

# State Vector Reduction as a Shadow of a Noncommutative Dynamics

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June 18, 1999

## Abstract

A model, based on a noncommutative geometry, unifying general relativity and quantum mechanics, is further developed. It is shown that the dynamics in this model can be described in terms of one-parameter groups of random operators. It is striking that the noncommutative counterparts of the concept of state and that of probability measure coincide. We also demonstrate that the equation describing noncommutative dynamics in the quantum mechanical approximation gives the standard unitary evolution of observables, and in the “space-time limit” it leads to the state vector reduction. The cases of the spin and position operators are discussed in details.

## 1 Introduction

There are many attempts to create a quantum theory of gravity, or at least a generalized version of the theory of general relativity, based on a noncommutative geometry (for example, [1, 2, 4, 9, 19, 20, 21, 26]). The starting

idea of all these works consists in transforming the space-time manifold into a noncommutative space. In the series of works (Refs. 12-15) we have proposed another strategy. It turns out that if one replaces space-time  $M$  with a groupoid  $G$  “over”  $M$ , one can construct a consistent model unifying general relativity and quantum mechanics. The idea is to define a noncommutative algebra  $\mathcal{A}$  of complex valued functions on the groupoid  $G$  (with convolution as multiplication) such that the algebra  $\mathcal{A}$ , if narrowed to (a subset of) its center  $\mathcal{Z}(\mathcal{A})$ , is isomorphic with the algebra  $C^\infty(M)$  of smooth functions on a space-time  $M$  (with the usual pointwise multiplication). Then a noncommutative (derivation based) differential geometry is developed in terms of the algebra  $\mathcal{A}$ , and a noncommutative generalization of general relativity is constructed. It turns out that, after this construction is done, the algebra  $\mathcal{A}$  can be completed to the  $C^*$ -algebra (it is called *Einstein  $C^*$ -algebra*). The next step is to quantize the system in close analogy with the standard  $C^*$ -algebraic method. Details of this approach are summarized in Section 2.

Noncommutative geometry, which in this approach is supposed to model the pre-Planck epoch, is strongly nonlocal with no local concepts (such as that of space point or time instant) having any meanings. In Sections 3 and 4 it is shown that, in spite of this, in the noncommutative regime, the true albeit generalized dynamics is available. It is only during the “phase transition” from the noncommutative geometry to the usual (commutative) geometry that space, time and other local structures emerge. It can be demonstrated that some nonlocal phenomena, known from quantum mechanics and cosmology, such as Einstein-Podolsky-Rosen experiment [14] or the horizon problem [17], can be explained as “shadows” or remnants of the primordial totally global regime. It also turns out that in the noncommutative regime the singularity concept loses its meaning (the algebra  $\mathcal{A}$  does not distinguish between singular and nonsingular states), and classical singularities are but products of the transition to standard physics of the commutative era [11, 16].

The main goal of the present paper is to show that the probabilistic character of quantum mechanics is a direct consequence of our model, and that the generalized noncommutative dynamics unifies in itself both the unitary evolution of quantum observables and the “reduction of the state vector”. The key point is that the von Neumann algebra  $\mathcal{M}$ , generated by the algebra  $\mathcal{A}$ , is both a “dynamical object” and a noncommutative generalization of the probability measure [3, 5]. On the strength of the Tomita-Takesaki theorem we introduce one-parameter groups which serve to define the generalized

evolution of von Neumann operators (i. e. elements of  $\mathcal{M}$ ). We demonstrate that these operators correspond to random operators on a Hilbert space and that generalized dynamics of these operators, in the quantum mechanical approximation, gives the unitary evolution of observables known from quantum mechanics (Section 5), and in the “space-time limit” it leads to the reduction of the state vector; the cases of the spin and position operators are analyzed in details. (Section 6). It looks as if the source of the “measurement problem” in quantum mechanics was the fact that quantum processes do not occur in space-time whereas the act of measurement is, out of its very nature, a spatio-temporal event. As a byproduct of this analysis we show that the generalized dynamics of our model need not be assumed a priori (by postulating an additional equation describing this evolution, as it was done in our previous works), but it can be deduced from the model under rather mild conditions. In Section 7, we collect our main results.

## 2 A Noncommutative Unification of General Relativity and Quantum Mechanics

In this section we briefly summarize our model in a quasi-axiomatic way; the model presented earlier (Refs. 12-15) could be considered as a special instant of this more general scheme.

1. Let us consider a product  $G = E \times \Gamma$ , where  $E$  is a smooth manifold (or, more generally, a differential space of constant dimension [10]), and  $\Gamma$  a Lie group acting on  $E$  (to the right).  $G$  can be given the structure of a smooth groupoid (see [3, p. 99], [23]). We additionally assume that there is a subset  $\tilde{\mathcal{Z}}$  of the center  $\mathcal{Z}(\mathcal{A})$  of the algebra  $\mathcal{A}$  such that  $\tilde{\mathcal{Z}}$  is isomorphic with  $C^\infty(M)$  where  $M$  is the space-time manifold (or its differential space generalization).

2. We define the involutive algebra  $\mathcal{A} = (C_c^\infty(G, \mathbf{C}), +, *, *)$  of compactly supported, complex valued functions on the groupoid  $G$  where “+” is the usual addition, and the multiplication is defined to be the convolution

$$(a * b)(\gamma) = \int_{G_q} a(\gamma_1)b(\gamma_2),$$

where  $\gamma = (q, qg) \in G$ ,  $\gamma = \gamma_1 \circ \gamma_2$ , and  $G_q$  is the fiber of  $G$  over  $q \in E$ ;

the integral is taken with respect to the (left) invariant Haar measure. The involution is defined in the following way:  $a^*(\gamma) = \overline{a(\gamma^{-1})}$ . Let us notice that instead  $\gamma = (q, qg)$  we can simply write  $\gamma = (q, g)$ .

3. Let  $\text{Der}\mathcal{A}$  be the set of all derivations of the algebra  $\mathcal{A}$ ; it has the structure of a  $\mathcal{Z}(\mathcal{A})$ -module. We define the differential algebra  $(\mathcal{A}, V)$  where  $V$  is a  $\mathcal{Z}(\mathcal{A})$ -submodule of  $\text{Der}\mathcal{A}$  of the form  $V = V_E \oplus V_\Gamma$  with  $V_E$  being the set of all derivations “parallel” to  $E$  and  $V_\Gamma$  the set of all derivations “parallel” to  $\Gamma$ .

