WIGGLERS

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Abstract
The main effects of wigglers on the operation of electron storage rings are discussed, including the closed orbit, linear focusing effects and matching conditions, non-linear beam dynamics, and effects on beam properties arising from the emission of synchrotron radiation.

1 INTRODUCTION

A wiggler is a special magnetic device consisting of a sequence of dipoles with alternating polarity, arranged so that there is no net deflection or displacement of the beam. Such a device in general does not have a major influence on the ring optics and so can be added later and operated almost independently of the rest of the ring in order to enhance particular features of the operation of the ring.

There are several reasons for installing wigglers in electron rings. Firstly, wigglers can be useful as a means of changing various electron beam properties. They contribute to both radiation damping and quantum excitation processes, leading to a new equilibrium with modified bunch length, energy spread, and emittance (larger or smaller) [1]. Several different types of wiggler have been proposed with different field distributions. The basic dipole wiggler, also called a "damping wiggler", gives rise principally to changes in equilibrium emittance, energy spread and damping times [2]. Use of a gradient field leads mainly to a change of damping partition, and was first proposed by Robinson as a means of counteracting radial antidamping in combined function machines [3]. A device of this kind was first used in the CEA in order to allow operation as a storage ring [4]. Finally, non-linear wigglers have been proposed with higher order fields such as sextupole and octupole in order to modify the bunch shape and energy distribution [5-7].

A second use of wigglers is to control electron spin polarization [2]. Due to the emission of Synchrotron Radiation electrons tend to become polarized with their spins aligned in the vertical direction (Sokolov-Ternov effect) [8]. Wigglers can be used to increase the level of polarization, which is of interest for particular high energy physics experiments.

The major use of wigglers however is as specialized sources of synchrotron radiation, in which context they are generally referred to as "insertion devices" (IDs). At present more than fifty IDs are in operation in synchrotron radiation sources world-wide and many more are being built for the most recent "third generation" rings, the largest of which will eventually contain between 30 and 40 IDs. Compared to the radiation emitted by the more conventional bending magnet sources, insertion devices have the following advantages:

i) Higher photon energies. The critical energy of the radiation emitted by the wiggler depends on the magnetic field strength, which can be greater than that of the ring bending magnets. The first device to be proposed for this application consisted of a single high field strength pole with half-strength poles of opposite polarity on either side in order to produce a single bump of the electron trajectory. It was termed a "wavelength shifter" because of its harder spectrum compared to the bending magnet radiation [9]. Since then in order to reach the highest possible magnetic field superconducting devices have been developed. For example a 3.5 T device was installed on VEPP3 in 1979 [10], and a 5 T wiggler on the SRS in 1982 [11]. Superconducting wigglers are currently operating in several synchrotron radiation facilities.
SRS (England), UVSOR and Photon Factory (Japan), NSLS X-ray Ring (U.S.A.) and DCI (France).

ii) Increased photon flux. Using many emitting poles the total photon flux increases proportionally. The first "multipole wiggler" devices were electromagnetic. For example, a 1.8 T, 5-pole device was installed in SPEAR (U.S.A.) in 1979 [12]. Since then permanent magnet devices have been developed with many more poles, for example the 55-pole device installed in SPEAR in 1983 [13].

iii) Increased brightness (photon flux per unit area and solid angle) and quasi-monochromatic spectrum. Under certain circumstances, particularly when the wiggler has a small field strength and period length, interference effects can become important which produces a radiation spectrum consisting of a series of lines, rather than the smooth spectrum characteristic of bending magnet sources. The radiation also has a higher degree of collimation and hence "brightness". The name "undulator" is often used for devices which make use of this interference effect, although there is no essential difference to a multipole wiggler. An undulator was first proposed [14], and later used, by Motz to generate millimetre and visible radiation with electron energies of 5 MeV and 100 MeV respectively [15]. It also formed the basis of the subsequent development of coherent radiation sources, such as the ubitron [16] and free-electron laser [17]. The first use of an undulator in a storage ring for generation of synchrotron radiation was at SSRL in 1980 [18]. A parallel development took place using the VEPP-3 storage ring at Novosibirsk [19]. A device of this kind has also been used in the SPS to generate visible light as a diagnostic tool for the proton and anti-proton beams [20].

iv) Different polarization characteristics. There is increasing interest in using synchrotron radiation with circular or elliptical polarization, which can be generated by wigglers of the "asymmetric" type, or devices with helical or elliptical electron trajectories [21].

In this lecture we consider the main effects of standard wigglers, i.e. "plane" wigglers with midplane symmetry, on the operation of electron storage rings. We examine firstly the electron trajectory in the device and the effects on the closed orbit, then the effects on linear and non-linear beam dynamics, and finally the effects on beam properties arising from the emission of synchrotron radiation.

2 TRAJECTORY AND CLOSED ORBIT

The equation of motion in the standard fixed co-ordinate system \{z, x, s\} [22] in the case of small angular deflection \((x' \ll 1, z' \ll 1)\) is given by:

\[
\begin{align*}
x'' &= \frac{e}{\gamma mc} (B_z - z' B_x) \\
z'' &= \frac{e}{\gamma mc} (x' B_z - B_x)
\end{align*}
\]  

(1)

In the median plane only the vertical field component, \(B_z\), exists and so the horizontal motion can therefore be obtained by simple integration:

\[
\begin{align*}
x'(s) &= \frac{e}{\gamma mc} \int ds' B_z(s') \\
x(s) &= \frac{e}{\gamma mc} \int ds' \int ds'' B_z(s'')
\end{align*}
\]  

(2)
In the periodic part of a wiggler the field is in many cases well described by a sinusoidal distribution: \( B_z = B_z \cos(k_z s) \), where \( k_z = 2\pi/\lambda_o \) and \( \lambda_o \) is the wiggler period length. The trajectory in the periodic part then becomes:

\[
x'(s) = \frac{\sin(k_z s)}{\rho_0 k_z} \quad x(s) = -\frac{\cos(k_z s)}{\rho_0 k_z^2}
\]

where \( 1/\rho_0 = eB_z/\gamma mc \).

![Diagram](image)

Fig. 1 Projection of the change in beam position and angle introduced by a wiggler to the centre of the device.

In considering the net change in angle (\( \phi \)) and position (\( \delta \)) introduced by the device it is convenient to calculate these quantities at the centre of the device \( (s = 0) \) rather than at the end (see Fig. 1) i.e. \( \phi = x'(L/2) \) and \( \delta = x(L/2) - x'(L/2)L/2 \). We have therefore, after an integration by parts for \( x(s) \):

\[
\phi = \frac{e}{\gamma mc} \int_{-L/2}^{L/2} B_z \, ds \\
\delta = -\frac{e}{\gamma mc} \int_{-L/2}^{L/2} sB_z \, ds
\]

The usual requirement is for zero net change in angle or position of the beam and therefore \( \phi = \delta = 0 \). The most common arrangement employs a field distribution that is symmetric about the centre of the device i.e. an odd number of poles. In the ideal case, neglecting field errors, it follows directly from Eq. (4) that \( \delta = 0 \). Zero angular deflection is obtained by a suitable arrangement of end-poles, for example using half-strength poles at the ends. Figure 2(a) illustrates the sequence of pole strengths \( (1, -2, 2, -2, 1) \) and the resulting trajectory for poles with a rectangular shaped field distribution; this and the following results are however valid also for the more common case of a half-sinusoid field shape. A more complicated sequence \( (1, -3, 4, -4, -3, 1) \) that eliminates the offset of the oscillation axis is illustrated in Fig. 2(b). Anti-symmetric arrangements with an even number of poles are also possible. In this case by symmetry \( \phi = 0 \) and zero change in position is obtained by a suitable end pole arrangement. One general sequence is that of Fig. 2(c), \( (1, -3, 4, -4, -3, -1) \) which is similar to that of Fig. 2(b).

So far we have referred only to the changes in angle and position of an electron trajectory, which enters with zero position and angle. When placed in a ring however residual errors in either plane will give rise to a distortion of the closed orbit. In the case of a localized angular deflection the change to the closed orbit is well known:


\[ u(s) = \frac{\phi}{2} \sqrt{\beta(s)\beta_w} \frac{\cos(\pi Q - \mu(s))}{\sin \pi Q} \]

\[ u'(s) = \frac{\phi}{2} \sqrt{\frac{\beta_w}{\beta(s)}} \frac{\sin(\pi Q - \mu(s)) - \alpha \cos(\pi Q - \mu(s))}{\sin \pi Q} \]

(5)

Fig. 2 Various magnetic field distributions (solid lines) resulting in no net change in position or angle of the beam trajectory (dashed lines).

