Exponential and Power-Law Hierarchies from Supergravity

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Abstract

We examine how a $d$-dimensional mass hierarchy can be generated from a $d+1$-dimensional set up. We consider a $d+1$–dimensional scalar, the hierarchon, which has a potential as in gauged supergravities. We find that when it is in its minimum, there exist solutions of Hořava-Witten topology $R^d \times S^1/Z^2$ with domain walls at the fixed points and anti-de Sitter geometry in the bulk. We show that while standard Poincaré supergravity leads to power-law hierarchies, (e.g. a power law dependence of masses on the compactification scale), gauged supergravity produce an exponential hierarchy as recently proposed by Randall and Sundrum.
1 Introduction

A puzzling feature in all efforts to extend the Standard Model (SM) is the hierarchy $m_{EW}/M_{Pl}$ of the electroweak scale $m_{EW} \sim 10^3$ GeV and the Planck scale $M_{Pl} = G_N^{1/2} \sim 10^{18}$ GeV. A proposal for explaining this hierarchy has been made in [1] realizing recent ideas on the size of the compactification, the string scale and the coupling constants [2]. According to this proposal, the higher $4+n$ dimensional theory with $n \geq 2$ has a $4+n$ dimensional Planck mass $M_{Pl(4+n)}$ at the TeV scale while the scale $R_c$ of the extra $n$ dimensions is less than a millimeter. The proposal has been designed in such a way as to generate the hierarchy $m_{EW}/M_{Pl}$. However, it suffers from another hierarchy, that of $m_{EW}R_c$.

In [3], an alternative scenario was proposed for generating the hierarchy without large extra dimensions. According to this, the four-dimensional metric is not factorizable but rather is multiplied by a warp factor with exponential dependence on a transverse coordinate which has finite range. The overall space is in fact a portion of a five-dimensional anti-de Sitter space-time (AdS). Masses of the four-dimensional world are also multiplied with this warping factor which can generate hierarchies for not necessarily compactification radius.

Along these lines, we will try to push forward the idea that mass hierarchies in four dimensions can be smoothed out in a higher-dimensional setting. We will show that exponential hierarchy can actually be produced by a scalar in five dimensions with a potential like the one in gauged supergravities. We will call this scalar hierarchon for obvious reasons. In fact it is one of the scalars which appear in gauged supergravities and have recently been discussed in connection with the renormalization group flow [4, 5, 6] in the context of AdS/CFT correspondence [7, 8, 9]. Recalling that according to the general belief, solutions of gauged supergravities are true compactifications of type IIB or eleven-dimensional supergravity, our solutions should also correspond to ten or eleven-dimensional supersymmetric backgrounds.

The potential of the hierarchon is such that it has negative value at its minimum and effectively produce a cosmological constant. In this case, there are solutions which describe flat domain walls separated by a bulk AdS geometry and correspond to an $S^1/Z_2$ compactification of the radial AdS coordinate. Similar solutions in the context of M-theory have been found in [11]. An induced exponential hierarchy on the walls is then generated as in [3] which depend on the value of the potential at its critical points. In the examples we study, exponential hierarchy is generated on one of the boundaries only while on the other we find a power-law hierarchy. Finally, when there is no such scalar as in the usual ungauged Poincaré supergravity, the hierarchy is always power-law and a scenario with large extra dimensions should be taken as in [1].

\footnote{For the four-dimensional $SO(8)$ gauged supergravity this has been proven in [10].}
2 Exponential hierarchy in gauged supergravity

We will start by recalling some results from gauged-supergravity theories. These theories exist only in $d = 3, \ldots, 7$ dimensions and they are ultimately related to the existence of AdS supergroups in these dimensions [12]. For their construction, one usually starts with the ungauged $d$-dimensional Poincaré supergravity which, generally, contains a set of scalars parametrizing a coset $G/H$ and a set of (abelian) vectors $A^I_M$ transforming in a specific representation $r$ of $G$. The gauged supergravity is then obtained from the ungauged one by gauging an appropriate non-abelian subgroup $K \subset G$. The gauge group $K$ is such that the decomposition of $r$ in representations of $K$ contains the adjoint. The construction proceeds by replacing the abelian vectors with non-abelian ones. This replacement clearly violates supersymmetry which, however, can be recovered by adding terms in the action and changing appropriately the supersymmetry transformation rules. This procedure generates a potential for the scalar fields with non-trivial critical points which is the characteristic of gauged supergravities.

