Theoretical and Phenomenological Aspects of Superstring Theories

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Abstract

We discuss aspects of heterotic string effective field theories in orbifold constructions of the heterotic string. We calculate the moduli dependence of threshold corrections to gauge couplings in (2,2) symmetric orbifold compactifications. We perform the calculation of the threshold corrections for a particular class of abelian (2,2) symmetric non-decomposable orbifold models, where the internal twist is realized as a generalized Coxeter automorphism. We define the limits for the existence of states causing singularities in the moduli space in the perturbative regime for a generic vacuum of the heterotic string. The 'proof' provides evidence for the explanation of the stringy 'Higgs effect'. Furthermore, we calculate the moduli dependence of threshold corrections as target space invariant free energies for non-decomposable orbifolds, identifying the 'Hauptmodul' functions for the relevant congruence subgroups. The required solutions provide for the \( \mu \) mass term generation in the effective low energy theory and affect the induced supersymmetry breaking by gaugino condensation. In addition, we discuss one loop gauge and gravitational couplings in (0,2) non-decomposable orbifold compactifications. In the second part of the thesis, the one loop correction to the Kähler metric for a generic \( N = 2 \) orbifold compactification of the heterotic string is calculated as solution of a partial differential equation. In this way, with the use of the one loop string amplitudes, the prepotential of the vector multiplets of the \( N = 2 \) effective low-energy heterotic strings is calculated in decomposable toroidal compactifications of the heterotic string in six-dimensional \( N = 1 \) string vacua. This method provides the solution for the one loop correction to the prepotential of the vector multiplets for the heterotic string compactified on the \( K_3 \times T^2 \) manifold. Moreover, using modular properties of the one loop prepotential, we calculate it for \( N=2 \) heterotic strings coming from \( N=1 \) orbifold compactifications of the heterotic string based on non-decomposable torus lattices. These sectors appear in \( N = 1 \) orbifold compactifications of the heterotic string on non-decomposable torus lattices. In the third part of the thesis we discuss supersymmetry breaking through gaugino condensates in the presence of the subgroups of the modular group \( SL(2, \mathbb{Z}) \). We examine the way we can modify a known semirealistic model to incorporate S- and T- dualities in the superpotential of its effective action. We show how the discrete isometries of the effective theory restrict the Kähler potential and the superpotential of the effective theory together with implications for the globally supersymmetric case. Finally, we discuss the effect of our one-loop computation of the first part of the thesis on the \( \mu \) term in orbifold compactifications.
Where the * sign appears, is a sign of indication that at this part of the thesis, new results are obtained.

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CHAPTER 1
1. Introduction

String theory still remains, after a period of over ten years of development, the only candidate for a quantum theory of gravity, which we believe to finally describe consistently and accurate the particle world. A great obstacle, in the interpretation of properties of string theory, that could exhibit the novel futures of the formalism and allow us to extract the physical information, remains the mathematical structure and our ability to improve the calculability of perturbative and non-perturbative aspects of the theory. However, string theory is unique. It gives partial or complete answers to the biggest problems of particle physics. Namely, spacetime supersymmetry breaking, cosmological constant, strong and weak CP violation, flavour changing neutral currents, determination of the unbroken gauge group of the theory. In this Thesis, we will be discussing in detail some of the above problems, while we will mention the rest.

In addition, in string theory there is another another problem, the determination of the underlying principle able to choose the correct vacuum, of the final theory, among the huge degeneracy of string vacua\cite{8, 9, 10, 11, 80}. Determination of the correct vacuum will give the realistic three generation model which may be extension of the standard model and may include the new physics beyond the electroweak scale. This problem is largely unanswered, as string vacua provide us with a huge number of possible semirealistic models, which all but failed to satisfy all the phenomenological requirements. In the most popular scenaria of recent years\cite{92, 94, 809, 311} the minimal supersymmetric standard model(MSSM) was used a priori as an effective low energy theory of string vacua. Its presence gives quantative results for string vacua since it has not been proved to be coming from a particular string vacuum. We should note that in all the model constructions up to now, where the gauge group contains part of the standard model gauge group, the particle content of string models is not in any case of the MSSM. When it does appear it is corrected by additional particles.

The problem of proliferation of the string vacuum is left untouched from string perturbation theory. In particular, supersymmetric models coming from superstring vacua appear in great numbers and there is no underlying principle to distinguish between different categories apart from phenomenological criteria\cite{92, 94}.

The simplest string model, the classical bosonic string represents a generalization of the
one-dimensional point particle action $S = -m \int ds$, to an n dimensional object, which sweeps an $(n + 1)$ worldvolume as it moves into space, described by the metric $h_{\alpha\beta}(\sigma)$ and the action $S = -(T/2) \int d^{n+1}\sigma \sqrt{h} h^{\alpha\beta}(\sigma) g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$. Here, $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$ and $\sigma^0(\tau) = \tau$ and $\sigma^i, i = 1, \ldots, n$ is an n-dimensional object $^1$, while $T$ is the string tension with coefficient of $(\text{mass})^{n+1}$. Especially, for strings the tension $T$ is given in terms of the Regge slope $T \propto \frac{1}{\alpha'}$. In this Thesis, we will discuss properties of oriented closed strings (OCS). For oriented closed string the physical spectrum is invariant under the level matching constraint $^3$. For non-orientable clodes strings physical states must also be invariant under the operator $F$, which exchanges e.g, for the bosonic string, the $\alpha$ and $\tilde{\alpha}$ oscillators of the left and right moving number operators $N = \sum_{i=1}^{\infty} n\alpha^k_i \alpha^k_i$, $\tilde{N} = \sum_{i=1}^{\infty} n\tilde{\alpha}^k_i \tilde{\alpha}^k_i$, respectively.

In the early days of its development string theory was used as a theory of hadrons. The important step in realizing string theory as a theory of fundamental interactions was taken in $^1$, $^2$ where it was shown that the effective action of a massless spin two state is in the zero slope limit given by the Einstein-Hilbert lagrangian. However, it was not until 1984 when Green and Schwarz $^4$ proved that the type I theories $^5$ are free of anomalies which made the physics community realized the importance of superstring theories. They proved that, by adding non-gauge invariant local counterterms for the two form $B$ in the effective low energy theory $^6$ as

$$\delta B = tr(AdA) - tr(\omega d\theta), \quad (1.1)$$

where $\Lambda, \theta$ are infinitesimal gauge and Lorentz transformations of the ten dimensional background gauge field and spin connection $A$ and $\omega$ respectively, the theory remains free of gauge, gravitational and mixed gauge and gravitational anomalies when the Yang-Mills gauge group is $SO(32)$ or $E_8 \times E_8$. While type I theories with group $SO(32)$ where known at the time, the theory corresponding to $E_8 \times E_8$ was not known. It was soon after the heterotic string was build, where gauge symmetries coming from the group $E_8 \times E_8$ were observed, even not chiral, that a Kaluza-Klein origin of gauge symmetries as isometries of the internal manifold was suggested $^8$. However,

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$^1$ where the invariant integral is given by $ds^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu$.

$^2$ For $n = 0$ we have the point particle, for $n = 1$ we have the string, while for $n = 2$ we have the membrane and so on . . . .

$^3$See the footnote following 3.31.

$^4$Unoriented open and closed strings with $N = 1$ supersymmetry.

$^5$ The ten dimensional $N = 1$ supergravity coupled to matter has anomalies coming from heptagon diagrams.
compactification of the heterotic string on toroidal backgrounds produces unwanted non-chiral models with extended $N = 4$ supersymmetry. Soon after [144], by examining the low energy lagrangian of $N = 1$ supergravity, the low energy limit of the heterotic string when the Regge slope $\alpha' \to 0$, supersymmetric solutions in four dimensions were found. They correspond to classical solutions of the string equations\textsuperscript{6} where the internal manifold is a smooth manifold, the so called Calabi-Yau manifold. The phenomenologically interesting $N = 1$ supersymmetry comes by demanding\textsuperscript{7} that the manifold has $SU(3)$ holonomy\textsuperscript{8}. The theory is subject to gauge symmetry breaking by twisting the boundary conditions in a way that does not preserve charges corresponding to the broken symmetries of the world sheet. The problem of determination of the gauge group of the theory, consistently attached to the grand unified theories [17] and their supergravity successors [256, 257] in string models is solved. However, it remains the problem of determining the derivation of the standard model from it. A number of different methods of producing consistent compactifications of the heterotic string has been constructed [80, 83, 11, 11, 208, 211, 31, 8, 9] which all but one will not be reviewed in this Thesis. The research carried out in this Thesis, will be based in the theory of orbifolds [80]. In this Thesis, we will discuss the results of our research of phenomenological and theoretical aspects of the orbifold constructions of the heterotic string.

In chapter two, we discuss elements of the basic theory which is extensively well known [80]. Orbifolds are constructed, by twisting boundary conditions to break Lorentz symmetries, so that spacetime coordinates of the $x^\mu(\sigma)$ are not periodic functions of $\sigma$ but periodic up to Lorentz transformations. Mathematically, this means that compactifying the heterotic string on a six dimensional torus $T$ and dividing by a non freely acting symmetry group $G$. There are sectors in the theory, called twisted sectors, where the coordinates of the string obey not $X^i(\sigma + 2\pi) = X^i(\sigma)$ but $X^i(\sigma + 2\pi) = gX^i(\sigma)$ for $g \in G$.

Superstring theory offers a large number of theoretical arguments which single out its uniqueness. It combines quantum field theory with general relativity. It possesses a minimal number of parameters, namely the string scale\textsuperscript{4}, and in addition a number of fields called moduli, whose vacuum expectation values enter the calculation of the basic functions which determine the fermion

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\textsuperscript{6} They examined the field theory limit of the heterotic string compactified on the space $M^4 \times K$, where $K$ a compact manifold.

\textsuperscript{7} see chapter three

\textsuperscript{8} at one loop level the string scale is corrected by moduli dependent effects
and boson terms of the low energy energy $N = 1$ lagrangian. The low energy limit of orbifold models\cite{8} is that of $N = 1$ supergravity coupled to super Yang-Mills modified by moduli dependent effects characterizing the orbifold vacuum\cite{144, 301, 236, 137}.

In this way, the running gauge couplings\cite{58} exhibit a specific moduli dependence characterized from invariances of the moduli variables. In chapter three, we discuss the calculation of threshold corrections for a particular class of non-decomposable orbifold models. We discuss as well, some aspect of the gauge couplings of the theory related, as the string analog of Seiberg-Witten theory\cite{142}, to special points of gauge symmetry enhancement. It has to do with the appearance of gauge symmetry enhancement at special points in the moduli space and its contribution to the gauge coupling constants for different regions in the moduli space. In general, when in a region of moduli space there is a point where a previously massive states become massless, then the effective gauge couplings exhibit a behavior like in eqn.\cite{3.21}.

In addition, in chapter three we discuss the calculation of target space free energies\cite{132, 120, 67} as moduli dependent threshold corrections, coming from integration of massive compactification modes. The calculation uses the free energy, the effective action coming from the integration of the massive compactification modes. We use this result to calculate the physical effects of the integration of the massive compactification modes to the calculation of the threshold corrections for the gauge and gravitational coupling constants in orbifold models. This method is alternative to the calculation of the effective action coming from a direct string loop calculation, and it must give the same result if the associated sums are performed exactly. Unfortunately at the moment, complete regularization of the sums is not possible.

The general method of determining the exact string effective action of the massless modes comes from the calculation of string diagrams which have been performed e.g for the heterotic string\cite{19, 21} or for orbifolds\cite{301, 71, 56, 236}. The low energy theory of $N = 1$ orbifolds is fixed in terms of the Kähler potential $K$, the superpotential $W$ and the function $f$ which determines the kinetic terms for the gauge fields. In $N = 1$ locally supersymmetric theories the superpotential $W$ calculates the Yukawa couplings of the chiral matter superfields to the Higgs scalars. The superpotential of the effective theory of $(2,2)$ $N = 1$ orbifolds depends on the moduli fields which have a flat potential to all orders of string perturbation theory. As a result their values remain undetermined and the superpotential is not renormalized to all orders of
string perturbation theory due to non-renormalization theorems\cite{14} but is corrected from world sheet instantons\cite{15,301}. As a result moduli dependent contributions to the superpotential may be come from non-perturbative effects, infinite genus effects. While non-perturbative techniques are not yet available, it is possible to calculate non-perturbative contributions to the superpotential of the effective superstring action below the string unification scale. By integrating out the massive modes coming from compactification modes\cite{132,67}, and taking into account the singularities in the moduli space of vacua, we will obtain non-perturbative contributions to the superpotential of $(2,2) N = 1$ orbifold compactifications of the heterotic string in chapter three. These contributions are coming at the level of the effective superstring theory from locally supersymmetric F-terms involving more than two spacetime derivatives\cite{236}.

Toroidal compactifications of the heterotic string have in four dimensions $N = 4$ supersymmetry and contain among other fields, the dilaton $\Phi$, the antisymmetric tensor $B_{\mu\nu}$ which transforms after a duality trasformation to the axion $\tilde{\alpha}$ and moduli scalars described by a matrix $M_{ab}$, which parametrize the coset $SO(6,22)/SO(6) \times SO(22)$. The spectrum and interactions of the perturbative phase of the effective theory are invariant under the ”duality” symmetry $R \leftrightarrow \alpha'/R$, which survives in all orders in string perturbation theory\cite{13}.

In orbifold models the physical quantities, coming from the compactifications of the heterotic string, depend on the moduli parameters. For any orbifold constructions of the heterotic string associated with toroidal compactifications, the spectrum in four dimensions contains, among other fields, the U modulus is associated with the complex structure of the torus and complex modulus field $T$ given by $T = R^2/\alpha' + iB$, where the vacuum expectation value of the $T$ modulus is associated with the size $R$ of the compactification and the antisymmetric background field $B$.

The low energy world observed at energies of the electroweak scale is in perfect agreement with the standard model predictions. The standard model does not have any supersymmetry. However, in the following, we insist on supersymmetric models as supersymmetry solves technically the hierarchy problem\cite{134}. Take for example the standard model. Its lightest scale, corresponding to the spontaneously breaking of the $SU(2) \times U(1)$ into $U(1)_{em}$ giving mass to the $W^\pm$ and $Z^0$ gauge boson carriers of the weak force and to quarks and leptons, is about 100 GeV. In the case that the standard model is embedded in a grand unified theory (GUT) with a bigger gauge group, there is another scale in the theory the grand unified scale which can be of order
If the standard model is coming from a superstring vacuum then the additional scale is the superstring scale which can be two orders of magnitude higher than GUT scale. The question then arises of hierarchy of scales, why the electroweak scale is so small compared with the other scales. In addition, non-supersymmetric theories involving Higgs scalars, like the standard model, receive quadratic quantum corrections to the Higgs sectors of their theory which can drive the low energy scale as high as e.g the GUT scale \[134\]. In this form, renormalized low energy scale \( M_r \) receives quadratic quantum corrections in the form
\[
M_r^2 = C \alpha_r^2 M_{GUT}^2 + M^2,
\]
where \( C \) a number of order \( 10^{0\pm1} \), \( \alpha_r \) a coupling constant and \( M_{GUT} \) the high energy scale of the theory. The quadratic quantum corrections for a typical GUT scale are of order \( 10^{30} \) GeV, something which is unphysical. It requires in order to keep the corrections to \( M_r \) under control large fine tuning of parameters \( C, \alpha_r \). Supersymmetry gives a solution to the problem of quadratic divergencies, since supersymmetric models are free of quadratic divergencies and the contributions to \( M_r \) are
\[
M_r^2 = M^2(1 + C \alpha_r^2 \ln \frac{M_{GUT}^2}{M^2}).
\]
However, in rigid supersymmetric lagrangians we can add the so called soft terms which break supersymmetry without introducing new quadratic divergencies. These terms arise naturally, to the models coming from locally supersymmetric lagrangians as it is the heterotic string or the type II superstring in their field theory limits.

Moreover, in the simplest effective rigid theory, the minimal supersymmetric standard model, the Higgs potential of the theory has a \( \mu \) term which mixes the two Higgs doublets. This term when the low energy theory comes from a \((2,2)\) orbifold compactifications of the heterotic string receives\[236\] contributions, beyond the expected ones coming from the general presence of mixing terms between the 27 and \( \bar{27} \)'s in the Kähler potential of the theory. They are coming from the so called higher weight interactions\[236\] and represent higher derivative couplings of auxiliary fields and scalars. The phenomenologically interesting case of candidate \textit{non-perturbative superpotentials} which contribute to the value of \( \mu \) term and to soft terms in \((2,2)\) orbifold compactifications is exhibited appropriately in chapter five, by taking into account contributions coming from the calculation of the threshold corrections as target space free energies of chapter three. The proposed solutions are solutions to the \( \mu \) problem along the lines of \[230, 120\] and can easily form alternative scenarios for solutions of the \( \mu \) problem along the lines of \[321\]. The proposed superpotentials may come from gaugino condensation\[321\] and respect transformation properties originating from the invariance of the one-loop superstring effective action in the linear representation of the dilaton. The proposed candidate superpotentials provide us with the necessary \( \mu \)
terms which solve the strong CP problem and are required from the low-energy superpotential of
the theory to give masses to d-type quarks and e-type leptons and to avoid the massless axion.

We know that phenomenologically interesting models must have \( N = 1 \) or \( N = 0 \) supersymmetry.\(^9\) However, we will only be interested to models coming from \((2,2)\) symmetric orbifolds with \( N = 1 \) space-time supersymmetry, since non-supersymmetric models suffer from stability problems\(^{260}\) related to tunneling of the cosmological constant to negative values. The cosmological constant \( \Lambda \) amounts to the introduction of a general constant \( \Lambda \) into the effective action in D-dimensions in the form
\[
-\frac{1}{k_{\text{grav}}^2} \int d^D s \sqrt{-g} \Lambda,
\]
where \( g \) is the determinant of the metric and \( k_{\text{grav}} \) the gravitational coupling constant in D dimensions. Here, \( k_{\text{grav}}^2 = 8\pi G \), where \( G \) is the Newton’s constant in D dimensions. The one loop contribution to the cosmological constant \( \Lambda \) in string theory is given by
\[
\Lambda = \frac{1}{2} k_{\text{grav}}^2 \sum_i (-1)^F \sum_i \int d^D p \log(p^2 + M_i^2),
\]
where \( F \) is the fermion number operator and the sum is over all particles. Because the value of \( \Lambda \) from dimensional reasons has upper limit of order \( M_{\text{Plank}}^4 \approx 10^{76} \text{ GeV} \), while the astronomical upper limit is \( 10^{-47} \text{ GeV}^4 \), there is a huge discrepancy of order of magnitude \( 10^{123} \) between the theoretical and cosmological considerations. This creates the cosmological constant problem\(^{18}\). In supersymmetric vacua we expect \( \Lambda \) to vanish. Because in supersymmetric vacua we have equal number of fermions and bosons at each mass level this contribution vanishes. However, because at low energies we should recover the standard model which is not supersymmetric, supersymmetry must be broken in these models, and the cosmological constant is different from zero. The breaking of spacetime supersymmetry due to the presence of the gravitino in the effective low energy theory of \((2,2)\) \( N = 1 \) symmetric orbifolds must be spontaneous. In general, we expect the cosmological constant to be different from zero after supersymmetry breaking. In models coming from heterotic strings the cosmological constant may be different form zero when supersymmetry is broken spontaneously\(^{259}\) or even when the model is non-supersymmetric\(^{13, 260}\). If the cosmological constant \( \Lambda \) for a particular string vacuum is different from zero this means that, there is a non-vanishing dilaton one point function, the background is not a solution to the string equations of motion. Recently, a different mechanism was proposed to set the cosmological constant to zero\(^{10}\). By starting with the three dimensional local theory in three dimensions which has zero cosmological constant\(^{10}\), we can claim that the four dimensional theory maintains the same property. It was proved that it happens in three dimensions\(^{11}\) and it was claimed in \(^{10}\) that

\(^9\)Tachyon free non-supersymmetric models have been constructed from the heterotic string in \(^{12, 13}\).
by sending the coupling constant of the three dimensional theory to \( \infty \), we get a four dimensional theory with zero cosmological constant.

In chapter five, we will discuss supersymmetry breaking. Supersymmetry breaking, with or without vanishing cosmological constant, is one of the main unsolved problems of string theory but contrary to grand unified theories or supergravity theories, string theory can suggest that its origin may come from stringy non-perturbative effects. In conventional field theoretical approach to supersymmetry breaking, supersymmetry breaking appears as a field theoretical phenomenon, coming from \textit{gaugino condensation}\cite{273, 274, 283, 298} in a pure Yang-Mills sector of the theory.

In this case, the nonperturbative superpotential \( W \) appears as \( W = (1/4)U(f + 2/3\beta \log U) \), where \( U \) is the gaugino bilinear superfield and \( \beta \) is the \( \beta \)-function of the theory. In general, gaugino condensation can occur in the pure Yang-Mills theory. The theory is asymptotically free, and the gauge coupling becomes strong at some scale which is the gaugino condensation scale. Take, for example, the auxiliary fields of \( N = 1 \) supergravity

\[
F_i = e^{-G/2}(G^{-1})^j_i G_j + \frac{1}{4} f_{\gamma \delta \xi} (G^{-1})^j_i \lambda^\gamma \lambda^\delta,
\]

where \( G \) is the function of eqn. (5.5) and \( \lambda \) is the gaugino field. When gauginos condense, then \( < \lambda \lambda > \neq 0 \) and the auxiliary field gets a non-vanishing expectation value, and thus breaks supersymmetry. The scale of supersymmetry breakdown is then \( M_S^2 \propto <\lambda \lambda>/M^2 \), where \( M \) is the Planck mass \( M = 1/k_{grav} \).

Toroidal compactifications of the heterotic string have in four dimensions \( N = 4 \) supersymmetry and contain among other fields, the dilaton \( \Phi \), the antisymmetric tensor \( B_{\mu \nu} \) which transforms after a duality trasformation to the axion \( \tilde{\alpha} \) and moduli scalars described by a matrix \( M_{ab} \), which parametrize the coset \( SO(6, 22)/SO(6) \times SO(22) \).

The spectrum and interactions of the perturbative phase of the effective theory are invariant under the ”duality” symmetry \( R \leftrightarrow \alpha'/R \), which survives in all orders in string perturbation theory\cite{13}. In \( N = 1 \) orbifolds\cite{80}, the perturbative duality symmetry is known under the name T-duality, and the spectrum and interactions are parametrized in terms of the T moduli, mentioned earlier, of the unrotated \( N = 2 \) complex planes. The spectrum and the interactions of the theory are invariant under the symmetry \( R \leftrightarrow \alpha'/R \) and the form of the effective action is strongly constrained by the T-duality symmetry\cite{258}. If we freeze all moduli except the T-modulus then under the general \( PSL(2, Z)_T \) trasformations \ref{5.6}, we get e.g that the super-
potential has to transform with modular weight -1. We describe the constraints imposed by the physical symmetries of string theory, on the basic quantities of the low energy effective $N = 1$ supergravity theory extending earlier results \[258\].

Let us combine the axion and the dilaton into a complex scalar $\lambda = \tilde{\alpha} + i \exp(-\phi)$ with the expectation value $<\lambda> = \theta/2\pi + i/g^2 = \lambda_1 + i\lambda_2$, where $\theta$ is the vacuum angle and $g$ the coupling constant. Then the action coming from compactification of the ten dimensional heterotic string on a 6dimensional torus can be written in the form

$$S = \frac{1}{32\pi^2} \int d^4x \sqrt{-g} \left\{ R - \frac{1}{2(\lambda_2^2)} G^{\mu\nu} \partial_\mu \lambda \partial_\nu \lambda - \lambda_2 F^T_{\mu\nu} \cdot LML \cdot F^{\mu\nu} + \lambda_1 F^T_{\mu\nu} \cdot L \cdot \tilde{F}^{\mu\nu} + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu ML \partial_\nu ML) \right\},$$  \hspace{1cm} (1.3)

where $M$ a $28 \times 28$ matrix satisfying $[258]$:

$$M^T = M, \quad M^T LM = L, \quad \text{with} \quad L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}.$$  \hspace{1cm} (1.4)

In addition, $A_{\mu}^{(a)}$, $a = 1, \ldots, 28$ is a set of 28 abelian vector fields and $F_{\mu\nu}$ a 28 dimensional vector $\tilde{F}^{\mu\nu}$ the dual of $F^{\mu\nu}$. The action (1.4) is invariant under the group $O(22,6)$. The moduli fields obeying (1.4) parametrize the Narain lattice $O(22,6)/O(22) \times O(6)$. The equations of motion derived from the previous action are invariant under

$$\lambda \rightarrow \lambda + 1, \quad F_{\mu\nu} \rightarrow M, \quad G_{\mu\nu} \rightarrow G_{\mu\nu}, \quad \text{and}$$

$$\lambda \rightarrow \lambda' = -\frac{1}{\lambda}, \quad F_{\mu\nu} \rightarrow F'_{\mu\nu} = -\lambda_2 ML \tilde{F}_{\mu\nu} - \lambda_1 F_{\mu\nu}, \quad M \rightarrow M, \quad G_{\mu\nu} \rightarrow G_{\mu\nu}.$$  \hspace{1cm} (1.5)

The transformations involving $\lambda_1$ generates $[19]$ the $SL(2,Z)$. So the equations of motion, and not the action, coming from the complex dilaton field are invariant under the $SL(2,Z)$ transformations

$$\lambda \rightarrow a\lambda - ib \frac{i}{ic\lambda + d}.$$  \hspace{1cm} (1.7)

\[10\] In reality the action (1.3) the equations of motion are invariant under shifts in the vacuum parameter which give as $\lambda \rightarrow \lambda + c$, where $c$ a real number, but it is believed that it is broken $[26]$ to $SL(2,Z)$ by world sheet instantons.
It was claimed[251] that the $SL(2, Z)$ S-duality invariance group of the equations of motion in (1.4) has to be promoted[11] to an exact symmetry group of string theory and be active in the $N = 1$ superstring vacua coming from compactifications of the heterotic string. We should note, that $SL(2, Z)_S$ is largely conjectural, as this group appears only as an invariance of the equations of motion of toroidal compactification of the heterotic string and not as an invariance of the action[252]. Since this symmetry inverts the coupling constant is non-perturbative in nature. Moreover, since S-duality represents strong weak coupling duality, it can be used to constrain the form of the effective action coming from non-perturbative effects. S-duality in string theory is associated with supporting evidence in $N = 1$ non-abelian supersymmetric Yang-Mills[28] (SYM)or $N = 4$ non-abelian SYM[163]. We should say here, that at the limit $\alpha' \to 0$, the $N = 4$ SYM appears as the low energy limit of the toroidaly compactified heterotic string.

In chapter five we use S-duality, to allow for S-dual superpotentials which use a single composite bilinear gaugino condensate chiral superfield. They are used to break supersymmetry in order to allow the dynamical determination of the dilaton vacuum expectation value, which represents the string tree level coupling constant[12]. This extends an earlier result[263, 266], related to S-duality invariant gaugino condensates in the effective Lagrangian approach. The effective low-energy $N = 1$ supergravity theory associated with the proposed generalized S-T dual superpotentials exhibit the same target space duality modular symmetry groups as the one’s appearing in the non-decomposable $(2,2)$ symmetric orbifold models. The appropriate use of the the bilinear condensates in an action invariant under the same S-duality as target space duality group, tacitly conjectures the existence of $\Gamma^0(n)$ or $\Gamma_0(n)$ S-duality invariance of the low-energy effective action. In the spirit of[285], we accept that this class of theories in its final form, including non-perturbative contributions, must respect $\Gamma^0(n)$ or $\Gamma_0(n)$ respectively, as an exact symmetry of string theory at the quantum level. Nevertheless, such a limit will exist, and unless someone proves that $SL(2, Z)$ is singled out as an exact symmetry of string theory such a conjecture may be expected to hold.

The ten dimensional heterotic, fundamental, string admits five brane solutions which corre-

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11In exact analogy as electric-magnetic duality in the $N = 4$ super-Yang Mills the spectrum and interactions of the elementary string states are identical to those of the monopole sector of the theory, claimed to occur[250] with the spectrum and interactions of $N = 4$ super-Yang Mills[27].

12In perturbative string theory, the vacuum expectation value of the dilaton is the true expansion parameter[24] around a specific vacuum.
spond to a dual formulation of the heterotic string. After compactification to six dimensions the dual heterotic string has a dual dilaton \( \tilde{S} \), which obeys the string equations of motion with the opposite sign and is related to \( S \) of the fundamental string, as \( \tilde{S} = -S \). Especially, for compactifications to four dimensions the roles of \( SL(2, \mathbb{Z})_T \) and \( SL(2, \mathbb{Z})_S \) dualities are interchanged \( T \leftrightarrow S \) and the \( SL(2, \mathbb{Z})_S \) invariance group appears as a subgroup of the target space duality group of the dual string. In addition, the \( SL(2, \mathbb{Z})_S \) group which for the fundamental theory was the invariance group of the equations of motion, now becomes the target space duality group \( SL(2, \mathbb{Z})_T \). This is equivalent to the fact that \( SL(2, \mathbb{Z})_T \) target space duality for the dual string is equivalent to \( SL(2, \mathbb{Z})_S \) duality for the fundamental string. We should note that we must expect the S-duality of \( N = 4 \) heterotic string constructions to hold, i.e in heterotic strings compactified on a four torus and its dual IIA on \( K_3 \), since a one loop partition function test of \( N = 4 \) super Yang-Mills(SYM) has been performed.

It appears that S-duality holds in a twisted \( N = 4 \) SYM in the following sense: the modular transformations exchange the partition function for the gauge groups \( SU(2) \) with \( SO(3) \) and the partition function \( Z(S) \) of the theory transforms with modular weight \( k \), \( Z(-1/S) = (S^2w)Z(S) \), where \( w \propto \chi \) the Euler characteristic. S-duality holds only when we introduce the modified partition function \( \tilde{Z}(S) \equiv e^{-w\log \eta}Z(S) \).

We refer to this version of SYM because it provides evidence for the appearance of the groups \( \Gamma_o(n) \) in SYM and indirectly to a possible string theory limit. For example for \( N = 4 \) SYM with \( SU(2) \) gauge group the modular symmetry group \( \Gamma_o(2) \) appears. The cusp at \( \tau = +\infty \) corresponds to the \( SU(2) \) instanton expansion while the cusp at \( \tau = 0 \) to its dual theory \( SO(3) \).

Phenomena related to duality invariance do appear at the string theory level in different heterotic compactifications and in different spacetime dimensions than four or ten. These phenomena involve equivalences between different string theories, under which different regions of the moduli space of the two theories are matched to each other for particular limits of their coupling constants. Such phenomena are widely known as dualities and for a particular version of dual of theories are the subject of chapter four.

However, it may be that the complete solution of all the problems, mentioned up to now, may come from determining the way that we break spacetime supersymmetry. However, this involves

\textsuperscript{13}Unfortunately, all string theories have \( N = 4 \) supersymmetry
determination of the complete effective action of the string theory vacuum. Recently, there is accumulating evidence that type I, type II and Heterotic string theories may be complementary descriptions of a more fundamental theory originating from higher dimensions, which in dimensions greater than ten is coming from M[168] of F-theory[151]. In chapter four, we examine this duality phenomena. We discuss perturbative aspects of string theory, based on orbifold[80] and manifold[148] compactifications of the heterotic string. In particular, we reexamine a recently proposed equivalence of moduli spaces between the $N=2$ heterotic string compactified on the $K_3 \times T_2$, we will call it A theory, and the $N=2$ type II superstring compactified on a Calabi-Yau three fold[148], which we will call it B theory. It has been conjectured[148] that moduli spaces of A, B theories coincide when all corrections including perturbative and non-perturbative one’s are taken into account. In this way, $N=2$ physical quantities like, expressed in $N=1$ language, Yukawa couplings, gauge couplings which are impossible to be calculated in the weakly coupled phase of the heterotic string are mapped to superstring vacua of type II, where their expressions are well known[57, 185, 186, 187].