In the above \( W \) refers to the wiggler location, and \( u \) refers to either \( x \) or \( z \). The invariant for such a closed orbit, \( \varepsilon_{c.o.} = \gamma u^2 + 2\alpha u' + \beta u'^2 \), is as follows:

\[ \varepsilon_{c.o.} = \beta_w \left( \frac{\phi}{2 \sin \pi Q} \right)^2 \]

(6)

Similarly for a localized displacement, \( \delta \), the closed orbit is given by:

\[ u(s) = \frac{\delta}{2} \sqrt{\frac{\beta(s)}{\beta_w}} \frac{\sin(\pi Q - \mu(s)) + \alpha \cos(\pi Q - \mu(s))}{\sin \pi Q} \]

\[ u'(s) = -\frac{\delta}{2\sqrt{\beta_w \beta(s)}} \frac{[1 + \alpha_w \alpha(s)]\cos(\pi Q - \mu(s)) - [\alpha_w - \alpha(s)]\sin(\pi Q - \mu(s))}{\sin \pi Q} \]

(7)

and hence the invariant is:

\[ \varepsilon_{c.o.} = \gamma_w \left( \frac{\delta}{2 \sin \pi Q} \right)^2 \]

(8)

Thus, while the sensitivity to angle errors depends on the \( \beta \) function at the wiggler, the effect of displacement errors depends on the \( \gamma \) function and may therefore be an important effect for wigglers located at positions with a small beta function. In most cases however \( \beta > 1 \) and so an anti-symmetric structure with an even number of poles is sometimes used to reduce the effect on the closed orbit arising from non-perfect field integral compensation.

In synchrotron radiation sources it is usually required that the beam position and angle is stable to some fraction of its natural rms size and divergence in order to maintain the high
brightness of the synchrotron radiation beams. In other words it is required that $\varepsilon_{e,0} < \varepsilon$, where the latter is the natural electron beam emittance in the relevant plane. This condition must be met also during switch-on or tuning of a wiggler field if other users are not to be disturbed. For example, with an emittance of $\varepsilon = 10^{-9}$ m rad, typical of a modern synchrotron radiation source, and with $Q = 0.25$, $\beta_W = 10$ m ($\alpha_W = 0$), this condition requires that the perturbations introduced by a wiggler are limited to the following values: $\phi < 14$ $\mu$rad and $\delta < 140$ $\mu$m. At 2 GeV for example this corresponds to field integrals, Eq. (4), of less than $10^{-4}$ Tm and $10^{-3}$ Tm$^2$ respectively. Such small field integrals are difficult to achieve in practice without active correction elements either forming part of the wiggler, or external to it. It should be noted that the limits on the field integrals decreases with energy, making wiggler operation in low energy rings even more critical.

3 LINEAR BEAM DYNAMICS

We consider firstly the strongest effect that occurs in a standard wiggler, namely the second-order focusing in the vertical plane. The analysis is extended in section 3.2 to include focusing in the horizontal plane arising from transverse field variations and in section 3.3 a more general approach is presented that is valid also in the case of more complex wigglers which lack midplane symmetry. The effects of the focusing are examined in section 3.4 and ways of dealing with them in section 3.5.

3.1 Vertical plane focusing

A simple approach to the linear dynamics of particles in wigglers is a "hard-edge model" in which each pole is treated as a region of constant magnetic field, and hence bending radius ($\rho$), as shown in Fig. 3 [9,23,24]. Each pole can then be modelled using the standard transport matrix for a dipole magnet, which can be multiplied together to form the matrix for a complete wiggler. In the usual case of magnet poles arranged with parallel entrance and exit faces, i.e. rectangular magnets, the beam enters and leaves each pole with a non-zero angle ($\phi$) that gives rise to edge focusing. The edge effect cancels the weak focusing term in the horizontal plane leaving no net effect, but in the vertical plane there remains a net focusing which adds up from each pole edge and which is proportional to $\phi/\rho$.

![Fig. 3 Hard-edge model for a wiggler with regions of constant magnetic field](image)

The physical origin of the vertical focusing can be understood more clearly from Fig. 4. It can be seen that off-axis vertically there is a longitudinal $B_z$ field component that couples to the angular deflection in the horizontal plane ($\chi'$) to give a force which acts always towards the beam axis. Near the median plane $B_z = (dB_y / dz) z = (dB_z / ds) z$, which inserted in Eq. (1) together with the expression for $\chi'(s)$, Eq. (2), gives:
\[ z'' = \left( \frac{e}{\gamma mc} \right)^2 \left( \int B_z \frac{dB_z}{ds} \right) z \]  

(9)

Averaging over the wiggler length using integration by parts, the result becomes:

\[ \int z'' ds = -z \left( \frac{e}{\gamma mc} \right)^2 \int B_z^2 ds = -z \int \frac{1}{\rho^2} ds \]  

(10)

i.e. in the vertical plane the wiggler behaves as a quadrupole whose average focusing parameter is given by \( K_z = \langle 1/\rho^2 \rangle \).

![Field lines and beam trajectory in a wiggler](image)

Fig. 4 Field lines and beam trajectory in a wiggler

A third approach is to consider the motion with respect to the oscillating trajectory in the device. We define the local field gradient (see Fig. 5):

\[ \frac{dB_z}{d\xi} = -\frac{d}{ds} \chi' \]  

(11)

where \( \chi' \) is the angle of the reference trajectory and where we have assumed \( dB_z/dx = 0 \). The usual focusing parameter is then:

\[ k = \frac{e}{\gamma mc} \frac{dB_z}{d\xi} = -\frac{e}{\gamma mc} \frac{dB_z}{ds} \chi' \]  

(12)

Averaging over the wiggler length as above we obtain effectively the same result as before:

\[ \int k ds = \left( \frac{e}{\gamma mc} \right)^2 \int B_z^2 ds = \int \frac{1}{\rho^2} ds \]  

(13)

The standard linear equations of motion are therefore:

\[ x'' + K_z x = 0 \]  

\[ z'' + K_z z = 0 \]  

(14)

with

\[ K_z = \langle (1/\rho^2 - k) \rangle = 0 \]  

\[ K_z = \langle k \rangle = \langle 1/\rho^2 \rangle \]  

(15)
Fig. 5 Definition of the local co-ordinate perpendicular to the beam trajectory

It can be seen from the results above that the standard wiggler behaves like a drift space in the horizontal plane and as a focusing quadrupole magnet in the vertical plane. It should be noted that the focusing is a second-order effect being proportional to the square of the field and inversely proportional to the square of the energy. In the case of a sinusoidal wiggler with amplitude $B_0$ the focusing parameter, averaged over each magnet period, becomes:

$$K_x = \left( \frac{e}{\gamma mc} \right)^2 \frac{B_0^2}{2} \frac{1}{2\rho_0^2}$$

(16)

The averaging can however be carried out over the whole magnet, if the integration is carried out between points with zero field. The integrated focusing parameter is therefore $K_sL = \int 1/\rho^2 ds$ for any arbitrary field distribution $B_z(s)$. The only restrictions on the validity are that the magnet has midplane symmetry and no transverse field variation.

3.2 Horizontal plane focusing

In the previous section we have assumed that there is no transverse variation (i.e. in the $x$ direction) of the $B_z$ field component. However such a variation can arise due to the finite dimensions of the magnet poles, or can even be introduced deliberately by curving the pole faces. With such an alternating quadratic field variation, the beam oscillating from side to side sees an effective quadrupole field (see Fig. 6). In the case where the field amplitude increases away from the axis this gives rise to focusing in the horizontal plane and defocusing in the vertical plane.

Fig. 6 Field gradient arising from a quadratic transverse field variation and the oscillating beam trajectory
Extending the analysis of section 3.1, the local gradient becomes:

\[ \frac{dB_z}{dx}\bigg|_x = \frac{dB_z}{ds}\bigg|_s x' \quad (17) \]

In the above the subscript indicates that the derivatives are to be evaluated on the electron trajectory \( x(s) \). Assuming no linear field gradient at the origin we can approximate the above as follows:

\[ \frac{dB_z}{dx^2}\bigg|_o = \frac{d^2B_z}{ds^2}\bigg|_o x' \quad (18) \]

Using Eq. (2) for the trajectory, and integrating by parts, assuming \( x'(\infty) = x(\infty) = 0 \), we obtain:

\[ \int k \, ds = \frac{e}{\gamma mc} \int \frac{dB_z}{d\xi} = \left( \frac{e}{\gamma mc} \right)^2 \frac{1}{2} \frac{d^2}{dx^2} \left( \int B_z ds \right)^2 ds \quad (19) \]

The integrated focusing parameters for the horizontal and vertical plane therefore become:

\[ \int K_x \, ds = \int (1/\rho^2 - k) \, ds = \left( \frac{e}{\gamma mc} \right)^2 \frac{1}{2} \frac{d^2}{dx^2} \left( \int B_z ds \right)^2 ds \quad (20) \]

\[ \int K_z \, ds = \int k \, ds = \left( \frac{e}{\gamma mc} \right)^2 \frac{1}{2} \frac{d^2}{dz^2} \left( \int B_z ds \right)^2 ds \]

where use has been made of the relationship:

\[ \frac{1}{2} \frac{d^2}{dx^2} \left( \int B_z ds \right)^2 ds + \frac{1}{2} \frac{d^2}{dz^2} \left( \int B_z ds \right)^2 ds = \int B_z^2 ds \quad (21) \]