The effective action of the $d = 4, 7$ and $d = 5$ gauged supergravities can be obtained by KK compactification of eleven-dimensional supergravity and ten-dimensional type IIB supergravity on $S^4, S^7$ and $S^5$, respectively. Since we are mainly interested in the $d = 5$ case, let us describe it in more details.

The toroidal compactified type IIB string theory in five dimensions has global $E_{6(+6)}$ and a local $USp(8)$ symmetry. The massless spectrum fills the $\mathcal{N} = 8$ five-dimensional graviton multiplet which consists of a graviton, 8 gravitini, 27 vectors, 48 gauginos and 42 scalars [13]. The scalars parametrize the non-compact symmetric space $E_{6(+6)}/USp(8)$. An $SO(6) = SU(4)$ subgroup of $E_{6(+6)}$ can be gauged leading to the $\mathcal{N} = 8$ gauged supergravity [14, 15]. The potential of the latter is rather involved, it is $SU(4)$-invariant and all the 42 scalars appear in it except the dilaton and the axion. There is no classification of the critical points of the potential although some of the vacua are known [5, 6, 14, 15].

Turning to the general case, the symmetry group $K$ can be broken to a subgroup $K_0 \subset K$ by the expectation value of a scalar $\lambda$ which is a $K_0$ singlet. We will call the scalar $\lambda$ which takes an expectation value hierarchon since, as we will see, it will set up the hierarchy. The bosonic part of the gauged supergravity action for the graviton and the hierarchon is then of the form

$$S = \frac{1}{2\kappa^2 d} \int d^d x \sqrt{G} \left( R - \frac{1}{2} \partial_M \lambda \partial^M \lambda - V(\lambda) \right), \quad (1)$$

where $\kappa^2$ is the gravitational constant in $d$-dimensions. The potential $V(\phi)$ has, generically, critical points for $\lambda = \lambda_0 = \text{const}$ and the value of $V$ at these points can conveniently be parametrized as $V(\phi_0) = -\frac{(d-2)(d-1)}{L^2}$. For example, in the case of five-dimensional gauged supergravity, the potential for the $SO(5)$-invariant hierarchon is

$$V(\lambda) = -\frac{1}{32} g^2 \left( 15e^{2\lambda} + 10e^{-4\lambda} - e^{-10\lambda} \right), \quad (2)$$
where $g$ is the gauge-coupling constant. The above potential has an $SU(4)$ symmetric minimum at $\lambda_0 = 0$ where $V(\lambda_0) = -3g^2/4$ with $\mathcal{N} = 8$ supersymmetry. There is also an other minimum at $\lambda_0 = -1/6 \log 3$ with $V(\lambda_0) = -3^{5/3}g^2/8$ with $SO(5)$ symmetry and no supersymmetry, $\mathcal{N} = 0$. Similarly, if $\lambda$ is taken to be an $SU(3) \times U(1)$ singlet, the potential turns out to be

$$V(\lambda) = \frac{3}{32} g^2 \left( \cosh(4\lambda)^4 - 4 \cosh(4\lambda) - 5 \right). \quad (3)$$

Here, the $SU(3) \times U(1)$-symmetric minimum exist at $\lambda_0 = 1/4 \cosh^{-1} 2$ with $V(\lambda_0) = -27/32g^2$ [16]. The above potentials have been employed in [4] for the discussion of the renormalization-group flow in the AdS/CFT correspondence.

