belong to E_6 fundamental and anti-fundamental representations respectively, to the $E_6 \times U(1)$ algebra, T_A ($A = 1, \dots, 78$) and T : $\mathcal{E}_7 = \mathcal{E}_6 \oplus \mathcal{U}(1) \oplus \mathbf{27} \oplus \bar{\mathbf{27}}$. By the same manner as we have constructed E_6 algebra in the last appendix, their commutation relations are obtained as follows:

$$\begin{aligned} [T_A, T_B] &= if_{AB}{}^C T_C, & [T, T_A] &= 0 \\ [T_A, E^i] &= \rho(T_A)^i{}_j E^j, & [T_A, \bar{E}_i] &= -\rho(T_A)^T{}_i{}^j \bar{E}_j \\ [T, E^i] &= \sqrt{\frac{2}{3}} E^i, & [T, \bar{E}_i] &= -\sqrt{\frac{2}{3}} \bar{E}_i \\ [E^i, E^j] &= [\bar{E}_i, \bar{E}_j] = 0, & [E^i, \bar{E}_j] &= \rho(T_A)^i{}_j T_A + \sqrt{\frac{2}{3}} \delta^i{}_j T. \end{aligned} \quad (\text{E.1})$$

Here $\rho(T_A)$ is a fundamental representation matrix and $f_{AB}{}^C$ are structure constants of E_6 , whose explicit forms have been obtained in the last section. The $U(1)$ charge of E^i is decided from the difference of x and y^i etc. in Eq. (3.50) and \bar{E}_i as its conjugate. In the last equation, the coefficient of the second term coincides with the $U(1)$ charge of E^i due to the anti-symmetry of the structure constants of E_7 .

And the first term has been decided by the Jacobi identity, $[\bar{E}, [E, E]] + (\text{cyclic}) = 0$ and the nontrivial identity for the E_6 fundamental representation [29],

$$\Sigma_A \rho(T_A)^{[i}_j \rho(T_A)^{k]}_l = -\frac{2}{3} \delta^{[i}_j \delta^{k]}_l. \quad (\text{E.2})$$

This is satisfied under the notation $\text{tr} \rho(T_A)^2 = 6$.

E.2 Invariant tensors of E_7

From the tensor product of fundamental representations [26], $\mathbf{56} \otimes \mathbf{56} = \mathbf{1}_a \oplus \dots$ and $\mathbf{56} \otimes \mathbf{56} \otimes \mathbf{56} \otimes \mathbf{56} = \mathbf{1}_s \oplus \dots$, there exist the rank-2 anti-symmetric tensor $f_{\alpha\beta}$ and the rank-4 symmetric tensor $d_{\alpha\beta\gamma\delta}$ as E_7 invariant tensors. To find their components, we construct a linear combination of $E_6 \times U(1)$ invariants of quartic order and require the invariance under E or \bar{E} as the last appendix. The result is

$$\begin{aligned} I_4 &= d_{\alpha\beta\gamma\delta} \phi^\alpha \phi^\beta \phi^\gamma \phi^\delta \\ &= -\frac{1}{2} (xw - y^i z_i)^2 - \frac{1}{3} w \Gamma_{ijk} y^i y^j y^k - \frac{1}{3} x \Gamma^{ijk} z_i z_j z_k \\ &\quad + \frac{1}{2} \Gamma^{ijk} \Gamma_{ilm} z_j z_k y^l y^m. \end{aligned} \quad (\text{E.3})$$

Here, I_4 is invariant due to the Springer relation for E_6 invariant tensor, Eq. (D.6). The components can be read from this invariant. Since we do not use the anti-symmetric tensor $f_{\alpha\beta}$, we do not construct it.

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