In the case of a sinusoidal field distribution, the complete expression for the field components near the axis including the transverse variation is as follows:

\[
\begin{align*}
B_z &= B_o \cosh(k_o x) \cosh(k_o z) \cos(k_o s) \\
B_x &= -k_x/k_o B_o \sinh(k_o x) \sinh(k_o z) \cos(k_o s) \\
B_y &= -k_y/k_o B_o \cosh(k_o x) \sinh(k_o z) \sin(k_o s)
\end{align*} \quad (22)
\]

where, \( k_x^2 + k_y^2 = k_o^2 = (2\pi/\lambda_o)^2 \). Evaluating Eq. 20 in this case, the average focusing parameters are as follows:

\[ \langle K_x \rangle = \langle 1/\rho^2 - k \rangle = \frac{k_x^2}{2\rho^2 k_o^2} \quad (23) \]

\[ \langle K_z \rangle = \langle k \rangle = \frac{k_z^2}{2\rho_o^2 k_o^2} \]

With the given constraint of midplane symmetry it follows directly from Eq. (14) that the sum of the two focusing parameters depends only on the square of the vertical field component:
\[ \langle K_x \rangle + \langle K_z \rangle = \langle 1/\rho^2 \rangle \]  

(24)

In the usual case with finite pole width \( k_x^2 \) is negative; the \( \cosh(k_x x) \) terms in Eq. 22 are then replaced by \( \cos(k_x x) \). This gives rise to a defocusing in the horizontal plane and an increase in the vertical focusing. With pole faces curved so as to increase the field away from the axis, there is focusing in both planes. Such a situation is of particular interest in free-electron laser applications [25].

3.3 General approach to second-order effects

In the case of wigglers with an arbitrary field distribution the analysis of section 3.2 is no longer valid and a more general approach is needed [26]. Starting from Eq. (1) we can write the first-order solution using the on-axis field components directly as follows:

\[
\begin{align*}
x'_o(s) &= \frac{e}{\gamma mc} \int ds B_z, \quad x_o(s) = \frac{e}{\gamma mc} \int ds B_z ds \\
z'_o(s) &= -\frac{e}{\gamma mc} \int ds B_x, \quad z_o(s) = -\frac{e}{\gamma mc} \int ds B_x ds
\end{align*}
\]

(25)

The second-order solution can be obtained by substituting in Eq. (1) and expanding the field components about the trajectory \((x_o, z_o)\):

\[
\begin{align*}
x'' &= \frac{e}{\gamma mc} (x_o \frac{dB_z}{dx} + z_o \frac{dB_x}{dz} - z_o' B_z) \\
z'' &= \frac{e}{\gamma mc} (x_o' B_z - x_o \frac{dB_z}{dx} - z_o \frac{dB_x}{dz})
\end{align*}
\]

(26)

Integration then yields the second-order angular deflections. After integration by parts and some simplification, assuming \( x_o(\infty) = z_o(\infty) = 0 \), we obtain:

\[
\begin{align*}
\Delta x' &= -\frac{1}{2} \left( \frac{e}{\gamma mc} \right)^2 \frac{d}{dx} \left\{ \left( \int B_z ds \right)^2 \right\} ds \\
\Delta z' &= \frac{1}{2} \left( \frac{e}{\gamma mc} \right)^2 \frac{d}{dz} \left\{ \left( \int B_z ds \right)^2 \right\} ds
\end{align*}
\]

(27)

In some cases the magnet geometry may lead to second-order deflections, which can be calculated directly using the above formulae. Taking the first derivatives with respect to \( x \) and \( z \) yields the focusing parameters:

\[
\begin{align*}
\int K_x \, ds &= \frac{1}{2} \left( \frac{e}{\gamma mc} \right)^2 \frac{d^2}{dx^2} \left\{ \left( \int B_z ds \right)^2 \right\} ds \\
\int K_z \, ds &= \frac{1}{2} \left( \frac{e}{\gamma mc} \right)^2 \frac{d^2}{dz^2} \left\{ \left( \int B_z ds \right)^2 \right\} ds
\end{align*}
\]

(28)

Further differentiation yields the integrated non-linear terms.
3.4 Linear effects in a storage ring

A general focusing element can be represented by the following standard matrix:

\[
M = \begin{pmatrix}
\cos \theta & \beta \sin \theta \\
-\sin \theta/\beta & \cos \theta
\end{pmatrix}
\]

(29)

where \( \theta = \sqrt{K} L \), \( \beta = L/\theta = 1/\sqrt{K} \) and where \( K \) represents either \( K_z \) or \( K_x \). Such a matrix can be used directly in any of the standard programs that calculate the linear properties of a magnet lattice. To proceed further analytically it is preferable to use the effective "thin-lens" (i.e. zero length) matrix, \( M_{\text{eff}} \), that acts at the centre of the device which can be obtained as follows [24]:

\[
M_{\text{eff}} = \begin{pmatrix}
1 & -L/2 \\
0 & 1
\end{pmatrix}
< 
\begin{pmatrix}
\cos \theta & \beta \sin \theta \\
-\sin \theta/\beta & \cos \theta
\end{pmatrix}
< 
\begin{pmatrix}
1 & -L/2 \\
0 & 1
\end{pmatrix}
\]

(30)

After carrying out the multiplication the result can be expressed in a similar form to that of matrix \( M \), i.e.

\[
M_{\text{eff}} = \begin{pmatrix}
\cos \theta^* & \beta^* \sin \theta^* \\
-\sin \theta^*/\beta^* & \cos \theta^*
\end{pmatrix}
\]

(31)

where,

\[
\cos \theta^* = \cos \theta + \frac{\theta \sin \theta}{2}
\]

(32)

and

\[
\beta^* = L \left( \frac{1}{\theta^2} - \frac{\cot \theta}{\theta} - \frac{1}{4} \right)^{1/2}
\]

(33)

Figure 7 shows the variation of \( \beta^* \) with bending radius for two different lengths of a standard wiggler. It can be seen that at large \( \rho \) it tends to a constant value, given by \( L/\sqrt{12} \). As \( \rho \) reduces there is only a small increase until the point is reached when \( \theta \) approaches a value of \( \pi \). At this limit the wiggler matrix equals the negative of the identity matrix, and \( \beta^* \) goes to infinity.

![Variation of \( \beta^* \) function with bending radius, for two wiggler lengths](image)

Fig. 7 Variation of \( \beta^* \) function with bending radius, for two wiggler lengths

The effect on the tune can be found in the standard way by calculating the modified one-turn matrix, by multiplying the unperturbed one-turn matrix by \( M_{\text{eff}} \). The resulting expression for the modified phase advance with the wiggler activated, \( \mu_W \), becomes in the general case:
\[ \cos \mu_w = \cos \mu \cos \theta^* - \sin \mu \sin \theta^* \left( \gamma \beta^* + \frac{\beta}{\beta^*} \right) \]  

(34)

where \( \mu \) is the one-turn phase advance in the absence of the wiggler. If the wiggler is located at a symmetry point the result can be written as follows:

\[ \cos \mu_w = \cos(\mu + \theta^*) - \sin \mu \sin \theta^* \left( \beta^* - \frac{(\beta^* - \beta)^2}{2\beta \beta^*} \right) \]  

(35)

A second effect of the wiggler is to modify the betatron function. For simplicity we consider only the case where the wiggler is located at a symmetry point and that the ring has at least two-fold symmetry. We then evaluate the modified one-turn matrix starting from the symmetry point on the opposite side of the ring:

\[
\begin{pmatrix}
\cos \mu/2 & \beta \sin \mu/2 \\
-\gamma \sin \mu/2 & \cos \mu/2
\end{pmatrix}
\begin{pmatrix}
\cos \theta^* & \beta^* \sin \theta^* \\
-\sin \theta^*/\beta^* & \cos \theta^*
\end{pmatrix}
\begin{pmatrix}
\cos \mu/2 & \beta \sin \mu/2 \\
-\gamma \sin \mu/2 & \cos \mu/2
\end{pmatrix}
\]

(36)

Evaluating the element \( m_{12} \) of the above matrix we obtain the result:

\[ \beta_w \sin \mu_w = \beta \sin \mu \cos \theta^* + (\beta^* \cos^2 \mu/2 - \beta^2 \sin^2 \mu/2) \sin \theta^*/\beta^* \]

(37)

where \( \beta_w \) is the modified \( \beta \) value at the point opposite the wiggler. The beta values at other points around the ring can be found from:

\[ \frac{\Delta \beta(s)}{\beta(s)} = \frac{(\beta_w - \beta)}{\beta} \left[ 1 - \left( \sin^2 \mu(s) \left( 1 + \frac{\beta}{\beta_w} \right) \right) \right] \]

(38)

from which it follows that the maximum relative change in betatron function at any point is given by:

\[ \left( \frac{\Delta \beta(s)}{\beta(s)} \right)_{\text{max}} = \frac{(\beta_w - \beta)}{\beta}, \quad \text{if } \beta_w > \beta \]

(39)

\[ = \frac{(\beta - \beta_w)}{\beta_w}, \quad \text{if } \beta > \beta_w \]

A third consequence of the wiggler is that a stopband is produced. The stopband is defined as the interval of \( \mu \) for which \( |\cos \mu_w| > 1 \). The boundary points of the stopband can be calculated from the following [24]:

\[ \tan \left( \frac{\mu}{2} \right) = -\tan \left( \frac{\theta^*}{2} \right) \left[ h \pm \sqrt{h^2 - 1} \right] \]

(40)

where \( h = (\beta^2 + \beta^2) / 2\beta \beta^* \).