4. A metric on  $V$  is defined to be a  $\mathcal{Z}(\mathcal{A})$ -bilinear nondegenerate symmetric mapping  $g : V \times V \rightarrow \mathcal{A}$ . For our model we choose the metric

$$g = pr_E^* g_E + pr_\Gamma^* g_\Gamma \quad (1)$$

where  $g_E$  and  $g_\Gamma$  are metrics on  $E$  and  $\Gamma$ , respectively, and  $pr_E$  and  $pr_\Gamma$  are the obvious projections. It has been demonstrated by Madore and Mourad [20] that for a broad class of derivation based differential algebras the metric is essentially unique. This is the case for the  $\Gamma$ -part of metric (1), but the  $E$ -part of this metric is determined by the Lorentz metric on the space-time  $M$ .

5. Now, we develop the noncommutative differential geometry as in [12, 15]: we define the linear connection (with the help of the Koszul formula), the curvature and the Ricci operator  $\mathbf{R} : V \rightarrow V$ , and we write the *noncommutative Einstein equation*

$$\mathbf{G} = 0 \quad (2)$$

where  $\mathbf{G} = \mathbf{R} + 2\Lambda\mathbf{I}$  with  $\mathbf{R}$  being the Ricci operator,  $\Lambda$  a constant related to the usual cosmological constant, and  $\mathbf{I}$  the identity operator;  $\ker\mathbf{G}$  is evidently the solution of this equation. Because of the form of metric (1) eq. (2) can be written in the form

$$\mathbf{G}_E + \mathbf{G}_\Gamma = 0$$

where  $\mathbf{G}_E$  is the part parallel to  $E$ , and  $\mathbf{G}_\Gamma$  is the part parallel to  $\Gamma$ . Since in the  $\Gamma$ -direction there is essentially one metric, the equation  $\mathbf{G}_\Gamma = 0$  should be solved for derivations  $v \in \ker\mathbf{G}_\Gamma \subset V_\Gamma$ . The equation  $\mathbf{G}_E = 0$  is a “lifting”

of the usual Einstein equation in the space-time  $M$  (therefore, it should be solved for the metric); all derivations  $v \in V_E$  satisfy it, and all derivations  $v \in V_\Gamma$  satisfy it trivially. It can be easily seen that  $\ker \mathbf{G} = \ker \mathbf{G}_E \oplus \ker \mathbf{G}_\Gamma$  is a  $\mathcal{Z}(\mathcal{A})$ -submodule of  $V$  (see [12, 15]).

6. Let  $\pi_q : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  be a representation of the algebra  $\mathcal{A}$  in the Hilbert space  $\mathcal{H} = L^2(G_q)$ , where  $\mathcal{B}(\mathcal{H})$  denotes the algebra of bounded operators on  $\mathcal{H}$ , given by the formula

$$(\pi_q(a)\psi)(\gamma) = (a_q * \psi)(\gamma), \quad (3)$$

$a_q$  is here a restriction of  $a \in \mathcal{A}$  to the fiber  $G_q$ ,  $q \in E$ , and  $\gamma \in G$ ,  $\psi \in \mathcal{H}$ . The completion of  $\mathcal{A}$  with respect to the norm  $\|a\| = \sup_{q \in E} \|\pi_q(a)\|$  is a  $C^*$ -algebra (see [3, p. 102]). This algebra will be called *Einstein  $C^*$ -algebra*.

7. We quantize the above system with the help of the algebraic method based on classical works by Jordan, von Neumann, and Wiener [18], Segal [24, 25], Haag and Kastler [8]. Let  $\mathcal{S}$  denote the set of states on the Einstein  $C^*$ -algebra  $\mathcal{A}$ . We assume that elements of  $\mathcal{S}$  represent states of the system and pure states of  $\mathcal{S}$  represent pure states of the system. Let  $a \in \mathcal{Z}(\mathcal{A})$  be a Hermitian element of  $\mathcal{A}$  and let  $\varphi \in \mathcal{S}$ . In such a case,  $\varphi(a)$  is the expectation value of the observable  $a$  if the system is in the state  $\varphi$ .

Let us, for simplicity, assume that  $\Gamma$  is a compact group (general case is discussed in [15]). Two fibres  $G_p$  and  $G_q$  of  $G$ ,  $p, q \in E$ , are said to be *equivalent* if there is  $g \in \Gamma$  such that  $q = pg$ . The set of all functions of  $\mathcal{A}$  which are constant on the equivalence classes of fibres of this equivalence relation are called *projectible functions*; they form a subalgebra of  $\mathcal{A}$  denoted by  $\mathcal{A}_{proj}$ . It can be easily seen that  $\mathcal{A}_{proj} \subset \mathcal{Z}(\mathcal{A})$ , and consequently  $\mathcal{A}_{proj}$  is a commutative algebra. In fact, it is isomorphic with the algebra  $C^\infty(M)$  of smooth functions on the space-time  $M$ . In this way, we recover the usual general relativity (in Geroch's formulation [7]). In subsequent sections we shall show that the standard quantum mechanics is also incorporated into our model.

We have computed the above presented model for the case in which  $G = E \times D_4$  where  $E$  is the total space of the frame bundle over the Minkowski space-time and  $D_4$  a group of 4 rotations and 4 reflections (it is a noncommutative subgroup of  $SU(2)$ ) [15] and, more generally, for the case when  $\Gamma$  is a finite group [12]. These cases should be regarded as “toy models” demonstrating the consistency of our approach.

### 3 Random Operators

Let  $\mathcal{A} = C_c^\infty(G, \mathbf{C})$  (in fact, in what follows we can assume that  $\mathcal{A}$  is the Einstein algebra). For each  $a \in \mathcal{A}$  there is a function  $\rho_a$  on  $E$  with values in the space of operators given by

$$\rho_a(p) = \pi_p(a)$$

for  $p \in E$ . The function  $\rho_a$  is said to be  $\Gamma$ -invariant if, for every  $g \in \Gamma$ ,  $\rho_a(pg) = \rho_a(p)$ .