The effective low energy theory of the $N=2$ heterotic compactifications is described by the language of special geometry[227, 230]. For the part of the moduli space described by vector superfields the effective theory is determined completely by the knowledge of one particular function, the prepotential. The prepotential $F$ of $N=2$ compactifications of the heterotic string receives perturbative corrections up to one loop[172]. The third derivative of $F$ with respect of the T moduli appearing in two dimensional torus compactifications, which in $N=1$ language represents Yukawa couplings, was calculated with the use of modular symmetries in [172] while directly via the one loop corrections of the Kähler metric in [173]. The calculation on both cases was based on non-decomposable orbifolds. The same result was shown [130] to be coming from the indirect, ansatz, calculation of $F$ when the heterotic string is compactified on the $K_3 \times T_2$. In chapter four, we find through a string one loop calculation the general differential equation obeyed by $F$ and which is valid for any compactification of the heterotic string on $K_3 \times T_2$. In addition, we derive the differential equation for the Yukawa couplings with respect of the U moduli, corresponding to the complex structure of the $T_2$.

Our final hope is that when all perturbative and non-perturbative corrections will be taken into account that the vacuum potential will be such that, it does not only calculate the values of the moduli fields but it fixes simultaneously the value’s of the matter fields allowed in the theory
and break the gauge group into that of the standard model.
CHAPTER 2
2. Compactification On Orbifolds

In this chapter of the Thesis, we will mainly review the orbifold compactifications of the heterotic string. It is illustrative to consider before we describe the orbifold construction, the compactification on a D dimensional torus $T^D$. The purpose of doing so is not meaningless since the orbifolds that we are interested in the bulk of this work, are coming from the toroidal compactification of the heterotic string modded out by the appropriate action of a point group. Our presentation is organized as follows: In section 2.1 we will discuss toroidal compactifications of the closed bosonic string. In section 2.2 we will discuss toroidal compactifications of the heterotic string together with orbifold constructions of the heterotic string. In section 2.3 we describe the redundancy under global parametrizations in the moduli space of string vacua, in particular the duality invariance in toroidal and orbifold compactifications.

2.1 Toroidal Compactifications

We are considering bosonic strings, living in twenty six dimensions, propagating in a toroidal background. If we compactify D bosonic coordinates on a D dimensional torus we end up with a 26-D dimensional flat space. The D-dimensional torus $T^D$ is parametrized so that points on its $R^D$ space are identified as

$$X^I = X^I + \sqrt{2\pi} \sum_{i=1}^{D} n_i R_i e_i^I, \quad i = 1, \ldots, D, \quad n_i \in \mathbb{Z}. \quad (2.1)$$

Here, $e_i^I$ are D basis vectors with the property $e_i^2 = 2$. In addition, the quantities $L^I = (1/2)^{1/2} n_i R_i e_i^I$ can be considered as lattice vectors on a D-dimensional lattice $\Lambda^D$ with basis vectors $(1/2)^{1/2} R_i e_i^I$. For compactifications on a circle $S^1$, (2.1) can simplified as $X^1 = X^1 + 2\pi RL$, that we identify points which differ by an integer L times $2\pi R$, where R the circle radius.

The torus is identified as $T^D = R^D / 2\pi \Lambda^D$. The momentum vectors are defined as $p^I = \sqrt{2\pi} m_i^{1/2} e_i^I$, where $e_i^* I$ are the dual vectors on the lattice with the properties $e_i^I e_j^* I = \delta_{ij}$, $e_i^I e_j^* J = \delta^{IJ}$ and $e_i^2 = 1/2$. The basis vectors on the dual lattice $\Lambda^*$ are defined as $\sqrt{2\pi} R_i e_i^*$. For closed bosonic

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14 In the following we follow the convention that repeated indices are summed, e.g $e_i^I e_j^* I \equiv \sum_{i=1}^{D} e_i^I e_j^* I$. Compact coordinates will be denoted with capital superscript letters.
strings the compact dimensions, on the torus, satisfy

\[ x^I(\sigma + 2\pi, \tau) = x^I(\sigma, \tau) + 2\pi L^I. \]  (2.2)

Splitting (2.2) into left and right moving coordinates\(^{15}\) (respectively) as

\[ X^I_L = \frac{1}{2} x^I + (p^I + \frac{1}{2} L^I)(\tau + \sigma) + i \sum_{n \neq 0} a_n L^I e^{-in(\tau + \sigma)} , \]

\[ X^I_R = \frac{1}{2} x^I + (p^I - \frac{1}{2} L^I)(\tau - \sigma) + i \sum_{n \neq 0} a_n L^I e^{-in(\tau - \sigma)} , \]  (2.3)

we obtain for the mass formula, of the left and right movers in the uncompactified coordinates

\[ m^2_L = \frac{1}{2} (p^I + \frac{1}{2} L^I)^2 + N_L - 1 = \frac{1}{2} p^2_L + N_L - 1, \text{ and} \]

\[ m^2_R = \frac{1}{2} (p^I - \frac{1}{2} L^I)^2 + N_R - 1 = \frac{1}{2} p^2_R + N_R - 1, \]  (2.4)

where \( N_L, N_R \) the number of the left and right moving oscillators. The total mass operator in the uncompactified 26 \(- D\) dimensions is defined as

\[ m^2 = M^2_L + M^2_R = N_R + N_L - 2 + \sum_{i=1}^{D} (p_i^L p_i^R + \frac{1}{4} L_i^L L_i^R) \]

\[ = N_L + N_R - 2 + \sum_{i,j=1}^{D} (m_i g_{ij}^* m_j + \frac{1}{4} n_i g_{ij} n_j) , \]  (2.5)

where \( g_{ij} \) and \( g_{ij}^* = g_{ij}^{-1} \)

\[ g_{ij} = \frac{1}{2} \sum_{i=1}^{D} R_i e_i^I R_j e_j^I, \quad g_{ij}^* = 2 \sum_{i=1}^{D} \frac{1}{R_i} e_i^I \frac{1}{R_j} e_j^I \]  (2.6)

are the metrics on the lattices \( \Lambda^D \) and \( \Lambda^{*D} \). Defining the signature of the metric on the lattice in the form \((+1)^D, (-1)^D\), meaning \( P \cdot P' = (p_i^L p_i^R)' - p_i^L (p_i^R)' \), the vectors \( P = (p_L, p_R) \) build an even, \( P \cdot P' \in 2Z \), self-dual lattice \( \Gamma_{D,D} \). In addition, the theory satisfy the reparametrization invariance constraint\(^{16}\) \( N_R - N_L = \sum_{i=1}^{D} m_i n_i \). Lets us now consider the spectrum in some detail. Set the momentum and the winding numbers equal to zero. Then the tachyon comes from \( N_R = N_L = 0, m^2 = -2, \) which we identify as \( |0> \). The graviton, dilaton, antisymmetric tensor

\(^{15}\) where \( p_i \) and \( L_i \) the momentum and the winding numbers respectively,

\(^{16}\) see the comment in chapter three, after relation (3.31).
and the dilaton come from states at the next, massless, level, \( a_{-1}^{\mu} \bar{a}_{-1}^{\nu} |0 > \). Let us decompose \( a_{-1}^{\mu} \bar{a}_{-1}^{\nu} \) into irreducible representations of the transverse rotation\(^{17}\) group \( SO(24 - D) \) in the light cone gauge, as

\[
a_{-1}^{\mu} \bar{a}_{-1}^{\nu} |0 > = \{ a_{-1}^{\mu} \bar{a}_{-1}^{\nu} \} |0 > + \left\{ \frac{1}{(24 - D)} \delta^{\mu\nu} a_{-1}^{\rho} \bar{a}_{-1}^{\rho} \right\} |0 > + \left\{ \frac{1}{(24 - D)} \delta^{\mu\nu} a_{-1}^{\rho} \bar{a}_{-1}^{\rho} \right\} |0 > . \tag{2.7}
\]

The brackets in the first term denote symmetrization of indices, while the parenthesis in the second term denote antisymmetrization of indices. Each of the three terms corresponds to a different kinds of particles, namely the first term to the graviton, the second term to the antisymmetric tensor and the last one to the dilaton.

In addition, we have 2D vectors in the form \( |A_1 > = a_{-1}^{\mu} \bar{a}_{-1}^{I} |0 > \) and \( |A_2 > = a_{-1}^{\mu} \bar{a}_{-1}^{I} |0 > \) associated with the gauge bosons of the \( U(1)_L \times U(1)_R \) gauge symmetry of the torus. Moreover, there are \( D^2 \) massless scalars \( \phi^{IJ} = \alpha_{I}^{I} a_{-1}^{J} |0 > \), which correspond to the moduli of the toroidal compactification. The number of \( D^2 \) parameters can be interpreted as corresponding to the presence of the following background fields, whose vacuum expectation values on the torus are given by \( \frac{D}{2} (D + 1) \) and \( \frac{D}{2} (D - 1) \) remaining parameters, corresponding to the metric \( g_{IJ} \) and the antisymmetric tensor \( B_{IJ} \) respectively. The action corresponding to the toroidal compactification via the background field interpretation is

\[
\int d^2 \sigma (\sqrt{g} \eta^{\mu\nu} \partial \mu X^I \partial \nu X^J g_{IJ} + \epsilon^{\mu\nu} \partial \mu X^I \partial \nu X^J B_{IJ}), \tag{2.8}
\]

where \( \epsilon \) the antisymmetric invariant in two dimensions. Because of the interpretation of the moduli parameters associated with \( B_{IJ} \), the mass operator of eqn.(2.4) now must reanalysed such that \( [52] \) it receives contributions from the non zero \( B_{IJ} \) in the form, \( p_{L,R}^2 \propto \frac{1}{2} B_{IJ} L_J \). The Lorentzian even self dual lattices build from different values of the \( D^2 \) parameters are obtained from each other under \( SO(D,D) \) rotations of some reference lattice, which for convenience can correspond to the lattice with \( B_{IJ} = 0 \) and \( g_{IJ} = \delta_{IJ} \). The exact moduli space of torus compactification is build, in its exact form, after we consider the invariance of the spectrum under the rotations \( SO(D)_L, SO(D)_R \) of the vectors \( p_L, p_R \). All even self dual lattices

\(^{17}\)Lorentz invariance requires that physical states are physical states are arranged into representations of the little group of the Lorentz group \( SO(d - 1,1) \). This is \( SO(d - 2) \) for massless particles.
in the form discussed in this section are invariant under $SO(D, D; R)$ rotations. However, the Hamiltonian of the toroidal compactifications $\frac{1}{2}(p_L^2 + p_R^2)$ is invariant under $SO(D)_L \times SO(D)_R$. This means that the scalar fields parametrizing the moduli space take values on the coset space $SO(D, D; R)/SO(D) \times SO(D)$. In addition, for special values of the momentum and the winding numbers $m_i = n_i = \pm 1$, $g_{ij} = 2\delta_{ij}$ we can get the additional massless vectors, $p_L^2 = 2, N_L = 0$, $p_R = 0, N_R = 1$ and the gauge symmetry is enhanced to $SU(2)_L \times U(1)_R$. This corresponds to $R_i = \sqrt{2}$, and $e_i = \sqrt{2}\delta^I_i$, the torus is compactified on D circles with radii $R = \sqrt{2}$.

2.2 General theory of orbifold compactifications

In the main bulk of this section, we will be concentrating in the study of strings propagating in $Z_N$ orbifold backgrounds \cite{80, 83, 308, 11, 301}. The orbifold compactifications that we are interested in this Thesis, will come from toroidal compactifications of the heterotic string\cite{19}. The heterotic string is a construction of a left moving twenty six dimensional bosonic string together with a right moving ten dimensional superstring. In the bosonic construction, as left moving coordinates we have ten uncompactified bosonic fields $X^\mu_L(\tau + \sigma), \mu = 0, \ldots, 9$ and sixteen internal bosons which live on a sixteen dimensional torus. The right moving degrees of freedom consist of ten uncompactified bosons $X^\mu_R(\tau - \sigma), \mu = 0, \ldots, 9$ and their fermionic superpartners the Ramond-Neveu-Schwarz(RNS) right moving fermions $\Psi^\mu_R(\tau - \sigma)$. The sixteen internal fields are compactified on a torus of fixed radii and the momenta $P^I$ span an even self-dual lattice $\Gamma^{16}$. In sixteen dimensions there are two such lattices, the root lattice of $E_8 \times E_8$ and the weight lattice of $Spin(32)/Z_2$. Both the appearance of the two groups as well as spacetime SUSY are enforced by modular invariance of the left and right moving sectors. The Hilbert space of the theory is build by direct products of the left and right moving states. The physical massless modes correspond to ten dimensional $N = 1$ supergravity coupled to $E_8 \times E_8$ (or $Spin(32)/Z_2$). The theory contains no tachyon as in the case of the bosonic string.

The most general compactification of the heterotic string\cite{82} involves compactification on the torus $R^{10-d, 10-d+16}/\Gamma_{10-d, 10-d+16}$, where d dimensions are uncompactified. Here, $\Gamma_{10-d, 10-d+16}$ is the lattice coming from the $SO(10 - d, 10 - d + 16)$ rotation of the lattice $\Gamma_{16} \otimes (P_2)^{10-d}$, where $P_2$ is the two dimensional Lorentz lattice with signature $((+), (-))$ and $\Gamma_{16}$ is the $E_8 \times E_8$ (or $Spin(32)/Z_2$). However, toroidal compactifications of the heterotic string in four dimensions
give $N = 4$ space time supersymmetry which is non-chiral and thus disaster for phenomenology. To obtain supersymmetry breaking down to $N = 1$ or $N = 0$ we have to consider different compactification schemes. Here, we will consider orbifolds.

In general, if the action of a discrete group on a manifold acts freely, without fixed points, we obtain another manifold, if not then the resulting space is an orbifold $\mathcal{O}$. Because the discrete group should preserve the metric of the space, then if the space is Euclidean the discrete group must be a subgroup of the Euclidean group consisting of translations and rotations. In this case the discrete group is the space group $S$. The general element of $S$ is represented by $g = (\theta, v)$, where $\theta$ represents rotations and $v$ translations. In this way, each point is identified with its orbit under the action of the space group, and thus the name orbifold. The action of the space group element $g = (\theta, v)$ on a vector $x \in \mathbb{R}^d$ is $gx = \theta x + v$, while that of the inverse element is defined as $g^{-1} = (\theta, v)^{-1} = (\theta^{-1}, -\theta^{-1}v)$ and $(\theta, v)(k, j) = (\theta k, v + \theta j)$. The subgroup of $S$, consisting of elements of the form $(1, v)$, forms the $d$-dimensional lattice $\Lambda$ of $S$. The point group $P$ is defined as the subgroup of $S$ consisting of pure rotations $\theta$, such that $(\theta, v) \in S$ for some $S$. Take for example two different elements $(\theta, u)$ and $(\theta, v) \in S$. Then $(\theta, v)(\theta, u)^{-1} = (1, v - u)$. So the point group has a well defined action on the torus and its action on the torus is unique up to lattice translations. We can conclude that we can identify its action on the torus as $\bar{P} = S/\Lambda$. There are two equivalent ways of defining orbifolds. This means that either the orbifold can be viewed as the $D$ dimensional Euclidean space $\mathbb{R}^D$ divided by the action of the space group, e.g $\mathcal{O} = \mathbb{R}^D/S$ or as the torus divided by $\bar{P}$. Symbolically,

$$\mathcal{O} = \mathbb{R}^d/S = T^d/\bar{P}. \quad (2.9)$$

In the bosonic formulation, a four dimensional string is obtained, by compactifying the heterotic string with six right-moving and twenty two left-moving coordinates on a torus $T_R^6 \otimes T_L^{22}$, where -(in obvious notation) the right handed coordinates are compactified on the torus $T^6$, while the left coordinates are compactified on the torus $T^{22}$. Modding out this torus by a discrete group, an isometry of the torus, we obtain an orbifold. In general, there are different ways that the modding out operation can be realized. A asymmetric orbifold can be constructed

\[^{18}\text{In a fermionic formulation, the compactified coordinates can be described by worldsheet fermions.}\]
by modding the left and right coordinates as
\[ O = \frac{T_R^6}{P_R} \otimes \frac{T_L^6}{P_L} \otimes \frac{T_L^{16}}{G}. \] (2.10)

Here, we modded the extra six dimensions with different point groups P. The extra sixteen coordinates were modded by G, which we will call it the gauge twisting group. A symmetric orbifold can be obtained by modding the extra six dimensions with the same point group, \( P_L = P_R \). Orbifolds with non-abelian point group are called non-abelian orbifolds. However, in the case where the point group \( P \) is abelian the space group in general is not. This means that if we embed the space group in a non-abelian way in the gauge degrees of freedom then we obtain a non-abelian action in the gauge degrees of freedom, even if the point group is abelian.

Here, we will be interesting in the case where \( T_L^6 = T_R^6 \) and \( T_L^{16} = E_8 \times E_8 \). In this case we will represent our orbifold as
\[ O = \frac{R^6_{L+R}}{S} \otimes \frac{T_L^{E_8 \times E_8}}{G}, \] (2.11)

where P is associated with the point group action on the corresponding torus and S the space group in the gauge degrees of freedom. The meaning of (2.11) is that the bosonic closed strings boundary conditions are modded out by the element \( w = (\theta, v^i; \Theta, V^I) \)

\[ X^i(\sigma = 2\pi) = \theta X^i(\sigma = 0) + v^i, \quad i = 1, \ldots, 6, \]
\[ X^I(\sigma = 2\pi) = \Theta X^I(\sigma = 0) + V^I, \quad I = 1, \ldots, 16. \] (2.12)

Here, \( \theta, \Theta \) are automorphisms of the corresponding torus and \( E_8 \times E_8 \) lattices respectively. The corresponding lattice shifts are given by \( v \) and \( V \).

Orbifolds have two types of closed strings, untwisted and twisted. An untwisted string is the one which is closed in the torus, even before twisting starts. Their boundary conditions are

\[ X^i(\sigma = 2\pi) = X^i(\sigma = 0) + w^i, \quad i = 1, \ldots, 6, \] (2.13)

where \( w^i \) is a vector on the lattice. For \( w^i = 0 \) the string is obviously closed. The general coordinate expansion of the closed string for \( w^i \neq 0 \) has the form

\[ X^i(\sigma, \tau) = X^i_u + p^i \tau + \frac{w^i \sigma}{2\pi} + \frac{i}{2\pi} \sum_{n \neq 0} \left( \frac{\alpha^i_n}{n} e^{-i(n-\sigma)} + \frac{\tilde{\alpha}^i_n}{n} e^{-i(n+\sigma)} \right), \] (2.14)
where $X^i_u$ is the centre of mass coordinate, and $p^i$, $w^i$ are momentum and winding numbers. Strings which are not closed on the torus but are closed on the orbifold are called twisted strings. From the general form (2.12) of the twisted strings we can conclude that they don’t have any momentum or winding numbers in the twisted directions. However, when the strings are twisted\cite{29} by the element $\theta = 1$ and in addition there is a lattice shift the boundary conditions have the form

\[ X^i(\sigma, \tau) = X^i_t + \frac{1}{2} \sum_{m=0}^{\infty} \left( \frac{\alpha^i_{m+\frac{k}{n}}}{m + \frac{k}{n}} e^{-i(m+\frac{k}{n})(\tau-\sigma)} + \frac{\tilde{\alpha}^i_{m-\frac{k}{n}}}{m - \frac{k}{n}} e^{-i(m-\frac{k}{n})(\tau+\sigma)} \right). \] (2.15)

Here, $n$ is the order of the twist and the centre of mass $X^i_t$ is quantized. However here we confine our study of orbifolds to the effect of abelian $\mathbb{Z}_N$ twists on the compactified heterotic strings \textsuperscript{19}. In this case, the action of the general element of the space group $S$, $(\theta^k, \nu)$ on the closed strings takes the form

\[ X^i(\sigma + 2\pi, \tau) = SX^i(\sigma, \tau) = \theta^k X^i(\sigma, \tau) + \nu^i, \] (2.16)

with mode expansions

\[ X^i_k(\sigma, \tau) = x^{(k,f)i} + \frac{1}{2} \sum_{m \neq 0} \left( \frac{\alpha^i_{m+\frac{k}{n}}}{m + \frac{k}{n}} e^{-i(m+\frac{k}{n})(\tau-\sigma)} + \frac{\tilde{\alpha}^i_{m-\frac{k}{n}}}{m - \frac{k}{n}} e^{-i(m-\frac{k}{n})(\tau+\sigma)} \right). \] (2.17)

The position of the centre of mass $x^{(k,f)i}$, if $\theta^k \neq 1$, is not quantized and corresponds to the fixed points of the orbifold. In general, the action of the space group on the twisted strings forces the centre of masses to be the fixed points of the orbifold. From (2.12) we deduce that the ”fixed points” of the orbifold obey $X^{(k,f)i} = (1 - \theta^k)^{-1} \nu^i$.

We are interested in the effect of twistings $\theta$ which leave unbroken one space-time supersymmetry. To specify the orbifold exactly we have to choose a lattice in which the automorphism is acting. Different lattices may be chosen with the same point group. We will be mainly concerned with six dimensional Lie algebra lattices, where the point group is generated by automorphisms of these lattices.

We will now start our description of $\mathbb{Z}_N$ Coxeter orbifolds. The construction of Coxeter orbifolds involves the action of the point group generated from a Coxeter element on the six dimensional $\mathbb{R}^6$ torus. Propagation on the six-dimensional torus is associated with boundary

\textsuperscript{19}The case of $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold twists have been discussed in [34].
conditions in the form

\[ X^i(\sigma = 2\pi) = X^i(\sigma = 0) + 2\pi m^i e^i, \quad i = 1, \ldots, 6, \quad (2.18) \]

where the \( e^i_a \) are a set of simple roots of the root lattice \( \Gamma \) of the six dimensional torus \( T_6 \) defined by \( T_6 = \frac{\mathbb{R}^6}{\Lambda} \). The \( m^i_a \) are integers. The general form of the \( X^i \) fields is

\[ X^i(\sigma, \tau) = q^i + \frac{1}{2} p^i \tau + m^i_a e^i_a \sigma + \text{oscillators}, \quad i = 1, \ldots, 6. \quad (2.19) \]

The canonical momenta \( p^i \) take their values on the dual lattice \( \Gamma^* \) with \( p_i = n_a e^{i*}_a \). The \( e^{i*}_a \) represent basis vectors on the lattice \( \Gamma^* \) and the \( n_a \) are integer valued.

The point group automorphisms of six-dimensional lattices for the Coxeter orbifolds is generated from Weyl reflections in the form

\[ S_i(x) = x - \frac{2(e_a, x)}{|e_a|^2}. \quad (2.20) \]

For the phenomenologically interesting \( N = 1 \) supersymmetric models it is useful to represent the action of the point group in the complex basis

\[ Z^i = \frac{1}{2}(X^{2i+1} + iX^{2i+2}), \quad \tilde{Z}^i = \frac{1}{2}(X^{2i+1} - iX^{2i+2}) : i = 1, \ldots, 3, \quad (2.21) \]

by the Coxeter element

\[ \theta = \text{diag}(\exp[2\pi i (\eta^1, \eta^2, \eta^3)]), \quad (2.22) \]

where \( \eta^i \) are integers with \( 0 < \eta^i < 1 \), e.g. for the \( Z_3 \) orbifold a possible Coxeter element is \( \theta_{Z_3} = (1/3)(1,1,-2) \). Because, \( \theta \) is a automorphism of the six dimensional lattices, i.e it may act crystallographically. It must transform e.g \( \theta \rightarrow A \theta A^{-1} \) with \( A \in GL(2, R) \). The last condition, determines the crystallographic condition, namely that the order of \( \theta \) is 1, 2, 3, 4 or 6.

The action of the point group produces on the orbifold two types of closed strings, the untwisted and the twisted strings. The untwisted strings are given by the expressions of eqn’s (2.19) and (2.18) while for the twisted strings the following expressions hold

\[ Z^i(\sigma = 2\pi) = \theta^k Z^i(\sigma = 0) \mod 2\pi \Gamma, \quad (2.23) \]

\( \Lambda \) is the the subgroup of the space group, consisting of pure translations.
which look like

\[ Z^i(\sigma, \tau) = z_i^{(k,f)i} + \frac{1}{2} i \sum_{n \in \mathbb{Z} + k\eta_i} \frac{1}{n} b_n^i e^{-in(\tau - \sigma)} + \frac{1}{2} i \sum_{n \in \mathbb{Z} - k\eta_i} \frac{1}{n} c_n^i e^{-in(\tau + \sigma)} \]

\[ Z^i(\sigma, \tau) = \bar{z}_i^{(k,f)i} + \frac{1}{2} i \sum_{n \in \mathbb{Z} + k\eta_i} \frac{1}{n} \bar{b}_n^i e^{-in(\tau - \sigma)} + \frac{1}{2} i \sum_{n \in \mathbb{Z} - k\eta_i} \frac{1}{n} \bar{c}_n^i e^{-in(\tau + \sigma)}. \]

The position of the center of mass is quantized and the center of mass is found from

\[ q^{(k,f)i} = (C^k q^{(k,f)})^i + 2 \pi m^a e^{i}. \]

The value of \( k \) represents the twisted sector and \( k = 0, \ldots, N - 1 \). For the untwisted sector \( k = 0 \).

The twisted strings in this way are fixed by the automorphisms of the lattice of the orbifold. We can consider now the action of the space group in the NSR fields. By representing the action of the point group on the lattice with the shift \((1, u^t)\) we get that the NSR fields are given by,

\[ \phi = q^t + \frac{1}{2} (p^t + ku^t)(\tau - \sigma) + \frac{1}{2} \sum_{(n \neq 0)} \frac{1}{n} a_n^i e^{-in(\tau - \sigma)}, t = 1, \ldots, 4. \]

Here, we used the bosonized form of the NSR fermions, with \( p^t \) taking values on the weight lattice\[21\] of \( SO(8) \). As a result, the mass formulas for the spectrum of the physical particles from the k-twisted sectors, are found to be

\[ \frac{1}{8} (m_{R}^{(k)})^2 = \frac{1}{2} \sum_{i=3}^{8} (p_L^i)^2 \delta_{k,0} + \frac{1}{2} \sum_{i=1}^{4} (p^t + ku^t)^2 + N_{R}^{(k)} - \frac{1}{2} + c_k, \quad i = 1, \ldots, 6 \]

(2.27)

and

\[ \frac{1}{8} (m_{L}^{(k)})^2 = \frac{1}{2} \sum_{i=3}^{8} (p_L^i)^2 \delta_{k,0} + \frac{1}{2} \sum_{i=1}^{16} (P^t + kV^t)^2 + N_{L}^{(k)} - 1 + c_k, \quad i = 1, \ldots, 6 \]

(2.28)

where \( p_L^i \) and \( p_R^i \) represent the left and right moving momenta respectively of the \( X^i \), while \( N_{R}^{(k)} \) and \( N_{L}^{(k)} \) are the number operators in the k-twisted sectors. The shift embedding \( V^t \) in the gauge degrees of freedom is a automorphism of the \( E_8 \times E_8 \) and represents the embedding of the space group in the gauge degrees of freedom. In general S is embedded by an automorphism and/or shift in the gauge degrees of freedom. If the automorphism is in the Weyl group of \( E_8 \times E_8 \) then the twist is realized through a shift\[12\]. In general, it is possible to have additional

\[21\]The weight lattice of \( SO(8) \) can be constructed from its vector and spinor representations.
background field parameters corresponding to topologically non-trivial directions in the gauge
and torus directions defined by

$$\int A^I_i dx_i = A^I_i e^i_a \equiv \alpha^i_a, \quad I = 1, \ldots, 16, \quad i, a = 1, \ldots, 6,$$

(2.29)

where $e^i_a$ are the basis vectors on the six torus. In the case that the Wilson lines commute with
the general gauge group element, we can realize the gauge element and the Wilson lines through
shifts in the $E_8 \times E_8$ lattice. In this case, the shift in the momentum in the twisted sectors will
be modified to $p^I + V^I + \sum_{a=1}^6 m^I_a \alpha^i_a$, where $m^I_a$ integers is associated with the fixed points. If
the Wilson lines do not commute with shifts on the lattice, they can lower the rank of the gauge
group. We will not describe, additional properties of the Wilson lines since we will not need
them in this Thesis.

The numbers $c_k$ represent the contribution of the oscillators to the zero point energy in the form

$$c_k = \frac{1}{2} \sum_{i=1}^3 (|k\eta_i| - \text{Int}(|k\eta_i|))(1 - |k\eta_i| + \text{Int}(|k\eta_i|)), \quad (2.30)$$

where $\text{Int}(|k\eta_i|)$ denotes the integer part of the expression in parenthesis. In addition, physical
states have to satisfy criteria that are coming from the requirement that the physical Hilbert
space of the theory must keep states which are only $Z_N$ invariant. In the untwisted sector, we
will have to project to $S \times P$ invariant states in order to implement the orbifold projection of
the heterotic string spectrum.

The right moving excitations of the heterotic string are described by the ten dimensional
superstring (SP). The tachyon is projected out by the GSO projection, and the vacuum states
are those of the SP. As a result, the chirality of the physical states in the Hilbert space comes
form the right moving sector. The latter are the ten dimensional supersymmetry charges surviving the orbifold projection are given by the condition

$$[V(v), P_\phi] = 0, \quad V(u) = e^{2i\eta^i T^i}, \quad P_\phi = e^{-2\pi i u^i p^i}, \quad (2.31)$$

where $V(v)$ is the supercharge and $u^i \in 8c$, and $P_\phi$ is a translational operator acting on
the bosonic fields and represents a $Z_N$ rotation on the bosonic coordinates, $X^i(\sigma + 2\pi, \tau) =

\text{representing the gauge group element in the form } G \times P, \text{ where } G, P \text{ denote the gauge and point group}
elements, \text{ we may have } g \rightarrow (\Theta, 0; 1, V^I), \quad A \rightarrow (1, \epsilon^i_a; 1, \alpha^i_a). \text{ Then } [g, A] = 0.

\text{The mass operator for the heterotic string in the right moving sector is that of the NS-R superstring, namely } N = \sum_{n=-\infty}^{\infty} (a^i_n \bar{a}^i_n + nS^{i, -n} S^i_n), \text{ where vector } h^i |0 > \text{ and spinor } |S^a >.
\[e^{2\pi i u^i} X^i(\sigma, \tau).\]

As a result the number of supersymmetries is equal to the number of supercharges which satisfy the previous condition. The number of unbroken supersymmetries is given by the number of vectors which satisfy (2.31), i.e \(\Sigma_{i=1}^{4} u^i v^i = \text{integer}\) with \(v^i \in 8_c\). The 8c corresponds to the conjugate spinor representation of \(SO(8)\). For vacua with \(N = 1\) supersymmetry unbroken the previous condition becomes \(\pm u^1 \pm u^2 \pm u^3 \pm u^4 = 0\), where we must set \(u^1 = 0\) since space time degrees of freedom are not rotated. Now the point group is embedded in the standard way in a \(SO(6)\) subgroup of \(E_8\). Because its eight eigenvalues acting on the spinor of \(SO(8)\) are in the form \(e^{i\pi (u^1 \pm u^2 \pm u^3)}\), there are at least two zero modes for the right handed spinor fermions in the Green-Schwarz formalism. So at least one unbroken supersymmetry in four dimensions remains.