Provided the wiggler strength is not too large \( (\theta \leq 0.5) \) the following approximations can be made: \( \cos \theta^* = 1, \sin \theta^* = \theta^*/\sqrt{12}, \beta^* = L/\sqrt{12} \). The thin-lens wiggler matrix then becomes:
\[ M_{\text{eff}} = \begin{pmatrix} 1 & \theta^2 L/12 \\ -\theta^2/L & 1 \end{pmatrix} \] (41)

The tune shift \((\Delta Q = \Delta \mu / 2\pi)\) is given by:

\[ \Delta Q = \frac{KL\beta}{4\pi} \left( 1 + \frac{L^2}{12\beta^2} \right) \] (42)

the change in the betatron function by:

\[ \frac{\Delta \beta}{\beta} = -\frac{KL\beta}{2\sin\mu} \left( 1 - \frac{L^2}{12\beta^2} \right) \] (43)

and the tune stopband width is given by:

\[ \Delta Q = \frac{KL\beta}{2\pi} \left( 1 - \frac{L^2}{12\beta^2} \right) \] (44)

Note that in the limit \(L \to 0\) the standard expressions for a localized gradient error of magnitude \(KL\) are recovered. The effects in this small \(\theta\) limit are thus clearly proportional to \(K_{xz}\) and hence \((B_0/E)^2\). Figure 8 shows the measured vertical tune shift introduced by the SRS 5-T wiggler, which is in very good agreement with the calculated values, and clearly demonstrates the quadratic dependence on field strength [11].

**Table 1**

Tune shift, stopband width and maximum change in beta function produced in the vertical plane by various wigglers at different energies, assuming a residual tune value of 0.1, and \(\beta_x = 10\) m.

<table>
<thead>
<tr>
<th>Device</th>
<th>(E) [GeV]</th>
<th>(\theta)</th>
<th>(\beta^*)</th>
<th>(\Delta Q)</th>
<th>(\Delta Q_{\text{stopband}})</th>
<th>((\Delta \beta/\beta)_{\text{max}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1</td>
<td>0.8</td>
<td>0.33</td>
<td>0.73</td>
<td>3.1 \times 10^{-2}</td>
<td>6.7 \times 10^{-2}</td>
<td>3.3 \times 10^{-1}</td>
</tr>
<tr>
<td>W1</td>
<td>2.0</td>
<td>0.13</td>
<td>0.72</td>
<td>5.5 \times 10^{-3}</td>
<td>1.1 \times 10^{-2}</td>
<td>5.8 \times 10^{-2}</td>
</tr>
<tr>
<td>W1</td>
<td>6.0</td>
<td>0.04</td>
<td>0.72</td>
<td>6.2 \times 10^{-4}</td>
<td>1.2 \times 10^{-3}</td>
<td>6.6 \times 10^{-3}</td>
</tr>
<tr>
<td>W1</td>
<td>20.0</td>
<td>0.01</td>
<td>0.72</td>
<td>6.0 \times 10^{-5}</td>
<td>1.0 \times 10^{-4}</td>
<td>5.9 \times 10^{-4}</td>
</tr>
<tr>
<td>W2</td>
<td>0.8</td>
<td>0.99</td>
<td>0.82</td>
<td>1.8 \times 10^{-1}</td>
<td>3.3 \times 10^{-1}</td>
<td>2.7</td>
</tr>
<tr>
<td>W2</td>
<td>2.0</td>
<td>0.40</td>
<td>0.74</td>
<td>4.2 \times 10^{-2}</td>
<td>9.5 \times 10^{-2}</td>
<td>4.7 \times 10^{-1}</td>
</tr>
<tr>
<td>W2</td>
<td>6.0</td>
<td>0.13</td>
<td>0.72</td>
<td>5.5 \times 10^{-3}</td>
<td>1.1 \times 10^{-2}</td>
<td>5.8 \times 10^{-2}</td>
</tr>
<tr>
<td>W2</td>
<td>20.0</td>
<td>0.04</td>
<td>0.72</td>
<td>5.0 \times 10^{-4}</td>
<td>1.0 \times 10^{-3}</td>
<td>5.3 \times 10^{-3}</td>
</tr>
<tr>
<td>W3</td>
<td>0.8</td>
<td>0.46</td>
<td>0.10</td>
<td>3.4 \times 10^{-1}</td>
<td>4.0 \times 10^{-1}</td>
<td>7.3 \times 10^{-1}</td>
</tr>
<tr>
<td>W3</td>
<td>2.0</td>
<td>0.19</td>
<td>0.10</td>
<td>6.3 \times 10^{-2}</td>
<td>1.4 \times 10^{-1}</td>
<td>9.1 \times 10^{-2}</td>
</tr>
<tr>
<td>W3</td>
<td>6.0</td>
<td>0.06</td>
<td>0.10</td>
<td>8.4 \times 10^{-3}</td>
<td>1.7 \times 10^{-2}</td>
<td>8.4 \times 10^{-3}</td>
</tr>
<tr>
<td>W3</td>
<td>20.0</td>
<td>0.02</td>
<td>0.10</td>
<td>7.8 \times 10^{-4}</td>
<td>1.6 \times 10^{-3}</td>
<td>8.4 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Table 1 shows the effect of three different standard wigglers in rings of different energy, calculated using Eqs. (35, 37, 39, 40) above. The wiggler parameters are typical of those of
an undulator (W1), multipole wiggler (W2) and superconducting wavelength shifter (W3) used in synchrotron radiation sources. It is apparent that large perturbations can be introduced by the operation of high field wigglers in low energy rings. Although the tune shift may be easily overcome using the lattice quadrupoles, the beta function distortion and stopband remain and may in certain cases be large enough to prevent operation of the ring. One possibility then is to reduce the beta function at the wiggler location in order to reduce the perturbation, since all effects are roughly linear with the $\beta$ value. Alternatively, the wiggler can be matched into the lattice in such a way that the effects are eliminated or at least substantially reduced, as will be discussed in the following section. The effects decrease rapidly with energy, varying as $E^{-2}$, so that in high energy machines no compensation is usually necessary.

A final consequence of the focusing effect is a possible change in the closed orbit. For example, if the closed orbit is displaced in the vertical plane by an amount $z_o$ at the position of the wiggler when the wiggler is off, then switching on the wiggler will result in an angular deflection given by:

$$\phi = \frac{-K_i L z_o}{1 + \beta_i K_i L \cot(\pi Q) / 2}$$

(45)

![Graph showing the relationship between wigglers and tune shift](image)

Fig. 8 Measured (points) and calculated (line) vertical tune shift due to the SRS superconducting wiggler [11].

The denominator in the above expression expresses the fact that the angular kick is reduced due to the change in closed orbit at the wiggler location. The deflection can be quite a significant effect, for example about 63 $\mu$rad for a 1 mm offset in a wiggler of type W2 (see Table 1) at 2 GeV. The closed orbit must therefore be well corrected at the position of the wiggler, and the wiggler well aligned to the nominal beam axis, to minimize unwanted closed orbit changes when the wiggler is operated.

### 3.5 Compensation of linear optics effects

There are several reasons why it may be desirable to compensate for the linear optics effects introduced by wigglers:

- the tune shift could bring operation near a resonance
- the vertical beta distortion causes vertical beam size changes, which could affect the beam lifetime, and affect users of the synchrotron radiation
- the change of betatron phase advance between the lattice sextupoles could adversely affect the non-linear beam dynamics (dynamic aperture).
There exits one particular matching condition such that the lattice functions outside the wiggler remain unchanged when the wiggler is energised [24]. The condition can be deduced from Eqs. (35) and (37) and is simply that the betatron function at the wiggler location (with the wiggler off) must equal the wiggler "beta function" i.e. $\beta = \beta^*$. In this case with the wiggler off we have simply: $\mu_w = \mu + \theta^*$ and $\beta_w = \beta$, i.e. there is a tune change but no lattice function change. It can be seen from Eq. 40 that the stopband width is also zero. The matching condition is also evident in Eqs. (43) and (44), since in the limit of small $\theta$, there is zero beta function change and stopband width when $\beta = \beta^* = L/\sqrt{12}$. The matching condition may also be deduced by examining the lattice function changes introduced by the effective lens acting at the wiggler centre, Eq. (31) [27]. Since the matching condition depends on $\beta^*$, a perfect match is obtained at only one field value. As an example, consider the wiggler magnet that was installed in ADONE [24]. With a relatively high operating field of 1.8 T and low electron energy (0.54 GeV) the wiggler had a large effect on the linear optics. In order to reduce the effect, a new operating point was found which gave a betatron function closer to the matched value, $\beta^* = 0.96$ m, which substantially reduced the betatron function variation and stopband width, as shown in Table 2.