Having frozen the hierarchon field at its minimum, the rest of the field equations are simply

$$R_{MN} = -\frac{d - 1}{L^2} G_{MN}. \quad (4)$$

The supersymmetry transformations for the gravitino take the form

$$\delta \psi_M = \partial_M \epsilon - \frac{1}{2L} i \Gamma_M \epsilon. \quad (5)$$

The obvious solution to the above equations is the anti-de Sitter space $AdS_d$ which is the unique maximally symmetric space with negative cosmological constant. It is found as the vacuum solutions of gauged supergravity for $d = 7, 5, 4$ and describe M-theory vacua on $AdS_4 \times S^7, 4$ and type IIB on $AdS_5 \times S^5$. They are also realized as the near-horizon limit of various brane-configurations [17] and they are the supergravity duals of superconformal field theories [7, 8, 9]. The isometry group of $AdS_d$ spaces are $SO(2, d-1)$ which is the conformal group in $d-1$ dimensions. However, here we will describe solutions to the field equations (4) which are not maximally symmetric but rather invariant under the group $ISO(1, d-2) \times U(1)$ where $ISO(1, d-2)$ is the Poincaré group in $d-1$ dimensions. The ansatz for the $d$-dimensional metric is then of the form

$$ds^2 = H(z)^{2a} \eta_{\mu\nu} dx^\mu dx^\nu + H(z)^{2b} dz^2, \quad (6)$$

where $H(z)$ is a function of the transverse coordinate $z$. The constants $a, b$ can be determined by demanding supersymmetry, that is the solution to be annihilated by the supercharges. This is equivalent to the vanishing of all fermionic shifts. The integrability condition $\delta \psi_M = 0$ turns out to be

$$R_{MNAB} \Gamma^{AB} \epsilon = \frac{2}{L^2} \Gamma_{MN} \epsilon. \quad (7)$$

For the metric in (6), the integrability condition gives that $a^2 = 1, b = -1$. Moreover, $H(z)$ is a harmonic function in the transverse $z$-direction, i.e., it satisfies

$$H'' = 0 \quad \text{with} \quad H'^2 = 1/L^2, \quad (8)$$
where the prime (') denotes differentiation \( d/dz \). For \( a = 1 \) the solution turns out then to be

\[
\begin{align*}
    ds^2 &= H^2 \eta_{\mu \nu} dx^\mu dx^\nu + H^{-2} dz^2, \\
    H &= \frac{1}{L} z + c.
\end{align*}
\]  

(9) 

(10)

We will not consider the case \( a = -1 \) since it is a coordinate transformation of the \( a = 1 \) case. Clearly the metric above describes a space-time invariant under the Poincaré group \( ISO(1, d - 2) \) in the longitudinal \( d - 1 \) dimensions. In fact, if \( H(z) \) is continuous, the symmetry is \( SO(2, d - 1) \) since the metric (9) describes a d-dimensional anti-de Sitter space. On the other hand, if \( H(z) \) is piecewise continuous, it describes domain walls sited at the discontinuous points. Such solutions have been discussed in [18, 19, 20, 21, 22]. Another possibility is to consider the case in which, \( H(z) \) is piecewise continuous and periodic \(^2\). Among many possibilities, we will discuss two particular cases as depicted in figure 1 which solves (8).

Figure 1: The function \( H(z) \) for \( 0 \leq z < 2L \).

We will examine the two cases (a) and (b) separately.

**The (a) case**

Here, \( H(z) \) is of the form

\[
H(z) = \begin{cases} 
\frac{1}{L} z + c & 0 \leq z < L \\
-\frac{1}{L} z + c & L \leq z < 2L
\end{cases}
\]  

(11)

\( H(z) \) is discontinuous but nevertheless it has the right properties, namely it is periodic with period \( 2L \) as can easily be seen in its Fourier-series representation

\[
H(z) = \frac{1}{2} + c - \frac{4}{\pi^2} \sum_{n=1,3,5,...} \frac{1}{n^2} \cos \left( \frac{n \pi z}{L} \right),
\]

(12)

and satisfies the second equation in (8) everywhere while the first in (8) “almost” everywhere. By recalling the series representation of the \( \delta \)-function

\[
\delta(z) = \frac{1}{2L} + \frac{1}{L} \sum_{n=1}^{\infty} \cos \left( \frac{n \pi z}{L} \right),
\]

(13)