We will now describe constraints from modular invariance of the one loop vacuum amplitude. The space of physical states in the Hilbert space of the orbifolds is given by the direct product of the Hilbert spaces of the left and right sectors. The constraints of the spectrum coming from modular invariance are best described from the examination of the properties of the vacuum amplitude \(Z(g, h)\). At one loop the string world sheet describing the one loop amplitude with no external legs is a torus. The torus is described by the modular parameter \(\tau\). If we parametrize the torus by \((\sigma_1, \sigma_2)\), where a point on the torus corresponds to the complex number \(\sigma_1 + \tau \sigma_2\), then we can consider the bosonic string variables on the orbifold in the form

\[
X(\sigma_1 + 2\pi, \sigma_2) = h X(\sigma_1, \sigma_2), \quad X(\sigma_1, \sigma_2 + 2\pi) = g X(\sigma_1, \sigma_2), \quad 0 \leq \sigma_1, \sigma_2 \leq 2\pi.
\]  

(2.32)

Here, \(g\) and \(h\) represent twisted boundary conditions, which can be periodic or antiperiodic. For consistency of boundary conditions \(h\) and \(g\) must commute.

We are introducing the one-loop \(Z_N\) invariant vacuum amplitude as

\[
Z(g, h) = \int \frac{d^2 \tau}{(Im\tau)^2} \mathcal{H}(\tau).
\]  

(2.33)

Here, \(\mathcal{H} = \int DXD\Psi e^{-S}\) is the path integral of the action \(S\) on the world sheet torus over the fermionic \(\Psi\) and bosonic \(X\) degrees of freedom. In orbifolds, \(\mathcal{H}\) receives contributions from the
d

24 If we want to describe the group of global diffeomorphisms on the torus, the space of inequivalent tori, then the parameter space for

\(\tau\) is the group of modular tranformations under which \(\tau \rightarrow \frac{a \tau + b}{c \tau + d}\), with \(ad - bc = 1\), \(a\), \(b\), \(c\), \(d \in Z\), the modular group of the torus \(PSL(2, Z)\). See appendix A.
untwisted and the twisted sectors while its general form is

\[ \mathcal{H} = \frac{1}{|\mathcal{P}|} \sum_{(h,g)\neq 0} \epsilon(h, g) I(h, g)(\tau). \quad (2.34) \]

Here, \( \epsilon(h, g) \) are the discrete torsion phases\(^{[30]} \), while the interpretation of the sum in \( (2.34) \) is as follows. For abelian group \( G \), the path integral is calculated with boundary conditions \( h, g \) in the \( \sigma_1 (\sigma_2) \) directions. The various components of \( I(h, g) \) are related one another by modular transformations of the world sheet parameter \( \tau \). Under \( \tau \) modular transformations, i.e \( \tau \rightarrow \frac{a\tau + b}{c\tau + d} \), \( (h, g) \rightarrow (h^d h^a, h^b g^a) \) and \( \mathcal{H}(h, g) \rightarrow \mathcal{H}(h^d g^c, h^b g^a) \). For \( Z_N \) orbifolds with \( P \) elements \( \theta^m, m = 0, \ldots, N - 1 \), we have \( \epsilon(\theta^m, \theta^n) = 1 \) for every \( m, n \). The various contributions to \( I(h, g) \) are given by

\[ \text{Tr} \int \frac{d^2 \tau}{(16\pi^2)^2} (G(h)e^{2\pi i L^h_0} e^{-2\pi i L^l_0}), \quad (2.35) \]

where the \( \text{Tr} \) is over all modes while the \( L^h_0 \) and \( L^l_0 \) are the Virasoro operators for the \( h \)-twisted sectors, right and left respectively. Eqn.\((2.35)\) is associated with the sum over \( g \) twisted sectors, twisted by \( h \). Moreover,

\[ \hat{G}^{(h)} = \frac{1}{N} \sum_{g=0}^{N-1} \tilde{\psi}(h, g)(\Delta_g)^{h}, \quad (2.36) \]

where, \( N \) is the order of the point group, \( \hat{G}^{(h)} \) implements the projection operator \( (\Delta_g)^{h} \) in the \( g \) twisted sector. For \( Z_N \) orbifolds,

\[ \hat{G}^{(\theta^n)} = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\psi}^{(\theta^m, \theta^n)} \Delta_g(\theta^m)^n, \quad (2.37) \]

\[ \Delta_g(\theta^m) = e^{(2\pi i \frac{1}{2} n (\sum_{-u^2} + \sum_{(V^h)^2} + \sum_{(P^l + m V^f)^2} - \sum_{u^2} u^n))}, \quad (2.38) \]

Here, \( V \) is the embedding of the point group in the gauge degrees of freedom. In the case, that there are massless states with left oscillators \( (2.38) \) must be modified with the addition of the term \( e^{2\pi i \xi} \), which is the eigenvalue of the oscillator under \( \theta \). The number of fixed points under \( \theta \) gives us the degeneracy of the corresponding twisted vacuum and appears in the one loop partition function as an overall factor. The factor \( \tilde{\psi}^{(\theta^m, \theta^n)} \) is the degeneracy factor. The explicit formula for the \( \tilde{\psi} \) reads

\[ \tilde{\psi}^{(\theta^m, \theta^n)} = \psi^{(\theta^m, \theta^n)}, \quad \text{for } \psi^{(\theta^m)} \neq 0, \]

\[ \tilde{\psi}^{(\theta^m, \theta^n)} = \frac{\psi^{(\theta^m, \theta^n)}}{\prod_j \psi_j^{(\theta^n)}}, \quad \text{for } \psi^{(\theta^m)} = 0, \ n \neq 0 \]

\[ \tilde{\psi}^{(\theta^m, \theta^n)} = \prod_k \psi_k^{(\theta^m n)}, \quad \text{for } \psi^{(\theta^m)} = 0 \text{ and } n = 0, \quad (2.39) \]
where $\psi(\theta^m, \theta^n)$ is the number of points simultaneously fixed by the point group elements $\theta^m, \theta^n$ and the subscript $j$ runs over untwisted components and $k$ over twisted components. If the rotation $\theta^m$ leaves fixed tori, $\psi(\theta^m) = 0$. We should notice that the number of points fixed under the automorphism $\theta^\rho$ depends only on the automorphism, not on the specific lattice and can be calculated using the Lefschetz fixed point theorem \[n = \psi(\theta^\rho) = det(1 - \theta^\rho) = \Pi_j 4 \sin^2 m \eta_j,\] (2.40)

where the determinant is in the vector 6 representation of $SO(6)$ for compactifications that preserve one four-dimensional supersymmetry.

For $m = 0$, $\tilde{\psi}(1, \theta^m) = 1$, since we are considering untwisted strings. For example, the total partition function can be written as

$$P = P_{\text{Untwisted}} + P_{\text{Twisted}}, P_U = \sum_{n=0}^{n-1} I(1, \theta^n), P_T = \sum_{n=0}^{n-1} \sum_{[h,g]=0,m \neq 1} I(\theta^m, \theta^n). \quad (2.41)$$

The invariance of the vacuum amplitude $(\theta, 1)$ under modular transformations $\tau \rightarrow \tau + N$ gives\[30, 31\]

$$N \left( \sum (u^t)^2 - \sum (V^t)^2 \right) = \text{mod } n, \quad (2.42)$$

$$N \sum_{i=1}^{4} u^t = N \sum_{J=1}^{8} V^J = N \sum_{J=9}^{16} V^J = 0 \text{ mod } 2. \quad (2.43)$$

The level matching constraint (2.42)\[51, 80\] holds for $n$ even, while for $n$ odd becomes mod $2n$.

Up to know we have discussed properties of the space group of the orbifold. In general $S$ will be embedded in the gauge degrees of freedom by acting with an automorphism $\theta$ and or shift $V$ on the gauge degrees of freedom. In the absence of Wilson line backgrounds a Weyl automorphism is equivalent to a lattice shift\[25, 26\].

If one acts on the gauge degrees of freedom with exactly in the same way as in the $SU(3) \subset SO(6)$ subgroup of extra six dimensions this is called the standard embedding. This scheme is reminiscent of the identification of the spin connection with the gauge connection\[144\] in Calabi-Yau manifolds. There, by setting the H field equal to zero\[26\] the Bianchi identity, coming from compactification of the heterotic string in a six dimensional Calabi-Yau manifold, becomes

\[25\] The reverse argument is not valid.

\[26\] See some relevant discussion in chapter four.
\[ F_{[MN}^{a} F_{KL]}^{a} = R_{[MN}^{cd} R_{KL]}^{cd}, \]
where the capital letters take values in the tangent space of the ten-dimensional space time. Picking up as a solution, for manifolds of \( SU(3) \) holonomy the identification of the \( SU(3) \) spin connection with the \( SU(3) \) gauge field \( A_{M} \), breaks the \( E_{8} \) down to \( E_{6} \). In the same way, in orbifolds we can embed the \( SO(6) \), in the fermionic representation of \( E_{8} \) where the \( E_{8} \) degrees of freedom are described by sixteen world-sheet fermions transforming as \( 16 \) under the \( SO(16) \subset E_{8} \), action of \( P \) in \( S \) such that the embedding of the \( SO(6) \) in its Cartan subalgebra, i.e \( e^{\pi i(aJ_{12}+bJ_{34}+cJ_{56})} \) is identified with a shift \((a,b,c,0^5)\) on the group torus\[80].\]
For the \( Z_{3} \) orbifold, coming by the standard embedding, choosing the the point group embedding \( r_{1} = \frac{1}{3}(1,1,-2) \) and the gauge group embedding \( V = \frac{1}{3}(1,1,,0^5) \) the gauge group breaks to \( E_{6} \times SU(3) \times E_{8} \). The gauge group of the theory will come from the untwisted sector while the matter supermultiplets come from the twisted sectors. The gauge and matter content is a consequence of the masslessness condition \((2.27,2.28)\) and the orbifold projection \((2.38)\).

Our main interest is to the \( Z_{N} \) twists of six-dimensional Lie-algebra lattices - with the point group to be generated by automorphisms of these lattices - which leave unbroken space-time supersymmetry. These automorphisms are realized as inner or outer. The inner automorphisms are given by the Weyl group of the algebra. A special class of inner automorphisms are the Coxeter elements, which can be written as products of Weyl reflections with respect of all the simple roots, that will be discussed in more detail in chapter three.

### 2.3 Duality symmetries

The simplest way to get a four-dimensional (or more generally a D-dimensional, \( D < 10 \)) theory from the ten-dimensional heterotic string is the compactification of six (or \( d \)) space dimensions on a torus \[82\]: \( R^{10} \rightarrow M^{D} \times T^{d} \), where
\[ T^{d} = \frac{R^{d}}{\Lambda}, \quad d = 10 - D, \quad (2.44) \]
and \( \Lambda \) is a \( d \)-dimensional lattice with basis \( \{ e_{i} \} \). A state on the compactified sector is labeled by its continuous space-time momentum \( p^{\mu}, \mu = 1,\ldots, D \), its oscillation state, its gauge quantum numbers \( p^{I}, I = 1,\ldots, 16 \) and by its winding vector
\[ w = w^{i}e_{i} \in \Lambda, \quad i = \ldots, d \quad (2.45) \]
describing how the string wraps around the internal dimensions. In addition, it is labeled by its
discrete internal momentum
\[ p = p_i e^{*i} \in \Lambda^*, \quad i = 1, \ldots, d, \]  
where \( \{e^{*i}\} \) is the standard basis of the dual lattice \( \Lambda^* \) of \( \Lambda \). The metrics of the two lattices are
\[ G^{ij} = e^{*i} \cdot e^{*j} \quad \text{and} \quad G_{ij} = e_i \cdot e_j, \quad \text{where} \quad G^{ij} G_{jk} = e^{*i} \cdot e_k = \delta_k \text{.} \]  
All the internal quantum numbers of the D–dimensional theory can be combined into the left–
and right–moving momenta, which form the momentum lattice \( \Gamma_{16+d,d} \):
\[ (p_L; p_R) = (p_L^I e_I + p_L^i e_i; p_R^i e_i) \in \Gamma_{16+d,d}, \]  
where \( A^i \) are the Wilson line background fields and the \( e_I \) are basis vectors for the self-dual
lattice \( \Gamma_{16+6;6} \). Toroidal compactification of the heterotic string with p left moving and (16 +
p) right moving coordinates compactified, gives a moduli space parametrized from p(16 + p)
components. The antisymmetric tensor corresponds to \( (1/2)p(p - 1) \) components, the metric
tensor \( g_{ij} \) to \( (1/2)p(p + 1) \) components. The remaining 16p parameters are associated with the
Wilson lines \( A_I^i \). The presence of the Wilson lines are neccesary since even if the condition for
finding a vacuum solution for the Yang-Mills strength is \( F_{ij} = 0 \), the Yang-Mills field on the
torus can still have a non-trivial holonomy associated with the Wilson lines. Of course, this is
in exact analogy with the instanton solitions of Yang-Mills equations in gauge theories.

In the case of vanishing Wilson lines \( A_I^i \), I can write for the momenta
\[ p_L = \frac{1}{2} m + (G - B)n \quad \text{,} \quad p_R = \frac{1}{2} m - (G + B)n, \]  
where \( G, B \) represent the background metric and the antisymmetric tensor field. The moduli
space of heterotic string compactified on a D-dimensional torus \( \mathbb{R}^2 \) is locally isomorphic to the
coset manifold \( \frac{SO(D,D)}{SO(D) \times SO(D)} \). However in order to reveal the global geometry of the moduli space
we have to find how the discrete modular symmetries modify its coset structure. In this part of
the thesis we will describe the way, that the six-dimensional part of the compactification lattice fixes the background deformation parameters, namely we will find the conditions imposed to the background fields from the requirement of duality invariance on the physical spectrum of orbifold compactifications.

The target space duality symmetries now are defined as those discrete transformations acting on the the quantum numbers which leave the spectrum and the interactions invariant.

Duality in its simplest version, is the compactification of a closed bosonic string on a circle, represents the invariance of the spectrum under the inversion of the radius \( R \to \frac{1}{2R} \) and a simultaneous interchange of the momentum and winding numbers \( m \leftrightarrow n \). This result can be seen directly from the discussion in section 2.1, but for simplicity let us review this result. Compact bosonic strings with one of the compact dimensions, e.g. \( X^{25} \) compactified on \( S^1 \), a circle of radius \( R \), means that \( X^{25} \equiv X^{25} + 2\pi R n \). In this case, the wave equation for the compact coordinate splits into left and right movers

\[
X_R = x_R - \frac{1}{2} p_R (-\tau + \sigma) + \frac{1}{2} \sum_{k \neq 0} \alpha_k e^{-ik(\tau - \sigma)}, \quad (2.53)
\]

\[
X_L = x_L - \frac{1}{2} p_L (\sigma + \tau) + \frac{1}{2} \sum_{k \neq 0} \tilde{\alpha}_k e^{-ik(\sigma + \tau)}, \quad (2.54)
\]

where \( \alpha_k, \tilde{\alpha} \) are oscillators. The left and right moving momenta the Hamiltonian and the spin are given by

\[
p_L = \frac{m}{2R} + nR, \quad p_R = \frac{m}{2R} - nR, \quad H = \frac{m^2}{4R^2} + n^2 R^2, \quad S = \frac{p_L^2}{2} - \frac{p_R^2}{2} = mn. \quad (2.55)
\]

Obviously, the spectrum is invariant under the transformations \( R \leftrightarrow \frac{1}{2R} \) and \( m \leftrightarrow n \).

For the heterotic string, the Hamiltonian and the spin of the vertex operator are defined by

\[
H = \frac{1}{2} \left( p_L^\tau G^{-1} p_L + p_R^\tau G^{-1} p_R \right) = \frac{1}{2} \left( u^t \Xi u \right), \quad (2.56)
\]

\[
S = \frac{1}{2} \left( p_L^\tau G^{-1} p_L - p_R^\tau G^{-1} p_R \right) = \frac{1}{2} \left( u^t \eta u \right), \quad (2.57)
\]

where

\[
u = \begin{pmatrix} n \\ m \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 2 (G - B) G^{-1} (G + B) & BG^{-1} \\ -BG^{-1} & \frac{1}{2} G^{-1} \end{pmatrix}. \quad (2.58)
\]
and $1_d$ the identity matrix in $d$-dimensions. The duality transformations $\Omega$ act as

$$\Omega : u \rightarrow S_\Omega (u) = \Omega^{-1}u.$$  \hfill (2.59)

Invariance of eqn's. (2.56,2.57) under target space duality transformations gives the conditions for the background $H$, $S$ to remain invariant namely :

$$\Omega^T \eta \Omega = \eta$$  \hfill (2.60)

$$\Xi \rightarrow \Omega^T \Xi \Omega,$$  \hfill (2.61)

so that $\Omega$ is an element of $O(d,d,Z)$ and (2.61) defines the action of the duality group on the moduli fields. In the case of toroidal orbifolds \[83\] the quantum numbers transform as

$$u \rightarrow u' = Ru, \quad R^N = 1.$$  \hfill (2.62)

Because the point group acts on the compactified coordinates $x^\mu$ as $x^\mu(2\pi, \tau) = \theta^\mu_\nu x^\nu(\sigma, \tau) + 2\pi w^\nu$, where $\mu = 1, \ldots, d$ and the $\theta^\mu_\nu$ the twist matrix in the space time basis, we have that

$$R \overset{\text{def}}{=} \begin{pmatrix} Q & 0 \\ 0 & Q^* \end{pmatrix},$$  \hfill (2.63)

where $w^\nu$ is the winding number and $Q$ by definition is

$$\theta^\mu_\nu e_i^\nu \rightarrow e^\mu_i Q_{ij}$$  \hfill (2.64)

Because the twist is an automorphism of the lattice, $Q$ must have integer entries.

Finally the condition that the point group to be an a lattice automorphism gives \[102\]

$$Q^T G Q = G,$$  \hfill (2.65)

$$Q^T B Q = G.$$  \hfill (2.66)

We will end the discussion of the toroidal duality symmetries in orbifold models by discussing one more condition that have to be satisfied by the momentum and winding numbers in an orbifold background. For the $Z_N$ orbifolds the target space duality symmetries of the untwisted sector that are surviving the orbifold projection, i.e the torus has to commute with the twist, have to satisfy \[103\], $\Omega R - R^k \Omega = 0, k = 0, 1, \ldots N$, while if the point group action $\theta^k$ leaves a complex plane invariant $Q^k n = n, Q^k m = m$, where * denotes inverse and transpose. For twisted sectors of $Z_N$ Coxeter orbifolds modular symmetries were examined in \[117, 118\]. In reality there are three ways to find the modular symmetry. They are listed in \[120\].
CHAPTER 3
3. Aspects of Threshold corrections to Low energy Effective string theories compactified on Orbifolds

3.1 Introduction

Four-dimensional \([8, 9, 10, 80, 208, 211]\) superstrings \([27]\) represent at present our best ever candidates for a theory which can consistently unify all interactions including gravity, even if at present, there is no second quantized formalism. In order for the string theory at the Planck scale to make contact with the observed world at the weak scale, one needs to find the effective low energy theory of the superstring theory (ELET).

Corrections to the general theory of relativity (GR) coming from the massless modes of the superstring theory appear in two different forms. One is associated with the use of background field contributions of the infinite tower of massive modes leading to corrections in \(\alpha'\), while the other corresponds to quantum loop effects. In the first form, the effective lagrangian of the massless modes can be studied in the \(\sigma\) model approach \([20]\). An example action involving bosonic backgrounds only is the following:

\[
S = \frac{1}{2\pi\alpha'} \int d^2z \left[ \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(x) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(x) + \alpha' \sqrt{g} R^{(2)} \Phi(x) + \sqrt{g} f(x) \right],
\]

(3.1)

where \(G_{\mu\nu}(x), B_{\mu\nu}(x), \Phi\) and the vacuum expectation values of the usual background fields, namely metric, antisymmetric tensor, dilaton and tachyon field respectively \([20]\). In the conformal gauge \(\sqrt{g} g^{\alpha\beta} = \delta^{\alpha\beta}\), imposing conformal invariance, namely traceless stress energy tensor, guarantees decoupling of negative norm states. In this case, conformal invariance, i.e vanishing \(\beta\)-functions for the background fields, leads into the correct \([4]\) equations of motion for the massless fields.

\[
\beta^\Phi = \frac{1}{\alpha'} \frac{d - 26}{48\pi^2} + \frac{1}{16\pi^2} \left[ 4(\nabla^2 \Phi)^2 - 4(\nabla^2 \Phi) - R + \frac{1}{12} H^2 \right] + O(\alpha')
\]

\[
\beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H^\xi_\mu H_{\nu\xi} + 2 \nabla_\mu \nabla_\nu \Phi + O(\alpha')
\]

\[
\beta^B_{\mu\nu} = \nabla_\xi H^\xi_{\mu\nu} - 2(\nabla_\xi \Phi) H^\xi_{\mu\nu} + O(\alpha'), \quad H_{\xi\mu\nu} = 3 \nabla_\xi B_{\mu\nu}
\]

\(27\) derived from Heterotic string constructions \([13]\).

\(28\) By expanding the stress energy tensor to leading order in the loop coupling constant \(\alpha'\).
In the other form, corrections to GR come via the S-matrix approach\[21\]. In the S-matrix approach, the calculation of the effective lagrangian of the massless modes proceeds through the calculation of string scattering amplitudes. It has been shown\[21\], that the tree level action of the heterotic string corresponds to the bosonic part of the Chapline-Manton lagrangian with the field strength of the antisymmetric tensor field appropriately generalized to include Chern-Simons three forms. The latter account for the cancellation of the gauge and gravitational anomalies in the 10D heterotic string action.

For the $N = 1$ heterotic string the massless sector consists of the Yang-Mills supermultiplet $\left(A_\mu^\alpha, \chi^\alpha\right)$ in the adjoint representation of the gauge group and the $N = 1$ supergravity multiplet(SM) consisting partly of the graviton, the dilaton scalar $\phi$, and the antisymmetric tensor $B_{\mu\nu}$. It’s effective lagrangian in 10D describes $N = 1$ supergravity coupled to supersymmetric Yang-Mills. The low energy lagrangian coming from the calculation of superstring scattering amplitudes describes the massless mode excitation dynamics of the heterotic string\[52, 53\] at the string unification scale.

The effects of string theory in our low energy world at the weak scale, are becoming apparent via the running of physical couplings through the evolution of the Renormalization group equations(RGE). Particular role in this respect is played by the gauge coupling constants whose properties we will examine later at this chapter. In string theory physical couplings and masses are field dependent. They depend explicitly on the vevs of some massless scalar fields, the so called moduli\[30\]. From the point of view of effective low energy theory the moduli are massless neutral scalar fields with a flat potential to all orders\[22, 35\] of perturbation theory with classically undetermined vevs that can be as large as $M_{\text{plank}}$. Neglecting non-perturbative effects, the moduli fields give an infinite degeneracy of string vacua. In addition, the global structure of the the moduli space $M$ is affected by the invariance under some discrete reparametrizations of the moduli fields $\Phi$

$$\Phi_i \rightarrow \tilde{\Phi}_i (\Phi_i) \in M,$$

(3.3)

\[29\]Here $A_M^\alpha, \chi^\alpha$ represent the gauge field and its gaugino in 10 dimensions respectively.

\[30\]The continuous deformations, of a superstring solution constitute its moduli space. At the level of conformal field theory they correspond\[232\] to integrably marginal operators $\Phi$, i.e. BRST invariant operators, that we can add to the world sheet lagrangian without affecting the equations of conformal invariance i.e. the value of the $\beta$-function remains zero. The general form of the deformation appears as $\sum_i g_i \int d^2z \Phi_i (z, \bar{z})$, where the constants $g_i$ correspond to the coordinates of the moduli space, i.e they are moduli.
the so called target space duality transformations. They change the geometry of the internal space and leave invariant the massive spectrum and interactions. They are of great importance, since by lifting the degeneracy by perturbative string theory or non-perturbative effects we will be able to see clearly the effects of string theory on the physical observables. On the other hand, the moduli dependence of the effective action is important since non-perturbative effects like gaugino condensation can provide a potential for moduli fields which can lift the vacuum degeneracy and provide a mechanism for supersymmetry breaking. Unfortunately, a non-perturbative formulation of string theory is still lacking.

In this chapter we will explicitly discuss the one-loop moduli dependence of effective gauge couplings. String theory demands that the massless spectrum of physical particles originate from superstring excitations that are massless at the string unification scale. Below this scale the effective gauge couplings evolve according to the usual RGEs which at the one loop level receive a threshold correction from the ultrahigh energy theory, string theory. The values of the physical couplings at the string unification scale $M_{\text{string}}$ represent the boundary conditions of our RGEs. At tree level the gauge interactions $g_a$ are all connected to a single scale, the string mass $M_{\text{string}} \approx 0.527 \times 10^{18}$ GeV, as follows

$$g_a^2 k_a = 4 \pi \alpha'^{-1} G_N = g_{\text{string}} = G_N M_{\text{string}}^2 = \frac{\kappa^2}{2 \alpha'},$$

where $\kappa$ represents the gravitational coupling, $k_a$ the kac-Moody level of a gauge group factor, $\alpha'^{-1/2}$ the string tension and $G_N$ the Newton constant. The gauge coupling constant at a mass scale $\mu$, receives up to one-loop level corrections according to

$$\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_{\text{string}}^2} + \frac{b_a}{16\pi^2} \ln \frac{M_{\text{string}}^2}{\mu^2} - \frac{1}{16\pi^2} \Delta_a,$$

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31 See chapter 5.
32 The most satisfactory solutions to supersymmetry breaking up to know are coming from non-perturbative gaugino condensation mechanisms even if they fail to fix correctly the dilaton—see chapter 5.
33 For a review see reference (51).
34 At the level of perturbative string theory.
35 Effective quantum field theories involve two kinds of couplings. The Wilsonian gauge couplings, which are gauge couplings of an effective lagrangian from which the massive modes have been integrated out and depend on the cut-off scale. In addition, there are the EGC which depend on the momentum scale. Since don’t depend on the cut-off scale, they do not correspond to any local effective lagrangian.
36 We assume that the gauge group at the string unification scale is a product of group factors $G = \Pi_a G_a$. 

with $\Delta_a$ given by
\[ \Delta_a = \int \frac{d^2 \tau}{\tau_2} \left(B_a(\tau, \bar{\tau}) - b_a\right). \tag{3.6} \]

Here $b_a$ is the field theoretical $\beta_a$ function coefficient of the gauge group factor $G_a$ of the effective theory of massless modes contributing to the threshold corrections. The quantity $B$ will be defined in detail later in eqn.(3.10). $\Delta_a$ denotes the string theoretical threshold correction\(^{37}\), of the factor $G_a$ of the gauge group $G = \Pi_a G_a$, to the gauge coupling constants. In addition, $\tau = \tau_1 + i\tau_2$ is the modulus of the world-sheet torus and the integration is over the fundamental domain $\mathcal{F} = \{\tau_2 > 0, |\tau_1| < 1/2, |\tau| > 1\}$.

The inclusion of threshold corrections is necessary to test the old ideas of unification of gauge interactions in the grand unified models, particularly since LEP measurements\(^{17}\) support the possible existence of a grand unified theory at an energy of $10^{16}$ GeV. Matching the correct values of the electroweak data at $M_Z$, by running the RGEs down to energies of the electroweak scale, can support the existence of Higgs scalars in the adjoint, necessary to break the grand unified gauge group to the standard model. Unfortunately the gauge interactions in string theory depend on the Kac-Moody level “$k$” and in the most popular searches at $k = 1$\(^{73}\) there is no way that adjoint scalars can appear\(^{74}\), except the construction of flipped SU(5)\(^{78}\) where the breaking of the GUT group happens without adjoint scalars. The appearance of adjoint Higgs at higher Kac-Moody levels \(^{72, 73}\) becomes possible, via the existence of string models with a grand unified group such as i.e, SU(5) or SO(10) \(^{72}\). This of course makes more appealing the testing of GUT’s but complicates the proliferation of a specific string vacuum if any\(^{38}\), since the proliferation of the vacuum is reduced to the old GUT problem of gauge symmetry breaking.

One of the big problems of string theory at the moment, is that the value of string unification

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\(^{37}\)More details will be given in section (3.2.1).

\(^{38}\)If after compactification we get a gauge group as $G_1 \times G_2 \times \ldots$ then from the total contribution of the central charge to the left moving sector, we get the constraint $c_G = \sum_i \leq 22$. From this relation $c_G = \sum_i = \sum_i \frac{k_i \dim G_i}{k_i + \rho_i} \leq 22$, where $\dim G_i$ and $\rho_i$ are respectively the dimension and the dual Coxeter number of $G_i$ ($\rho = N$ for SU($N$); $\rho = 12$ for $E_6$), we can easily derive that SO(10), $E_6$ can be at most realized at levels 7 and 4 respectively.
scale which is calculated up to one loop at string level\textsuperscript{53} in the $\bar{D}R$ scheme to be
\[
M_{\text{string}} \overset{\text{def}}{=} \left( \frac{2e^{(1-\gamma/2^3-3/4)}}{(2\pi\alpha')^{1/2}} \right) \equiv \frac{e^{(1-\gamma/2^3-3/4)}}{4\pi}g_{\text{string}}M_{\text{plank}} \equiv 0.527 \, g_{\text{GUT}} \times 10^{18} \, \text{GeV},
\]

is in apparent disagreement with the success of the gauge coupling unification of minimal supersymmetric standard model (MSSM)\textsuperscript{70} at an energy of about $10^{16} \, \text{GeV}$. Since MSSM gives us full agreement with the LEP measurements the reason for disagreement caused a lot of excitement and several reasons were invoked to resolve the discrepancy. In the case of additional massless chiral fields on top of the spectrum of MSSM\textsuperscript{75, 76, 61, 98}, one needs an additional intermediate scale $M$ at $\sim 10^{12-14} \, \text{GeV}$, to lower the string unification scale down to $10^{16} \, \text{GeV}$. An alternative way of lowering the string unification scale, is to consider variations of the hypercharge normalization\textsuperscript{77}, which for the case of the $U(1)$ gauge group is identical to $k_1$. In this case we find agreement as long as the $k_1 \sim 1.4$.

A different option uses the target space modular invariant constraints necessary for cancellation of target space $\sigma$-model duality anomalies of the effective lagrangian to find the necessary range of modular weights of matter fields which account for anomaly cancellation associated with completely rotated planes\textsuperscript{41} and minimal string unification. In \textsuperscript{92} it was found that for (0,2) orbifolds only the $Z'_8$ and $Z_N \times Z_M$ orbifold survive this test. Unfortunately the values of the moduli which satisfy the constraints of duality cancellation and minimal string unification have values near 16, very far from the values obtained from gaugino condensation at their self-dual points. Of course it remains to be seen if further superstring corrections to physical quantities of interest will improve this analysis. Finally, we would like to mention the recent attempts \textsuperscript{94, 95} which use the soft terms\textsuperscript{42} and minimal unification of gauge coupling constants to calculate at the weak scale the masses. In such an approach, one uses as effective low-energy particle content of the theory the MSSM and the minimal unification approach, to make predictions for the low-energy $\alpha_{\text{em}}$ and the particle masses. This will test string theory in the near future.

\textsuperscript{39}$\gamma_e \simeq 0.57722$ is the Euler-Mascheroni constant. Normalization of the string coupling is as $g_{\text{string}}^{\text{tree level}} = g_{\text{GUT}}$. In this case\textsuperscript{53}, $\alpha' M_{\text{plank}}^2 g_{\text{string}}^2 = 32\pi$.

\textsuperscript{40} which is by itself contradictory since string theory is a theory of only one scale.