**Lemma 3.1** *The function  $\rho_a$  is  $\Gamma$ -invariant if and only if  $a \in \mathcal{A}_{proj}$ .*

**Proof.** Let us notice that  $\rho_a(pg) = \rho_a(p)$  is equivalent to  $a_{pg} * \xi_{pg} = a_p * \xi_p$  which gives  $a_{pg} = a_p$ .  $\square$

**Lemma 3.2** *If  $\rho_a = \rho_b$  then  $a = b$  (almost everywhere).  $\square$*

Therefore, we have two equivalent descriptions of our noncommutative geometry: one in terms of the algebra of smooth, compactly supported, complex valued functions (with convolution as multiplication) on the groupoid  $G$ ; another in terms of the algebra of operator valued functions on  $E$ . The first description is, in many cases, easier to work with, and gives us the direct contact with better known commutative functional algebras (such as the algebra  $C^\infty(M)$  on a manifold  $M$ ); the second description gives us better insight when the underlying space is strongly singular.

Let us notice that the groupoid  $G = E \times \Gamma$  has the natural structure of foliation; this foliation will be denoted by  $\mathcal{F}$ . The fibres  $G_p$ ,  $p \in E$ , are leaves of the foliation  $\mathcal{F}$  (for simplicity, we assume that  $G$  is a smooth manifold). Let the space of leaves be denoted by  $Y$ . We should notice that  $Y$  is in the bijective correspondence with  $E$ . Let  $\lambda : G \rightarrow Y$  be the natural projection of the element  $\gamma \in G$  onto the leaf containing  $\gamma$ , i. e.,  $\lambda(\gamma) = G_p$  where  $\gamma = (p, g)$ ; we shall also write  $p = \text{beg}\gamma$ .

Let us consider a “bundle” of Hilbert spaces  $(L^2(G_{\lambda(\gamma)}))_{\gamma \in G}$ . Sections of this bundle form one-parameter families  $(\xi_\gamma)_{\gamma \in G}$  such that, for every  $\xi_\gamma$ ,  $\xi_\gamma \in L^2(\lambda(\gamma))$ .

**Definition 3.1** *The random operator is a one-parameter family  $r = (r_p)_{p \in E}$  of operators  $r_p \in \text{End}(L^2(G_p))$ ,  $p \in E$ , satisfying the following measurability condition: for any sections  $(\xi_\gamma)_{\gamma \in G}$  and  $(\eta_\gamma)_{\gamma \in G}$  of  $(L^2(G_{\lambda(\gamma)}))_{\gamma \in G}$  the function  $G \rightarrow \mathbf{C}$ , given by  $\gamma \mapsto (\pi_{\text{beg}(\gamma)}(a)\xi_\gamma, \eta_\gamma)$ , is measurable (in the usual sense).*

This definition is an application of the general concept of random operator [3, p. 51] to our case.

The norm of the random operator  $r$  is defined as  $\|r\| = \sup_{p \in E} \|r_p\|$ . The equivalence classes of random operators, modulo almost everywhere, equipped with the obvious algebraic operations, form the von Neumann algebra [3, p. 52] which will be denoted by  $\mathcal{N}$ . It is also called the *von Neumann algebra of the foliation*  $\mathcal{F}$ .

Let us notice that any function  $a \in \mathcal{A}$  determines the random operator  $\rho_a$  given by  $\rho_a = \pi_p(a)$  for  $p \in E$  (but not every random operator must be determined by a function  $a \in \mathcal{A}$ );  $\rho_a$  is in fact a one-parameter family of operators parametrized by the elements of the set  $E$ , and it is easy to check that it satisfies the condition of Definition 3.1.

**Proposition 3.1**  $\bigoplus_{p \in E} \pi_p(\mathcal{A})$  is a subalgebra of the von Neumann algebra  $\mathcal{N}$  of the foliation  $\mathcal{F}$ , and  $(\bigoplus_{p \in E} \pi_p(\mathcal{A}))''$  is a von Neumann subalgebra of  $\mathcal{N}$ .  $\square$

## 4 State Dependent Evolution of Random Operators

First, let us remind some well known concepts. Let  $A$  be a  $*$ -subalgebra of the algebra  $\mathcal{B}(\mathcal{H})$  of bounded operators on a Hilbert space. A vector  $\xi \in \mathcal{B}(\mathcal{H})$  is *separating* for the algebra  $A$  acting on  $\mathcal{H}$  if, for every  $T \in A$ ,  $T\xi = 0$  implies  $T = 0$ . A vector  $\xi \in \mathcal{B}(\mathcal{H})$  is *cyclic* for  $A$  acting on  $\mathcal{H}$  if  $A\xi$  is dense in  $\mathcal{H}$ . The fact that  $\xi$  is cyclic for  $A$  acting on  $\mathcal{H}$  implies that  $\xi$  is separating for  $A'$ . If  $A$  is a von Neumann algebra the reverse is also true (see [6, Appendix A14]). Now we go back to our case.

**Lemma 4.1** *If  $\mathcal{A}$  is an algebra with unit then the vector  $\xi = \mathbf{1} \in L^2(G_q)$  is cyclic for the von Neumann algebra  $(\pi_q(\mathcal{A}))''$  acting on  $L^2(G_q)$ .*

**Proof.** We have the obvious equality:  $\mathcal{A}_q = \pi_q(\mathcal{A})\mathbf{1}$  where  $\mathcal{A}_q = \{a_q : a \in \mathcal{A}\}$ . The functional space  $\mathcal{A}_q$  contains polynomial functions, therefore  $\mathcal{A}_q$  is dense in  $L^2(G_q)$ , and consequently  $\xi$  is cyclic in  $L^2(G_q)$ .

Let  $T = \pi_q(a)$ ,  $a \in \mathcal{A}$ , be an operator such that  $T\xi = 0$ . In such a case,  $a_q * \mathbf{1} = 0$  which implies  $a_q \cdot \mathbf{1} = 0$ , and  $T = 0$ .  $\square$

**Corollary 4.1** *If  $\mathcal{A}$  is the algebra with unit then the vector  $\xi = (\mathbf{1}_q)_{q \in E}$ , where  $\mathbf{1}_q \in L^2(G_q)$  and  $\mathbf{1}_q$  is the constant function equal to one, is cyclic and separable for the von Neumann algebra  $\mathcal{M} = (\bigoplus_{q \in E} \pi_q(\mathcal{A}))''$  acting on  $\bigoplus_{q \in E} L^2(G_q)$ .  $\square$*

This allows us to formulate the following theorem.