**Table 2**

Parameters of the ADONE wiggler at two different working points [24]

<table>
<thead>
<tr>
<th>$Q_z$</th>
<th>$\beta$</th>
<th>$\Delta Q_z$</th>
<th>$\Delta \beta/\beta$</th>
<th>Stopband ($\Delta Q$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>3.1</td>
<td>0.20</td>
<td>0.64</td>
<td>0.27</td>
</tr>
<tr>
<td>5.2</td>
<td>0.92</td>
<td>0.15</td>
<td>0.05</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Figure 9 illustrates the fact that in the case of a perfect match for the above example, the beta function changes only in the region of the wiggler and not outside it.

![Fig. 9 Vertical beta function variation in the matched case with the wiggler off (solid line) and on (dashed line).](image)

The above matching scheme places a severe constraint on the lattice optics, requiring in general very low vertical beta functions, and is therefore not suitable (or necessary) in the majority of cases. A more flexible approach is usually required, suitable for overcoming the effects of several different devices, using a series of quadrupoles located in the same straight section as the wiggler, or nearby. To compensate fully for the focusing effects in both planes would require in the general case adjustment of six independent parameters $- \alpha_x, \alpha_z, \beta_x, \beta_z, \mu_x, \mu_z$ requiring therefore six independent quadrupoles. If the wiggler is located at a symmetry point, this could be reduced to four independent quadrupole pairs, however in practise rings have at most two or three independent quadrupole pairs for each insertion device.
straight section. Since all parameters cannot then be corrected, priority is usually given to the tune and secondly to the $\beta$ function variation. In the following we consider the various schemes that have been proposed.

The simplest scheme to overcome the vertical focusing effects of a standard wiggler is to adjust a single defocusing quadrupole located near the wiggler in order to correct the vertical tune and at the same time reduce the vertical beta-function variation [28]. A beta variation is introduced in the horizontal plane, but since at the quadrupole $\beta_z > \beta_x$, this is generally small. A small radial tune change is also introduced that can be compensated globally. Such a scheme is used in the SRS in order to overcome the effects of the superconducting wiggler [29]; Fig. 10 shows that the change in beta function introduced by the wiggler is much reduced.

![Graph](image)

**Fig. 10** Vertical beta function modulation due to the SRS wiggler with global (a) and local (b) tune correction. Computed curve and experimental points [29].

More complex lattice designs allow greater scope for overcoming the effects of wigglers. For example, with the wiggler located at a symmetry point two local quadrupole families can be used to correct the tunes in both planes and also reduce the beta-function variation, as used for example at BESSY [30]. Alternatively, the two quadrupole pairs can be used to eliminate the beta modulation in both planes, using a scheme called "alpha matching", see Fig. 11. In this technique the quadrupole strengths are modified to achieve $\alpha = 0$ in both planes at the centre of the wiggler starting from the un-modified lattice functions outside the quadrupoles. In this way the lattice functions remain unaltered outside the region of the quadrupole pairs. The remaining tune shift can then be corrected in the usual way using global quadrupole families. With three local quadrupole families there is further flexibility to either correct both tunes and minimize beta variations, correct the vertical tune and both beta variations, or make a compromise between tune and beta correction.

A different approach was taken in overcoming the significant effects of the vertical superconducting wiggler in the Photon Factory [31]. Before correction the wiggler introduces tune shifts of $\Delta Q_x = 0.081$, $\Delta Q_z = 0.023$ and beta modulations of $\Delta \beta_x/\beta_x = 70\%$, $\Delta \beta_z/\beta_z = 15\%$. A solution was sought using an arbitrary number and distribution of quadrupoles to correct both tunes and minimize the beta function variation. After correction
using three quadrupoles the tune shifts were corrected to zero and the beta modulation to less than 10%, as shown in Fig. 12.

![Graph showing beta function variation](image)

Fig. 11 "Alpha matching" scheme using two pairs of quadrupoles to correct the beta function variation in both planes due to the wiggler.

![Graphs showing beta function modulation](image)

Fig. 12 Beta function modulation due to the PF vertical superconducting wiggler before (left) and after (right) correction [31]

Finally, another special matching condition is worth mentioning that can be of use in particular situations. It follows from the equations that describe the transport of the beam envelope or Twiss parameters through an element that in the case of a focusing element when \( \alpha = 0 \) and \( \beta = 1/\sqrt{K} \) at the entrance, the Twiss parameters remain constant through the device. Such a condition is of special interest for free-electron lasers since it results in an optimum match of the electron beam to the wiggler to minimize the effects of beam emittance on spectral broadening and gain [27]. In the case of a plane magnet the matching condition is therefore \( \beta'_x = \sqrt{2}\rho_e \). The condition can be applied also in the horizontal plane given suitably shaped magnet pole faces, as mentioned in Section 3.2 above.
4 NON LINEAR BEAM DYNAMICS

4.1 Non-linear motion in the vertical plane

We consider firstly the equation of motion in the vertical plane, assuming the sinusoidal field model, Eq. (22), with \(B_x = 0\). Equation (1) with

\[
x'(s) = \frac{e}{\gamma mc} \frac{B_z}{k_o} \cosh(k_o z) \sin(k_o s)
\]

becomes:

\[
z''(s) = -\left(\frac{e}{\gamma mc}\right)^2 \frac{B_z^2}{2} \frac{\sinh(k_o z)}{k_o} \cosh(k_o z)
\]

(47)

Expanding the \(\sinh\) and \(\cosh\) terms gives:

\[
z''(s) = -\frac{1}{2\rho^2} \left( z + 2 \frac{2}{3} k_o^2 z^3 \right)
\]

(48)

The first term is the linear focusing discussed previously. The second term is a non-linear "octupole-like" term, whose magnitude is proportional to \(B_z^2/E^2\lambda_o^2\).

The equivalent octupole field strength that gives the same cubic term as the wiggler \((B_x = B_3 z^3)\) is given by:

\[
B_3 = \frac{e}{\gamma mc} \frac{B_z^2}{3} \left( \frac{2\pi}{\lambda_o} \right)^2
\]

(49)

For example, the BESSY undulator with \(\lambda_o = 70\) mm, \(B_o = 0.5\) T, \(E = 800\) MeV has an equivalent octupole field at \(z = 25\) mm, of only \(3.9\) mT. Nevertheless this is sufficient to lead to observable effects on the beam such as a tune shift with vertical amplitude and excitation of octupole resonances [32].

4.2 Equations of motion

A more accurate analytic treatment of the effects of wigglers on beam dynamics has been made in Ref. [33] using the Hamiltonian formalism. Here we summarize the method and its results. A Hamiltonian for the motion can be written as follows:

\[
H = \frac{1}{2} \left( p_x^2 + [p_z - A_z]^2 + [p_z - A_z]^2 \right)
\]

(50)

where the vector potential for the field distribution, Eq. (22), is given by:

\[
A_z = \frac{1}{k_o \rho_o} \cosh(k_o x) \cosh(k_o y) \sin(k_o z)
\]

\[
A_z = \frac{k_z}{k_o k_o \rho_o} \sinh(k_o x) \sinh(k_o y) \sin(k_o z)
\]

(51)

Making a canonical transformation to variables which are locally perpendicular to the trajectory, averaging over the wiggler period, and expanding to fourth order in \(x\) and \(z\) results in the following:
\[ H = \frac{1}{2}(p_x^2 + p_z^2) + \frac{1}{4k_x^2\rho_o^2}(k_x^2x^2 + k_x^2z^2) + \frac{1}{12k_x^2\rho_o^2}(k_x^4x^4 + k_x^4z^4 + 3k_x^2k_y^2x^2z^2) \]
\[ - \frac{\sin(k_xz)}{2k_x\rho_o} \left[p_x(k_x^2x^2 + k_x^2z^2) - 2k_x^2p_xz \right] \]

(52)

The equations of motion can then be derived as follows:

\[ x'' = -\frac{k_x^2}{2k_x^2\rho_o^2} \left[ x + \frac{1}{6}k_x^2x^3 + \frac{1}{2}k_x^2x^2z \right] \]
\[ + \frac{\cos(k_xz)}{\rho_o} \left[ \frac{1}{2}k_x^2x^2 + \frac{1}{2}k_x^2x^2z^2 + \frac{1}{4}k_x^2k_y^2x^2z^2 + \frac{1}{24}(k_x^4x^4 + k_x^4z^4) \right] \]
\[ + \frac{\sin(k_xz)}{\rho_o} k_xz \left[ 1 + \frac{1}{2}k_x^2x^2 + \frac{1}{6}k_x^2z^2 \right] \]

(53)

\[ z'' = -\frac{k_x^2}{2k_x^2\rho_o^2} \left[ z + \frac{1}{6}k_x^2z^3 + \frac{1}{2}k_x^2x^2z \right] \]
\[ - \frac{\cos(k_xz)}{\rho_o} \left[ \frac{1}{2}k_x^2xz + \frac{1}{6}k_x^2k_y^2xz^3 + \frac{1}{6}k_x^4x^3z \right] \]
\[ - \frac{\sin(k_xz)}{\rho_o} k_xz \left[ 1 + \frac{1}{2}k_x^2x^2 + \frac{1}{6}k_x^2z^2 \right] \]

(53)

The same result can be obtained by substituting directly in Eq. (1) a solution of the form:

\[ x = \delta_s - \frac{\cos(k_xz)}{\rho_o k_x^2} \]

(54)

in order to calculate the motion with respect to the reference trajectory.