\textsuperscript{41}Planes for which the eigenvalue of the point group embedding is equal to -1

\textsuperscript{42} Being left over after\textsuperscript{99} the spontaneous supersymmetry breaking of the effective supergravity theory of our heterotic vacuum.
Various calculations of string threshold corrections to the gauge coupling constants have been performed in the literature for different classes of heterotic strings. Initially calculations were performed \[53\] for \(Z_3\) models with \((2,2)\) supersymmetry and with the presence of no Wilson line background fields. Further investigation of the one-loop moduli dependence of the gauge coupling constants was performed in \[71\]. Application to fermionic constructions\[79\] was performed for the \(Z_2 \times Z_2\) orifolds and especially for the flipped \(SU(5)\) model, where the corrections were incompatible with minimal unification. An analogous calculation was applied in the case of type-II superstrings in \[62\] where the moduli sit at the enhanced symmetry point \[43\]. The same investigation was applied for various Calabi-Yau manifolds in \[57, 58\]. In addition, for symmetric \((2,2)\) decomposable orbifold compactifications in \[71\] and for the non-decomposable orbifolds in \[59\].

The value of the \(g_{\text{string}}\) in eqn. (3.6) includes a universal-gauge group independent and moduli dependent contribution \(\Delta_{\text{universal}}\) in the form\[58\]

\[
g_{\text{string}} = \text{Re} S + \frac{1}{16\pi^2} \Delta_{\text{universal}}(\phi, \bar{\phi}),
\]

which was discarded in all the previous calculations. By the way, the practical use of eqn. (3.5) was in the calculation of the one-loop threshold corrections coming by taking differences between different gauge groups. In this way the value of the universal term didn’t really matter. The infrared regulator part and the contribution due to gravity of were recently calculated in \[123\]. In \[63, 104\] a numerical calculation of the value of the gauge group independent universal term \(Y\) term\[44\] was reported for a variety of backgrounds with Wilson lines. The value of the universal term for \(Z_2 \times Z_2\) was recently calculated \[129\] using the work of reference \[123, 130\]. It’s value represents the exact contribution to the threshold corrections since in this case the underlying fields \(F_{a \mu \nu}\) are exactly marginal and therefore their deformation behaviour exactly calculable. The value of the universal term reflects the contribution\[58\] of the gravitational back-reaction to the Einstein equations of motion when the non-zero \(F_{\mu \nu}\) background field is turned on.

Furthermore the full moduli dependence for orbifolds where the underlying lattice is assumed to decompose into a direct sum of a two dimensional and a four dimensional sublattices namely

\[43\] This is just an example of a CFT since type-II superstring cannot incorporate the standard model \[99\].
\[44\] Writing the threshold corrections \(\Delta_a = b_a \Delta + k_a Y\), where \(\Delta\) the moduli dependent contribution, \(k_a\) the Kac-Moody level and \(b_a\) the \(N = 2\) \(\beta\)-function.
$\Lambda_2 \bigoplus \Lambda_4$ with the unrotated plane lying in $\Lambda_2$ together with the inclusion of Wilson line background fields have been derived in [65].

the inclusion of the Wilson line dependence on the one-loop string threshold corrections, can provide us with the correct values of the Weinberg’s angle and strong coupling constant $\alpha_s$ at the scale $M_Z$ [96]. Nevertheless, it is introducing gauge group dependence. It is unlikely that a proliferation of the string vacua will involve any Wilson-line dependence since in this case there must be a principle to proliferate over the particular choice of Wilson lines which breaks the (0, 2) compactification to the $SU(3) \times SU(2) \times U(1)$ gauge group at $M_Z$.

An interesting development is the direct calculation of the threshold corrections in [130] of N=2 theories exhibiting exactly the behaviour of the threshold corrections at the enhanced symmetry points, as was predicted on the basis of symmetry arguments in [120] [46]. It is interesting therefore to perform the same kind of calculation for the case of non-decomposable orbifolds.

In this part of the thesis we will calculate the full moduli dependence of the threshold corrections for the case of the $Z'_8$ non-decomposable orbifold [66], for the case where no Wilson line background fields are involved. This particular class of orbifold models was singled out from the list of $Z_N$ orbifolds, in the analysis of [92] on the basis of satisfying the constraints from modular anomaly cancellation and unification of coupling constants, if the low-energy particle content was that of the MSSM. Of course there is always the counter-argument: why is there no string theory with just this particle content?

3.2 Threshold corrections to gauge couplings

3.2.1 Introduction

The calculation of the threshold corrections to the gauge coupling constants in N=1 orbifold compactifications was performed explicitly in [71]. For the calculation of the threshold corrections to gauge couplings for the $Z'_8$ non-decomposable orbifold [66], we need the full expression of the

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45In my opinion.

46via the calculation of the topological free energy.
quantity in eq. (3.2).

This quantity appears in the calculation of the one-loop amplitude involving two gauge bosons. In analytic form,

\[ B_a(\tau, \bar{\tau}) = \frac{2}{|\eta(\tau)|^4} \sum_{even \ s} \frac{(-1)^{s_1+s_2}}{2\pi i} \frac{dZ_\psi(s, \bar{\tau})}{d\tau} \times tr_{s_1} \left[ Q_a^2 (-1)^{s_2} q^H q^\bar{H} \right]_{int}. \]  (3.10)

The factor \( 1/|\eta(\tau)|^4 \) is associated with the contribution of the light cone partition functions of the space-time bosonic coordinates \( X_\mu \), while \( Z_\psi \) is the corresponding quantity for the space-time bosons \( \Psi_\mu \).

The spin structure of the fermions is denoted by \( s = (s_1, s_2) \), where \( s_1, s_2 \in \{0, 1\} \). A zero (one) refers to anti-periodic (periodic) boundary condition on the torus. The partition function for one complex fermion is given\(^{47} \) by

\[ Z_\psi(s, \tau) \equiv \frac{1}{\eta(\tau)} \theta_{i}^{[s_2]_{s_1}}(\tau), \]  where \( i = 3 \) for \( \bar{s} = (0, 0) \), \( i = 4 \) for \( \bar{s} = (0, 0) \), \( i = 2 \) for \( \bar{s} = (0, 0) \) and \( i = 0 \) for \( s = (1, 1) \).

\(^{47}Z_\psi(s, \tau) \equiv \frac{1}{\eta(\tau)} \theta_{i}^{[s_2]_{s_1}}(\tau), \]  where \( i = 3 \) for \( \bar{s} = (0, 0) \), \( i = 4 \) for \( \bar{s} = (0, 0) \), \( i = 2 \) for \( \bar{s} = (0, 0) \) and \( i = 0 \) for \( s = (1, 1) \).
the heterotic string, this trace decomposes into sectors with boundary conditions \((g, h)\) along the cycles of the world-sheet torus as:

\[
Tr_{s_1}(Q_a^2(\cdots)^s_2Fq^{-11/12}\bar{q}^{-3/8})_{\text{int}} = \frac{1}{|G|} \sum_{g, h \in G} Tr_{(g, s_1)}(Q_a^2(\cdots)^s_2Fq^{-11/12}\bar{q}^{-3/8}).
\] (3.12)

Here, \(L_o\) and \(\bar{L}_o\) represent the generators of dilatations in the complex plane for the left and right moving sectors respectively. Only sectors which are not completely rotated from orbifold twists are nonvanishing in the sum. For the \(N = 4\) sectors where all the fermions have to be considered as untwisted, the sum over spin structures decomposes as

\[
\sum_{s_1, s_2 = 0, 1} (-1)^{s_1 + s_2}\theta^{(s_1)}(s_2)\frac{d}{d\tau}[\sum_{s_1, s_2} (-1)^{s_1 + s_2}\theta^{(s_1)}(s_2)] = 0,
\] (3.13)

where \([\sum_{s_1, s_2} (-1)^{s_1 + s_2}\theta^{(s_1)}(s_2)] = \theta^2_2(0|\tau) - \theta^4_3(0|\tau) + \theta^4_1(0|\tau) = 0\), because of the zero identity of the \(\theta\) functions. The only terms that give non-vanishing contributions to the moduli dependent sum in eqn.(3.12) are the sectors, where the point group \(G\) which divides the six dimensional torus is a subgroup of \(SU(2)\), i.e. sectors with \(N = 2\) supersymmetry. The union of all twists associated with \(N = 2\) planes of the abelian \(Z_N\) orbifolds form the little groups \(G_i \subset G\) of the unrotated planes. In this case, the moduli dependent threshold corrections become

\[
\Delta_a = \frac{b'_a |G'|}{|G|} \Delta'_a,
\] (3.14)

where \(\Delta'_a\) is the moduli dependent contribution corresponding to the \(N = 2\) \(T_6/G'\) orbifold.

In addition, for orbifold compactifications where the internal torus decomposes in the form \(T_6 = T_2 \oplus T_4\), the internal sector sum on \(B_a\) splits into different factors coming from the internal superconformal field theories (SCFT) with central charges \((c, \bar{c}) = (20, 6)\) and \((2, 3)\). Remember, that any heterotic vacuum is obtained by tensoring in the light-cone gauge two left moving free bosons \(X_\mu\) together with their right moving fermionic superpartners, corresponding to the space-time coordinates and the internal SCFT with central charge \((22, 9)\).

In this case, \(B_a\) becomes

\[
B_a = Z_{\text{torus}}K_a, \quad Z_{\text{torus}} = \sum_{(P_L, P_R) \in \Gamma(2, 2)} q^{P_L^2} \bar{q}^{P_R^2},
\] (3.15)
\[ K_a \equiv \eta(\tau)^{-4} Tr \left( \frac{1}{2} \left( - \right)^F Q^2 a q^{L_0-5/6} \bar{q}^{\bar{L_0}-1/4} \right)_{(c,\bar{c})=(20,6)} \]  

(3.16)

It is obvious from eqn.(3.15) that the moduli dependence of the threshold corrections is included in the term \( Z_{\text{torus}} \). To be precise, the exact form of the threshold corrections to gauge couplings is given\[55\] by the equation

\[ B_a(\tau, \bar{\tau}) = \frac{2}{|\eta(\tau)|^4} \sum_{\text{even } s} \frac{(-1)^{s_1+s_2}}{2\pi i} \frac{dZ_\psi(s, \bar{\tau})}{d\tau} \operatorname{tr} s_1 \left[ \left( Q^2_a - \frac{k_a}{8\pi \tau_2} \right)(-1)^{s_2} F^H \bar{q} q^H b_a \right]_{\text{int}}. \]  

(3.17)

Comparing the eqns.(3.10) and (3.17) we notice the presence of the additional term \(-\frac{k_a}{8\pi \tau_2}\). This term was included in the calculation of the threshold corrections in \[55\]. We will comment on this term in chapter four, where we will explain its connection to the universal term.

The general form of the moduli dependent threshold corrections is

\[ \Delta_\alpha = \int_\Gamma \frac{d^2 \tau}{\tau_2} \sum_{(g,h)} b_{(g,h)} Z_{(g,h)}(\tau, \bar{\tau}) - \int_{\text{calF}} \frac{d^2 \tau}{\tau_2}, \]  

(3.18)

where the \( Z_{(g,h)} \) refer to the moduli dependent part of the \( N = 2 \) sector of the \((g,h)\) orbifold invariant under the group \( SL(2, Z) \) as it happen for the decomposable orbifolds. The integration is over the fundamental domain \( F \) of the inhomogenous modular group \( PSL(2, Z) \).

The sum is over the \( N = 2 \) orbit of the orbifold sectors created by the \( N = 2 \) sectors.

For non-decomposable orbifolds eqn.(3.18) can be rewritten as

\[ \Delta_\alpha = \sum_{(g_0,h_0)\in \mathcal{O}} b_{(g_0,h_0)}^{(h_0,g_0)} \int_{\tilde{F}} \frac{d^2 \tau}{\tau_2} Z_{(h_0,g_0)}(\tau, \bar{\tau}) - b_{(g_0,h_0)}^{(h_0,g_0)} \int_{\tilde{F}} \frac{d^2 \tau}{\tau_2}. \]  

(3.19)

Here, \((g_0,h_0)\) denotes the set of twisted sectors which be created from the representative fundamental element \( Z_{(h_0,g_0)} \), by exactly those modular transformations which create the modular group of \( Z_{(h_0,g_0)} \) from the fundamental region of \( PSL(2, Z) \).

For non-decomposable orbifolds the moduli dependent sum in eqn.(3.19) is invariant under the modular group but under some congruence of \( \Gamma \), namely \( \Gamma_0(n) \) or \( \Gamma^0(n) \).

Here, \( \tilde{F} \) is the enlarged region defined as a left coset decomposition of the fundamental region \( F \), namely \( \tilde{F} = \bigcup a_i F \). For the group \( \Gamma_0(p) \) the union \( \cup a_i F \) is represented

\[ ^{48}\text{This dependence will be elaborated in section (3.6).} \]

\[ ^{49}\text{For a definition see appendix A.} \]
from the set of transformations \[109\]

\[ a_i = \{1, S, ST, \ldots, ST^{p-1}\}. \] (3.20)

### 3.3 * Low Energy Threshold Effects and Physical Singularities*

In general, if one wants to describe globally the moduli space and not just the small field deformations of an effective theory around a specific vacuum solution, one has to take into account the number of massive states that become massless at a generic point in moduli space. This is a necessary, since the full duality group \(SO(22, 6; Z)_T\) mixes massless with massive modes \[44\]. It happens because there are transformations of \(O(6, 22, Z)\) acting as automorphisms of the Lorentzian lattice metric of \(\Gamma^{(6, 22)} = \Gamma^{(6, 6)} \oplus \Gamma^{(0, 16)}\) that transform massless states into massive states \[50\].

Let us consider the \(T_2\) torus, coming from the decomposition of the \(T_6\) orbifold into the form \(T_2 \oplus T_4\). The \(T_2\) torus can be defined on a two dimensional lattice \(\Gamma^{(2, 2)}\) which is generated from the basis vectors \(\vec{e}_1\) and \(\vec{e}_2\). The metric \(G_{ij} \equiv \vec{e}_1 \cdot \vec{e}_2\) has three independent components, while the antisymmetric tensor \(B \equiv b\epsilon_{ij}\) one. In total we have four independent real components which define the moduli of the string compactification on \(T^2\). The moduli can be further combined in the form of two complex moduli as \(U = |\vec{e}_1| e^{if} \) and \(T = 2(b + iA)\), with \(0 \leq f < \pi\) the angle between the basis vectors and \(A = \sqrt{|\det G|}\) is the area of the unit cell of the lattice \(\Gamma\). At the large radius limit It was noticed in \[138\] that in the presence of states that become massless at a point in moduli space e.g. when the \(T \rightarrow U\), the threshold corrections to the gauge coupling constants receive a dominant logarithmic contribution \[138\] in the form,

\[
\Delta_a(T, \bar{T}) \approx b_a' \int_\Gamma \frac{d^2 \tau}{\tau_2} e^{-M^2(T)\tau_2} \approx - b_a \log M^2 (T),
\] (3.21)

where \(b_a\) is the contribution to the \(\beta\)-function from the states that become massless at the point \(T = U\).

Strictly speaking the situation is sightly different. We will argue that if we want to include in the string effective field theory large field deformations and to describe the string Higgs effect \[114, 112, 113\] and not only small field fluctuations, eqn. (3.21) must be modified.

\[50\] Construction of effective actions invariant under the \(O(6, 22, Z)\) duality group reproducing \(N = 4\) low energy effective actions of the heterotic string were constructed in \[90\].
We will see that massive states which become massless at specific points in the moduli space do so, only if the values of the untwisted moduli dependent masses are between certain limits. In [138] this point was not emphasized and it was presented in a way that the appearance of the singularity had a general validity for generic values of the mass parameter. We will complete the picture by giving more details on the exact behaviour of the contribution to the threshold corrections to the gauge coupling constants. We introduce the function Exponential Integral

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt \quad (|\arg z| < \pi), \quad (3.22)$$

with the expansion

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-)^n z^n}{n(n)!}. \quad (3.23)$$

It can be checked that for values of the parameter $|z| > 1$, the $\ln z$ term is not the most dominant, while for $0 < |z| < 1$ it is. In the latter case [106] the $E_1(z)$ term is approximated as

$$E_1(z) = -\ln(z) + a_{00} + a_{11}z + a_{22}z^2 + a_{33}z^3 + a_{44}z^4 + a_{55}z^5 + \epsilon(z). \quad (3.25)$$

Take now the form of eqn.(3.21) explicitly

$$\Delta(z, \bar{z}) = b'_a \int_{|\tau_1|<1/2} d\tau_1 \int_{|\tau_2|>1} d\tau_2 e^{-M^2(T)\tau_2}. \quad (3.26)$$

Then by using eqn.(3.22) in eqn.(3.26), we can see that the $-b'_a \ln M^2(T)$ indeed arise.

Notice now, that the limits of the integration variable $\tau_1$ in the world-sheet integral in eqn.(3.21) are between $-1/2$ and $1/2$. Then especially for the value $|1/2|$ the lower limit in the integration variable $\tau_2$ takes its lowest value e.g $(1 - \tau_2^2)^{1/2} = (1 - (1/2)^2)^{1/2} = \sqrt{(3)/2}$. Use now eqn.(3.23). Rescaling the $\tau_2$ variable in the integral, and using the condition $0 < z < 1$ which is necessary for the logarithmic behaviour to be dominant, we get

$$0 < M^2(T) < \frac{4}{\sqrt{3}a'}. \quad (3.27)$$

\[51\text{where}\]

$$\alpha_{00} = -0.577 \quad \alpha_{10} = 0.999 \quad \alpha_{20} = -0.249$$

$$\alpha_{33} = -0.551 \quad \alpha_{44} = -0.009 \quad \alpha_{55} = 0.001 \quad (3.24)$$

and $\epsilon(z) < 2 \times 10^{-7}$.\[52\text{Restoring units in the Regge slope parameter } a'.\]
This means that the dominant behaviour of the threshold corrections appears in the form of a logarithmic singularity, only when the moduli scalars satisfy the limit $M^2 < 4/(\sqrt{3}a')$.

We know that for particular values of the moduli scalars, the low energy effective theory appears to have singularities, which are due to the appearance of charged massless states in the physical spectrum. At this stage, the contribution of the mass to the low energy gauge coupling parameters is given by

$$M^2 \to -n_H |T - p|^2,$$

where the $n_H$ represents the number of states $\phi_H$ which become massless at the point $p$. This behaviour is consistent with large field deformations of the untwisted moduli. It is obvious at this point that the behaviour of the threshold effects over the whole area of the moduli space can not consistently described by the behaviour of the latter equation. The singular limit of this expression $T = p$ "can not be reached".

Of course, and as a consequence neither can the enhanced symmetry point. The parameter $(a'\sqrt{3}/4)M^2$ must always be between the limits zero and one in order that the contribution of the physical singularity to $\Delta$ become dominant. Therefore, the complete picture of the threshold effects reads

$$\frac{1}{g^2_a(\mu)} = \frac{k_a}{g^2_{string}} + \frac{b_a}{16\pi^2} \ln \frac{M^2_{string}}{\mu^2} - \frac{1}{16\pi^2} |\Delta_a(T_i)|^2 - \Theta(-M^2 + \frac{4}{\sqrt{3}a'})b'_a \log M^2(T)$$

where $\Theta$ in eqn.(3.29) is the step function. The logarithmic contribution at this stage is actually the threshold effect of the contribution of the states which become light. Their direct effect on the low energy effective theory is the appearance of the automorphic functions of the moduli dependent masses, after the integration of the the massive modes.

The same threshold effect dependence on the $\Theta$ function, takes place in Yang-Mills theories, via the decoupling theorem. The contribution of the various thresholds decouples from the full theory, and the net effect is the appearance of mass suppressed corrections to the physical quantities.

So far, we have seen that the theory can always approach the enhanced symmetry point behaviour from a general massive point on the moduli space under specific conditions. For

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53 When it is reached perturbation theory is not valid. This is a signal that new states become massless at this point.
"large" values of the moduli masses the enhanced symmetry point can only be reached if its mass is inside the limit (3.27). Remember that at the point $T = p$ eqn.(3.29) breaks down, since at this point perturbation theory is not valid any more. The upper limit (3.27) is in the strongly coupled regime of the perturbative $\sigma$-model coupling expansion parameter.

The previous result is particularly important in view of the fast development of the subject of dualities in superstring theory. It should be noted that in the case of Calabi-Yau manifolds, the appearance of singularities in the target space can provide for the web of connectivity [150, 153] through the entire moduli space.

Interestingly enough the presence of the logarithmic term was confirmed in the calculation of the target space [120] free energies for the massive modes and recently with an exact [130] calculation with the calculation of threshold effects from BPS states. An interesting example of the appearance of the singularity in heterotic strings will be described in the next section when a non-decomposable lattice is involved in the (2, 2) symmetric orbifold compactifications.

3.4 * Target space automorphic functions from string compactifications

In this part of the thesis, we will discuss the contribution of massive moduli dependent masses of the heterotic compactification to the threshold corrections of the gauge coupling constants.

In addition, we discuss the appearance of the extended non-abelian gauge group in particular Narain orbifolds, namely those that the internal lattice involved is a $Z_N$ non-decomposable orbifold. Initially, we will be describe how the moduli dependence becomes visible in the mass operators, in untwisted subspaces of orbifold compactifications of the heterotic string. Later on, we will focus our attention to the calculation of the moduli dependent threshold corrections, coming from direct integration of the massive untwisted states of the compactification.

For orbifold compactifications, where the underlying internal torus does not decompose into a $T_6 = T_2 \oplus T_4$, the $Z_2$ twist associated with the reflection $-I_2$ does not put any additional constraints on the moduli $U$ and $T$. As a consequence the moduli space of the untwisted subspace is the same as in toroidal compactifications. Orbifold sectors which have the lattice twist acting as a $Z_2$, give non-zero threshold one-loop corrections to the gauge coupling constants in $N = 1$
supersymmetric orbifold compactifications.

When the heterotic string is compactified on a six dimensional torus, the physical states have their mass given by

\[ \frac{\alpha'}{2} M^2 = N_L + N_R + \frac{1}{2}(P_L^2 + P_R^2) - 1, \]  
(3.30)

where \( P_L(R) \), \( N_L(R) \) are the left and right moving momentum and number operators respectively. In addition, invariance of the one-loop vacuum amplitude under the modular transformations \( T \to T + 1 \) gives the level matching constraint

\[ \frac{\alpha'}{2} M_L^2 = N_L + \frac{1}{2} P_L^2 - 1 = N_R + \frac{1}{2} P_R^2 = \frac{\alpha'}{2} M_R^2. \]  
(3.31)

From the above equations we deduce

\[ \frac{\alpha'}{2} M^2 = P_R^2 + 2N_R. \]  
(3.32)

For the calculations which we will describe in this chapter, we will need the moduli dependence of the mass operator. In order to display the moduli dependence of (3.32), we need the general form of the Narain lattice vector \( P_R \) in the presence of the Wilson lines as given in chapter two.

This is manifestly exhibited by expressing \( P_R \) in terms of the quantum numbers of the Lorentzian Narain lattice \( \Gamma_{22,6} \) and then projecting into an orthonormal basis. In the orthonormal basis, the untwisted moduli space factorizes into factors corresponding to the different twist eigenvalues.

The Narain vector \( P_R \) of the untwisted sector of the \( N = 1 \) orbifold is then parametrized in the usual way, by expressing it in terms of the 28 integer quantum numbers, namely the winding numbers \( n_i \), the momentum numbers \( m_i \) of the compactification and the charges \( q_I \) of the sixteen dimensional Euclidean even-self dual lattice of the leftmoving current algebra.

Initially, we project the momentum vectors

\[ P = q^I l_I + n^i k_i + m_j k_j \quad i = 1, \ldots, 6 \quad I = 1, \ldots, 16 \]  
(3.33)

---

54 This is equivalent to the argument to the following argument: in closed string theory, physical states must be invariant under global shifts in the space-like coordinate \( \sigma \) of the world-sheet. The operator \( e^{i\lambda(L_0 - \bar{L}_0)} \) satisfies, \( U_\lambda X^\mu(\sigma, \tau)U_\lambda = X^\mu(\sigma + \lambda, \tau) \) which means that it generates translations in the world-sheet variable \( \sigma \). However, in closed string theory there is no distinguished point in the world-sheet. This condition forces us to define the condition \( (L_0 - \bar{L}_0)|_{\text{phys}} = 0 \) or \( L_0 - \bar{L}_0 \).

55 In the following we will follow closely the article of (120).
into the vectors $\mathbf{e}_\mu^R = (0_{16}, 0_6, \mathbf{e}_\mu)$ of the orthonormal basis of $R^6$ with the result

$$\mathcal{P}_R = (q^I, n^i, m_j) \begin{bmatrix} l_I \cdot \mathbf{e}_\mu^{(R)} \\ \bar{k}_i \cdot \mathbf{e}_\mu^{(R)} \\ k^j \cdot \mathbf{e}_\mu^{(R)} \end{bmatrix} \mathbf{e}_\mu^R. \quad (3.34)$$

In this way, the norm of the right moving momentum factorises as

$$\mathcal{P}_R^2 = v^T \Phi \Phi^T v,$$  

where

$$v^T = (q^I, n^i, m_j) \in M(1, 28, \mathbf{Z}) \cong \mathbf{Z}^{28} \quad (3.36)$$

and

$$\Phi = \begin{bmatrix} l_I \cdot \mathbf{e}_\mu^{(R)} \\ \bar{k}_i \cdot \mathbf{e}_\mu^{(R)} \\ k^j \cdot \mathbf{e}_\mu^{(R)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -AE^* \\ DE^* \\ E^* \end{bmatrix} \in M(28, 6, \mathbf{R}) \quad (3.37)$$

which contains all the moduli dependence. Here $A$ represents the Wilson lines namely

$$A = (A_I) = (l_I \cdot \mathbf{A}_i) \in M(16, 6, \mathbf{R}), \quad (3.38)$$

while $D$ is the moduli matrix in the lattice basis

$$D = (D_{ij}) = 2( B_{ij} - G_{ij} - \frac{1}{4} \mathbf{A}_i \mathbf{A}_j) \in M(6, 6, \mathbf{R}) \quad (3.39)$$

and the 6-bein of the dual lattice $\Lambda^*$ is

$$E^* = (E^i_\mu) = (\mathbf{e}^i \cdot \mathbf{e}_\mu). \quad (3.40)$$

With the heterotic string further compactified on an orbifold, the action of the twist on the moduli is subject to compatibility conditions$^{[83, 108, 111]}$. These equations which are satisfied from the continuous parts of the metric, antisymmetric and Wilson line background fields are as follows:

$$D_{ij} \theta^j_k = \theta^j_i D_{jk}, \quad A_{ij} = \theta^j_i A_{jk}. \quad (3.41)$$

$^{56}$Here M represents a $1 \times 28$ matrix with integer coefficients.
The quantities $\theta_{ij}^j$ and $\theta_{jk}^j$ represent the twist action on the $\Gamma_{6;6}$ and its dual, while $\theta_I$ represents the action of the gauge twist in the gauge degrees of freedom of the $E_8 \times E_8$ current algebra. The moduli variable $\Phi$ satisfies the well-known equation for the $SO(22,6)$ coset space

$$\Phi^T H_{22,6} \Phi = -I_6,$$  \hspace{1cm} (3.42)

with $H_{22,6}$ the pseudo-euclidean lattice metric of the Narain lattice $\Gamma_{22,6}$

$$H_{22,6} = \begin{pmatrix} C_{(16)}^{-1} & 0 & 0 \\ 0 & 0 & I_6 \\ 0 & I_6 & 0 \end{pmatrix}. \hspace{1cm} (3.43)$$

The solution of the compatibility conditions (3.41) for the moduli in the orthogonal basis, results in the decomposition of the untwisted moduli space of the orbifold into factors corresponding to different twist eigenvalues.

Following this procedure, we now factorize the moduli space into subspaces corresponding to the different internal and gauge twist eigenvalues. We perform the change of variables from the lattice to an orthonormal basis by

$$\Phi = \frac{1}{2} \begin{pmatrix} -A_{ij} E^j \nu \\ D_{ij} E^j \nu \\ E^i \nu \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{E}^M_i & 0 & 0 \\ 0 & T_{\mu}^\nu & 0 \\ 0 & 0 & T_{\mu}^\nu \end{pmatrix} \hat{\Phi}, \hspace{1cm} (3.44)$$

creating the variable $\hat{\Phi}$ which exhibits no moduli dependence. In this form, the variable $\hat{\Phi}$ satisfies the equation

$$\hat{\Phi} \hat{H}_{22,6} \hat{\Phi} = -I_6$$  \hspace{1cm} (3.45)

with

$$\hat{H}_{22,6} = \begin{pmatrix} I_{16} & 0 & 0 \\ 0 & 0 & I_6 \\ 0 & I_6 & 0 \end{pmatrix}. \hspace{1cm} (3.46)$$

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57 This means that the untwisted moduli space of the $D = 4$, $N = 4$ toroidal compactifications of the heterotic string is the coset space $SO(22,6) \rightarrow SO(22) \times SO(6)$.

58 Here $C_{(16)}$ is the Cartan matrix of $E_8 \times E_8$.

59 The quantities $T_{\mu}^i$ and $T_{\mu}^i$ are moduli independent and obey the relation: $E^i \nu = S^i \nu T_{\mu}^j = T_{\mu}^i S^i \nu$, with $S$ a deformation matrix parameter connecting lattice to lattice($S^i_j$) or orthonormal($S^i_\nu$) basis.
In this way, the component form of the \( \hat{\Phi} \) variable becomes block diagonal with factors corresponding to the different twist eigenvalues, namely

\[
\hat{\Phi} \rightarrow \hat{\phi}^{(j)} \oplus \hat{\phi}^{(+1)} \oplus \hat{\phi}^{(-1)},
\]

while the mass operator factorises as \( \mathcal{P}_R^2 = \hat{\nu}^T \hat{\phi} \hat{\phi}^T \hat{\nu} \)

\[
\hat{\nu}^T = (q^I, n^i, m_j) \begin{pmatrix}
\mathcal{E}_i^M & 0 & 0 \\
0 & T_i^\mu & 0 \\
0 & 0 & T_i^\mu
\end{pmatrix}.
\]

The dimensions of the matrix variable \( \hat{\Phi} \), depend on the multiplicities of the eigenvalues of the internal and gauge twists in their block diagonal form. Especially, for the subspace of the twists corresponding to the eigenvalue \(-1\), the dimensions of the variable \( \hat{\Phi} \) are \((q + s, s)\).