**Theorem 4.1** *Let  $\mathcal{A}$  be the algebra with unit. With every state  $\varphi$  on the von Neumann algebra  $\mathcal{M} = (\bigoplus_{q \in E} \pi_q(\mathcal{A}))''$  there is associated the one-parameter group  $(\alpha_t^\varphi)_{t \in \mathbf{R}}$  of automorphisms of  $\mathcal{M}$  given by*

$$\alpha_t^\varphi(b) = \Delta^{-it} b \Delta^{it}, \quad (4)$$

for every  $b \in \mathcal{M}$ , where  $\Delta = S^*S$ , and the operator  $S: \mathcal{M} \rightarrow \mathcal{M}$  is defined by  $S(b)(\xi_\varphi) = b^*(\xi_\varphi)$  where  $\xi_\varphi = (\xi_{\varphi_q})_{q \in E}$  is cyclic in  $L^2(G_q)_{q \in E}$ .  $\square$

**Proof.** On the strength of Theorem 1 from [13] there exists the unique state  $\varphi = (\varphi_q)_{q \in E}$ , where  $\varphi_q = (\pi_q(a)\xi_q, \xi_q)$  with  $\xi_q$  cyclic in  $L^2(G_q)$ , such that

$$\pi_{\varphi_q}(a)[b] = [\pi_q(a)(b)], \quad (5)$$

$b \in \pi_q \mathcal{A}$ , is a GNS representation of the algebra  $\mathcal{A}$ . Here [...] represents the element of the quotient space with respect to the ideal  $N_{\varphi_q} = \{a \in \mathcal{A} : \varphi_q(aa^*)\}$ . Now, the Tomita-Takesaki theorem [27] asserts that the mapping  $\alpha_t^\varphi : \mathcal{M} \rightarrow \mathcal{M}$ ,  $t \in \mathbf{R}$ , is a one-parameter group of automorphisms of the von Neumann algebra  $\mathcal{M}$  with the desired properties.  $\square$

The one-parameter group  $\alpha_t^\varphi$ ,  $t \in \mathbf{R}$ , is called the *modular group* of the state  $\varphi$  on the von Neumann algebra  $\mathcal{M}$  which, by Proposition 3.1, is obviously a von Neumann subalgebra of the von Neumann algebra  $\mathcal{N}$  of the foliation  $\mathcal{F}$ .

With any element  $a \in \mathcal{A}$  there is associated the random operator  $\rho_a = (\pi_q)_{q \in E}$ . If  $\rho_a$  belongs to the von Neumann algebra  $\mathcal{M}$  we can consider the

function of the form  $t \mapsto \alpha_t^\varphi(\rho_a)$  which defines a one-parameter group of random operators representing the “evolution” of these operators starting from the “initial” random operator  $\rho_a = \alpha_0^\varphi(\rho_a)$ . Let us take the closer look at this evolution.

If we assume that  $\rho_a \in \mathcal{M}$ , we have

$$\alpha_t^\varphi(\rho_a) = e^{-it\Delta} \rho_a e^{it\Delta}.$$

After differentiating and multiplying by  $i\hbar$  this equation assumes the form

$$i\hbar \frac{d}{dt} \Big|_{t=0} \alpha_t^\varphi(\rho_a) = [\rho_a, -\hbar \ln \Delta]. \quad (6)$$

The “Hamiltonian”  $-\hbar \ln \Delta$ , through the dependence on the endomorphism  $S$ , depends on the state  $\varphi$ . This equation should be regarded as describing a noncommutative dynamics of random operators.

For any random operator  $r = (r_q)_{q \in E}$  we define its *eigenfunction*  $\kappa(q)$ ,  $q \in E$  by the equation

$$r_q \xi = \kappa(q) \xi \quad (7)$$

for any  $\xi \in L^2(G_q)$  (for simplicity, we consider a nondegenerate case). We also define the *eigenfunction*  $\kappa : E \times \mathbf{R} \rightarrow \mathbf{C}$  for the “evolution” of a random operator  $r$  by the following equation

$$(\alpha_\tau^\varphi(r))_{q \in E} \xi = \kappa(q, t) \xi, \quad (8)$$

for any  $\xi \in L^2(G_q)$  where  $\alpha_\tau^\varphi(r)$  is a random operator at the instant  $t = \tau$ .

In the noncommutative regime there is no time, and consequently there cannot be any dynamical equations (in the usual sense). However, as we have seen, a noncommutative counterpart of dynamics is encoded in (state dependent) modular groups of random operators. It is important to see how these modular groups project to the standard dynamics in the quantum mechanical case.

First, let us notice that if a random operator is of the form  $r = (\pi_q(a))_{q \in E}$ , where  $a \in \mathcal{A}_{proj}$ , then the function  $\kappa : E \rightarrow \mathbf{C}$  is  $\Gamma$ -invariant, i. e.,  $\kappa(qg) = \kappa(q)$  for every  $g \in \Gamma$ .

**Lemma 4.2** *If  $a \in \mathcal{A}_{proj}$  then the operator  $\rho_a$  can be identified with the function  $\rho_a : M \rightarrow \mathbf{C}$ . For any  $x \in M$ ,  $\kappa(x)$  is the eigenvalue of the operator  $\rho_a(p)$  where  $\pi_M(p) = x$ .*

**Proof.** We have

$$\rho_a(p)\xi = a_p * \xi.$$

Since  $a \in \mathcal{A}_{proj}$ ,  $a_p = \text{const}$  on the set  $\pi_M^{-1}(x)$ . For  $x \in M$ , such that  $\pi_M(p) = x$ , one has  $a_p = \kappa(x)$  where  $\kappa : M \rightarrow \mathbf{C}$  is a function on  $M$  such that  $a = \kappa \circ \pi_M$ . Therefore,

$$\rho_a(p)\xi = \kappa(x) \cdot \xi. \quad (9)$$

Consequently,  $\kappa(x)$  is the eigenvalue of the operator  $\rho_a$  and we can identify  $\rho_a(p)$  with  $\kappa$ .  $\square$