\(^2\)We thank L. Alvarez-Gaumé for pointing out this possibility
we find that
\[ H'' = \frac{2}{L} \left( \delta(z) - \delta(z + L) \right), \] (14)
and thus the discontinuity of \( H(z) \) can be attributed to sources at the discontinuous points. The form of the sources can be found by considering the energy-momentum tensor \( T_{MN} \) which in our case is given by
\[ T_{MN} = \frac{1}{\kappa_d^2} \left( R_{MN} - \frac{1}{2} G_{MN} R - \frac{(d - 2)(d - 1)}{2L^2} G_{MN} \right). \] (15)
Then, it is straightforward to verify that for the metric (9) the energy-momentum tensor is given by
\[ T_{\mu\nu} = \frac{d-2}{\kappa_d^2} H'' H^3 \eta_{\mu\nu}, \quad T_{zz} = 0, \] (16)
so that, by using (14)
\[ T_{\mu\nu} = \frac{2(d-2)}{L\kappa_d^2} \left( \delta(z) - \delta(z + L) \right) H^3 \eta_{\mu\nu}. \] (17)
Thus, our solution describes two domain walls placed at \( z = 0 \) and \( z = L \), respectively. In particular, the solution is invariant under \( z \to 2L - z \) as can be seen from (12) and there exist two fixed points, the \( z = 0 \) and \( z = L \). These are the points where our domain walls are sited. In this case we may restrict \( z \) to be in the interval \([0, L]\) corresponding to an \( S^1/\mathbb{Z}_2 \) orbifold of the transverse one-dimensional space. A similar solution has also been found in [11] in the context of M-theory where \( z \) is identified with the eleventh dimension.

The constant \( c \) in \( H(z) \) is determined in terms of the compactification radius \( R_c \) and the cosmological constant \( L \). From (9) we see that the compactification radius is
\[ R_c = \frac{1}{\pi} \int_0^L H^{-1} dz = \frac{L}{\pi} \ln \left( 1 + \frac{1}{c} \right), \] (18)
so that we get
\[ c = \left( e^{\pi R_c/L} - 1 \right)^{-1}. \] (19)

Let us now suppose that gauge theories live on the domain walls found above while gravity propagates in the bulk. Then for the case (a) in figure 1, we see from (9) that the masses \( m \) in the \( d \)-dimensional theory as measured from the domain wall flat metric \( \eta_{\mu\nu} \) at \( z = 0 \) and \( z = L \), respectively, are
\[ m_0 = H(0)m = \left( e^{\pi R_c/L} - 1 \right)^{-1} m, \] (20)
\[ m_L = H(L)m_0 = \left( e^{\pi R_c/L} - 1 \right)^{-1} m, \] (21)
since \( H(0) = c \) and \( H(L) = c + 1 \). For \( R_c > L/\pi \) we find that
\[ m_0 \approx e^{-\pi R_c/L} m, \quad m_L \approx m. \] (22)
Thus, while at \( z = L \) masses as measured in the full \( d \)-dimensional metric and the wall flat metric are of the same order, they are exponentially suppressed at \( z = 0 \). As a result, we get exponential hierarchy in one boundary as in [3].

The \((b)\) case

Let us now discuss the second example \((b)\) in figure 1 for the function \( H(z) \). Here, \( H(z) \) is

\[
H(z) = c + \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin \left( \frac{n\pi z}{L} \right),
\]

which is again periodic with period \( 2L \). By using (13), it is straightforward to verify that

\[
H'' = \frac{4}{L} \left( \delta(z + \frac{L}{2}) - \delta(z + \frac{3L}{2}) \right),
\]

and that the energy-momentum tensor is in this case

\[
T_{\mu\nu} = \frac{4(d-2)}{L\kappa_d^2} \left( \delta(z + \frac{L}{2}) - \delta(z + \frac{3L}{2}) \right) H^3 \eta_{\mu\nu}.
\]

Thus, again the solution (23) describes two domain walls placed at \( z = L/2, 3L/2 \) respectively. The constant \( c \) can also be determined in terms of the cosmological constant \( \sim 1/L^2 \) and the compactification radius \( R_c \) which in this case is found to be

\[
R_c = \frac{L}{2\pi} \int_0^{2L} H^{-1} dz = \frac{L}{2\pi} \ln \left( \frac{c+1}{c-1} \right).
\]