To average the motion over a magnet period, it is not sufficient to set the terms in \( \sin(kz) \) and \( \cos(kz) \) to zero. The correct expression is as follows:

\[ x'' = -\frac{k_x^2}{2k_x^2\rho_o^2} \left[ x + \frac{2}{3}k_x^2x^3 \right] - \frac{k_x^2xz^2}{2\rho_o^2} \]
\[ - \frac{k_x^2}{2k_x^2\rho_o^2} \left[ z + \frac{2}{3}k_x^2z^3 \right] - \frac{k_x^2x^2z}{2\rho_o^2} \]

(55)

i.e. linear focusing plus cubic "octupole-like" terms. The same result can be obtained by differentiating Eq. (27). In the case \( k_x = 0 \) we recover the result of section 4.1.

4.3 Effects on non-linear dynamics

The effects on the non-linear beam dynamics of inserting wigglers in a storage ring have been studied at many synchrotron radiation laboratories, see for example Refs. [34-37]. As a result of these studies it is clear that the effects on the dynamic aperture arise from two distinct processes:
i) The linear focusing of the device destroys the lattice symmetry, changing the phase advances between the chromaticity correction sextupoles and so exciting third order resonances. In the usual case where \( k_x \) is small, these effects are proportional to \( 1/\rho^2 \) i.e. \( B_\omega^2/E^2 \).

ii) The non-linear terms in the equation of motion derived in Section 4.2 introduce amplitude dependent tune shifts and excitation of resonances. In particular the systematic octupole terms can excite fourth-order resonances, while the oscillating sextupole-like terms can also excite third-order resonances. In this case the effects are proportional to \( k^2/\rho^2 \) i.e. \( B_\omega^2/\lambda_\omega^2E^2 \).

The relative effect of these two processes has been demonstrated by carrying out dynamic aperture calculations with the complete equation of motion in the wiggler, and with the wiggler replaced simply by a vertically focusing element in the vertical plane [38]. The effect of non-linearities for a given wiggler has been shown to be important by repeating calculations with different period lengths while keeping the field strength and total length, and hence focusing effect, constant [35].

The beam dynamics effects were first studied by discretizing the equation of motion and inserting in a tracking program, for example BETA and RACETRACK. Problems arose however with the amount of computer time required, since a large number of steps were needed for each magnet period in order to obtain consistent results, and the fact that it was not rigorously symplectic. To overcome these difficulties an improved scheme based on a canonical integration method was later developed and included in RACETRACK [39]. More recently further improvements have been made using an algebraic mapping routine [40].

The results of various tracking studies indicate that:

i) the effect of introducing wigglers into a storage ring is in general similar in magnitude to the effect of random multipole errors in the main lattice magnets and is therefore not a limiting factor for the operation of the ring;

ii) significant reduction in dynamic aperture can be caused even by a single device, however the dynamic aperture does not continue to deteriorate linearly with an increasing number of devices;

iii) the effects are stronger in low energy rings;

iv) compensation of the tune shift introduced by the wiggler is important, since the dynamic aperture is sensitive to the tune values, but the results are not very sensitive to which scheme is used for the correction or whether the lattice functions are also corrected;

v) both linear and non-linear effects are in general important factors in dynamic aperture reduction.

Some examples of the effect of wigglers on dynamic aperture are shown in Fig. 13, showing in particular a much larger effect in the lower energy ring.

Fig. 13 Effect of wigglers on the dynamic aperture in the ESRF (left) and ALS (right) [34,36]
4.4 Effects of field errors

So far we have considered only the intrinsic non-linear effects of wigglers with an ideal periodic field distribution. In practise however a wiggler is likely to contain multipole field errors that can give rise to further beam dynamics effects. Earlier wiggler magnets have contained a number of field imperfections:

i) Skew-quadrupole field: the BESSY undulator gives rise to a change of vertical beam size with wiggler strength and an increase in the stopband width of the sum resonance \( Q_x + Q_z = \) integer [32].

ii) Sextupole field: two undulators installed in SUPERACO contain sextupole errors which lead to changes in the chromaticity, since they are located in straight sections with finite dispersion [41].

iii) Skew-sextupole field: measurements of the effect of a 15-period, 1.45 T wiggler in SPEAR indicate the presence of such a field error [42].

iv) Skew-octupole field: resonances of the type \( 3Q_x + Q_z = \) integer and \( 3Q_x + Q_z = \) integer have been observed both at SPEAR, with the first permanent magnet undulator [43], and with two separate undulators on SUPERACO [41].

It should be noted that the most disturbing effects due to wiggler field errors have arisen in low energy rings, such as BESSY and SUPERACO which both have an energy of 800 MeV. Effects on the chromaticity and vertical beam size due to wiggler operation have also been seen at ALADDIN, at 0.8 and 1 GeV [44].

Progress with methods of wiggler construction, including better magnetic measurement and field correction techniques, should eliminate such errors from the newer generation of devices.

4.5 Measurement of non-linear effects

A very useful technique for determining the non-linear effects of a wiggler on the beam is to measure the tune shift as function of the horizontal and vertical position of the beam with respect to the wiggler axis. The presence of a sextupole field error is indicated by a linear tune shift with horizontal position (x); a skew-sextupole would give a tune shift with vertical position (z). Similarly, an octupole field would give a dependence of the tune shift with \( x^2 \) or \( z^2 \), while a skew-octupole would give a dependence on \( xz \). This technique was used at SPEAR, where a skew-sextupole error was discovered [42], and at BESSY, in order to confirm the magnitude of the "pseudo-octupole" field [45], see Fig. 14.

The effect of real field errors can be distinguished from second-order effects such as these, by determining the scaling with energy \( E \) - the latter scales as \( 1/E^2 \), whereas the effects of field errors scale as \( 1/E \).

Another method consists of performing a "resonance scan" i.e. moving the tune values so as to cross particular resonance lines, while measuring the effect on the lifetime of the stored beam. At BESSY this technique was used to determine the effect of the skew-quadrupole field errors on the \( Q_x + Q_z = \) integer resonance, and of the pseudo-octupole field on the \( 4 Q_z = \) integer, see Fig. 15 [32]. Similar scans at SUPERACO identified both octupole and skew-octupole driven resonances [41].
Fig. 14 Variation of vertical tune with vertical position in the wigglers at BESSY (left) and SPEAR (right) [45,42].

Fig. 15 Measurements of beam lifetime as function of vertical tune in BESSY with various wiggler strengths [32].

5 EFFECTS DUE TO THE EMISSION OF SYNCHROTRON RADIATION

5.1 Standard wigglers

For simplicity, we will consider the case of a wiggler with a sinusoidal field, with \( k_x = 0 \) and will neglect the field distribution at the end of the magnet. Then:

\[
\frac{1}{\rho} = \frac{1}{\rho_0} \cos(k_0 s) \quad k = \frac{1}{\rho_0^2} \sin^2(k_0 s)
\]

(56)

The dispersion function is the sum of the dispersion that exists in the straight section without the wiggler \((D_0, D'_0)\) together with a "self-dispersion" generated by the trajectory in the wiggler. Since the displacement, \( x \), is inversely proportional to energy, \( E \), we have \( dx = -x \) \((dE/E)\), and hence \( D = -x \), and similarly \( D' = -x' \). Thus the dispersion is given in general by:

\[
D(s) = D_0 + D'_0 s + \frac{\cos(k_0 s)}{\rho_0 k_0^2}
\]

\[
D'(s) = D'_0 - \frac{\sin(k_0 s)}{\rho_0 k_0}
\]