**Corollary.** For  $a \in \mathcal{A}_{proj}$  the operator  $\pi_p(a) = \rho_a(p)$ ,  $p \in E$ , is a homothety with the constant  $\kappa(x)$ , and consequently its eigenspace is the whole space  $L^2(G_p)$ .  $\square$

The function  $\rho_a$  is, in fact, the spectrum of the operator  $a$ . If  $\rho_a$  is a random operator, the function  $\rho_a : M \rightarrow \mathbf{C}$  is measurable in the usual sense (because of the measurability condition in Definition 3.1) The function  $\kappa$  (or  $\rho_a(p)$  understood as a function on  $M$ ) is an eigenfunction of  $a$ . Of course, if  $a$  is Hermitian, the eigenvalues  $\kappa(x)$  are real.

Lemma 4.2 is expressed in terms of operator valued functions on  $E$ . However, it can be equivalently expressed in terms of the algebra  $\mathcal{A} = C^\infty(G, \mathbf{C})$ . It then says that with every  $a \in \mathcal{A}_{proj}$  there is the canonically associated function (measurable in the usual sense)  $\tilde{a} : \mathcal{M} \rightarrow \mathbf{C}$  such that  $\tilde{a} \circ \pi_M = a$ . For any  $x \in M$ ,  $a(x)$  is an eigenvalue of the operator  $\rho_a(q)$  where  $\pi_M(q) = x$ .

Let  $\mathcal{M} = (\bigoplus_{q \in E} \pi_q(\mathcal{A}))''$  be the von Neumann subalgebra of the von Neumann algebra  $\mathcal{N}$  of the foliation  $\mathcal{F}$ . It is easy to check that the mapping  $\rho : \mathcal{A} \rightarrow \mathcal{M}$  given by  $\rho(a) = \rho_a$ , for  $a \in \mathcal{A}$ , is a homomorphism of algebras, and consequently we have  $\rho(\mathcal{Z}(\mathcal{A})) \subset \mathcal{Z}(\mathcal{M}) \subset \mathcal{Z}(\mathcal{N})$ . It follows that if  $a \in \mathcal{A}_{proj} \subset \mathcal{Z}(\mathcal{A})$  then the one-parameter group  $\alpha_t^\varphi(a)$  is constant. Therefore, if we go to the space-time approximation (if we restrict to  $\mathcal{A}_{proj}$ ) the noncommutative dynamics is switched off. Let us notice, however, that this is valid only for a given state  $\varphi$ . It is not unlike in the Schrödinger picture of quantum mechanics in which operators are constant and all time dependence goes to the state vectors. We should expect that the dynamics reappears in the quantum mechanical approximation.

Such an approximation is obtained if we narrow the algebra  $\mathcal{A}$  to its subalgebra

$$\mathcal{A}_\Gamma := \{f \circ pr_\Gamma : f \in fC_c^\infty(\Gamma, \mathbf{C})\}$$

where  $pr_\Gamma : G \rightarrow \Gamma$  is the obvious projection. For any  $a \in \mathcal{A}_\Gamma$ , the random operator  $\rho_a = (\pi_q(a))_{q \in E}$  is a family of operators which can be identified with each other (because of the natural isomorphism of leaves of the foliation  $\mathcal{F}$ ). In this sense any random operator  $\rho_a$ , with  $a \in \mathcal{A}_\Gamma$ , is a constant family projectible to the “typical leaf”  $\Gamma$ . In such a case, the operator, to which the random operator  $\rho_a$  projects, will be denoted by  $a_\Gamma$ ; it belongs to  $\text{End}(L^2(\Gamma))$ . Let us notice that  $a_\Gamma$  is not a random operator since random operators are defined on the foliated space and not on the “typical leaf”. Now, eq. (6) assumes the form

$$i\hbar \frac{d}{dt} \Big|_{t=0} \alpha_t^\varphi(a_\Gamma) = [a_\Gamma, -\hbar \ln \Delta f]. \quad (10)$$

The modular group  $\alpha_t^\varphi(a_\Gamma)$  is here defined with respect to the von Neumann algebra  $(\pi_q(\mathcal{A}_\Gamma))''$  where  $q$  is any element of  $E$ . Eq. (10) describes the evolution depending on the state  $\varphi$ ; we shall return to this problem in the subsequent section.

It is interesting to notice that the modular group  $\alpha_t^\varphi$ ,  $t \in \mathbf{R}$ , determines the derivation  $v \in \text{Der} \mathcal{M}$  of the von Neumann algebra  $\mathcal{M}$ . We define

$$v(\pi_{\varphi_q}(a)) := \frac{d}{dt} (\alpha_t^\varphi(\pi_{\varphi_q}(a)))$$

where  $a \in \mathcal{M}$ , and the representation  $\pi_{\varphi_q}$  is defined by eq. (5). After simple calculations (see [13]), from eq. (4) we obtain

$$v(\pi_{\varphi_q}(a)) = i[\pi_{\varphi_q}(a), \ln \Delta] = i \text{ad}_{\ln \Delta}(\pi_{\varphi_q}(a)).$$

The one-parameter groups  $\alpha_t^\varphi$ ,  $t \in \mathbf{R}$ , for which there exists a derivation  $v \in \ker \mathbf{G}$  such that

$$v(\pi_{\varphi_q}(a)) = \frac{d}{dt} (\alpha_t^\varphi(\pi_{\varphi_q}(a)))$$

deserve to be called *integral curves* of the noncommutative Einstein equation.

## 5 Unitary Evolution of Random Operators

In this Section we show that eq. (6), in the quantum mechanical approximation, gives the usual unitary evolution of quantum observables.

Let us first notice that the (above defined) homomorphism of algebras  $\rho: \mathcal{A} \rightarrow \mathcal{M}$  is a monomorphism. Indeed, if  $\rho(a) = 0$  then  $(\pi_q(a))_{q \in E} = 0$  which implies that  $a_q = 0$  for each  $q \in E$ , and this means that  $a = 0$ . Therefore, we have proved the following lemma.