However, the coset space structure of the moduli space becomes obvious in the standard metric \( \eta_{++} \), i.e \((+)22(-)^6\). The transition to this metric can be made obvious by an appropriate transformation\(^{[20]}\) on the \( \hat{\phi} \rightarrow \tilde{\phi} \) variable and \( \hat{\nu} \rightarrow \tilde{\nu} \), in such a way that \( \tilde{\nu}^T \tilde{\phi} = \tilde{\nu}^T \tilde{\phi} \). The \( \tilde{\phi} \) variable satisfies the equation

\[
\tilde{\phi}^T \eta_{++} \tilde{\phi} = -I_2,
\]

while the introduction of complex variables as

\[
\phi_c = \begin{pmatrix}
\phi_1^1 + i\phi_1^2 \\
\phi_2^1 \\
\phi_{q+2}^1 + i\phi_{q+2}^2
\end{pmatrix}
\]

restructures the (3.49) equations into the constraint equations for the \( SO(q + 2, 2) \) coset, namely

\[
\phi_c^\dagger \eta_{++} \phi_c = -2, \quad \phi_c^T \eta_{++} \phi_c = 0.
\]

The direct result is that the mass takes the form

\[
\mathcal{P}_R^2 = \tilde{\nu}^T \tilde{\phi} \tilde{\phi}^T \tilde{\nu} = \tilde{\nu}^T \phi_c \phi_c^\dagger \tilde{\nu} = |\tilde{\nu}^T \phi_c|^2.
\]

\(^{60}\)For the gauge twist, we assume an orthogonal decomposition into subspaces corresponding to the the complex eigenvalues \( e^{\pm k_j} \) and the real eigenvalues \(-1\) and \(+1\), with multiplicities \( R_j \), \( q \), \( p \) correspondingly. For the internal twist, we assume an orthogonal decomposition into subspaces corresponding to the complex eigenvalues \( e^{\pm \rho_j} \) and the real ones \(-1, +1\). The corresponding multiplicities are assumed to be \( Q_j, s, \xi \).
The solution of the coset equations \((3.51)\) eliminates the redundant degrees of freedom. By defining \(y \in \mathbb{C}^{q+4}\), the eqn’s \((3.51)\) become

\[
\sum_{i=1}^{q+2} |y_i|^2 - |y_{q+3}|^2 - |y_{q+4}|^2 = -2Y \\
\sum_{i=1}^{q+2} y_i^2 - y_{q+3}^2 - y_{q+4}^2 = 0.
\]  

Then e.g for the coset \(SO(4,2)\), the derivation of the mass operator for the untwisted subspace results from the solution of the constraint equations with the ansatz

\[
y_1 = (B_1 + C_1), \quad y_2 = (B_1 - C_1) \\
y_3 = \frac{1}{2}(T - 2U), \quad y_4 = -i(1 - \frac{1}{4}(2TU - 2BC)) \\
y_5 = \frac{1}{2}(T + 2U), \quad y_6 = i(1 + \frac{1}{4}(2TU - 2BC))
\]

and

\[
Y = \frac{1}{2}(T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{C})(C + \bar{B})
\]

giving the mass formula

\[
\frac{\alpha'}{2} M^2 = \frac{|\tilde{\rho}^T y|^2}{Y}.
\]  

The general contribution\([120, 141]\) to the mass formula for the \(Z_2\) orbifold plane\([62]\)

\[
p_R^2 = |m_2 - iUm_1 + iTn_1 - (TU - BC)n^2 + r_1f_1(k_i)(B + C) + r_2f_2(k_i)(B - C)|^2,
\]

where \(r_1, r_2 \in R\) and \(f_1(k_i), f_2(k_i)\) functions of the gauge quantum numbers. The above formula involves perturbative BPS states which preserve 1/2 of the supersymmetries, which belong to short multiplet representations of the supersymmetry algebra.

In the study of the untwisted moduli space, we will assume initially that under the action of the internal twist there is a sublattice of the Narain lattice \(\Gamma_{22,6}\) in the form \(\Gamma_2 \oplus \Gamma_4 \supset \Gamma_{22,6}\) with

\(^{61}\)It is associated with the two dimensional torus lattice \(T_2\) of the untwisted subspace of a \(Z_2\) orbifold, for which a two component Wilson line is turned on. The \(T_2\) is a sublattice decomposition of the Narain lattice as \(\Gamma_2 \oplus \Gamma_4 + \ldots\), with the internal twist acting as \(-I_2\) on \(\Gamma_2\).

\(^{62}\)for a general twist embedding in the gauge degrees of freedom with \(k_1, \ldots, k_d\) gauge lattice quantum numbers in the invariant directions.
the twist acting as $-I_2$ on $\Gamma_2$. In the general case, we assume that there is always a sublattice $\Gamma_{q+2;2} \oplus \Gamma_{r+4;4} \subset \Gamma_{16;6}$ where the twist acts as $-I_{q+4}$ and with eigenvalues different than $-I$ on $\Gamma_{r+4;4}$.

In this case, the mass formula for the untwisted subspace $\Gamma_{q+2;2}$ depends on the factorised form $P^2_R = v^T \phi \phi^T v$, with $v^T$ taking values as the row vector

$$v^T = (a^1, \ldots, a^q; n^1, n^2; m_1, m_2). \quad (3.59)$$

The quantities in the parenthesis represent the lattice coordinates of the untwisted sublattice $\Gamma_{q+2;2}$, with $a^1 \ldots a^q$ the Wilson line quantum numbers and $n^1, n^2, m_1, m_2$ the winding and momentum quantum numbers of the two dimensional subspaces.

When Wilson lines are present, the variable $\phi = \phi^{-1}$ satisfies the coset equation

$$\phi^T H_{q+2;2} \phi = -I_2. \quad (3.60)$$

The meaning of the previous equation is that the untwisted moduli space is that of an $SO(q+2,2)/(SO(q+2) \times SO(2))$ coset. For decomposable orbifolds with continuous Wilson lines turned on, the untwisted moduli space is $SO(q+2,2)/SO(q+2) \times SO(2)$ when the twist has two eigenvalues $-1$.

Shortly, we will discuss the case of $Z_6 - IIb$ non-decomposable orbifold. In this example\[^66\] the untwisted moduli space is as before, i.e in the form $SO(4,2)/(SO(4) \times SO(2))$. The internal twist acting on the 6-torus has two eigenvalues $-1$. The action of the internal twist can be made to act as $-I_2$ on a $T_2$ by appropriate parametrization of the momentum quantum numbers.

The moduli metric $H_{q+2;2}$ in (3.60) is given by the matrix

$$H_{q+2;2} = \begin{pmatrix} C_q^{-1} & 0 & 0 \\
0 & 0 & I_2 \\
0 & I_2 & 0 \end{pmatrix} \quad (3.61)$$

\[^{63}\]this does not correspond to a decomposition of the Narain lattice as $\Gamma_{22;6} = \Gamma_{q+2;2} \oplus \ldots$ since the gauge lattice $\Gamma_{16}$ is an Euclidean even self-dual lattice. So the only way for it to factorize as $\Gamma_{16} = \Gamma_q \oplus \Gamma_r$, with $q + r = 16$, is when $q = r = 8$.

\[^{64}\]on $\Gamma_{q+2;2}$

\[^{65}\]in the case that the untwisted subspace where the twist acts as $-I_{q+4}$.

\[^{66}\]with continuous Wilson lines turned on
where the matrix variable $\phi$ in (3.59) is a $(q + 2, 2)$ matrix with integer values and $C_q^{-1}$ the lattice metric for the invariant directions in the gauge lattice.

Let us consider first the generic case of an orbifold where the internal torus factorizes into the orthogonal sum $T_6 = T_2 \oplus T_4$ with the $Z_2$ twist acting on the 2-dimensional torus lattice. We will be interested in the mass formula of the untwisted subspace associated with the $T_2$ torus lattice. We consider as before that there is a sublattice of the Euclidean self-dual lattice $\Gamma_{22,6}$ as $\Gamma_{q+2,2} \oplus \Gamma_{20-q,4} \subset \Gamma_{22,6}$. In this case, the momentum operator factorizes into the orthogonal components of the sublattices with $(p_L; p_R) \subset \Gamma_{q+2,2}$ and $(P_L; P_R) \subset \Gamma_{20-q,4}$. And as a result the mass operator (3.32) factorizes into the form

$$\frac{\alpha'}{2} M^2 = p_R^2 + P_R^2 + 2N_R.$$ 

(3.62)

On the other hand, the spin operator $S$ for the $\Gamma_{q+2,2}$ sublattice becomes

$$p_L^2 - p_R^2 = 2(N_R + 1 - N_L) + \frac{1}{2} P_R^2 - \frac{1}{2} P_L^2 = 2n^T m + b^T C b,$$

(3.63)

where $C$ is the Cartran metric operator for the invariant directions of the sublattice $\Gamma_q$ of the $\Gamma_{16}$ even self-dual lattice. The spin $S$ can be expressed more elegantly in matrix form as

$$p_L^2 - p_R^2 = \frac{1}{2} u^T \eta u,$$

(3.64)

where

$$u = \begin{pmatrix} b \\ n \\ m \end{pmatrix}, \quad \text{and} \quad \eta = \begin{pmatrix} 0 & 0 & C_q \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}$$

(3.65)

Let us now consider the $Z_6 - IIb$ orbifold. For this particular orbifold we will discuss a number of issues. In particular, gauge symmetry enhancement and calculation of the modular orbits resulting in the generation of the non-perturbative superpotential. This orbifold is defined on the torus lattice $SU(3) \times SO(8)$ and the twist in the complex basis is defined as $\Theta = exp(2, -3, 1)\frac{2\pi i}{6}$.

67Compatibility of the untwisted moduli with the twist action on the gauge coordinates comes from the non-trivial action of the twist in the gauge lattice. This means that that the untwisted moduli of the orbifold have to the equation $M A = A Q$, where $Q, A$ and $M$ represent the internal, Wilson lines and gauge twist respectively.

68at the end of this discussion we will comment on the difference of the mass operator for the non-factorizable case.
The twist in the lattice basis is defined as

\[
Q = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 1 & 1 & -1 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
\end{pmatrix}.
\] (3.66)

This orbifold is non-decomposable in the sense that the action of the lattice twist does not decompose in the orthogonal sum \( T_6 = T_2 \oplus T_4 \) with the fixed plane lying in \( T_2 \). The orbifold twists \( \Theta^2 \) and \( \Theta^4 \), leave the second complex plane unrotated. The lattice in which the twists \( \Theta^2 \) and \( \Theta^4 \) act as a lattice automorphism is the \( SO(8) \). In addition there is a fixed plane which lies in the \( SU(3) \) lattice and is associated with the \( \Theta^3 \) twist.

Consider now a k-twisted sector of a six-dimensional orbifold of the heterotic string associated with the twist \( \theta^k \). If this sector has an invariant complex plane then its twisted sector quantum numbers have to satisfy

\[
Q^k n = n, \quad Q^{*k} m = m, \quad M^k l = l,
\] (3.67)

where \( Q \) defines the action of the twist on the internal lattice and \( M \) defines the action of the gauge twist on the \( E_8 \times E_8 \) lattice. In general, if \( E_\alpha, \ \alpha = 1, 2 \) is a set of basis vectors for the fixed directions of the orbifold and \( E_\mu, \ \mu = 1, \ldots, d \) is a set of basis vectors for the fixed directions in the gauge lattice then the momentum and winding numbers for the invariant directions of the twisted states, are found to have the general form

\[
P = \tilde{m}_1 \tilde{E}_1 + \tilde{m}_2 \tilde{E}_2, \quad L = \tilde{n}_1 E_1 + \tilde{n}_2 E_2, \quad (\tilde{m}_1, \tilde{m}_2, \tilde{n}_1, \tilde{n}_2 \in \mathbb{Z}),
\] (3.68)

with \( \tilde{E}_1, \tilde{E}_2 \) particular linear combinations of the dual basis \( e_i^* \) and \( \tilde{E}_a \cdot E_b \overset{\text{def}}{=} \rho_{ab} \). This means that with

\[
n = \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix}, \quad m = \begin{pmatrix} \tilde{m}_1 \\ \tilde{m}_2 \end{pmatrix}, \quad l = \begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \end{pmatrix}
\] (3.69)

assuming that the internal lattice has basis vectors \( e_i, \ i = 1, \ldots, 6 \) and dual \( e_i^* \) with \( e_i^* \cdot e_i = \delta_{ij} \).
the momenta take the form

\[ P_L' = \left( \frac{\hat{m}}{2} + (G_\perp - B_\perp - \frac{1}{4} A_\perp^i C_\perp A_{\perp}) \tilde{n} - \frac{1}{2} A_\perp^i C_\perp \tilde{I} + A_{\perp} \tilde{n} \right) = (p_L', p_{R}') \] (3.70)

\[ P_R' = \left( \frac{\hat{m}}{2} + (G_\perp - B_\perp + \frac{1}{4} A_\perp^i C_\perp A_{\perp}) \tilde{n} - \frac{1}{2} A_\perp^i C_\perp \tilde{I}, 0 \right) = (p_{R}', 0). \] (3.71)

Here \( \rho \) is the \( \rho_{ab} \) matrix, \( C_\perp \) is the Cartran matrix for the fixed directions and \( A_{\perp}^i \) is the matrix for the continuous Wilson lines in the invariant directions \( i = 1, 2, I = 1, \ldots, d \). \( G_\perp \) and \( B_\perp \) are \( 2 \times 2 \) matrices and \( \tilde{n}, \tilde{m}, \tilde{l} \) are the quantum numbers in the invariant directions.

For the \( Z_6 - IIb \) orbifold, \( \rho = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \) In addition, the values of the momentum and winding numbers parametrizing the \( \theta^2 \) subspace are

\[
\begin{pmatrix}
0 \\
0 \\
n^1 \\
0 \\
n^1 - n^2 \\
n^2
\end{pmatrix}
\] with \( n^1, n^2 \in \mathbb{Z} \),

\[
p = \begin{pmatrix}
0 \\
0 \\
m_1 \\
-m_1 \\
m_2 \\
m_1 - m_2
\end{pmatrix}
\] with \( m_1, m_2 \in \mathbb{Z} \). (3.72)

The mass formula \[120, 132\] for the \( \Theta^2 \) subspace reads

\[ m^2 = \sum_{m_1, m_2} \frac{1}{Y} | - T U' n^2 + i T n^1 - i U' m_1 + 3 m_2^2 |_{U' = U - 2i}^2 = |M|^2 / (Y/2), \] (3.73)

with

\[ Y = (T + \bar{T})(U + \bar{U}). \] (3.74)

The quantity \( Y \) is connected to the Kähler potential, \( K = - \log Y \). The target space duality group is \( \Gamma^0(3) \times \Gamma^0(3) \) where \( U' = U - 2i \).

In eqns\[3.62, 3.63, 3.64\], we discussed the level matching condition in the case of a \( T_6 \) orbifold admitting an orthogonal decomposition. Mixing of these equations gives us the following equation

\[ p_L^2 - \frac{\alpha'}{2} M^2 = 2(1 - N_L - \frac{1}{2} P_L^2) = 2n^T m + q^T C q. \] (3.75)

The previous equation gives us a number of different orbits invariant under \( SO(q + 2, 2; \mathbb{Z}) \) transformations:
i) the untwisted orbit with

\[ 2n^T m + q^T C q = 2. \]  

(3.76)

In this orbit, \( N_L = 0, P^2_L = 0 \). When \( M^2 = 0 \), this orbit is associated with the “stringy Higgs effect”. The “stringy Higgs effect appears as a special solution of the equation (3.75) at the point where \( p^2_L = 2 \), where additional massless particles may appear.

ii) the untwisted orbit where

\[ 2n^T m + q^T C q = 0. \]  

(3.77)

Here \( 2N_L + P^2_L = 2 \). This is the orbit relevant to the calculation of threshold corrections to the gauge couplings, without the need of enhanced gauge symmetry points, as may happen in the orbit i).

iii) The massive untwisted orbit with

\[ 2N_L + P^2_L \geq 4 \]  

(3.78)

Here always \( M^2 \geq 0 \).

Let us now consider\(^{70}\) the modular orbit associated with the ’stringy Higgs effect’. It corresponds to certain points in the moduli space where singularities associated with the additional massless particles appear and have as a result gauge group enhancement. This point correspond to \( T = U \) with \( m^2 = n^2 = 0 \) and \( m^1 = n^1 = \pm 1 \). At this point the gauge symmetry is enhanced to \( SU(2) \times U(1) \). In particular, the left moving momentum for the two dimensional untwisted subspace gives

\[ p^2_L = \frac{1}{2T_2U_2}|\bar{T}U n_2 - iT n_1 - iU' m_1 + 3m_2|^2 = 2, \]  

(3.79)

while

\[ p^2_R = \frac{1}{2T_2U_2'}| - T U' n^2 + iT n^1 - iU' m_1 + 3m_2|^2 = 0. \]  

(3.80)

At the fixed point of the modular group \( \Gamma^o(3), \frac{\sqrt{3}}{2}(1 + i\sqrt{3}) \), there are no additional massless states, so there is no further enhancement of the gauge symmetry.

We will now describe the calculation of threshold corrections from the target space free energies of the massive untwisted states. In quantum field theory, when we are interested in

\(^{70}\)for the orbifold \( Z_6 - IIb \)
the calculation of the effective lagrangian, then we have to deal with the generator of the 1PI Feynman diagrams\textsuperscript{71}, the generating function $\Gamma$. In general, if our theory contains a number of fields, including light fields $\phi$ and heavier fields $\Phi$, then the effective action $\hat{\Gamma}$ for the light fields $\phi$, is the sum of the 1PI Feynman diagrams with respect of the light fields $\phi$. In other words, the effective action in this case, is given by

$$e^{-\hat{\Gamma}} = \int [d\Phi] e^{-I(\phi, \Phi)}$$

where the ”superheavy” fields $\Phi$ are ”integrated out”. The previous quantity $e^{-\hat{\Gamma}}$, is known to be related to the topological free energy through the definition

$$e^{-\hat{\Gamma}} = e^F$$

For example from the definition of the bosonic free energy, by expansion, we get

$$e^{F_{(\text{bosonic})}} = \int [D\phi] e^{-\phi M_\phi^2/2 + ...}$$

Here the ellipsis represent the usual derivative expansion terms of the effective action i.e $i \frac{\partial}{\partial M_\phi}$ and higher order $\phi$ terms. The action $\hat{\Gamma}$ contains an infinite number of non-renormalizable interactions\textsuperscript{121}, suppressed at energies $E \ll M_\phi$ by powers of $\frac{E}{M_\phi}$.

The fermionic free energy - for supersymmetric backgrounds coming from the integration of massive fermions - is defined as the negative of the bosonic free energy

$$F_{\text{fermionic}} = \log \det M^\dagger M.$$  

Here $M$ represents the fermionic mass matrix. Working in this way, we define the free energy as the one coming from the integration of the massive compactification modes, i.e. Kaluza-Klein and winding modes. We exclude non-compactification modes like massive oscillator modes. In this sense, the free energy is topological\textsuperscript{132, 67}.

Of particular importance to us, will be the calculation of the non-perturbative superpotential. We will calculate it, through its identification with the target space free energy coming from the\textsuperscript{72}.

\textsuperscript{71}Feynman diagrams that cannot become disconnected by cutting off one of their internal lines. See for example [?].

\textsuperscript{72}There is a distinction between bosonic or fermionic free energy, depending on whether the functional integration is over bosonic or fermionic states $\Phi$, respectively.
massive compactification modes. In particular it was argued\cite{132}, that the target space partition
function $Z$, defined as

$$Z = e^{-F_{\text{fermionic}}} = -\text{det}(M^\dagger M) = -\frac{|W|^2}{Y}, \quad (3.85)$$

when appropriately regularized\cite{73} provides us with the non-perturbative superpotential coming
from the integration of the massive chiral compactification modes.

Here $M$ is the mass matrix of all chiral masses of the particles involved in the compactification process and $Y$ is connected to the Kähler potential $K$, via the relation $K = -\log Y$ (see eqn. (3.74)). For the two-dimensional toroidal compactification with moduli space $SO(2, 2)/SO(2) \times SO(2)$ and modular group $SL(2, Z)$, the topological free energy is equal to $F = \sum_{\text{mom. and wind. numbers}} \log M^\dagger M$
where $M^\dagger M = (M^2/Y)$ and $M = |- TU n^2 + iT n^1 - iU m_1 + m_2|$.

In this way, the non-perturbative superpotential is identified as

$$W = \text{det} M \quad (3.86)$$

Here $W$ is the mass matrix $M$ of the chiral masses of the compactification modes.

Especially, for the case where the calculation of the free energy is that of the moduli space of the manifold $SO(2, 2)/SO(2) \times SO(2)$, in a factorizable 2-torus $T_2$, the topological\cite{132} bosonic free energy is exactly the same as the one, coming from the string one loop calculation in \cite{71}.

We will describe now the low-energy behaviour of the $N = 1$ orbifold string effective field theory.

Recall the general form of the bosonic non-local effective lagrangian in four-dimensions, up
to two space-time derivatives\cite{253, 71}

$$\mathcal{L}_{\text{eff}} = \frac{R}{2k^2} + \left( \frac{1}{4g^2(\phi)} \right)_{ab} F_{\mu\nu}^a F_{\mu\nu}^b + \frac{i}{32\pi^2} F_{\mu\nu}^a F_{\mu\nu}^b + \frac{1}{2} G_{ij}(\phi) \partial_\mu \phi^i \partial_\mu \phi^j + V(\phi). \quad (3.87)$$

The matrices $g_{\alpha\beta}^2(\phi)$ and $\Theta_{ab}(\phi)$ are field dependent inverse gauge couplings and vacuum angles respectively, $G_{ij} = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^i \partial \bar{\phi}^j}$, with $K(\phi, \bar{\phi})$ the Kähler potential, is the metric on the Riemannian manifold of the scalar fields, $V$ is the scalar potential, and $R$ is the scalar curvature of the space-time metric $G_{\mu\nu}$.

\footnote{We comment on the regularization procedure after relation (3.94).}
If the low energy theory is that of a $D = 4$, $N = 1$ supergravity\(^74\), then it is completely determined from the knowledge of three functions, namely the Kähler potential, the superpotential\(^75\) and the gauge kinetic function $f$. The latter is defined as

$$f_{ab}(\tilde{\phi}) = \left( \frac{1}{g^2(\tilde{\phi})} \right)_{ab} - i \frac{\Theta_{ab}(\tilde{\phi})}{8\pi^2} \tag{3.88}$$

and it has to be a holomorphic function of the complex coordinates $\tilde{\phi}$ of the Kähler scalar manifold. For $N = 1$ supersymmetric orbifolds the dependence on the complex scalar coordinates arises from the moduli fields, defined later. The study of the $f_{ab}$ function is particularly important, in view of the fact that its derivatives are involved in various non-renormalizable interactions\(^75\).

A general problem of a quantum field theory that involves massless particles\(^71\) appears when we expand the Wilson’s effective action $1\Pi$ diagrams, in terms involving a power series in particles momenta. Because the radius of convergence of the series is that of the lightest particle, if there are massless particles the radius is equal to zero. In this case, there is no local effective lagrangian, and the effective renormalized gauge couplings $1/g^2_a(p^2 = 0)$, cannot be defined.

However, by studying the running gauge couplings $\left( \frac{\partial g_a^{-2}}{\partial \Phi_i} \right)$ at some off-shell momentum $p^2 \neq 0$, this problem is avoided and the $p^2 = 0$ limit can be reached. For this reason, the supersymmetric one loop graph involving two gauge fields and one scalar field, with charged fermions contributing in the loops, gives the following relations\(^71\) between the gauge coupling constants and the effective axionic couplings at one-loop

$$8\pi^2 \left( \frac{\partial g_a^{-2}}{\partial \Phi_i} \right)^{(1-\text{loop})} = -i \left( \frac{\partial \Theta_a}{\partial \Phi_i} \right)^{(1-\text{loop})} = -\frac{1}{2} Tr \left( Q_a \frac{\partial M}{\partial \Phi_i} M^\dagger \frac{1}{MM^\dagger + O(p^2)} \right). \tag{3.89}$$

Here $Q_a$ is the generator of the gauge group $\alpha$. Furthermore, as matter as it concerns the integrability conditions, the following relations hold\(^71\):

1) the integrability condition for the running axionic couplings\(^76\)

$$\Theta_{\alpha,ij}(p^2; <\tilde{\phi}>) \neq \Theta_{\alpha,ji}(p^2; <\tilde{\phi}>) \tag{3.90}$$

\(^74\)Their effect on the physical parameters will become obvious in chapter five.

\(^75\)A number of $f_{ab}(\phi)$ derivatives, contributes to the effective superpotential which may cause supersymmetry breaking, through formation of gaugino condensates\(^122\).

\(^76\)The subscripts denote derivatives with respect to moduli fields, namely $\Theta_{\alpha,i} \equiv \frac{\partial}{\partial \phi^i} \Theta_{\alpha}$. 
is satisfied only in the $p^2 = 0$ limit. This means, that there is no well defined running axionic $\Theta$ coupling off-shell. In general, integrability is retained for $p^2 \ll \text{mass}^2$ of the lightest charged fermion, bearing in mind that the gauge symmetry must not be chiral (there must not be massless fermions) in order for the $M^\dagger M$ matrix to be invertible.

In the $p^2 = 0$ limit

$$\left(\Theta_\alpha^2\right)_{p^2=0}^{1-\text{loop}} = \text{Tr} (Q_\alpha^2 \text{Im} M) + \text{constant} = 8\pi^2 \text{Im} f_\alpha. \tag{3.91}$$

For QCD the above relation becomes the $\Theta$ angle, $\Theta = \text{ArgDet}(M_{\text{quark}})$.

\textit{ii) } As a result of the non-integrability of the axionic $\Theta$ couplings, the one-loop corrections to the gauge couplings are non-holomorphic.

By further integration of (3.89)-at the infrared limit $p^2 = 0$ limit- we obtain

$$16\pi^2 \left( g_\alpha^{-2}\right) = -\text{tr} (\text{log}(Q_\alpha^2 M^\dagger M)) + \text{constant}. \tag{3.92}$$

Here $M$ is the field dependent mass matrix for the charged fermions.

The equation (3.92) is the supersymmetric version of Weinberg’s formula [121] for the one-loop gauge coupling constants.

The stringy version of the Weinberg’s formula for the one-loop correction to the gauge couplings constant may come from the relation

$$\Delta_0 = \sum_{n,m} \text{log} \mathcal{M}^2 \overset{\text{def}}{=} \sum_{n,m} \text{log} \mathcal{M} + \sum_{n,m} \text{log} \mathcal{M}^\dagger. \tag{3.93}$$

We will now use eqn. (3.93) to calculate the stringy one-loop threshold corrections to the gauge coupling constants coming from the integration of the massive compactification modes with $(m, m', n, n') \neq (0, 0, 0, 0)$. The total contribution to the threshold corrections, coming from modular orbits associated with the presence of massless particles, is connected with the existence of the following orbits [132, 120]

$$\Delta_0 = \sum_{2n'+q^T C_q = 0} \text{log} \mathcal{M}|_{\text{reg}}$$

$$\Delta_1 = \sum_{2n'+q^T C_q = 0} \text{log} \mathcal{M}|_{\text{reg}}. \tag{3.94}$$

\textsuperscript{77} The same relation was used in the calculation of the threshold corrections as target space free energies in [120].

\textsuperscript{78} We calculate only the $\sum \text{log} \mathcal{M}$ since the $\sum \text{log} \mathcal{M}^\dagger$ quantity will give only the complex conjugate.
In the previous expressions, a regularization procedure is assumed that takes place, which renders the final expressions finite, as infinite sums are included in their definitions. The regularization is responsible for the subtraction of a moduli independent quantity from the infinite sum e.g. \( \sum_{n,m \in \text{orbit}} \log\mathcal{M} \). We demand that the regularization procedure for \( \exp[\Delta] \) has to respect both modular invariance and holomorphicity.

In eqn. (3.94), \( \Delta_0 \) is the orbit relevant for the stringy Higgs effect. This orbit is associated with the quantity \( 2n^T m + q^T C q = 2 \) where \( n^T m = m_1 n_1 + 3m_2 n_2 \) for the \( Z_6 - IIb \) orbifold. This specific orbit will be used as well, in the second part of the thesis, to discuss the threshold correction contribution to the gravitational couplings from the point of view of extended gauge group enhancement.

The total contribution from the previously mentioned orbit is

\[
\Delta_0 \propto \sum_{n^T m + q^2 = 1} \log\mathcal{M} = \sum_{n^T m = 1, q^T C q = 0} \log\mathcal{M} + \sum_{n^T m = 0, q^T C q = 2} \log\mathcal{M} + \sum_{n^T m = -1, q^T C q = 4} \log\mathcal{M} + \ldots \tag{3.95}
\]

We must notice here that we have written the sum over the states associated with the \( SO(4,2) \) invariant orbit \( 2n^T m + q^T C q = 2 \) in terms of a sum over \( \Gamma^0(3) \) invariant orbits \( n^T m = \text{constant} \). We will be first consider the contribution from the orbit \( 2n^T m + q^T C q = 0 \). We will be working in analogy with calculations associated with topological free energy considerations. From the second equation in eqn. (3.94), considering in general the \( SO(4,2) \) coset, we get for example that

\[
\Delta_1 \propto \sum_{n^T m + q^2 = 0} \log\mathcal{M} = \sum_{n^T m = 0, q = 0} \log\mathcal{M} + \sum_{n^T m = -1, q^2 = 1} \log\mathcal{M} + \ldots \tag{3.96}
\]

Consider in the beginning the term \( \sum_{n^T m = 0, q = 0} \log\mathcal{M} \). We are summing up initially the orbit with \( n^T m = 0; n, m \neq 0 \), i.e \( \Delta_1 \)

\[
\mathcal{M} = 3m_2 - im_1 U' + in_1 T + n^2 (-U'T + BC) + q \text{ dependent terms.} \tag{3.97}
\]

More details of this procedure can be found in [132].