(57)
The contributions to the five Synchrotron Radiation Integrals [46] for a wiggler of length $L$ can then be derived as follows:

\[
\begin{align*}
\Delta I_1 &= \frac{\rho^2}{\rho} \int \frac{\cos^2 (k_s s)}{\rho^2 k_s^2} = \frac{L}{2 \rho^2 k_s^2} \\
\Delta I_2 &= \frac{1}{\rho^2} \int \frac{\cos^2 (k_s s)}{\rho^2} = \frac{L}{2 \rho^2} \\
\Delta I_3 &= \frac{1}{\rho^3} \int \frac{\cos^3 (k_s s)}{\rho^2} = \frac{4L}{3 \pi \rho^3} \\
\Delta I_4 &= \frac{D}{\rho^3} - \frac{2 kD}{\rho} = -\int \frac{\cos^4 (k_s s)}{\rho^2 k_s^2} + 2 \int \frac{\sin^2 (k_s s) \cos^2 (k_s s)}{\rho^2 k_s^2} = -\frac{L}{8 \rho^4 k_s^2} \\
\Delta I_5 &= \frac{H}{\rho^3} \text{ where } H(s) = \gamma D^2 + 2 \alpha D + \beta D^2
\end{align*}
\]

(58)

To evaluate $\Delta I_5$ a numerical calculation is generally needed, however two extreme cases can be considered:

i) large natural dispersion. In this case the self-dispersion can be neglected and so after averaging $1/\rho^3$ over the wiggler length one obtains:

\[
\Delta I_5 = \frac{4}{3 \pi} \frac{\langle H \rangle L}{\rho^3}
\]

(59)

ii) zero natural dispersion. The dominant term in the expression for $H$ is $\beta D^2$ and so:

\[
\Delta I_5 = \frac{\beta D^2}{\rho^3} = \left(\beta\right) \frac{\sin^2 (k_s s) \cos^3 (k_s s)}{\rho^2 k_s^2} = \frac{4}{15 \pi} \frac{\langle \beta \rangle L}{\rho^3}
\]

(60)

Making use of the above formulae we can now deduce the effects on the beam due to changes in the SR integrals:

i) Momentum compaction factor: $\alpha_r = I_1 / I_{tot}$, where $I_{tot}$ is the ring circumference. The change introduced is therefore:

\[
\Delta \alpha_r = \frac{\Delta I_1}{I_{tot}} = \frac{L}{2 \rho^2 k_s^2}
\]

(61)

The effect in most cases is negligible, for example with $L_W = 5$ m, $L_{tot} = 100$ m, $B_0 = 1.5$ T, $E = 1.0$ GeV, $\lambda_0 = 0.1$ m we have $\Delta \alpha = 1.3 \times 10^{-6}$.

ii) Damping partition numbers: $J_z = 1 - (I_4 / I_2)$, $J_e = 2 + (I_4 / I_2)$

The change to the $J_x$ damping partition number can be written as follows:

\[
J_x = \frac{1 + \frac{\Delta I_2 - \Delta I_4}{I_2 - I_4}}{1 + \frac{\Delta I_2}{I_2}}
\]

(62)
From the expressions above it can be seen that $\Delta I_4 / \Delta l_2 = 1 / 4 \rho_0^2 k_0^2$ and hence $\Delta I_4$ can be neglected compared to $\Delta l_2$. Also, in most rings $I_4$ is very small, and hence $J_x$ is close to unity. In most cases therefore $J_{\chi}$, and hence $J_z$ and $J_{\varepsilon}$, are unchanged by the action of the wiggler.

iii) Damping times: $\tau_i = 3T_e / r_i \gamma^2 J_i I_2$, where $i = x, z$ or $\varepsilon$.
Since the effect of the wiggler is to increase $I_2$, all damping times are thereby reduced.

iv) Energy spread: $\left( \frac{\sigma_E}{E} \right)^2 = C_q \gamma^2 \frac{I_5}{2I_2 + I_4}$, where $C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.84 \times 10^{-13}$ m

The change in energy spread can therefore be written as follows:

$$\left( \frac{\sigma_E}{\sigma_E} \right)^2 = \frac{1 + \Delta I_5}{1 + \frac{\Delta I_2 + \Delta I_4}{2I_2 + I_4}}$$ (63)

Neglecting as before $\Delta I_4$ and $I_4$, then in the case of a sinusoidal wiggler:

$$\left( \frac{\sigma_E}{\sigma_E} \right)^2 = \frac{1 + \frac{4L}{3\pi} \frac{\rho^3}{\rho_w}}{1 + \frac{4L}{2\pi} \frac{\rho^2}{\rho_w}}$$ (64)

Thus, the energy spread is increased if $\rho / \rho_w > 3\pi / 8$, i.e. roughly when the wiggler field exceeds that of the bending magnets. Assuming constant synchrotron oscillation frequency, the bunch length is changed by the same factor as the energy spread. Figure 16 shows the variation of $(\sigma_E / \sigma_E)$ as a function of magnetic field strength for wigglers of two different lengths in 1.5 and 6 GeV rings. The result is independent of the wiggler period length. It can be seen that the change in energy spread increases with increasing wiggler field strength and length, and reduces with higher electron beam energies.

v) Emittance:

$$\varepsilon_z = C_q \gamma^2 \frac{I_5}{I_2 - I_4}$$

The change in emittance can therefore be written as follows:

$$\varepsilon_z' = \frac{1 + \Delta I_5}{1 + \frac{\Delta I_2 - \Delta I_4}{I_2 - I_4}}$$ (65)

Neglecting $I_4$ and $\Delta I_4$ it can be seen that the emittance is increased if $\Delta I_5 / I_5 > \Delta I_2 / I_2$.

Figure 17 shows the emittance ratio $(\varepsilon_z / \varepsilon_x)$ as a function of field strength for wigglers of different period length and total length in rings of 1.5 and 6 GeV, assuming zero natural dispersion at the wiggler location. At low field strength the emittance reduces, due to increase of the $I_2$ term. The reduction therefore depends only on the total length, not the period length. At higher fields the $I_5$ term which causes emittance growth becomes increasingly important. This is particularly true for longer period lengths (curves a and c), in agreement with Eq. (60).
Fig. 16  Variation of energy spread with wiggler field strength in 1.5 GeV (left) and 6 GeV (right) rings. Ring parameters: \( B = 1.2 \, \text{T}, J_x = 1.0 \). Wiggler parameters: (a) \( L = 1 \, \text{m} \), (b) \( L = 5 \, \text{m} \).

The condition for emittance increase is given by:

\[
\frac{4}{3\pi} \frac{\langle H \rangle_w L}{\rho_w^3} \frac{\rho^3}{2\pi\rho} > \frac{L}{2\rho_w^2} \frac{\rho^2}{2\pi\rho}
\]

which can be expressed as follows:

\[
\lambda_\varepsilon^2 B_o^3 > 5.87 \times 10^9 \frac{\varepsilon \, E_{(\text{GeV})}}{\langle \beta_z \rangle}
\]

and which therefore depends on the period length, independent of the wiggler length. At higher energy the effect on the emittance is reduced, and the point at which the emittance starts to increase is shifted to higher field strengths, in accordance with Eq. (67).

It follows from the expression above that in most practical situations the emittance reduces, except for very high field wiggler in low energy, low emittance rings. Wiggler have therefore been proposed as a means of reducing the emittance, for example in PEP, for operation as a low emittance synchrotron radiation source [47]. Wiggler are also proposed in many new damping ring designs, to reduce damping times and emittance.

Figure 18 shows the effect of non-zero dispersion \( (D_o) \) in the wiggler straight and it can be seen that with sufficiently large dispersion the emittance increases. The condition for the natural dispersion to have a larger effect than that of the wiggler self-dispersion is given by:

\[
\langle H \rangle_w > \frac{\beta}{5k_o \rho_w^2}, \text{ or in terms of the dispersion: } D_o > \frac{\beta}{\sqrt{5k_o \rho_w}} \text{ or } D_o' > \frac{1}{\sqrt{5k_o \rho_w}}, \text{ assuming } \alpha = 0.
\]

Since \( 1/k \rho_w \ll 1 \) this condition is easily met. Thus even if the wiggler straight has a nominal zero-dispersion, the residual dispersion arising from errors can be the dominant effect. In this case the condition for emittance increase is given by:

\[
\frac{\langle H \rangle_w \rho}{\langle H \rangle \rho_w} > \frac{3\pi}{8}
\]

which can be written as follows:
\[
\langle H \rangle_w B_o > 2.68 \times 10^6 \frac{\varepsilon_x}{E_{\text{GeV}}}
\] (69)

The residual dispersion must therefore be carefully controlled in low emittance rings with high field wigglers to avoid any unwanted emittance increase. For example, for a 1.5-T wiggler in the 1.5-GeV ring of the above example, the dispersion must be kept below the following limits:

\[D_o < 0.29 \text{ m}, \quad D'_o < 0.029 \text{ m}.\]

![Graph showing variation of emittance with wiggler field strength in 1.5-GeV (left) and 6-GeV (right) rings.](image)