**Lemma 5.1**  $\rho: \mathcal{A} \rightarrow \rho(\mathcal{A})$  is an isomorphism of algebras.  $\square$

Let  $\mathcal{U} = \{u \in \mathcal{A} : uu^* = u^*u = 1\}$  be the unitary group of the algebra  $\mathcal{A}$ . Then  $\rho(\mathcal{U})$  is the unitary group of the algebra  $\rho(\mathcal{A})$ . Let us notice that for the subalgebra  $\mathcal{A}_\Gamma \subset \mathcal{A}$  we have  $\rho(\mathcal{A}_\Gamma) \subset \rho(\mathcal{A})$  and the unitary group of this subalgebra is of the form

$$\mathcal{U}_\Gamma = \{u \in \mathcal{A}_\Gamma : uu^* = u^*u = 1\}.$$

Evidently  $\mathcal{U}_\Gamma \subset \mathcal{U}$ , and correspondingly  $\rho(\mathcal{U}_\Gamma) \subset \rho(\mathcal{U})$ .

Let  $\mathcal{R} = (\rho(\mathcal{A}_\Gamma))''$  be the von Neumann algebra generated by the algebra of random operators  $\rho(\mathcal{A}_\Gamma)$ . In agreement with the general construction [5, 13], the automorphisms  $\alpha' : \mathcal{R} \rightarrow \mathcal{R}$  and  $\alpha'' : \mathcal{R} \rightarrow \mathcal{R}$  are said to be *inner equivalent* if there is an element  $u \in \rho(\mathcal{U}_\Gamma)$  such that

$$u\alpha''(b) = \alpha'(b)u$$

for every  $b \in \mathcal{R}$ . The set of equivalence classes of this relation is called the *group of outer automorphisms* and is denoted by  $\text{Out}\mathcal{R}$ . As well known, the one-parameter group  $\alpha_t^\varphi$ ,  $t \in \mathbf{R}$ , canonically projects onto the (nontrivial) one-parameter group  $\tilde{\alpha}_t$ ,  $t \in \mathbf{R}$ , in  $\text{Out}\mathcal{R}$  which is independent of the state  $\varphi$ .

From eq. (6) it follows that  $\tilde{\alpha}_t$  satisfies the following equation in  $\text{Out}\mathcal{R}$

$$\frac{d}{dt}\Big|_{t=0}[\alpha_t(a)] = i[[a], [\ln \Delta]]$$

where [...] denotes the equivalence class of inner equivalence. This equation can also be written in the form

$$i\hbar \frac{d}{dt}\Big|_{t=0}\tilde{\alpha}_t(\tilde{a}) = [\tilde{a}, H]$$

where  $H = -\hbar[\ln \Delta]$ . This equation, after being projected to the ‘‘typical leaf’’  $\Gamma$ , assumes the form

$$i\hbar \frac{d}{dt}\Big|_{t=0}\tilde{\alpha}_t(\tilde{a}_\Gamma) = [\tilde{a}_\Gamma, H] \quad (11)$$

which is the same as eq. (10) but now independent of the state  $\varphi$ . This is, in fact, the Schrödinger equation in the Heisenberg picture of quantum mechanics in which operators evolve but state vectors are time independent. In this way, by projecting to the “typical leaf”  $\Gamma$ , we recover from our model the unitary evolution of ordinary quantum operators.

## 6 Reduction of the State Vector

The product structure of the groupoid  $G = E \times \Gamma$  plays the essential role in our model. The “ $E$ -component” of the model is, in principle, responsible for its gravitational effects, whereas the “ $\Gamma$ -component” is responsible for quantum mechanical effects. In the quantum mechanical approximation we simply forget about the  $E$ -component effects. In this Section we show that precisely this fact leads to the effect which, from the  $\Gamma$ -perspective, looks like the reduction of the state vector. First, however, we must do some preparatory work.

Let us consider a function  $f : G \rightarrow \mathbf{C}$  on the groupoid  $G = E \times \Gamma$ . With fixed  $g \in \Gamma$  we obtain the function  $f_g : E \rightarrow \mathbf{C}$  given by

$$f_g(p) = f(p, g)$$

for  $p \in E$ . We recognize in it the eigenfunction  $\kappa(q)$  of equation (7). This function determines the one-parameter family of functions  $(f_g)_{g \in \Gamma}$ . For a fixed  $g \in \Gamma$  we obtain the sequence of the values  $(f_g(p))_{p \in E}$  of the function  $f$  on the fibre  $E \times \{p\}$  for each  $p \in E$ . In particular, for any Hermitian element  $a \in \mathcal{A}_{proj}$  we obtain the sequence of real values  $a_g(p), p \in E$ . In this case, the sequence  $(a_g(p))_{p \in E}$  does not depend of  $g \in \Gamma$  (since elements of  $\mathcal{A}_{proj}$  are constant on fibres  $G_p$  for every  $p \in E$ ). On the strength of Lemma 4.2 and the subsequent corollary, if  $a \in \mathcal{A}_{proj}$  is Hermitian then the random operator  $\rho_a = (\pi_p(a))_{p \in E}$  is a one parameter family of homotheties with constants  $a(p, g)$  where  $g$  is any fixed element of the group  $\Gamma$ , and the operator  $r_p = \pi_p(a)$ , for a fixed  $p \in E$ , satisfies the eigenvalue equation

$$r_p \xi = \kappa(p) \xi \tag{12}$$

for  $\xi \in L^2(G_p)$ . The eigenspace of the operator  $r_p$  is the whole Hilbert space. Since  $a \in \mathcal{A}_{proj}$  the eigenfunction  $\kappa(p) = a(p, g)$  assumes constant values on

the fibres  $\pi^{-1}(x)$ ,  $x \in M$ . Consequently, there is the real valued function  $\tilde{\kappa} : M \rightarrow \mathbf{R}$  such that  $\tilde{\kappa} \circ \pi_M = \kappa$ , and the random operator  $\rho_a = (r_p)_{p \in E}$  has the following set of eigenvalues

$$\{\kappa(p) : p \in E\} = \{\tilde{\kappa}(x) : x \in M\}.$$

Let us notice that in every act of measurement the measuring apparatus is always located at a given point in space-time  $x \in M$ . This automatically causes the function  $\tilde{\kappa}$  to “collapse” to its value  $\tilde{\kappa}(x)$  (in the examples below we shall see that this indeed is connected with the reduction of the state vector). Such a procedure is meaningful only with respect to operators which commute with the position operator since measuring the eigenvalue  $\tilde{\kappa}(x)$ , for a given  $x$ , presupposes the knowledge of  $x \in M$ .