We use a general embedding of the gauge twist in the gauge degrees of freedom.
We calculate the sum over the modular orbit \( n^T m + q^2 = 0 \). As in [120] we calculate initially the sum over massive compactification states with \( q_1 = q_2 = 0 \) and \( (n, m) \neq 0 \). Namely, the orbit

\[
\sum_{n^T m = 0, \, q = 0} \log \mathcal{M} = \sum_{(n, m) \neq (0, 0)} \log(3m_2 - im_1 U' + im_1 T + n_2 (-U'T)) + \frac{n_2}{3m_2 - im_1 U' + im_1 T - n_2 U'T} + O((BC)^2).
\]

The sum in relation (3.98) is topological (it excludes oscillator excitations) and is subject to the constraint \( 3m_2 n^2 + m_1 n^1 = 0 \). Its solution receives contributions from the following sets of integers:

\[
m_2 = r_1 r_2, \, n_2 = s_1 s_2, \, m_1 = -3r_2 s_1, \, n_1 = r_1 s_2
\]

(3.99)

and

\[
m_2 = r_1 r_2, \, n_2 = s_1 s_2, \, m_1 = -r_2 s_1, \, n_1 = 3r_1 s_2.
\]

(3.100)

So the sum becomes,

\[
\sum_{n^T m = 0} \log \left(3m_2 - im_1 U' + im_1 T - n_2 U'T\right) = \sum_{(r_1, s_1) \neq (0, 0)} \log \left(3(r_1 + is_1 U')\right) \times \sum_{(r_2, s_2) \neq (0, 0)} \log \left(r_2 + is_2 T\right)
\]

(3.101)

Substituting explicitly in eqn. (3.98), the values for the orbits in equations (3.99) and (3.100) together with eqn. (3.101), we obtain

\[
\sum_{n^T m = 0; q = 0} \log \mathcal{M} = \log \left(\frac{1}{3} \eta^{-2}(U')\eta^{-2}(T)\right) + \log \left(\frac{1}{3} \eta^{-2}(-\frac{T}{3})\eta^{-2}(T)\right) +
\]

\[
\left(BC \left( \sum_{(r_1, s_1) \neq (0, 0)} \frac{s_1}{r_1 + is_1 U'} \right) \left( \sum_{(r_2, s_2) \neq (0, 0)} \frac{s_2}{3r_2 + is_2 T}\right)\right) +
\]

\[
\left(BC \left( \sum_{(r_1, s_1) \neq (0, 0)} \frac{s_1}{3(r_1 + is_1 U')} \right) \left( \sum_{(r_2, s_2) \neq (0, 0)} \frac{s_2}{r_2 + is_2 T}\right)\right) + O((BC)^2)
\]

(3.102)

Notice that we used the relation

\[
\sum_{(r_1, s_1) \neq (0, 0)} \log 3 = \log \frac{1}{3}
\]

(3.103)
with \( \sum_{(r_1, s_1) \neq (0,0)} = -1 \). We substitute \( \sum_{(r_1, s_1) \neq (0,0)} \overset{\text{def}}{=} \sum' \) and \( \sum_{(r_2, s_2) \neq (0,0)} \overset{\text{def}}{=} \sum'' \). Remember that \( \sum' \log(t_1 + it_2 T) = \log \eta^{-2}(T) \), with \( \eta(T) = \exp \frac{\pi T}{12} \Pi_{n>0}(1 - \exp^{-2\pi n T}) \) This means that eqn.(3.102) can be rewritten as

\[
\log_{n^T m = 0; q = 0} \log \mathcal{M} = \log \left( \eta^{-2}(U') \eta^{-2} \left( \frac{T}{3} \right) \left( \frac{1}{3} \right) \right) + \log \left( \frac{1}{3} \eta^{-2} \left( \frac{U'}{3} \right) \eta^{-2} (T) \right) + \\
-BC \left( \left( \frac{U'}{3} \right) \sum \log(r_1 + is_1 U') \left( \frac{r_2 + is_2 T}{3} \right) \right)
\]

Finally,

\[
\sum_{n^T m = 0; q = 0} \log \mathcal{M} = \log \left( \eta^{-2}(T) \eta^{-2} \left( \frac{U'}{3} \right) \left( \frac{1}{3} \right) \right) + \log \left( \eta^{-2}(U') \eta^{-2} \left( \frac{T}{3} \right) \frac{1}{3} \right) - \\
-BC \left( \eta^{-2}(T) \eta^{-2} \left( \frac{1}{3} \right) \left( \frac{U'}{3} \right) + \eta^{-2} \left( \frac{T}{3} \right) \right) + O((BC)^2)
\]

So

\[
\sum_{n^T m = 0; q = 0} \log \mathcal{M} = \log \left[ \eta^{-2}(T) \frac{1}{3} \eta^{-2} \left( \frac{U'}{3} \right) \right] (1 - BC \left( \eta^{-2}(T) \eta^{-2} \left( \frac{1}{3} \right) \left( \frac{U'}{3} \right) \right) + \\
\eta^{-2} \left( \frac{T}{3} \right) ( \eta^{-2}(U') \eta^{-2} \left( \frac{1}{3} \right) ) + O((BC)^2)
\]

or

\[
\sum_{n^T m = 0; q = 0} \log \mathcal{M} = \log \left[ \eta^{-2}(T) \frac{1}{3} \eta^{-2} \left( \frac{U'}{3} \right) \right] (1 - 4 BC \left( \eta^{-2}(T) \right) \frac{1}{3} \eta^{-2} \left( \frac{1}{3} \right) ) + \\
\eta^{-2} \left( \frac{T}{3} \right) ( \eta^{-2}(U') \eta^{-2} \left( \frac{1}{3} \right) ) (1 - 4 BC \left( \eta^{-2}(U') \right) \eta^{-2} \left( \frac{1}{3} \right) )
\]
The last expression provides us with the non-perturbative \[132, 120\] generated superpotential \(W\), by direct integration of the string massive modes. In fact \[120\] the corresponding expression for the decomposable orbifolds, was found to be the same as the expression argued to exist in \[230\], for the non-perturbative superpotential. The latter was obtained from the requirement that the one loop effective action in the linear formulation for the dilaton be invariant under the full \(SL(2, Z)\) symmetry up to quadratic order in the matter fields. In exact analogy, we expect our expression in eqn.(3.106), to represent the non-perturbative superpotential of the \(Z_6 - IIb\) orbifold\(^{81}\). The contribution of this term could give rise to a direct Higgs mass in the effective action and represents a particular solution to the \(\mu\) term problem. In (2,2) compactifications of the heterotic string, a superpotential mass term in the form \(\mu_{\alpha\xi} D^\alpha E^\xi\) is generated\[236\] in the observable sector below the supersymmetry breaking scale. Here, \(D^\alpha, E^\xi\) correspond to singlet superfields(moduli), which are in one to one correspondence with the 27, 27 supermultiplets of matter fields of the \(E_6 \times E_8\) gauge group. The dependence of the \(\mu\) term on the non-perturbative superpotential appears through the relation

\[\mu \propto e^{G/2} W_{DE},\]  

(3.108)

where \(W_{DE}\) e.g represents the quantity

\[W_{DE} = -\frac{4}{\eta^2(T)\eta^2(U'3)}(\partial_T \log \eta(T)) \left(\frac{U'}{3}\right)\]  

(3.109)

and transforms correctly under the required modular transformations. Here, \(G\) is the gauge kinetic function. More details on the \(\mu\) term generation can be found in chapter 5.

The exact form of the non-perturbative superpotential for the \(Z_6 - IIb\) orbifold is given by(see chapter 5)

\[\mathcal{W} e^{-3S/2b} = \left[ (\eta^{-2}(T))(\frac{1}{3})\eta^{-2}(\frac{U'}{3})\right] (1 - BC) \left(\frac{U'}{3}\right) \left(\frac{1}{3}\right) \times \]

\[\eta^2\left(\frac{U'}{3}\right) \right] + \left[ (\eta^{-2}(U'))(\eta^{-2}(\frac{T}{3}))\frac{1}{3}\right] (1 - BC) \left(\frac{U'}{3}\right) \left(\frac{1}{3}\right) \times \]

\(^{81}\) Further discussion of our results and related matters will be presented in chapter 5, which is related to supersymmetry breaking mechanisms in string theory.
\[
(\partial_U \log \frac{1}{3} \eta^2 (\frac{U'}{3})) |\tilde{W} + O((BC)^2),
\]

(3.110)

where S is the dilaton and b the \(\beta\) function of the condensing gauge group, and \(\tilde{W}\) depends on the moduli of the other planes, e.g. the third invariant complex plane, when there is no cancellation of anomalies by the Green-Schwarz mechanism.

We will see later in chapter five that the exact form of the induced, \(\mu\)-term, depends explicitly on the details of the non-perturbative generated superpotential we propose.

We have calculated the topological free energy, as a sum of the effective theory of the massive compactification modes. Alternatively, the previous calculation could be performed directly at string theory level. The general result for a vacuum associated with \(D\) compactified coordinates, is that the free energy is the ratio of the world-sheet determinants of the \(\partial_k \partial_{\bar{k}}\) operator for the \(D\)-dimensional space \(R^D\) and the \(D\)-dimensional internal space \(M^D\). Explicitly

\[
F = \int \frac{d^2 \tau}{\pi (Im \tau)^2} \left( \frac{(det \partial_k \partial_{\bar{k}})_{R^D}}{V(R^D)(det \partial_k \partial_{\bar{k}})_{M^D}} - 1 \right).
\]

(3.111)

We turn now our discussion to the contribution from the first equation in (3.94) which is relevant to the stringy Higgs effect. Take for example the expansion (3.95. Let’s examine the first orbit corresponding to the sum \(\Delta_{0,0} = \sum n^m m=1, q=0 \log M\). This orbit is the orbit for which some of the previously massive states, now become massless. At these points the \(\Delta_{0,0}\) has to exhibit the logarithmic singularity. In principle we could predict, in the simplest case when the Wilson lines have been switched off the form of \(\Delta_{0,0}\). The exact form, when it will be calculated has to respect that the quantity \(e^{\Delta_{0,0}}\) has modular weight \(-1\) and reflects exactly the presence of the physical singularities of the theory. At this point it is appropriate to introduce the quantity \(\omega(T)\) where \(\omega(T)\) is given explicitly by

\[
\omega(T) = \left( \frac{\eta(T/3)}{\eta(T)} \right)^{12}\]

(3.112)

and represents the Hauptmodul for \(\Gamma^0(3)\), the analogue of \(j\) invariant for \(SL(2, \mathbb{Z})\). It is obviously automorphic under \(\Gamma^0(3)\) and possess a double pole at infinity and a double zero at zero. It

\[\text{[82]}\text{This point was not explained in [120] but it is obvious that it corresponds to the superpotential and thus transforming with modular weight } -1.\]
is holomorphic\cite{267} in the upper complex plane and at the points zero and infinity has the expansions

\[
\omega(T) = t^{-1}_\infty \sum_{\lambda=0}^\infty a_\lambda t^{\lambda}_\infty, \quad a_o \neq 0
\]

\[
\omega(T) = t^{-1}_o \sum_{\lambda=0}^\infty b_\lambda t^{\lambda}_o, \quad b_o \neq 0
\]

(3.13)

at \(\infty\) and 0 respectively with \(t = e^{-2\pi T}\).

For \(\Delta_{0,0}\) we predict

\[
\Delta_{0,0} \propto (\omega(T) - \omega(U'))^\gamma \times \ldots
\]

(3.14)

In full generality, the Hauptmodul functions for the \(\Gamma^0(p)\) are the functions\cite{109}

\[
\Phi(\tau) = \left(\frac{\eta(\tau)}{\eta(p\tau)}\right)^r
\]

(3.15)

Here, \(p=2,3,5,7\) or 13 and \(r = 24/(p - 1)\). For these values of \(p\) the function in eqn.\,(3.115) remains modular invariant, i.e it is a modular function.

The corresponding functions for the group \(\Gamma_0(p)\) are represented by the \(\left(\frac{\eta(\tau)}{\eta(p\tau)}\right)^r\).

\[
\Delta_{0,0} = \sum_{n^2m=1} \log(TU'n^2 + Tn^4 - U'm_1 + 3m_2) = \log((\omega(T) - \omega(U'))^6\eta(T)^2
\]

\[
\times \eta(U'/3)^{-2}) + \log((\omega(T) - \omega(U'))\eta(T/3)^{-2}\eta(U')^{-2} + \ldots
\]

(3.16)

The behaviour of the \(\Delta_0\) term reflects the\footnote{In the following we will be using the variable \(U\) instead of \(U'\).} fact that at the points with \(T = U\), generally previously massive states becoming massless, while the \(\eta\) terms are needed for consistency under modular transformations. Finally, the integers \(\xi, \chi\) have to be calculated from a string loop calculation or by directly performing the sum.

After this parenthesis, we continue our discussion by turning on, Wilson lines. When we turn the Wilson lines on, for the \(SO(4,2)\) orbit of the relevant untwisted two dimensional subspace \(\Delta_{0,0}\) becomes

\[
\Delta_{0,0} = \sum_{n^2m=1} \log\{3m_2 - im_1 U + in_1 T - n_2(UT - BC)\}
\]

(3.17)
The sum after using an ansatz similar to [120] and keeping only lowest order terms has the form

$$\Delta_{0,0} = \log (\omega(T) - \omega(U) + BC \, X(T,U))^\xi + \log \left(\eta(T)^{-2} \eta(U/3)^{-2} + BC \, Y(T,U)\right) + \log \left(\eta(T/3)^{-2} \eta(U)^{-2} + BC \, W(T,U)\right) + \ldots$$

(3.118)

The functions $X(T,U), Y(T,U), W(T,U)$ will be calculated by the demand of duality invariance.

Demanding invariance [236] of the first term in (3.118) under the target space duality transformations which leave the tree level Kähler potential invariant

$$U \rightarrow \frac{\alpha U - i\beta}{i\gamma U + \delta}, \quad T \rightarrow T - i\gamma \frac{BC}{i\gamma U + \delta}, \quad \alpha \delta - \beta \gamma = 1$$

$$B \rightarrow \frac{B}{i\gamma U + \delta}, \quad C \rightarrow \frac{C}{i\gamma U + \delta}, \quad \beta = 0 \mod 3,$$

we get that $X(T,U)$ has to obey - to lowest order in $BC$ - the transformation

$$\omega(T) - \omega(U) \xrightarrow{\Gamma^{(3)U}} \omega(T) - \omega(U) - i\gamma \frac{BC}{i\gamma U + \delta} \omega'(T).$$

(3.120)

As a consequence

$$X(T,U) \xrightarrow{\Gamma^{(3)U}} (i\gamma U + \delta)^2 X(T,U) + i\gamma(i\gamma U + \delta) \omega'(T).$$

(3.121)

In the same way, demanding invariance under $\Gamma^{(3)T}$ transformations we find that $X(T,U)$ has to transform as

$$X(T,U) \xrightarrow{\Gamma^{(3)T}} (i\gamma T + \delta)^2 X(T,U) - i\gamma(i\gamma T + \delta) \omega'(U).$$

(3.122)

Because, the first term in (3.118) has to exhibit the logarithmic singularity at the point $T = U$, $X(T,U)$ in turn has to vanish at the same point. All the previous mentioned properties, are properly exhibited from the function

$$X(T,U) = \partial_U \{\log \eta^6(U/3)\} \omega'(T) - \partial_T \{\log \eta^6(T/3)\} \omega'(U) + \beta \{\omega(T) - \omega(U)\} \eta^4(T/3) \eta^4(U/3) + O((BC)^2)$$

(3.123)

The $\beta$ is a constant which may be decided from a loop calculation. The exact calculation of the threshold corrections involving the presence of the logarithmic term may come from a calculation
similar to the one performed in [130]. Let us discuss now the term \(Y(T, U)\). Demanding the term \(\eta(T)^{-2} \eta(U/3)^{-2} + BC Y(T, U)\) to transform under the \(\Gamma^o(3)_U\) transformations of eqn. (3.119) with modular weight \(-1\), gives that \(Y(T, U)\) has to transform

\[
Y(T, U) \rightarrow \Gamma^o(3)_U (i\gamma U + \delta)Y(T, U) + i\gamma (i\gamma U + \delta)\eta(U/3)^{-2}(\partial_T \eta^{-2}(T)).
\] (3.124)

Under \(\Gamma^o(3)_T\), \(Y(T, U)\) has to transform as

\[
Y(T, U) \rightarrow \Gamma^o(3)_T (i\gamma T + \delta)Y(T, U) + i\gamma (i\gamma T + \delta)\eta(T)^{-2}.\]

(3.125)

The following function satisfies all requirements up to order \((BC)^2\),

\[
Y(T, U) = -\eta^{-2}(T)\eta^{-2}(U/3)(\partial_T \log \eta^2(T))(\partial_U \log \eta^2(U/3)) + \nu_1 \eta^4(T)\eta^4(U/3)
\]

\[
+ \nu_2 \eta^4(T)\eta^4(U).
\] (3.126)

The transformation behaviour under the proper modular transformations is not enough to determine the constants \(\nu_1\) and \(\nu_2\). They may be decided from a string loop calculation [130]. In a similar way, demanding the term \(\eta(T/3)^{-2}\eta(U)^{-2} + BCW(T, U)\) to transform with modular weight \(-1\), we find that \(W(T, U)\) has to transform as

\[
W(T, U) \rightarrow \Gamma^o(3)_U W(T, U)(i\gamma U + \delta) + i\gamma (\partial_T \eta^{-2}(T))\eta^{-2}(U),
\] (3.127)

and

\[
W(T, U) \rightarrow \Gamma^o(3)_T W(T, U)(i\gamma T + \delta) + i\gamma (\partial_U \eta^{-2}(U))\eta^{-2}(T/3).
\] (3.128)

The following function satisfies the requirements of eqn.(3.127) and (3.128),

\[
W = -\eta^{-2}(T/3)\eta^{-2}(U)(\partial_T \log \eta^2(T))(\partial_U \log \eta^2(U)) + \lambda_1 \eta^4(T/3)
\]

\[
\times \eta^4(U) + \lambda_2 \eta^4(T)\eta^4(U),
\] (3.129)

where \(\lambda_1, \lambda_2\) will be decided from the string loop calculation similar that in [130].

3.5 * Threshold corrections to gauge and gravitational couplings
3.5.1 * Threshold corrections to gauge couplings

Let us now complete our previous discussions, by considering the contributions of gravitational threshold corrections due to the integration of the massive modes of the heterotic string. We will be concentrating our discussion on \((2,2)\) symmetric non-decomposable orbifolds for which an explicit calculation of moduli dependence of the threshold corrections to the gauge coupling constants exists.

We will analyze the case of gravitational threshold corrections in the case of \(N = 2\) heterotic string compactifications, up to one loop and for the case of non decomposable orbifold compactifications of the heterotic string. Before we examine the threshold contributions to the gravitational threshold corrections, we will study their effect on the gauge coupling constants. It will help us to understand properly the connection between the calculation of the free energy we performed before, and the the effective gauge couplings.

When considering an effective locally supersymmetric field theory, we have to distinguish between two kinds of renormalized physical couplings involved in the theory. These are the cut-off dependent Wilsonian gauge couplings and the moduli and momentum dependent effective gauge couplings(EGC)\(^{296}\).

Let us follow a field theoretical approach for the calculation of contributions of the physical modes of our theory to the effective gauge couplings. We demand our physical theory at the high energy threshold, to be a product of several gauge groups namely \(G = \otimes G_a\). Then, the one loop corrected effective gauge couplings obey the following formula

\[
\frac{1}{g_a^2(p^2)} = \frac{k_a}{g_a^2(M_X^2)} + \frac{b_a}{16\pi^2} \log \frac{M_X^2}{p^2} + \frac{\Delta_a}{16\pi^2} + \frac{\tilde{\Delta}_a}{16\pi^2},
\]

when\(^\text{84}\)

\[
\Delta_a = [-2 \sum_i T_a(r) \log \text{det} g_e] + c_a K].
\]

In the formula (3.130) which is valid at energies \(p^2 \ll M_X^2\), we have tacitly assumed that the light particles of the theory are exactly massless, while the massive charged fields decouple at the high energy threshold \(M_X^2\).

\(^{84}\)For convenience, we will set the Kac-Moody level equal to one.
Here, the $N = 1 \beta$ function coefficient is given by $b_a = -3c(G_a) + \sum_\omega T_a(r_\omega)$, where $c(G_a)$ is the quadratic Casimir of the gauge group and the sum is over the massless charged chiral matter superfields transforming under the representation $r_\omega$ of the gauge group $G_a$. In addition, $c_a = (-c(G_a) + \sum_\omega T_a(r_\omega))$ and $T_a(r)$ is given by $Tr(T_a^\mu T_a^\nu) = \delta^{\mu\nu}T_a(r)$, where $T_a^\nu$ is the generator of the gauge group $G_a$ and the sum is over massless fermions transforming under $G_a$. Finally, $K$ is the Kähler potential of our low energy theory and $g_r$ is the $\sigma$-model metric of the massless subsector of the charged matter fields transforming in the representation $r$ of the gauge group.

The contributions $\tilde{\Delta}$ of the threshold corrections describe the tower of massive modes that decouple\textsuperscript{85} at the high energy threshold $M_X$. The non-holomorphic threshold contribution of the term in the brackets comes from the contributions of the Kähler and $\sigma$-model anomalies. Its contribution to the four dimensional effective one-loop string action is associated to triangle diagrams involving two [136, 137, 84, 86] gauge and moduli fields as external legs while massless particles running in the loops. The $\sigma$-model anomalies are similar to the local gauge anomalies but now one of the external legs of the triangle diagram is a composite $\sigma$-model connection or a Kähler connection. In the $\sigma$-model description\textsuperscript{255, 133} of $N = 1$ supergravity, fermion kinetic terms

$$\frac{i}{2}g_{ij}\bar{\psi}_j\gamma^\mu \partial_\mu \psi + \frac{i}{2}Re f_{ab}\bar{\lambda}_a^\mu \partial_\mu \lambda^b$$

are accompanied by the interaction terms

$$\left(\frac{i}{2}Re f_{ab}\bar{\lambda}_a^\mu \partial_\mu \lambda^b - \frac{i}{2}g_{ij}\bar{\psi}_L\gamma^\mu \partial_\mu \psi_L \right) \frac{1}{2}V_{Kähler} K^\mu_{\mu} + \left(\frac{i}{2}g_{ij}\bar{\psi}_L\gamma^\mu \partial_\mu \psi_L(-i\Gamma_{ikl}\partial_\mu z_k) + h.c\right),$$

with the $\sigma$-model connection is given by $\Gamma_{ijk} = \frac{\partial}{\partial \phi^j} g_{km}$ and the Kähler connection is given by

$$\alpha_\mu = -i \left[ \frac{\partial}{\partial \phi_i} K(\phi, \bar{\phi}) \partial_\mu \phi^i - \frac{\partial}{\partial \phi_i} K(\phi, \bar{\phi}) \partial_\mu \bar{\phi}^i \right].$$

The composite Kähler connection is analogous to $K(\phi, \bar{\phi})$ and\textsuperscript{86} couples to gauginos $\lambda_L$ as well to chiral matter fields $\psi_L \equiv A_\alpha(r_\omega)$. Its presence is a reflection of the tree level invariance of the theory under Kähler transformations.

\textsuperscript{85}In N=1 orbifold compactifications the high energy threshold coincides\textsuperscript{71, 53} with the string unification scale.

\textsuperscript{86}Here $K(\phi, \bar{\phi})$ represents the moduli field dependent part of the Kähler potential. Of course, we concentrate our discussion in the $N = 2$ sectors of the $N = 1 (0, 2)$ orbifold compactification of the heterotic string.
The contributions from the Kähler and σ model connections lead to the following one-loop modification of the tree level supersymmetric non-linear σ-model moduli Lagrangian:

$$L_{\text{non-local}} = \sum_\alpha \int d^2\theta \frac{1}{4} W^\alpha W_\alpha \{ S - \frac{1}{16\pi^2} \frac{1}{16} \Box^{-1} \mathcal{D}\mathcal{D}\mathcal{D} \}
$$

$$\left( [c(G_\alpha) - \sum_{r_\omega} T(r_\omega)] K(\phi, \bar{\phi}) + 2 \sum_{r_\omega} T(r_\omega) \log \det K_{\alpha\beta}(\phi, \bar{\phi}) \right) + \text{h.c.}, \quad (3.135)$$

with $K_{\alpha\beta}(A, \bar{A})$ the Kähler metric of the matter fields, the chiral superfield $W^\alpha \overset{\text{def}}{=} -(1/4) \mathcal{D}\mathcal{D} e^{-V} D_\alpha e^V$ and $V$ is the vector superfield. In this form the general field theoretical contribution to the threshold corrections appears to be

$$\frac{1}{g_a^2(p^2)} = \frac{1}{g_a^2(M_X^2)} + \frac{b_a}{16\pi^2} \log \frac{M_X^2}{p^2} + \frac{\Delta_a}{16\pi^2} \quad (3.136)$$

$$\Delta_a = 16\pi^2 \text{Re} f^{1\text{-loop}} = [-2 \sum_r T_a(r) \log \det g_r + c_a K] \quad (3.137)$$

and $\text{Re} f^{1\text{-loop}}$ is induced from the integration of massive modes that decouple at the scale $M_X^2$.

Notice that the general form of the gauge coupling dependence in a $N = 1$ supersymmetric gauge theory appears in the form

$$\frac{1}{4} \sum_a \int d^2\theta f_a(\phi)(W^\alpha W_\alpha)_a + \text{h.c.} = -\frac{1}{4} \left\{ \sum_a (\text{Re} f)_a(F_{\mu\nu} F^{\mu\nu})_a - \text{Im} f_a(F F)_a \right\}, \quad (3.138)$$

where the index $a$ labels the different group factors of the high energy gauge group $G = \otimes G_a$. Obviously

$$f_a^{\text{tree}} = \left\{ \frac{1}{g_a^2} - \frac{i\theta_a}{8\pi^2} \right\}^{\text{tree}} = k_a S, \quad (3.139)$$

with $k_a$ the Kac-Moody level.

By looking at eqn.(3.135), we can see that the σ-model lagrangian is not invariant under the duality Kähler transformations

$$K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + g(\phi) + g(\bar{\phi}), \quad (3.140)$$

87The Kähler metric of the matter fields, appears when we expand the Kähler potential for the matter fields in lowest order in the matter fields, as $K^\text{matter} = K^\text{matter}_{\alpha\beta} A_\alpha \bar{A}_\beta$.

88See for example [134, 135].
and reparametrizations which act on the matter metric as

\[ K_{\alpha\beta} \rightarrow h_{\alpha\gamma}(\phi)^{-1}h_{\beta\delta}(\bar{\phi})K_{\alpha\beta} \]  

(3.141)

The non-invariance of eqn. (3.135) is reflected in the presence of the additional term

\[ \delta \mathcal{L} = \frac{-1}{16\pi^2} \sum_\alpha \int d^2\theta \frac{W^\alpha W_\alpha}{4} \left( [c(G_\alpha) - \sum_{r_\omega} T(r_\omega) g(\phi) + 2 \sum_{r_\omega} T(r_\omega) \log \det h_{\alpha\beta}(\phi, \bar{\phi})] + h.c \right) \]  

(3.142)

Take for example (0, 2) abelian orbifolds. The Kähler potential for the matter fields, when it is expanded around the \(< A_\alpha = 0 >\) classical vaccum becomes

\[ K_{\text{matter}} = \delta_{\alpha\beta} \prod_{i=1}^{h_{(1,1)}} (T + T)_i^{n_i} \prod_{m=1}^{h_{(2,1)}} (U + U)_m^{l_m} A_\alpha \bar{A}_\beta \]  

(3.143)

which means that every matter field is characterized by \((h_{(1,1)} + h_{(2,1)})\) rational numbers, the modular weights, which are represented in vector form as \(\vec{n}_i^\alpha = (n_{i,1}^\alpha, n_{i,2}^\alpha, \ldots, n_{i,h_{(1,1)}}^\alpha)\) and \(\vec{l}_m^\alpha = (l_{m,1}^\alpha, l_{m,2}^\alpha, \ldots, l_{m,h_{(2,1)}}^\alpha)\). In addition, invariance of the kinetic energy for the matter fields under e.g. \(SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U\) target space duality transformations, produces the requirement

\[ A_\alpha \rightarrow A_\alpha \prod_{i=1}^{h_{(1,1)}} (icT + d)_i^{n_i^\alpha} \prod_{m=1}^{h_{(2,1)}} (icT + d)_m^{l_m^\alpha} \]. \]  

(3.144)

For abelian orbifolds and untwisted matter fields associated with the \(j - \text{th}\) complex plane the modular weights are given\(^{[22]}\) by \(n_j^i = -\delta_{ij}\) and \(l_j^i = -\delta_{ij}\), while for twisted states associated with the order N twist vector \(\tilde{\theta} = (\theta^1, \theta^2, \theta^3)\) \(0 \leq \theta^i < 1, \sum_{i=1}^{3} \theta^i = 1\) and having a complex plane not being fixed in two or all three complex planes, the modular weights are \(n_j^i = -(1 - \theta^i), l_j^i = -(1 - \theta^i), \theta^i \neq 0\) and \(n_j^i = l_j^i = 0, \text{ for } \theta^i = 0\).

Substituting explicitly in eqn. (3.135) the values of Kähler potential and the matter metric we get

\[ \mathcal{L} = \sum_\alpha \int d^2\theta \frac{W^\alpha W_\alpha}{4} \{ S - \frac{1}{16\pi^2} \frac{1}{16} \Box [ \sum_{i=1}^{h_{(1,1)}} b_i^\alpha \log (T + \bar{T})_i + \sum_{m=1}^{h_{(2,1)}} b_m^\alpha \log (U + \bar{U})_m] \} + h.c, \]  

(3.145)

with \(b_i^\alpha = -c(G_\alpha) + \sum_{r_\omega} T(r_\omega)(1 + 2n_{r_\omega}^i)\) and \(b_m^\alpha = -c(G_\alpha) + \sum_{r_\omega} T(r_\omega)(1 + 2l_{r_\omega}^i)\).
The final contribution to the gauge kinetic terms including one-loop corrections coming from the heavy modes that decouple at the string unification scale is found to be

\[ \mathcal{L} = \sum_\alpha \int d^2 \theta \frac{W^\alpha W_\alpha}{4} \left( S - \frac{1}{16\pi^2} \frac{1}{16} \frac{\partial^2}{\partial \vec{T}^2} \sum_{i=1}^{h_{(1,1)}} b^i_\alpha \log(T + \bar{T})_i \eta^4(T) + \sum_{m=1}^{h_{(2,1)}} b^m_\alpha \log(U + \bar{U})_m \eta^4(U) \right) + \text{h.c.} \quad (3.146) \]

It appears finally that the non-invariance of the lagrangian (3.135, 3.146) under \( SL(2, \mathbb{Z})_T \) modular transformations

\[ T \rightarrow \frac{aT - ib}{icT + d}, \quad ad - bc = 1; a, b, c, d \in \mathbb{Z}, \quad (3.147) \]

can only be compensated by the use of the Green-Schwarz(GS) mechanism.

In the presence of the Green-Schwarz mechanism in the four dimensional one-loop effective string action, the previous considerations have to be modified. In that case we will have to subtract the contribution of the Green-Schwarz term from the total anomaly coefficient. The Green-Schwarz mechanism in four-dimensions induces the following modification to the lagrangian (3.146)

\[ \mathcal{L} = \sum_\alpha \int d^2 \theta \frac{W^\alpha W_\alpha}{4} \left( \frac{1}{16\pi^2} \frac{1}{16} \frac{\partial^2}{\partial \vec{T}^2} \sum_{i=1}^{h_{(1,1)}} \delta^i_{GS} \log(T + T) \right) \]

\[ + \frac{1}{4\pi^2} \sum_{m=1}^{h_{(2,1)}} \delta^m_{GS} \log(U + \bar{U})_m \right] + \frac{1}{4\pi^2} \sum_{i=1}^{h_{(1,1)}} \left( b^i_a - \delta^i_{GS} \right) \log(T_i + \bar{T}_i) \eta(T_i)^4 + \]

\[ \sum_{m=1}^{h_{(2,1)}} \frac{1}{4\pi^2} \left( b^m_a - \delta^m_{GS} \right) \log(U_m + \bar{U}_m) \eta(U_m)^4 \right) \quad (3.148) \]

Notice that the presence of duality anomalies under the transformation of eqn. (3.147) can be cancelled by a transformation of the dilaton field as

\[ S \rightarrow S - \frac{1}{8\pi^2} \sum_{i=1}^{h_{(1,1)}} \delta^i_{GS} \log(icT + d)_i \quad (3.149) \]

\(^{89}\), used for the cancellation of anomalies of the ten dimensional theory or better to provide for a Fayet-Iliopoulos D-term in order to break supersymmetry.
In equation (3.148) the term in brackets involving the dilaton induces a mixing in the one-loop Kähler potential between dilaton and the moduli fields. This mixing corresponds to the following Kähler potential

\[ K = -\log[S + \bar{S} - \frac{1}{4\pi^2} \sum_{i=1}^{h_{(1,1)}} \delta_{GS} \log(T + \bar{T})]. \] (3.150)

The induced one-loop correction to the tree level Kähler potential corresponds to a supergravity lagrangian formulated in the linear representation\(^{90}\) of the dilaton.

We may now complete our discussion of the contributions to the gauge couplings by considering contributions from a \((2,2)\) symmetric non-factorizable orbifold theory. We will apply our discussion to the \(Z_6 - IIb\) orbifold. We examine the contributions from the subsector \((1,\Theta^3)\) only. Contributions coming from the \((1,\Theta^3)\) sector are invariant under \(SL(2,Z)_T \times SL(2,Z)_U\), they are identical to those listed in \([120]\), and we will not include them here as their result is additive to the gauge coupling constants.

We consider the contribution to the effective gauge coupling constants\(^{91}\).