Then we find from (26) that \( c \) is given in this case by

\[
c = \frac{e^{2\pi R_c/L} + 1}{e^{2\pi R_c/L} - 1}.
\]

Similarly to the case \((a)\), we find that masses \( m_{L/2} \) and \( m_{3L/2} \) measured with the flat \( \eta_{\mu\nu} \) metric at the domain walls in \( z = L/2 \) and \( z = 3L/2 \), respectively are related to the mass \( m \) measured with the metric (9) by

\[
m_{L/2} = \left( \frac{2e^{2\pi R_c/L}}{e^{2\pi R_c/L} - 1} \right)^{-1} m, \quad m_{3L/2} = \left( \frac{2}{e^{2\pi R_c/L} - 1} \right)^{-1} m.
\]

Thus, again for \( R_c > L/2\pi \) we get

\[
m_{L/2} \approx m, \quad m_{3L/2} \approx e^{-\frac{2\pi R_c}{L}} m,
\]

so that we have exponential hierarchy in one of the boundaries. This seems to suggest that this type of behaviour, i.e., exponential hierarchy at one of the two boundaries is universal.
3 Power-law hierarchies in ungauged supergravity

Let us now turn to standard supergravity in which there exist p-brane solutions involving the metric $G_{MN}$, the dilaton $\Phi$ and an antisymmetric form field strength of rang $n$. The p-brane is a charged object with electric charge if $p = n - 2$ and in this case the p-brane is elementary. If $p = d - n - 2$, where $d$ is the space-time dimensions, the p-brane has magnetic charge and it is solitonic. In the latter case we may also have $n = 0$ which means that there is no antisymmetric field strength but rather a cosmological-type term. The bosonic part of the supergravity action is then [18, 19]

$$S = \frac{1}{2\kappa_{d}^{2}} \int d^{d}x \sqrt{G} \left( R - \frac{1}{2} \partial_{M} \Phi \partial^{M} \Phi - \frac{2}{L^{2} \Delta} e^{-\alpha \Phi} \right), \quad (30)$$

where $\Delta = \alpha^2 - 2(d - 1)/(d - 2)$. The action (30) admits solitonic $d - 2$-brane solution of the form

$$ds^{2} = H^{\frac{4}{\Delta(\alpha - 2)}} \eta_{\mu \nu} dx^{\mu} dx^{\nu} + H^{\frac{4(d - 1)}{\Delta(\alpha - 2)}} dz^{2}, \quad (31)$$

$$e^{\Phi} = H^{\frac{2}{\Delta}}, \quad (32)$$

where, again $H = H(z)$ satisfies (8). By choosing $H$ as in figure 1, we get a background with two domain walls as in the gauged supergravity discussed above. For the case (a), the compactification radius $R_{c}$ turns out to be

$$R_{c} = \int_{0}^{L} H^{\frac{2(d - 1)}{\Delta(\alpha - 2)}} dz = \frac{\Delta L}{\pi \alpha} \left( (1 + c)^{\alpha^2/\Delta} - c^{\alpha/\Delta} \right). \quad (33)$$

Then, the masses $m_{L/2}$ as measured with the boundary flat metric $\eta_{\mu \nu}$ are related to the masses $m$ measured with the bulk metric (31) by $m_{L/2} = c^{\frac{2}{\Delta(\alpha - 2)}} m$ and a hierarchy may be generated for $c \ll 1$. From (33) we find then

$$m_{L/2} \approx \left( \frac{\pi R_{c}}{L} \right)^{\frac{\Delta}{\alpha^2}} \frac{\Delta}{\alpha^2} m \quad (34)$$

i.e., a power-law hierarchy. This is a general feature in this kind of solutions, namely, the $\alpha \neq 0$ case, which corresponds to domain wall solution in standard Poincaré supergravity, always leads to power-law hierarchy since the constant $c$ is a power of the compactification radius $R_{c}$. On the other hand, in gauged supergravities we may have $\alpha = 0$ leading to an exponential dependence of $c$ on $R_{c}$ and consequently to exponential hierarchies as explained in the previous section.

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Note added: While this work was in its final stage, we received [23] where backgrounds which realize exponential hierarchy was also constructed.
References