The above analysis is carried out from the perspective of the  $E$ -component of our model; to go back to the standard measurement interpretation in quantum mechanics we must see how the process looks like from the perspective of its  $\Gamma$ -component. Let us choose an orthonormal basis  $\{\psi_n(g)\}$  in the Hilbert space  $L^2(\Gamma)$ . We are looking for the operator  $\rho_\Gamma$  acting in this space the eigenvalues of which would be  $\kappa(p)$ . This does not necessarily mean that the spectrum of this operator is continuous. For instance, if the function  $\kappa(p)$  is constant there is only one eigenvalue. Let us first assume that the spectrum  $\kappa(p)$  is discrete. In this case, the looked for operator is

$$\rho_\Gamma = \sum_n \kappa_n P_{\psi_n},$$

where  $P_{\psi_n}$  is the projector onto the direction determined by  $\psi_n$  (for simplicity, we consider the nondegenerate case). In the case of continuous spectrum, we proceed analogously and use the corresponding spectral theorem (see below Example 2). In this way, we recover the standard formalism of quantum mechanics. If this formalism is taken separately, without paying attention to what happens in the  $E$ -direction, all interpretative problems of quantum mechanics immediately arise. The following examples show that these problems are naturally solved if the model is regarded in its totality.

**Example 1. Spin measurement.** In [14] it has been shown that to the usual z-component spin operator  $\hat{S}_z$  there correspond two elements  $s_1$  and  $s_2$  of the noncommutative algebra  $\mathcal{A}$  such that

$$\pi_p(s_1)\psi = +\frac{\hbar}{2}\psi \quad \text{if } \psi \in \mathbf{C}^+$$

and

$$\pi_p(s_2)\psi = -\frac{\hbar}{2}\psi \text{ if } \psi \in \mathbf{C}^-$$

where  $\mathbf{C}^+ = \mathbf{C} + \{0\}$  and  $\mathbf{C}^- = \{0\} + \mathbf{C}$ . Since  $s_1$  and  $s_2$  are observables we assume that they are Hermitian and elements of  $\mathcal{A}_{proj}$ ; consequently, they can be regarded as real valued functions on  $M$  defined by:  $s_1 = +\frac{\hbar}{2}$  and  $s_2 = -\frac{\hbar}{2}$ . Since  $s_1$  and  $s_2$  are constant functions they also belong to the subalgebra  $\mathcal{A}_\Gamma$ . Both  $s_1$  and  $s_2$  are homotheties, and consequently the entire Hilbert space  $L^2(G_p)$  is the eigenspace of these operators. This means that the results of the spin measurements are strictly predetermined (i. e., they are obtained with certainty), although at the present we do not know the mechanism of this predetermination. However, for the sake of concreteness, let us naively assume that it is given by the following random operator

$$r_p = \begin{cases} \pi_p(s_1) & \text{if } \pi^3(p) \geq 0, \\ \pi_p(s_2) & \text{if } \pi^3(p) < 0, \end{cases} \quad (13)$$

where  $\pi^3(p) = x^3$  is the projection onto the third space coordinate ( $z$ -coordinate). It is indeed the random operator since the mappings  $\gamma \mapsto (+\frac{\hbar}{2}\xi_\gamma, \eta_\gamma)$  and  $\gamma \mapsto (-\frac{\hbar}{2}\xi_\gamma, \eta_\gamma)$  are measurable.

To see what happens in the perspective of the observer performing the measurement we must situate the observer in space-time  $M$  (the  $E$ -component of our model). Let us suppose that the measuring apparatus is at a space-time point  $x = \pi_M(p)$ ,  $p \in E$ . The result of the measurement will be  $+\frac{\hbar}{2}$  or  $-\frac{\hbar}{2}$  (with probability 1) depending on whether  $\pi^3(p) \geq 0$  or  $\pi^3(p) < 0$  with  $\pi_M(p) = x$ . These two conditions are not known to the observer, therefore, in computing the probability of the result he uses the  $\Gamma$ -perspective (i. e., the standard machinery of quantum mechanics), and the outcome of the measurement looks for him as the ‘‘collapse of the wave function’’. We can see this by choosing two orthonormal vectors  $\psi_1, \psi_2 \in L^2(\Gamma)$  which span the subspace  $\langle \psi_1, \psi_2 \rangle_{\mathbf{C}} \subset L^2(\Gamma)$ ; then with the help of the spectral theorem we recover the usual spin operator

$$\hat{S}_z = +\frac{\hbar}{2}P_{\psi_1} - \frac{\hbar}{2}P_{\psi_2}$$

where  $P_{\psi_1}$  and  $P_{\psi_2}$  are projecting operators onto the directions determined by  $\psi_1$  and  $\psi_2$ , respectively. In this way, the operator  $\hat{S}_z$  is determined by the

random operator  $r_p$ . As usually, if the system is in the state  $\varphi$  the probability that the result of a measurement will give  $+\frac{\hbar}{2}$  is  $|\langle\varphi, \psi_1\rangle|^2$ , and analogously for  $-\frac{\hbar}{2}$ . Of course, in the act of measurement the state vector  $\varphi$  collapses either to the eigenvalue  $+\frac{\hbar}{2}$  or to the eigenvalue  $-\frac{\hbar}{2}$  in agreement with the standard procedure of quantum mechanics.

The conditions “either  $\pi^3(p) \geq 0$  or  $\pi^3(p) < 0$ ” in the formula (13) were put there by hand and we could easily imagine some other conditions which would do the job. However, it could be hoped that if the theory is more developed (i. e., if the concrete algebra  $\mathcal{A}$  and the concrete group  $\Gamma$  are chosen basing on physical grounds), the correct mechanism selecting either  $\pi_p(s_1)$  or  $\pi_p(s_2)$  will be determined by the theory itself. By now, we could only guess that this mechanism is connected with the random character of the operator  $r_p$  (formula (13)). The essential point is that since  $\pi_p(s_1)$  is a homothety the eigenspace of the eigenvalue  $+\frac{\hbar}{2}$  is the entire Hilbert space, and the result of the measurement must be strictly predetermined. The same is true for the eigenvalue  $-\frac{\hbar}{2}$ . This means that there is some “very well hidden mechanism” predetermining the outcome of the measurement.