The moduli will play the role of Higgs fields, breaking the original gauge group. The original gauge group of our theory is supposed to be sited at the string unification scale \(M_s\) and it is composed from gauge group factors \(G_a\), as \(G = \oplus G_a\). We assume that space-time supersymmetry remains unbroken while the moduli fields spontaneously break the original gauge group in the form \(\tilde{G} = \oplus \tilde{G}_a\). The running of the subgroups \(\tilde{G}_a\) includes the presence of additional threshold scales coming from gauge group enhancement at special points and as a reverse consequence previously massless particles become massive and decouple. In describing the running of effective gauge couplings of those subgroups, we substitute explicitly the tree level values for the gauge coupling constants at the string unification scale \((S + \bar{S})/2\), the value of the \(\sigma\)-model matter metric in the case of vanishing Wilson lines \(\left((T + \bar{T})(U + \bar{U})\right)^{-1}\). We find that the full contribution to the gauge couplings\(^{92}\) from the modular orbit \(2n^T m + q^T C q = 0\) is

\[
\frac{1}{g_{E_8}^2(p^2)} = \frac{S + \bar{S}}{2} + \frac{b_{E_8}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{c_{E_8}}{16\pi^2} \log \left((T + \bar{T})(U' + \bar{U}') - (B + \bar{C})(C + \bar{B})\right)
\]

\(^{90}\)See chapter five.

\(^{91}\)A similar methodology was followed in \([120]\) by turning on moduli fields \(B,C\) in the invariant subsector.

\(^{92}\)Remember that we consider a \((2,2)\) theory for which the gauge group is always \(E_8 \times E_6\) and the matter fields are consisting of \(h_{(1,1)}\) 27 multiplets and \(h_{(2,1)}\) \(\bar{27}\) multiplets in the \(E_6\) part of the gauge group.
\[
\begin{align*}
&+ \frac{c_{E_8}}{16\pi^2} \log \left( 9\eta^4(T)\eta^4(U') \right) \left( 1 - BC(\partial T \log \eta^2(T))(\partial U' \log \eta^2(U')) \right)^{-2} \\
&+ \frac{c_{E_8}}{16\pi^2} \log \left( 9\eta^4(U')\eta^4(T) \right) \left( 1 - BC(\partial T \log \eta^2(T))(\partial U' \log \eta^2(U')) \right)^{-2},
\end{align*}
\]

(3.151)

with \( b_{E_8} = -3c(E_8) \) and we have assumed for simplicity that \( \delta_{GS} = 0 \).

The expression (3.151) represents the gauge couplings which are not affected from the presence of the Wilson line moduli. These can be for example the gauge couplings which belong to the \( E_8 \) hidden sector. By making the correspondence with the general superfield behaviour of the charged matter in \( (2,2) \) theories, we conclude that a priori this has to be the case since the charged matter in \( (2,2) \) theories are always \( E_8 \) neutral. This built in property of the theory, can be used in the gaugino condensation approach to supersymmetry breaking by taking the hidden \( E_8 \) sector to be associated with the pure Yang-Mills gauge sector. In the previous description of the gauge couplings we have tacitly assumed that the original gauge group of the theory, spontaneously breaks at a product of subgroups and as a consequence previously massless particles decouple.

In the presence of the Green-Schwarz mechanism, (3.151) becomes

\[
\frac{1}{g_{E_8}^2(p^2)} = \Pi + \frac{b_{E_8} - \delta_{GS}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{c_{E_8} - \delta_{GS}}{16\pi^2} \log((T + \bar{T})(U' + \bar{U'}) - (B + \bar{C})
\times (C + \bar{B})) + \frac{c_{E_8} - \delta_{GS}}{16\pi^2} \log \left( 9\eta^4(T)\eta^4(U') \right) \left( 1 - BC(\partial T \log \eta^2(T))(\partial U' \log \eta^2(U')) \right)^{-2}
\]
\[
+ \frac{c_{E_8} - \delta_{GS}}{16\pi^2} \log \left( 9\eta^4(U')\eta^4(T) \right) \left( 1 - BC(\partial T \log \eta^2(T))(\partial U' \log \eta^2(U')) \right)^{-2},
\]

(3.152)

with

\[
\Pi = S + \bar{S} - \frac{\delta_{GS}}{8\pi^2} \log \left( (T + \bar{T})(U' + \bar{U'}) - (B + \bar{C})(C + \bar{B}) \right) + (\text{modular function}).
\]

(3.153)

We will give the unspecified modular function later, in chapter 4. We will only say at this stage that its value is universal, and gauge group independent. In the \( N = 2 \) part of the internal superconformal field theory, the gauge couplings depend on scalars belonging to vector multiplets and not on the hypermultiplet moduli.

We will now discuss the effect on the gauge couplings on the vector multiplets of the invariant subspace. In the case of six dimensional compactifications of heterotic string vacua, the moduli
of the invariant subspace belong to vector multiplets. In such a case, the gauge couplings of the
vector multiplets result in

$$\frac{1}{g_{U(1)}^2(p^2)} = \frac{1}{g_{ee}^2} + \frac{\hat{b}_{U(1)}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{(\hat{b}_{U(1)}-\hat{b}_{U(1)})}{16\pi^2} \log \left(\omega(T) - \omega(U')\right)^2 - \frac{a_{U(1)}}{16\pi^2}$$

$$\times \left\{ \log \left( (T + \bar{T})(U' + \bar{U}')|9\eta(T)|^4 \right) + \log \left( (T + \bar{T})(U' + \bar{U}')|9\eta(U')\eta(\frac{T}{3})|^4 \right) \right\} \quad (3.154)$$

with \( \hat{b}_{U(1)} = 0 \), since \( c_{U(1)} = 0 \) and there are no hypermultiplets charged under the \( U(1) \). Here, the without hat quantities correspond to the running of the gauge couplings between the string unification scale and the intermediate threshold \( M_I \), while the hat quantities correspond to the running of the gauge couplings between the threshold \( M_I \) and the momentum scale \( p^2 \), with \( p^2 < M_I^2 \). Explicitly,

$$\frac{1}{g_{U(1)}^2(p^2)} = \frac{1}{g_{ee}^2(M_I^2)} + \frac{\hat{b}_{U(1)}}{16\pi^2} \log |\omega(T) - \omega(U')|^2 + \frac{\hat{b}_{U(1)}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} +$$

$$+ \frac{\hat{\Delta}}{16\pi^2} \quad (3.155)$$

where \( \hat{\Delta} \) the contribution from Kähler and \( \sigma \)-model anomalies. The sum is over the chiral multiplets of massless particles between the thresholds. Furthermore, for the running of the couplings between the threshold \( M_I^2 \) and the string unification scale

$$\frac{1}{g_{U(1)}^2(M_I^2)} = \frac{1}{g_{U(1)}^2(M_{\text{string}}^2)} + \frac{b_{U(1)}}{16\pi^2} \log \frac{1}{|\omega(T) - \omega(U')|^2} + \frac{\Delta}{16\pi^2} \quad (3.156)$$

where \( \Delta \) are the contribution coming from integration of the massive modes. Combining eqn.(3.155) and eqn.(3.154), we obtain eqn.(3.154).

In the same way as in \[120\], \( a_{U(1)} = 0 = -c_{U(1)} + \sum_{rC} T_{rC} \), since the gauge group under the additional threshold scale \( M_I \) is abelian. The coefficient \( b_{U(1)} \) equals \( \hat{b}_{U(1)} + 2b_{vec} \), where \( 2b_{vec} \) the contribution from the two \( \beta \)-function coefficients of the two vector multiplets which are massless above the threshold scale. The additional threshold scale beyond the traditional string tree level unification scale is the one associated with the term \( \omega(T) - \omega(U) \). The threshold scale now becomes \( M_I = |\omega(T) - \omega(U')| M_{\text{string}} \) and is associated with the enhancement of the abelian part of the gauge group to \( SU(2) \). The appearance of the threshold scale is specific at the point where the non-abelian \( SU(2) \) factor is being broken to \( U(1) \).
3.5.2 * Threshold corrections to gravitational couplings

In this part of the thesis we will discuss briefly contributions to the running gravitational couplings in $(2,2)$ symmetric $Z_N$ orbifold constructions of the heterotic string.

For $(0,2)$ $Z_N$ orbifolds the effective low energy action of the heterotic string is

$$L = -\frac{1}{2} R + \frac{1}{4} g_{grav} C + \frac{1}{4} (RS) GB + \frac{1}{4} (3S) R_{abcd} R^{abcd}$$

(3.157)

and $RS \equiv (S + \bar{S})/2$, $3S \equiv (S - \bar{S})/2$. We have used as the gravitational couplings $1/g_{grav} \equiv RS$, while $GB$ is the Gauss-Bonnet combination

$$4GB = C^2 - 2R_{ab}^2 + \frac{2}{3} R^2, \quad C = R_{abcd} R^{abcd} - 2R_{ab} R^{ab} + \frac{1}{3} R^2,$$

(3.158)

and $C$ the conformal Weyl tensor $C_{abcd}$. The Weyl tensor $C$ can occur only through the Gauss-Bonnet (GB) combination in eqn.(3.158), since single $C^2$ terms coupled to Einstein gravity can violate unitarity. In other words, presence of powers of GB terms may guarantee the absence of ghost particles in the effective low energy limit of string theories. When the above relation is written in the form

$$L_{grav} = \Delta^{grav}(T, \bar{T})(R_{abcd} - 4R_{ab} + R^2) + \Theta^{grav}(T, \bar{T}) \epsilon^{abcd} R_{abcd} R_{ef}$$

(3.159)

then the one-loop corrections to the gravitational action, in the absence of Green-Schwarz mechanism, give $\Delta^{grav} = \frac{\tilde{\beta}^{grav}}{4\pi^2} \log(T + \bar{T})|\eta(iT)|^4$, e.g for a $Z_4$ orbifold. The gravitational $\beta$-function coefficient $\tilde{\beta}^{grav}$ equals

$$\tilde{\beta}^{grav} = \frac{1}{45}(N_S + \frac{7}{4} - 13N_V - \frac{113}{2} N_F^L N_F^R + 304 N_V^L N_V^R).$$

(3.160)

Here, $N_S, N_V, N_F$ is the number of scalars, vectors and Majorana fermions contributing to the gravitational $\beta$-function. The coefficients in front of $N_S, N_V, N_F$ represent the contributions of the various fields to the integrated conformal anomaly. This is 1 for scalars, $7/4$ for spin 1/2 fermions, 33 for vector bosons and $-233/4$ for gravitinos and 212 for the graviton. Finally, 304 is the contribution in the trace anomaly for the graviton, dilaton and antisymmetric tensor $B_{\mu\nu}$, while $N_F^L N_F^R$ is associated with the contribution of the gravitino together with a Majorana fermion.

The corrections to the gravitational couplings considered up to now in the literature, are concerned with the decomposable orbifolds. We will complete the discussion of corrections to
the gravitational couplings by examining non-decomposable orbifolds. We focus our attention to the case of $Z_6 - IIb$ orbifold. We consider the case of vanishing Wilson lines. Working in the field theoretical approach \[220\], in the presence \[120\] of the threshold $p^2 \ll M_I^2 \ll M_{\text{string}}^2$, we get

$$\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{S + \bar{S}}{2} + \frac{\hat{b}_{\text{grav}}}{16\pi^2} \log \frac{M_I^2}{p^2} - \frac{\hat{a}_{\text{grav}}}{16\pi^2} \log \frac{M_I^2}{M_{\text{string}}} - \frac{\hat{\Delta}_{\text{grav}}}{16\pi^2} \log[(T + \bar{T})(U' + \bar{U}')]$$

$$- \frac{a_{\text{grav}}}{16\pi^2} \log[\eta^2(T)\eta^2(U')/3] - \frac{a_{\text{grav}}}{16\pi^2} \log[\eta^2(T/3)\eta^2(U')/9].$$

(3.161)

Let me explain more on eqn.(3.161). Here, $\hat{b}_{\text{grav}}, \hat{a}_{\text{grav}}$ are associated with the running of the gravitational couplings below the additional threshold scale $M_I^2$, while the bare quantities, e.g $b_{\text{grav}}, a_{\text{grav}}$ are associated with the running in the area $M_I^2 \ll p^2 \ll M_{\text{string}}$. The following equation is valid for energies $p^2 \ll M_I^2$:

$$\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{1}{g_{\text{grav}}^2(M_I^2)} + \frac{\hat{b}_{\text{grav}}}{16\pi^2} \log \frac{M_I^2}{p^2} - \frac{\hat{a}_{\text{grav}}}{16\pi^2} \frac{\hat{\Delta}_{\text{grav}}}{16\pi^2},$$

(3.162)

where $\hat{a}_{\text{grav}}$ is given below and $\hat{\Delta}_{\text{grav}}$ is the moduli dependent contribution from the Kähler and $\sigma$-model anomalies. For decomposable orbifolds, this contribution \[55\] is the usual one as for the gauge coupling \[71\]. For energies $M_I^2 \ll p^2 \ll M_{\text{string}}^2$ the following equation is valid \[120, 220\]

$$\frac{1}{g_{\text{grav}}^2(M_I^2)} = \frac{S + \bar{S}}{2} + \frac{b_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{string}}}{M_I^2} - \frac{a_{\text{grav}}}{16\pi^2} \frac{\tilde{\Delta}}{16\pi^2},$$

(3.163)

where $\Delta$ is the moduli dependent contribution coming from the integration of massive modes. Substituting eqn.(3.163) in eqn.(3.162) we get

$$\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{S + \bar{S}}{2} + \frac{b_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{string}}}{p^2} + \frac{[-b_{\text{grav}} + b_{\text{grav}}]}{16\pi^2} \log |\omega(T) - \omega(U')|^2$$

$$- K_1 \frac{a_{\text{grav}}}{16\pi^2} \log[9\eta(T)\eta(U')/3] + K_2 \frac{a_{\text{grav}}}{16\pi^2} \log[9\eta^2(U')\eta^2(T/3)]^4$$

$$- K_1 \frac{\tilde{a}_{\text{grav}}}{16\pi^2} \log[(T + \bar{T})(U' + \bar{U}')] - K_2 \frac{\tilde{a}_{\text{grav}}}{16\pi^2} \log[(T + \bar{T})(U' + \bar{U}')].$$

(3.164)

The coefficients $K_1, K_2$ are numerical coefficients that may appear in front of the moduli dependent threshold corrections after performing the string calculation \[53\] of moduli dependence of threshold corrections in analogy to the calculation of the threshold corrections to gauge couplings in \[59\].
Here $\hat{a}_{\text{grav}}$ comes from non-holomorphic contributions from Kähler and $\sigma$-model anomalies and is given by $\hat{a}_{\text{grav}} = \frac{1}{24} (21 + 1 - \dim G + \gamma_M + \sum \hat{C}(1 + 2 n_{\hat{C}}))$. The $\hat{a}_{\text{grav}}, g_{\text{grav}}$ coefficients have been calculated in the absence of Green-Schwarz mechanism, as coefficients of the Gauss-Bonnet term in the gravitational action and represent the contribution of the completely rotated plane. In that case, $\hat{a}_{\text{grav}} \equiv a_{\text{grav}} = \tilde{\beta}_{\text{grav}}$.

The gauge couplings receive contributions from states appearing at points in the moduli space where $T = U'$, namely, one additional $N = 2$ vector multiplet, namely at the additional threshold scale $M_I = (\omega(T) - \omega(U')) M_{\text{string}}$. Furthermore, $b_{\text{grav}} - \hat{b}_{\text{grav}} = \delta^V_{\text{grav}} + \delta^C_{\text{grav}}$, where $\delta^V$ and $\delta^C$ are the contributions to the gravitational $\beta$-function coming from the N=1 vector and chiral decompositions of the $N = 2$ vector multiplet. A $N = 2$ vector multiplet of a supersymmetric gauge theory has in total $2^{2+1} = 8$ states and consists two vectors, two complex Majorana fermions and a complex scalar. It can decomposed into a $N = 1$ vector multiplet which has two vectors plus their fermionic superpartners and a $N = 1$ hypermultiplet with two Majorana fermions plus their superpartner, a complex scalar.

Rewriting eqn. (3.164), we obtain

$$
\frac{1}{g^2_{\text{grav}}(p^2)} = \frac{S + \bar{S}}{2} + \hat{b}_{\text{grav}} \frac{M^2_{\text{string}}}{p^2} - \frac{\delta^V_{\text{grav}} + \delta^C_{\text{grav}}}{16\pi^2} \log |\omega(T) - \omega(U')|^2 \\
- K_1 \frac{\hat{a}_{\text{grav}}}{16\pi^2} \log \{|9\eta(T)\eta(U')^3|/3\}^4 - K_2 \frac{\hat{a}_{\text{grav}}}{16\pi^2} \log \{|9\eta^2(U')\eta^2(T)/3\}^4 \\
- K_1 \frac{a_{\text{grav}}}{16\pi^2} \log \left((T + \bar{T})(U' + \bar{U}')\right) - K_2 \frac{a_{\text{grav}}}{16\pi^2} \log \left((T + \bar{T})(U' + \bar{U}')\right). (3.165)
$$

Finally,

$$
\frac{1}{g^2_{\text{grav}}(p^2)} = \frac{S + \bar{S}}{2} + \hat{b}_{\text{grav}} \frac{M^2_{\text{string}}}{p^2} - \frac{\delta^V_{\text{grav}} + \delta^C_{\text{grav}}}{16\pi^2} \log |\omega(T) - \omega(U')|^2 \\
- K_1 \frac{a_{\text{grav}}}{16\pi^2} \log \{(T + \bar{T})(U' + \bar{U}')|\eta(T)\eta(U')/3\}^4 \\
- K_2 \frac{a_{\text{grav}}}{16\pi^2} \log \{(T + \bar{T})(U' + \bar{U}')\eta^2(U')\eta^2(T)/3\}. (3.166)
$$

### 3.6 Threshold corrections for the $Z_8$ orbifold

We will present now the calculation of the threshold corrections for the class of orbifolds defined by the Coxeter twist $(e^{i\pi/4}, e^{3i\pi/4}, -1)$ on the root lattice of $A_3 \times A_3$. This orbifold is non-decomposable,
in the sense that the action of the lattice twist on the $T_6$ torus does not decompose into the orthogonal sum $T_6 = T_4 \oplus T_4$ with the fixed plane lying on the $T_2$ torus. Similar calculations\footnote{of threshold corrections for non-decomposable orbifolds} have been performed in \cite{59}. Our calculation completes the calculation of the threshold corrections for the list of orbifolds defined\footnote{In \cite{59}, a classification of six orbifold compactifications with $N = 1$ supersymmetry was performed. Similar calculations for non-decomposable $Z_N \times Z_M$ orbifolds were examined in \cite{221}.} in \cite{66}.

The twist can equivalently be realized through the generalised Coxeter automorphism $S_1 S_2 S_3 P_{35} P_{36} P_{45}$. The generalised Coxeter automorphism is defined as a product of the Weyl reflections\footnote{The Weyl reflection $S_i$ is defined as a reflection $S_i(x) = x - 2\frac{<x, e_i>}{<e_i, e_i>} e_i$ \eqref{3.167}} $S_i$ of the simple roots and the outer automorphisms represented by the transposition of the roots. A outer automorphism represented by a transposition which exchange the roots $i \leftrightarrow j$, is denoted by $P_{ij}$ and is a symmetry of the Dynkin diagram.

For the orbifold $Z_8$ there are four complex moduli fields. There are three $(1, 1)$ moduli due to the three untwisted generations $27$ and one $(2, 1)$-modulus\footnote{In Table five of \cite{66}, the number of the $h^{(2,1)}$ moduli was reported for the $Z_8$ and $Z_8'$ orbifolds to be zero and one respectively. They were misquoted. Clearly, these values may be interchanged. In our calculation for the $Z_8$ orbifold with the twist defined via the generalized Coxeter twist $S_1 S_2 S_3 P_{35} P_{36} P_{45}$ the value of the $h^{(2,1)}$ moduli is one, confirming the results of \cite{52}.} due to the one untwisted generation $\bar{27}$.

The realization of the point group is generated by

$$Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}. \quad \text{(3.168)}$$

Therefore the metric $g$ (defined by $g_{ij} = <e_i|e_j>$) has three and the antisymmetric tensor field $b$ an other three real deformations. The equations $gQ = Qg$ and $bQ = Qb$ determine
the background fields in terms of the independent deformation parameters. If the action of
the generator of the point group leaves some complex plane invariant then the corresponding
threshold corrections have to depend on the associated moduli of the unrotated complex plane.

Solving these equations one obtains for the metric

\[
G = \begin{pmatrix}
R^2 & u & v & -u & -2v - R^2 & -u \\
u & R^2 & u & v & -u & -2v - R^2 \\
v & u & R^2 & u & v & -u \\
-u & v & u & R^2 & u & v \\
-2v - R^2 & -u & v & u & R^2 & u \\
-u & -2v - R^2 & -u & v & u & R^2 \\
\end{pmatrix},
\]

with \( R, u, v \in \mathbb{R} \) and the antisymmetric tensor field:

\[
B = \begin{pmatrix}
0 & x & z & y & 0 & -y \\
-x & 0 & x & z & y & 0 \\
-z & -x & 0 & x & z & y \\
-y & z & -x & 0 & x & z \\
0 & -y & -z & -x & 0 & x \\
y & 0 & -y & -z & -x & 0 \\
\end{pmatrix},
\]

with \( x, y, z \in \mathbb{R} \). The N=2 orbit is given by these sectors which contain completely unrotated
planes, \( \mathcal{O} = (1, \Theta^4), (\Theta^4, 1), (\Theta^4, \Theta^4) \).

The element \((\Theta^4, 1)\) can be obtained from the fundamental element \((1, \Theta^4)\) by an \(S\)-transformation
on \( \tau \) and similarly \((\Theta^4, \Theta^4)\) by an \(ST\)-transformation. The partition functions for the zero mode
parts \( Z_{torus}^{(g,h)} \) of the fixed plane have the following form\[^{[56]}\]

\[
Z_{(1,\Theta^4)}^{torus}(\tau, \bar{\tau}, G, B) = \sum_{P \in (\Lambda_N^\bot)} q^{\frac{1}{2} P_L^2} q^{-P_R^2},
\]

\[
Z_{(\Theta^4,1)}^{torus}(\tau, \bar{\tau}, G, B) = \frac{1}{V_{\Lambda_N^\bot}} \sum_{P \in (\Lambda_N^\bot)^*} q^{\frac{1}{2} P_L^2} q^{\frac{1}{2} P_R^2},
\]

\[
Z_{(\Theta^4,\Theta^4)}^{torus}(\tau, \bar{\tau}, G, B) = \frac{1}{V_{\Lambda_N^\bot}} \sum_{P \in (\Lambda_N^\bot)^*} q^{\frac{1}{2} P_L^2} q^{\frac{1}{2} P_R^2} q^{i\pi(P_L^2 - P_R^2)},
\]

with \( \Lambda_N^\bot \) we denote the Narain lattice of \( A_3 \times A_3 \) which has momentum vectors
\[
P_L = \frac{p}{2} + (G - B)w, \quad P_R = \frac{p}{2} - (G + B)w.
\]

\(\Lambda_{N^\perp}\) is that part of the lattice which remains fixed under \(Q^4\) and \(V_{\Lambda_{N^\perp}}\) is the volume of this sublattice. The lattice in our case is not self dual in contrast with the case of partition functions \(Z_{(g,h)}^{torus}(\tau, \bar{\tau}, g, b)\) of [71]. Stated differently the general result is - for the case of orbifolds similar to our’s - exactly, that the modular symmetry group is some subgroup of \(\Gamma\) and as a consequence the partition functions \(\tau_2 Z_{(g,h)}^{torus}(\tau, \bar{\tau}, g, b)\) are invariant under the same subgroup of \(\Gamma\).

The subspace corresponding to the lattice \(\Lambda_{N^\perp}\) can be described by the following winding and momentum vectors, respectively:

\[
w = \begin{pmatrix}
    n^1 \\
n^2 \\
    0 \\
    0 \\
n^1 \\
n^2
\end{pmatrix}, \quad n^1, n^2 \in \mathbb{Z}
\quad \text{and} \quad p = \begin{pmatrix}
    m_1 \\
m_2 \\
-m_1 \\
-m_2 \\
m_1 \\
m_2
\end{pmatrix}, \quad m_1, m_2 \in \mathbb{Z}.
\]

They are determined by the equations \(Q^4w = w\) and \(Q^4p = p\). The partition function \(\tau_2 Z_{(1, \Theta^4)}^{torus}(\tau, \bar{\tau}, g, b)\) is invariant under the group \(\Gamma_0(2)\) which belongs to the congruence subgroups of \(\Gamma\). The integration of the contribution of the various sectors \((g, h)\) is over the fundamental domain for the group \(\Gamma_0(2)\) which is a three fold covering of the upper complex plane. By taking into account the values of the momentum and winding vectors in the fixed complex directions we get for \(Z_{(1, \Theta^4)}^{torus}\)

\[
Z_{(1, \Theta^4)}^{torus}(\tau, \bar{\tau}, g, b) = \sum_{(P_L, P_R) \in \Lambda_{\hat{N}}} q^{\frac{1}{2} P_L G^{-1} P_L} q^{\frac{1}{2} P_R G^{-1} P_R}
\]

\[
= \sum_{p, w} e^{2\pi i \tau p^t w} e^{-\pi \tau_2 (\frac{1}{2} p^t G^{-1} P_L G^{-1} Bw + 2w^t Bw) + 2w^t Gw - 2w^t B G^{-1} Bw - 2p^t w) }.
\]

Consider now the the following parametrization of the torus \(T^2\) [30], namely define the the

---

97By * we mean inverse transpose.

98see appendix A
(1, 1) $T$ modulus and the (2, 1) $U$ modulus as:

\begin{align*}
T &= T_1 + iT_2 = 2(b + i\sqrt{\det G_\perp}) \\
U &= U_1 + iU_2 = \frac{1}{G_{12}}(G_{112} + i\sqrt{\det G_\perp}).
\end{align*}

(3.175)

Here $g_\perp$ is uniquely determined by $w^tGw = (n^1n^2)G_\perp \binom{n^1}{n^2}$.

Here $b$ is the value of the $B_{12}$ element of the two-dimensional matrix $B$ of the antisymmetric field. This way one gets

\begin{align*}
T &= 4(x - y) + i8v \\
U &= i.
\end{align*}

(3.176)

(3.177)

The partition function $Z_{(1,\Theta^4)}^{torus}(\tau, \bar{\tau}, g, b)$ takes now the form

\begin{align*}
Z_{(1,\Theta^4)}^{torus}(\tau, \bar{\tau}, T, U) &= \sum_{m_1, m_2 \in \mathbb{Z}} e^{2\pi i\tau(m_1n^1 + m_2n^2)} e^{-\pi T_2^2 |TU n^2 + Tn^1 - Um_1 + m_2|^2}.
\end{align*}

(3.178)

By Poisson resummation on $m_1$ and $m_2$, using the identity:

\begin{align*}
\sum_{p \in \Lambda^*} e^{-\pi(p+\delta)^tC(p+\delta) + 2\pi ip^t\phi} = V_{\Lambda}^{-1} \frac{1}{\sqrt{\det C}} \sum_{l \in \Lambda} e^{[-\pi(l+\phi)^tC^{-1}(l+\phi) - 2\pi i\delta^t(l+\phi)]}
\end{align*}

(3.179)

we conclude

\begin{align*}
\tau_2 Z_{(1,\Theta^4)}^{torus}(\tau, \bar{\tau}, T, U) &= \frac{1}{4} \sum_{A \in \mathcal{M}} e^{-2\pi iT_2 \det A} T_2 e^{-\pi T_2^2 |(1, U)A(\cdot)|^2},
\end{align*}

(3.180)

where

\begin{align*}
\mathcal{M} &= \left( \begin{array}{c}
 n_1 \\
 \frac{1}{2} l_1 \\
 n_2 \\
 \frac{1}{2} l_2
\end{array} \right)
\end{align*}

(3.181)

and $n_1, n_2, l_1, l_2 \in \mathbb{Z}$.

From (3.57) one can obtain $\tau_2 Z_{(1,\Theta^4)}^{torus}(\tau, \bar{\tau})$ by an $S$–transformation on $\tau$. After exchanging $n_i$ and $l_i$ and performing again a Poisson resummation on $l_i$ one obtains

\begin{align*}
Z_{(1,\Theta^4)}^{torus}(\tau, \bar{\tau}, T, U) &= \frac{1}{4} \sum_{m_1, m_2 \in \mathbb{Z}} e^{2\pi i\tau(m_1n^1 + m_2n^2)} e^{-\pi T_2^2 |TU n^2 + Tn^1 - Um_1 + m_2|^2}.
\end{align*}

(3.182)
The factor 4 is identified with the volume of the invariant sublattice in (3.58). The expression
\( \tau_2 Z_{\text{torus}}(\Theta, T, U) \) is invariant under \( \Gamma^0(2) \) acting on \( \tau \) and is identical to that for the \( (\Theta^4, \Theta^4) \) sector. The subgroup \( \Gamma^0(2) \) of \( SL(2, \mathbb{Z}) \) is defined as with \( b = 0 \mod 2 \) instead of \( c = 0 \mod 2 \). Thus the contribution of the two sectors \( (\Theta^4, 1) \) and \( (\Theta^4, \Theta^4) \) to the coefficient \( b_a^{N=2} \) of the \( \beta \)-function is one fourth of that of the sector \( (1, \Theta^4) \).

The coefficient \( b_a^{N=2} \) is the contribution to the \( \beta \) functions of the \( N = 2 \) sectors of the \( N = 1 \) orbifold as we already described in the introduction of this chapter.

Including the moduli dependence of the different sectors, we conclude that the final result for the threshold correction to the inverse gauge coupling (3.3) reads

\[
\Delta_a(T, \bar{T}, U, \bar{U}) = -b_a^{(1, \Theta^4)} \ln \left| \frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}} T \right| \left| \eta \left( \frac{T}{2} \right) \right|^4 U_2 \left| \eta \left( (U) \right) \right|^4 \\
-\frac{1}{2} b_a^{(1, \Theta^4)} \ln \left( \frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}} T \right) \left| \eta \left( \frac{T}{2} \right) \right|^4 U_2 \left| \eta \left( (U) \right) \right|^4
\]

(3.183)

The value of \( U_2 \) is fixed and equal to one as can be easily seen from eqn.(3.177). In general for \( Z_N \) orbifolds with \( N \geq 2 \) the value of the U modulus is fixed. The final duality symmetry of (3.183) is \( \Gamma^0_f(2) \times \Gamma_U \) with the value of U replaced with the constant value i.