**Fig. 17** Variation of emittance with wiggler field strength in 1.5-GeV (left) and 6-GeV (right) rings. Ring parameters: \(B = 1.2 \text{ T}, J_x = 1.0, \varepsilon_x = 7 \times 10^{-9} \text{ m rad}, \beta_x = 10 \text{ m}.\) Wiggler parameters: (a) \(L = 1 \text{ m}, \lambda_o = 0.25 \text{ m},\) (b) \(L = 1 \text{ m}, \lambda_o = 0.05 \text{ m},\) (c) \(L = 5 \text{ m}, \lambda_o = 0.25 \text{ m},\) (d) \(L = 5 \text{ m}, \lambda_o = 0.05 \text{ m}.\)

![Graph showing variation of emittance with wiggler field strength and natural dispersion at the centre of the wiggler straight section (\(D_o\)) in 1.5-GeV (left) and 6-GeV (right) rings.](image)

**Fig. 18** Variation of emittance with wiggler field strength and natural dispersion at the centre of the wiggler straight section \((D_o)\) in 1.5-GeV (left) and 6-GeV (right) rings. Ring parameters as Fig. 17. Wiggler parameters: \(L = 2.5 \text{ m}, \lambda_o = 0.10 \text{ m}.\)

A practical example of the use of wigglers to change beam properties is in LEP [48]. A set of "damping wigglers", located in dispersion free regions, are used to increase the energy spread and bunch length by a factor 5-6 at injection energy to increase beam stability, while a series of "emittance wigglers", located in regions with non-zero dispersion, are used to increase the beam size at high energy in order to optimize the luminosity [49].

An emittance increase due to the operation of two superconducting wigglers is unavoidable in the SRS, since no zero-dispersion straights are available. The emittance
increase due to the latest device was kept as low as possible (50 \%) by minimizing $\Delta I_5$ and therefore $\int 1/r^3 \, ds$ in the magnet design [50].

5.2 Gradient and non-linear wigglers

In combined function strong focusing lattices ($l_4/l_2 \approx 2$, i.e. $J_x \approx -1$, and so the radial motion is anti-damped [1]. Gradient wigglers, or dipole-quadrupole wigglers, were proposed by Robinson as a means of overcoming this difficulty [3] and they were subsequently employed at CEA to permit operation as a storage ring [4], and also at the PS for operation with electrons [51]. It follows from Eq. (62) that a large negative change to the $l_{4}$ integral is required, which can be obtained in a wiggler with an alternating field and field gradient such that $k/r$ is positive, if located at a position of large positive dispersion. Gradient wigglers can also be used to reduce the emittance in a storage ring. For a wiggler located at a position of large dispersion and beta function the dominant effect in Eq. (65) is the reduction in $l_{4}$, or in other words the change in damping partition, $J_x$. Since the emittance can be expressed as $\epsilon_x = C_x r^2 I_x / J_x I_2$ and since the damping partition can be changed from 1 (for a conventional separated function lattice) to 2 without loss of damping in the longitudinal plane, it follows that a reduction in emittance of a factor of 2 is achievable by this means.

Non-linear wigglers have been proposed as a means of producing a dependence of the damping partition ($J_x$) with energy deviation, $\epsilon$ [5,6]. A dipole-sextupole wiggler can give a linear dependence, $J_x \approx \epsilon$, which can be used to increase the damping aperture by making $dJ_x/d\epsilon = 0$. A dipole-octupole wiggler gives a quadratic dependence, $J_x \propto \epsilon^2$, which alters the equilibrium energy and charge distribution. A quadrupole-sextupole wiggler has also been proposed [7].

To gain some insight into the effect of such devices on the beam, we consider a series of constant field regions with alternating polarity, but constant gradient, sextupole and octupole terms, located at a position with non-zero dispersion:

$$B_x(x) = B + B' x + \frac{B'' x^2}{2} + \frac{B''' x^3}{6}$$  \hspace{1cm} (70)

where $x = D\epsilon$ and $\epsilon$ is the relative energy deviation. Multiplying by $e/\gamma mc$ we have:

$$-\frac{1}{\rho}(x) = \frac{1}{\rho_w} + kD\epsilon + \frac{r}{2} D^2\epsilon^2 + \frac{q}{6} D^3\epsilon^3$$  \hspace{1cm} (71)

where

$$\frac{1}{\rho_w} = -\frac{e}{\gamma mc} B \quad k = \frac{e}{\gamma mc} B' \quad r = \frac{e}{\gamma mc} B'' \quad q = \frac{e}{\gamma mc} B'''$$  \hspace{1cm} (72)

The damping partition number is defined by:

$$J_x = \frac{dU}{d\epsilon} \frac{E_x}{U} = \frac{dU}{d\epsilon} \frac{1}{U}$$  \hspace{1cm} (73)

The energy loss per turn ($U$) depends on $\int 1/r^2$ and hence the total energy loss including the wiggler ($U_w$) is given by:

$$U_w = U_o + U_o \rho \frac{1}{2\pi} \int \left(-\frac{1}{\rho_w} + kD\epsilon + \frac{rD^2\epsilon^2}{2} + \frac{qD^3\epsilon^3}{6}\right) ds$$  \hspace{1cm} (74)
We have then:

\[
J_\epsilon = J_{\epsilon_0} + \frac{U_0}{U_w} \frac{\rho}{2\pi} \int \frac{d}{d\epsilon} \left( \frac{1}{\rho_w} + kD\epsilon + \frac{rD^2\epsilon^2}{2} + \frac{qD^3\epsilon^3}{6} \right)^2 ds
\]  

(75)

We can now examine the effect of the various proposed combinations of field components:

i) Dipole-quadrupole wiggler:

\[
J_\epsilon = J_{\epsilon_0} - \frac{\rho L}{2\pi} \left( \frac{2kD}{\rho_w} + 2k^2D^2\epsilon \right)
\]  

(76)

ii) Dipole-sextupole wiggler:

\[
J_\epsilon = J_{\epsilon_0} - \frac{\rho L}{2\pi} \frac{2rD^2\epsilon}{\rho_w}
\]  

(77)

iii) Dipole-octupole wiggler:

\[
J_\epsilon = J_{\epsilon_0} - \frac{\rho L}{2\pi} \frac{qD^3\epsilon^2}{\rho_w}
\]  

(78)

iv) Quadrupole-sextupole wiggler:

\[
J_\epsilon = J_{\epsilon_0} - \frac{\rho L}{2\pi} \left\{ 2k^2D^2\epsilon + 3krD^3\epsilon^2 + 6k^2D^2\epsilon^2 \right\}
\]  

(79)

In the above \(J_{\epsilon_0}\) represents the energy oscillation damping partition number in the absence of the wiggler, which is itself a function of energy deviation. A dipole-sextupole wiggler can thus be used to vary the linear dependence of \(J_\epsilon\) with \(\epsilon\), and make it zero if desired. A combination of dipole-quadrupole and dipole-octupole can result in a damping partition number variation of the form:

\[
J_\epsilon = -a + b\epsilon^2
\]  

(80)

i.e. anti-damped for particles of the nominal energy, but damped for particles of large energy deviation. As a result it is possible to modify the energy and longitudinal charge distributions, for example to reduce the peak charge density and so reduce the effects of instabilities, as shown in Fig. 19 [52].

5.3 Spin polarization

The polarization rate is [53]:

\[
\frac{1}{\tau_p} = \frac{5\sqrt{3}hr_0}{8m} \gamma S \frac{I_3}{L_{\text{los}}}
\]  

(81)

Since \(I_3\) is always increased by the effect of wigglers it follows that this can be used to increase the polarization rate. However, the asymptotic polarization rate is affected also. For an ideal machine this is given by:
\[ P_n = \frac{8}{5\sqrt{3}} \frac{I_{3a}}{I_3} \]  

(82)

where

\[ I_{3a} = \frac{1}{\rho} ds \]  

(83)

Fig. 19 Calculated longitudinal bunch profiles in LEP due to the action of a dipole-octupole wiggler [52].

In a conventional wiggler the positive and negative poles have equal strength and hence \( I_{3a} \) is approximately zero, which leads to a reduction of the asymptotic polarization rate. In order to increase the polarization rate while at the same time maintaining a high polarization level requires an "asymmetric" wiggler with different positive (B+) and negative (B-) field strengths. In order that the field integral for each period is zero, and so produces no net deflection, the pole lengths must also vary so as to satisfy: B+ L+ = B- L-. The ratio of field strengths is called the "asymmetry parameter", \( r = B_+ / B_- \). Thus if \( r \) is large the changes to \( I_3 \) and \( I_{3a} \) are dominated by the positive poles and so are similar in magnitude, and hence the asymptotic polarization level is not affected very much.

For example, LEP has 12 "polarization wigglers" with an asymmetry parameter \( r = 8.0 \) [54]. Calculations show that they should reduce the polarization time from 310 min to 36 min at 46 GeV with 88% asymptotic degree of polarization [53].

* * *

REFERENCES

[34] ESRF Foundation Phase Report, Feb. 1987, p. 82.