**Example 2. Position measurement.** Let, for simplicity,  $M$  be  $\mathbf{R}^4$ , and let  $\widetilde{pr}_k : \mathbf{R}^4 \rightarrow \mathbf{R}$ ,  $k = 0, 1, 2, 3$ , be the projection function defined by  $\widetilde{pr}_k(x^0, x^1, x^2, x^3) = x^k$ . One can see that  $pr_k = \widetilde{pr} \circ \pi_M \circ \pi_E$ , ( $\pi_E : G \rightarrow E$  being the obvious projection) is a Hermitian element of  $\mathcal{A}_{proj}$ . It can be easily guessed that  $pr_k$  is an observable corresponding to the position operator. Its eigenvalue equation is

$$\pi_p(pr_k)\xi = pr_k(x)\xi \quad (14)$$

for  $\xi \in L^2(G_q)$ , where  $x = \pi_M(p)$ ,  $p \in E$ . Hence we obtain

$$\pi_p(pr_k)\xi = x^k\xi, \quad (15)$$

as it should be (the position operator acts by multiplication). The spectrum of the position operator  $\pi_p(pr_k)$  is evidently  $\mathbf{R}$ . The operator  $\pi_p(pr_k)$  is a homothety, and consequently the entire Hilbert space  $L^2(G_p)$  is the eigenspace corresponding to the eigenvalue  $pr_k(x)$ . In other words, the result of the position measurement of a quantum object is always predetermined, although at the present stage of the development of the model the mechanism of this predetermination is not known. Therefore, the only thing we could do is to change to the  $\Gamma$ -perspective. We simply look for the operator acting on the Hilbert space  $L^2(\Gamma)$  the spectrum of which is equal to  $\mathbf{R}$ . By using the

spectral theorem we find (in the one-dimensional case)

$$\hat{X} = \int_{-\infty}^{+\infty} x dE(x)$$

where  $E$  is a suitable spectral measure (see, for instance, [22, pp.24-31]). Then we can write down the standard eigenvalue equation (which essentially is the same as eq. (15)) and compute the probabilities of the expected results. After completing the measurement we would say that the “wave function has collapsed”. However, if the entire model is taken into account there is no real collapse; the measurement result is strictly predetermined by the fact that  $\pi_p(pr_k)$  is a homothety.

## 7 Conclusions

The overview picture that emerges from the above analysis is the following. In general, the noncommutative regime is atemporal. The only meaning which we can ascribe to the term “dynamics” is through the fact that certain geometric quantities are expressed in terms of derivations of the algebra  $\mathcal{A}$  defining the considered noncommutative geometry. Derivations are counterparts of vector fields, and as such they can be regarded as modelling certain type of “global change”. The concept of dynamics improves if the algebra  $\mathcal{A}$  has properties admitting the existence of one-parameter modular groups (see [13]). Then the von Neumann algebra, generated by the algebra  $\mathcal{A}$ , becomes a dynamical object (see [5]), and modular groups describe the “evolution” of the corresponding operators. In what follows, we shall assume that the algebra  $\mathcal{A}$  admits the existence of modular groups.

In our model this means that the dynamics of the system is described by eq. (6). This dynamics (or “evolution”) depends on the state  $\varphi$  in which the system finds itself. It is especially interesting to consider the evolution of random operators. This is not a limitation since, by Lemma 5.1 there is an isomorphism between the algebra  $\mathcal{A}$  and the von Neumann algebra of random operators. This fact has its further important consequences. Random operators, as elements of the von Neumann algebra, are probabilistic objects albeit in a generalized sense. Let us remind that the noncommutative counterpart of the probability space is a pair  $(\mathcal{M}, \varphi)$  where  $\mathcal{M}$  is a von Neumann algebra and  $\varphi$  a (faithful and normal) state on  $\mathcal{M}$ . By the definition

of state,  $\varphi$  is positive (i. e.  $\varphi(aa^*) \geq 0$  for every  $a \in \mathcal{M}$ ), and normalized (i. e.,  $\varphi(\mathbf{1}) = 1$ ), in close analogy to the standard probability measure. It is striking that in the noncommutative regime the concept of state and that of probability measure coincide. If we go to the quantum mechanics approximation these concepts split but remain strictly interconnected. We are entitled to say that the probabilistic character of quantum mechanics is the consequence of the fact that the quantum mechanical observables are but “shadows” (projections) of random operators.

We have demonstrated that in the dynamical equation (6) there are encoded both the unitary evolution of observables of the standard quantum mechanics and the reduction of the state vector (“collapse of the wave function”) occurring in the act of measurement of a given observable. To obtain the unitary evolution of observables of the usual quantum mechanics two steps must be made. The first step is to project dynamical equation (6) to the “typical leaf”  $\Gamma$ . This leads to eq. (10). As the consequence of this procedure the random operator  $\rho_a$  changes into the ordinary operator  $a_\Gamma$ , but its evolution is still state dependent. In the second step, we form the group of outer automorphisms (see Section 5) and obtain one-parameter groups  $\tilde{\alpha}_t, t \in \mathbf{R}$ , which are now state independent. With this modification dynamical equation (10) changes into eq. (11) which is the usual Schrödinger equation (in the Heisenberg picture) of quantum mechanics describing the unitary evolution of observables. This process of the transition to quantum mechanics “truncates” the model leaving aside its  $E$ -component (in this sense the usual quantum mechanics is incomplete). We should notice that precisely the  $E$ -component of the model is responsible for all measurements (every measuring device as a macroscopic object is situated in space-time). We have shown that exactly this “truncation” is the reason why what is seen from the  $\Gamma$ -perspective looks like a sudden collapse of information in the act of measurement.

We could briefly summarize the situation by saying that the unitary evolution of quantum observables and the reduction of the state vector in the act of their measurements are but two different “projections” of the same process, namely, of the generalized dynamics in the noncommutative regime.

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