In the appendix A, we list details of the integration.
CHAPTER 4
4. Introduction

Inspired from the progress\cite{142,143} on the rigid supersymmetric Yang-Mills theories, recently much progress has been made towards understanding non-perturbative effects in string theory\cite{148,149}. At the level of $N=2$ supersymmetric $SU(r+1)$ Yang-Mills the quantum moduli space was associated with a particular genus $r$ Riemann surface parametrized by $r$ complex moduli and $2r$ periods $(\alpha_D, \alpha)$\footnote{This development was subsequently generalized\cite{143} for arbitrary $SU(n)$ gauge groups.}, while their effective theories up to two derivatives is encoded in the following $N=2$ effective supersymmetric lagrangian of $r$ abelian $N=2$ vector multiplets in the adjoint representation of the gauge group

$$\mathcal{L} = \frac{1}{4\pi} Im \{ \int d^4\theta \frac{\partial \mathcal{F}(\alpha)}{\partial A_i} \bar{A}_i + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}(A)}{\partial A_i \partial A_j} W_{\alpha_i} W_{\alpha_j} \}, \quad i,j = 1,\ldots,r. \quad (4.1)$$

When matter is not present it allows for generic values of the scalar field of the theory to be broken down to the Cartran sub-algebra and it is described from $r$ $N=2$ abelian vector supermultiplets, which can be decomposed into $r$ $N=1$ chiral superfields $\mathcal{A}$ and $r$ $N=1$ vector superfields $W_{\alpha}$. The theory is dominated from the behaviour of the holomorphic function $\mathcal{F}(\mathcal{A})$, namely the prepotential. The supersymmetric non-linear $\sigma$-model is described by the Kähler potential

$$K(\mathcal{A}, \bar{\mathcal{A}}) = Im \{ \frac{\partial \mathcal{F}(\alpha)}{\partial A} \bar{A} \},$$

while the metric in its moduli space $\tau(A) = Im \{ \partial^2 \mathcal{F}/\partial A^2 \}$ is connected to the complexified variable $\theta_{eff}/\pi + 8\pi i (g_{eff}^{-2}) \equiv \tau(A)$. The metric is connected\footnote{Here, $\alpha, \beta$ form a basis of the homology cycles of the hyperelliptic curve which has the same moduli space as $N=2$ supersymmetric Yang-Mills theory.} to the interpretation of the periods $\pi$

$$\pi = \left( \begin{array}{c} \alpha_D^i \\ \alpha^i \end{array} \right), \quad \alpha_D^i = \frac{\partial \mathcal{F}}{\partial \alpha^i}, \quad i = 1,\ldots,r \quad (4.2)$$

as an appropriate family of a meromorphic one-forms associated with $\lambda$,

$$\alpha_D^i = \oint_{\alpha_i} \lambda, \quad \alpha^i = \oint_{\beta_i} \lambda, \quad \tau = \frac{d\alpha_D^i/du}{d\alpha^i/du}. \quad (4.3)$$

$\alpha$, $\beta$ form a basis of the first homology group $H_1(E_g, \mathbb{Z}) = \mathbb{Z}^{2g}$, where $E_g$ a Riemann surface at genus $g$. The intersection of cycles in the canonical basis takes the form $(a_i, b_j) = -(b_j, a_i) = \delta_{ij}$.\footnote{The cycles $\alpha, \beta$ form a basis of the first homology group $H_1(E_g, \mathbb{Z}) = \mathbb{Z}^{2g}$, where $E_g$ a Riemann surface at genus $g$. The intersection of cycles in the canonical basis takes the form $(a_i, b_j) = -(b_j, a_i) = \delta_{ij}$.}
For the $SU(2)$ group, the hyperelliptic curve $E_u$ is

$$y^2 = (x - 1)(x + 1)(x - u), \quad (4.4)$$

and

$$\tau \equiv \tau_{E_u} = \frac{\mathcal{f}_{\alpha_1}}{\mathcal{f}_{\beta_1}} \lambda^1, \quad \lambda^1 = \frac{dx}{y} \quad (4.5)$$

The theory possess singular points with non-trivial monodromies $M$. The one-loop correction to the prepotential drives the local monodromy $M$ around a given singularity and transforms the section

$$(\begin{array}{c} \alpha_D \\ \alpha \end{array}) \to M \left( \begin{array}{c} \alpha_D \\ \alpha \end{array} \right), \quad M \in Sp(2r, R). \quad (4.6)$$

The metric in the moduli space

$$(ds)^2 = Im \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} da^i d\bar{a}^j = Im \, da^i d\bar{a}_i \quad (4.7)$$

is invariant under the monodromy transformations $M$. The one loop contribution to the holomorphic prepotential is due to the appearance of extra massless states at special points in the quantum moduli space. At a generic point of the moduli space the semiclassical monodromies split into non-perturbative monodromies. The pure gauge theory has singularities at the points $\infty, \pm \Lambda^2$, where $\Lambda$ is the dynamically generated scale where the gauge coupling becomes strong. The contributions to the prepotential at $\infty$ correspond to weak coupling and the semiclassical monodromy $M$ at $\infty$ splits as $M^\infty = M^{+\Lambda^2} M^{-\Lambda^2}$.

The theory yet possess stable dyonic points, labeled by magnetic $\nu_m$ and electric charges $\nu_e$, the so called BPS states. The masses of the stable particles are given by the BPS formula $M^2 = 2|Z|^2 = 2|\nu_e \alpha + \nu_m \alpha_D|^2$. At the strong coupling singularity $+\Lambda^2$ a magnetic monopole become massless with quantum numbers $(\nu_e, \nu_m) = \pm(0, 1)$. The corresponding cycle vanishes. At $-\Lambda^2$ a dyon become massless, a particle with both electric and magnetic charges, with $(\nu_e, \nu_m) = \pm(-2, 1)$. For the classical $SU(n)$ gauge theory the moduli space of the theory is parametrized by the parameter $u = Tr(\alpha^2)$.

The parameter $\alpha$, the complex scalar of eqn.(4.1), is in the adjoint representation of the gauge group. For non-trivial $u$ values, the gauge group is abelian. In the quantum theory the last picture is modified. The theory takes into account the singularities in the moduli space. The gauge group
around the singularities is abelian and the non-abelian gauge symmetry is never restored. Note, that the classical point $\alpha = 0$, where the gauge symmetry is non-abelian is missing in the quantum theory. Instead, the weak coupling point $\infty$ is available where perturbative calculations can be carried out\textsuperscript{102}. The general picture emerging from the study of the supersymmetric Yang-Mills is that the vacuum expectation values of the Higgs fields break the theory to its maximal abelian subgroup.

In general, there are only five different descriptions of string theories which all give consistent string vacua. The closed type IIA and type IIB superstrings, the open type I with gauge group $SO(32)$ and the closed heterotic string with gauge group $SO(32)$ or $E_8 \times E_8$. For these vacua we can perform perturbative calculations. This means that the any amplitude representing the scattering of $n$-particles can be expanded in the form\textsuperscript{103}

$$\hat{A} = \sum_{n=0}^{\infty} g_{\text{string}}^{-\xi} \hat{A}^n,$$

(4.8)

with $\xi$ the Euler number\textsuperscript{104} and $\hat{A}^n$ the scattering amplitude in genus $n$ Riemann surface. Eqn. (4.8) represents the fact that that the $n$-point amplitude as an expansion in the string coupling is equivalent to the sum over all worldsheet topologies. The variable $g_{\text{string}}$ is equal to $e^D$. Here, $D$ is the dilaton field of the bosonic part of the ten dimensional N=1 heterotic string in the string frame\textsuperscript{27}

$$\mathcal{L} = \sqrt{g} e^{-2D} \left\{ -\frac{1}{2\kappa^2} R - \frac{1}{4} \sum_a k_a (F_{\mu\nu} F^{\mu\nu})_a - \frac{2}{\kappa^2} \partial_m D \partial^m D + \frac{1}{16\kappa^2} H_m H^m - 2 G_{IJ} D_m \phi^I D^m \bar{\phi}^J - V(\phi, \bar{\phi}) \right\}.$$  

(4.9)

We have assumed that the gauge group of the low energy theory is a product of gauge group factors in the form $G = \otimes G_a$. Here, $\kappa$ is the gravitational coupling, $k_a$ the Kac-Moody level and the $H_m$ field strength contains Chern-Simons terms necessary for the anomaly cancellation of gauge and gravitational anomalies in ten dimensions. Rescaling\textsuperscript{317} the space time metric $g_{\mu\nu}$ and changing variables as $S = e^{-2D} + i\bar{a}$, $\bar{a}$ the axion, in the lagrangian (4.9) we bring

\textsuperscript{102} For various checks os Seiberg-Witten theory using traditional style perturbation techniques see \textsuperscript{147}.
\textsuperscript{103} See for example \textsuperscript{146}.
\textsuperscript{104} The Euler number is defined as $\left(1/4\pi\right) \int d^2 \sigma \sqrt{g} R^{(2)} = \xi = -2h - b + 2$, where $h$ is the number of loops, $b$ the number of external legs, $g_{\text{string}}$ the string coupling constant of the string vacuum.
the Einstein term in its canonical form. Direct comparison with eqn. (3.87) gives that in four-dimensions $K = -\kappa^{-2}\ln(S + \bar{S}) + \hat{K}(\phi, \bar{\phi})$ and $G_{IJ} = \partial_I \partial_J K$ and $\hat{K}$ a function of the scalars fields $\phi$.

Various equivalences between the different theories have been proposed and the picture emerging is that the different string theories are expansions of a more fundamental theory around different points in the moduli space of string vacua. We mention string-string duality where type IIA compactified on $K_3$ manifold with $N = 4$ supersymmetry has the same moduli space as the heterotic string on a $T^4$ torus $[167, 166, 168, 170, 160, 161]$ with $N = 4$ supersymmetry. For $N = 2$ type IIB the string generalization of Seiberg-Witten’s (SW) quantum theory is provided by the conifold transitions$^{105}$ of wrapped three-branes on Calabi-Yau$^{106}$ spaces. Type IIB in ten dimensions admits extremal black holes solutions in the RR sector of the theory. They represent BPS saturated p-brane solitons. Compactification of type IIB on a Calabi-Yau space produces $h^{(1,1)} + 1$ supermultiplet moduli with +1 associated with the dilaton and $h^{(2,1)}$ vector multiplets and the graviphoton. In addition, it gives the abelian gauge group $U(1)^{h^{(2,1)}+1}$. In general special geometry$^{107}$ applied to the compactification of type IIB on the Calabi-Yau space in four dimensions, requires that the scalar component $Z$ and the prepotential $F$ of the vector multiplets to be given by the period of the three form $\Omega$ over the canonical homology cycles $a_I, b_I$ as

$$Z_I = \int_{a_I} \Omega, \quad \frac{\partial F}{\partial Z_I} = \int_{b_I} \Omega, \quad I = 1, \ldots, h^{(2,1)}. \quad (4.10)$$

Here, $\Omega$ is the holomorphic three form describing the complex structure of the Calabi-Yau space. BPS states are $\propto \left| -\hat{\nu}_e^I Z_I + \hat{\nu}_m^I F_I \right|$. The integers $\hat{\nu}_e^I, \hat{\nu}_m^I$ are the electric and magnetic charges of the $^{105}$Moduli spaces of distinct Calabi-Yau(CY) manifolds touch each other along certain regions. These regions are called conifolds and they represent smooth manifolds apart from some singular points. In this sense CY’s can form a web of connected$^{153, 154, 155}$ manifolds. The neighbourhood of the singular region is described from the quadric $\sum_{i=1, \ldots, 4}(Z^i)^2 = 0$, where $Z^i$ complex variables in $C^4$.

$^{106}$Type IIB theory admits$^{156, 157}$ soliton solutions in the NS and RR sector of the theory called p-branes. They are extended objects in a theory with p spatial dimensions. They arise if there is $p + 1$ form in the theory coupled to the $p + 1$ dimensional world volume. For those p-branes associated with the RR sector, $p = -1, 1, 3, 5, 7$. Recently, the dynamics of the RR sector p-branes was$^{158}$ associated to objects referred as Dp-branes, extended objects with mixed boundary conditions(B.C) referring to as Neumann or Dirichlet at the worldsheet boundary, in the type I theory. Here D stands for Dirichlet B.C. Neumann B.C exist in p-dimensions and Dirichlet one’s in the remaining. D-branes are the carriers of the RR charge, predicted from string-string duality in six dimensions.

$^{107}$See next section.
threebrane wrapped around the three surfaces \(a_i, b_j\). The appearance of a logarithmic singularity in the Kähler metric at the conifold point \(Z=0\), involved in the compactification of type IIB on the Calabi-Yau space, is then identified with the extremal three brane black hole becoming massless. In analogy with SW theory, the three-brane becomes massless when the associated cycles vanish. The appearance of the singularity, when the corresponding 3-cycles along the 3-surfaces vanish, is then identified with the existence of a massless black hole solution in the metric of type IIB for the 3-brane. This is the analog of Seiberg-Witten appearance of the massless monopole singularity.

Among the various equivalences between the different perturbative string theories, we mention the S-duality conjecture for which more details will be given in chapter five.

In this chapter we are interested only in the proposal of which provided evidence for the exact nonperturbative equivalence of the heterotic string compactified on \(K_3 \times T_2\), with IIA compactified on a Calabi-Yau threefold. The proposal identifies the moduli spaces of heterotic string and its dual IIA as \(M^\text{heterotic}_V = M^{IIA}_V\) and \(M^\text{heterotic}_H = M^{IIA}_H\), where the subscripts refer to vector multiplets and hypermultiplets respectively. In this sense the complete prepotentials for the vector multiplets for the two ”different” theories match, including perturbative and non-perturbative corrections, and \(F^\text{het} = F^{IIA}\). Compactification of the heterotic string on \(K_3 \times T_2\) and of type IIA on a Calabi-Yau threefold produce models with \(N=2\) supersymmetry in four dimensions.

Lets us assume that we have a heterotic string compactified on the \(K_3 \times T^2\) manifold for which the dual type IIA compactified on a Calabi-Yau threefold exists. Then the following non-renormalization theorem holds:

Since in \(N = 2\) heterotic strings the dilaton is part of the vector multiplet, in reality of the vector-tensor multiplet as we will see in the next section, the prepotential of the vector multiplets for the heterotic model is getting corrected beyond tree level of perturbation theory while that the hypermultiplet superpotential is exact and equal to the tree level result. For the heterotic dual realization in type IIA, the prepotential of the vector multiplets at tree level is exact, while the hypermultiplet superpotential is getting corrected beyond tree level. Since there is no coupling between vector multiplets
and supermultiplets\[237\] at the perturbative level\[198\], we can extend\[150\] this argument at the non-perturbative level and conclude the following: The exact vector multiplet prepotential for a heterotic model, which has a type II dual, can be derived by the calculation of the tree level vector multiplet prepotential of the type IIA side. The exact supermultiplet prepotential for the heterotic side is equal to its tree level result.

The heterotic model receives perturbative and non-perturbative corrections to its prepotential of the vector multiplets in the heterotic side. In this chapter, we will calculate the one loop correction to the perturbative prepotential of the vector multiplets for the heterotic string compactified on a six dimensional orbifold. It comes from the solution of a partial differential equation. The one loop correction to the perturbative prepotential has already been calculated before in \[173\] from string amplitudes. Our procedure is complementary to \[173\] since we calculate, contrary to \[173\] where the third derivative of the prepotential with respect to the T moduli was calculated, the third derivative of the prepotential with respect to the U moduli. Furthermore, we establish a general procedure for calculating one loop corrections to the one loop prepotential, not only for heterotic strings compactified on six dimensional orbifolds, which has important implications for any compactification of the heterotic string having (or not) a type II dual. This procedure is an alternative way to the calculation of the prepotential which was performed in \[130\]. However, the procedure in \[130\] seems to us more complicated.

In addition, it was further proposed \[184\] that the existence of heterotic-type IIA duals\[148\] can be traced back to the $K3$ fiber structure or the elliptic fibration\[109\]. In section 4.4, we will discuss the issue of $K3$ fiber structure. Elliptic fibrations will be discussed in section 4.5.

A general result\[188\] concerning the geometry behind the existence of heterotic duals, is that the Calabi-Yau manifold in the IIA side can be written instead, as a fibre bundle with base $P^1$\[110\] generic fiber the $K3$ surface. The previous result was further elaborated in \[189\]. Existence of the IIA dual in the Calabi-Yau threefold phase with the dual heterotic string admitting a weakly coupled phase while the dual type IIA realization is in the strongly coupled phase, was proved that can happen only when \[188, 189\] the generic fibre is the $K3$ surface and the base is $P^1$.

\[108\] At the level of effective $N = 2$ supergravity, the vector multiplets are coordinates on a special Kähler manifold $M_V$ while the hypermultiplets parametrize\[237\] a quaternionic manifold $M_H$.

\[109\] These structures were further elaborated at \[188, 189, 190\].

\[110\] $P^1$ is the complex projective space with homogeneous coordinates $[x_0, x_1]$. 
At this chapter we will continue the work of [173, 56, 172, 173]. We will calculate the prepotential of \( N = 2 \) vector multiplets of heterotic string when the T,U moduli subspace exhibits an \( SL(2, Z)_T \times SL(2, Z)_U \times Z_2^{T-U} \) duality group\(^{111}\). The one loop Kähler metric in the moduli space of vector multiplets in \( N = 2 \) six dimensional orbifold\(^{30}\) compactifications of the heterotic string follows directly from this result. Furthermore, we calculate the prepotential of \( N = 2 \) vector multiplets of heterotic string for the case of \( N = 2 \) sectors in (2,2) symmetric non-decomposable Coxeter orbifolds.

For the description of general properties of the basic theory of \( N = 2 \) theories, we will follow closely in the beginning sections the work of [172] while in the description of calculating the prepotential of vector multiplets from string amplitudes\(^{112}\) we will follow the work of [173, 56]. In section 4.2 we will describe general properties of \( N = 2 \) heterotic strings. In addition, we describe properties of the moduli space of compactification of the heterotic string on a \( K_3 \times T_2 \) manifold. In section 4.3 we will discuss the special Kähler geometry describing \( N = 2 \) locally supersymmetric theory of the heterotic string with emphasis on the couplings of vector multiplets. In section 4.4 We give descriptions of the low energy theory for the classical and quantum theory for both the heterotic string compactified on a six dimensional orbifold. We also discuss the \( K_3 \times T_2 \) manifold and its dual type II on a Calabi-Yau 3-fold. In section 4.5 we describe our results for compactification of heterotic string on a six dimensional internal manifold. Here, we assume that the action of the lattice twist decomposes the torus in the form \( T^2 \oplus T^4 \). The calculation comes from the use of string amplitudes of [50, 173]. We will calculate one-loop corrections to the Kähler metric for the moduli of the usual vector multiplet T, U moduli fields of the \( T^2 \) torus appearing in \( N = 2 \) heterotic strings compactified on orbifolds. The calculation on the quantum moduli space takes into consideration points of enhanced gauge symmetry.

The one-loop correction to the prepotential for the vector multiplets then follows directly. In section 4.5, we will describe our results for the case of \( N=2 \) six dimensional orbifold compactification of the heterotic string, where the underlying torus lattice does not decompose as \( T^2 \oplus T^4 \). The moduli of the unrotated complex plane has a modular symmetry group that is a subgroup

\(^{111}\)However, due to factorization properties of the \( T^2 \) subspace of the heterotic Narain lattice, the same result can be applied to any heterotic string compactification on \( K_3 \times T^2 \), with different embeddings of the \( K_3 \) instantons with instanton number \( k \) on the gauge group.

\(^{112}\)We will draw heavily from these works.
of $SL(2, Z)$. In particular, we consider this modular symmetry to be one of those appearing in non-decomposable orbifold compactifications.

4.1 Properties of $N = 2$ heterotic string and Calabi-Yau vacua

A well known future of heterotic string vacua is the existence of their internal sector. The asymmetric nature of the heterotic string can be made apparent by the left-right asymmetry of the world-sheet degrees of freedom. In the critical dimension which is 10 for the heterotic and the type II, the number of the critical dimension comes from the fact that the the central charges of the matter system and the ghost system of the Virasoro algebra vanishes. For a generic vacuum of the heterotic string the contribution to the central charge from the ghost degrees of freedom is $-22$ for the left side and $-9$ for the right side. This has to be balanced from an appropriate internal CFT.

In general, for classical vacua of the heterotic string one replaces the internal manifold with a $c = (9, 9)$, $N = 2$ superconformally invariant theory on the world sheet together with four $N = (0, 1)$ free world-sheet superfields that give finally rise to the four dimensional space-time. Furthermore, we are left with a $c = 13$ trace anomaly to the left moving sector which is saturated from free bosons moving on a maximal torus $SO(10) \otimes E_8$ Kac-Moody algebra that is responsible for part of the gauge group. The list of $(2, 2)$ vacua includes the Calabi-Yau compactifications [206], orbifolds [80, 83], tensor products of minimal models [208] or generalizations [211]. We exclude the $(0, 2)$ models [212] since no corresponding type II theory exists.

For abelian $(2, 2)$ orbifolds constructed by twisting a six dimensional torus, the point group rotation is accompanied by a similar rotation in the gauge degrees of freedom. The four dimensional gauge group in this case is enlarged beyond $G = E_6 \otimes E_8$ by a factor that can be a $U(1)^2$, $SU(2) \times U(1)$, if $P = Z_4$ or $Z_6$, or $SU(3)$ if $P = Z_3$. If we symbolize by $h_{(1,1)}$ the number\footnote{For compactifications on Calabi-Yau manifolds, $h_{(1,1)}$ and $h_{(2,1)}$ represent the Hodge numbers of the manifold.} of $(1, 1)$ moduli in the untwisted sector then we have respectively $h_{(1,1)} = 3$, 5 and 9. Twisted moduli are not neutral with respect to $G$ and are not moduli of the orbifold. Abelian $(2, 2)$ orbifolds can flow to a Calabi-Yau vacuum, by blowing up the twisted moduli fields, by giving them vacuum expectation values [301].
On the other hand compactification of the heterotic string on a six dimensional compact manifold can put restrictions on the allowed manifolds, which demand on the number of supersymmetries we want to preserve in four dimensions. The supersymmetry transformations of the fermionic fields in ten dimensions give\cite{144,5}, assuming that the supersymmetry generator $\eta$ leaves the vacuum invariant,

$$\delta \psi_M = D_M \eta = 0$$

$$\delta \chi^\alpha = \Gamma^{MN} F^\alpha_{MN} \eta = 0, \quad M, N = 1, \ldots, 10,$$

where $\psi_M$ is the gravitino, $\eta$ is the supersymmetry generator, $F_{MN}$ the gauge field strength of the gauge fields and $D_M$ the covariant derivative. Eqn.(4.11) represents the fact that the spinor $\eta$ is covariantly constant. For compactifications of the ten dimensional target space in a manifold $M_4 \times K$, condition (4.11) imply, via the integrability condition $[D_M, D_N] = 0$, that $M_4$ space is Minkowskian. Furthermore, these conditions are associated with the existence of the compact manifold $K$ to be a complex K"ahler manifold. This means that it admits a metric of $U(N)$ holonomy. In local form, the metric comes from the equation $g_{ij} = \partial^2 K/((\partial z^i \partial \bar{z}^j)$, where $K$ is the K"ahler potential. Holonomy group $SU(n)$ implies that in the four dimensional space-time we get $N = 1$ supersymmetry and the complex manifold is a Calabi-Yau n-fold. $SU(2)$ holonomy is associated with the four dimensional $K_3$, while $G_2$ holonomy and Spin(7) one with seven and eight dimensional compactification manifolds respectively. Manifolds which satisfy the conformal invariance conditions admitting Ricci flat metrics are called Calabi-Yau manifolds\cite{144}. From the mathematical point of view, Ricci-flatness is associated with the vanishing first Chern class of the tangent bundle of the manifold. In the case of a six-dimensional orbifolds\cite{80}, by ”blowing up” the quotient singularities we recover the corresponding smooth Calabi-Yau manifold.

The $N = 2$ superconformal algebra - after we expand the fields in modes as $T(z) = \sum L_n z^{-n-2}, \mathcal{T}_F^\pm (z) = \sum G^\pm_r z^{-r-3/2}$ and $J(z) = \sum J_n z^{-n-1}$ - takes the form $(G^+_r = (G^-_{-r})^\dagger)$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},$$

$$[G^\pm_r, G^\pm_s] = 2L_{r+s} \pm (r - s) J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{(r+s,0)}, \quad [G^\pm_r, G^\pm_s] = 0.$$
\[
\begin{align*}
\{L_n, G^\pm_r\} &= (\frac{n}{2} - r) G^\pm_{n+r}, \quad [L_n, J_m] = -m J_{n+m} \\
\{J_n, J_m\} &= \left(\frac{c}{3} n \delta_{m+n,0}ight), \quad \{J_n, G^\pm_r\} = \pm G^\pm_{n+r}.
\end{align*}
\]

(4.15)

is generated by four generators: one conformal weight (CW) 2 energy momentum tensor \(T\), one abelian current \(J(z)\) and two fermionic conformal weight (CF) \(3/2\) supercurrents \(G^\pm(z)\) with an abelian charge \(\pm 1\). The current algebra \(J(z)\) corresponds to a free boson in two dimensions \(J(z) = \left(\frac{c}{3}\right)^{1/2} \partial_z \phi\) and \(c\) represents the trace anomaly. If \(r \in \mathbb{Z} + 1/2\), then the N=2 algebra describes the NS sector, while if \(r \in \mathbb{Z}\) we are in the Ramond sector. Acting on a state \(|\phi\rangle\) with the generators \(L_0, J_0\) and using the relations \(J_0 |\phi\rangle = q |\phi\rangle, \quad L_0 |\phi\rangle = h |\phi\rangle\), where \(q\) the \(U(1)\) charge and \(h\) the conformal weight, we get the constraints \(h \geq \frac{|q|^2}{2}\) for the NS sector and \(h \geq \frac{c}{24}\) in the Ramond sector. The left moving \(U(1)\) charge combines with the \(SO(10)\) Kac-Moody algebra to accomplish the left-moving gauge group enlargement group to the \(E_6\).

The last statement is really the demand that the operator product of the gravitino vertex operator

\[
\psi^i_\alpha(z) = e^{-\varphi/2} S^\alpha e^{i\sqrt{3}/2 \phi(z)}
\]

(4.17)

with a physical state to be local. The ghost field is \(\varphi\), \(S\) is the \(SO(4)\) space-time spin field and the exponential comes from the dependence from the internal sector, the Ramond ground states with conformal weight \(3/8\). If we have \(N\)-extended space-time supersymmetries then we have \(N\) supercharges \(Q^i_\alpha (i = 1, \ldots, N)\) which obey

\[
\{Q^i_\alpha, \bar{Q}^{\dot{j}}_{\dot{\alpha}}\} = 2i \delta^i_\dot{j} \gamma_{\alpha\dot{\beta}}^\mu P^\mu, \quad \{Q^i_\alpha, Q^j_\beta\} = 2C_{\alpha\beta} Z^{ij}.
\]

(4.18)

where \(Z^{ij}\) denotes the central charges. The supersymmetry charges are defined in general as

\[
Q^i_\alpha = \oint \frac{dz}{2\pi i} V^i_\alpha(z), \quad \bar{Q}^{\dot{i}}_{\dot{\alpha}} = \oint \frac{dz}{2\pi i} \bar{V}^{\dot{i}}_{\dot{\alpha}}(z),
\]

(4.19)

116 Take for example the bosonic string. Assuming that the energy-momentum tensor has an expansion in modes as \(T(Z) = \oint \frac{dz}{2\pi i} z^{-n+1} T(Z)\), we have \([L_n, \phi(z)] = z^n [z \partial_z + (n + 1)h] \phi(z)\). The conformal weight \(h\) is defined via the operator product expansion of the primary field with \(T(Z)\), namely

\[
T(Z) \phi(w) = \frac{\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{(z-w)}.
\]

(4.16)
where $V^{i}_{\alpha}(z)$ and $\bar{V}^{\dot{i}}_{\dot{\alpha}}(z)$ are the vertex operators in the $-1/2$ ghost picture.

When the heterotic vacuum has $N = 2$ space time supersymmetry the nature of the supersymmetry algebra implies that the right moving algebra splits into a sector with $c = 6$ and $N = 4$ SCFT and a free SCFT field theory $c = 3$ with $N = 2$ supersymmetry. On the other hand on the the left moving side of the heterotic string there is no world-sheet supersymmetry but instead we have a bosonic CFT, with a space-time sector consisting from four free bosons and an internal sector with $\tilde{c} = 22$. This is in contrast with the type II models where there is $N = 2$ world sheet supersymmetry in both sectors.

The massless spectrum of the $N=2$(space-time) $d=4$ heterotic string consists, among other particles, of the graviton $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton which are all created from vertex operators of the form $\propto \bar{\partial}X_{\mu}(\bar{z})\partial X_{\nu}(z)$ . In addition it contains two gravitini and two dilatini which are created from vertex operators of the form $\bar{\partial}X_{\mu}(\bar{z})V^{i}_{\alpha}(z)$ and the gauge boson generators of the group $U(1)^{2}_{R}$ with vertex operators $\bar{\partial}X_{\mu}(\bar{z})\partial X^{\pm}(z)$. In the supergravity multiplet there are included together with the graviphoton(the spin one gauge boson of the supermultiplet), the graviton and two gravitini. However, the dilaton is included in the vectortensor multiplet[172]. It contains the dilaton, the antisymmetric tensor, the two dilatini and a $U(1)$ gauge field. The vector-tensor multiplet consists of a N=1 vector multiplet and a $N = 1$ linear[175] multiplet. This on shell structure must be described as well from a $N = 2$ vector multiplet via a duality transformation on the antisymmetric tensor field. In the dual description, the antisymmetric tensor is converted to the axion $\tilde{a}$ via a supersymmetric duality transformation, and the dilaton D and the axion combine to form the complex scalar $S = e^{\phi'} + i\tilde{a}$. Off-shell the $8 + 8$ component structure of the vector-tensor multiplet can be realized[174, 176] in the presence of the central charge. In this case, after we linearize the Lagrangian for the vector-tensor multiplet we obtain[172]:

$$\mathcal{L} = -\frac{1}{2}(\partial \phi')^2 - \bar{\lambda}^{i}\phi \lambda_{i} + \frac{1}{2}H^2 - \frac{1}{4}F^2 + \frac{1}{2}D^2.$$  \hspace{1cm} (4.20)

Here, $D'$ is a real scalar auxiliary field, $\phi'$ is the dilaton, F is the self-dual field strength of the gauge field, H the antisymmetric two form antisymmetric tensor and $\lambda$ a doublet of Majorana spinors. The action (4.20) has the same degrees of freedom as the action for a vector multiplet.

Other particles that are present at the massless spectrum of the heterotic string include $N = 2$ vector multiplets with gauge bosons $A^{\mu}_{\alpha}$, together with their superpartners, the two gauginos $\lambda_{i}$.
and a complex scalar \( C^{\alpha} \). The vertex operator for a generic vector multiplet are

\[
\left( A^{\alpha}_{\mu}, \lambda_{i\alpha}^{a}, C^{a} \right) \sim \left( J^{a}(\bar{z}) \, \partial X_{\mu}(z), J^{a}(\bar{z}) \, V_{i\alpha}(z), J^{a}(\bar{z}) \, \partial X^{\pm}(z) \right)
\]  

(4.21)

and the multiplet itself and the currents \( J^{a} \) transform in the adjoint representation of the non-abelian gauge group \( G \) created from the zero modes of the Kac-Moody currents. In fact we will see later that the scalars of vector multiplets in the moduli space can be divided to moduli and matter. The maximal rank for \( G \) is 22.

Since we will be describing compactifications of heterotic vacua on a 2-torus \( T_{2} \), it is necessary to give some details. The moduli of the torus is parametrized from the relations \( T = 2(B + i\sqrt{G}) \) and \( U = 1/G_{11}(G_{12} + i\sqrt{G}) \) where \( G_{ij} \) is the metric of the \( T_{2} \), \( \sqrt{G} \) is its determinant and \( B \) the constant antisymmetric tensor background field. At the classical level, the moduli space of orbifold compactification of the N=2 heterotic string compactified in a six dimensional torus corresponds to the coset space \( \frac{SO(2,2)}{SO(2) \times SO(2)} | T;U \). The same type of moduli appears when we compactify\(^{174, 130}\) the heterotic string on the manifold \( K_{3} \times T_{2} \).

The subspace of the vector moduli space corresponding to the T, U moduli is associated with the lattice \( \Gamma^{2,2} \) part of the Narain lattice \( \Gamma^{22,6} \) based on compactifications of even Lorentzian lattices of the heterotic string. The gauge group, using the standard embedding by equating the spin connection with the gauge connection, gives at generic points in the moduli space of the torus, the gauge group \( E_{8} \times E_{7} \times U(1)^{4} \). The first \( U(1)^{2} \) is associated with the moduli of the \( T^{2} \) parametrizing the torus, while the other \( U(1)^{2} \) comes from the dilaton in the vector-tensor multiplet and the graviphoton. The heterotic prepotential for this model at the semiclassical limit \( S \rightarrow \infty \) has been calculated in\(^{172, 173, 130}\). Various tests have been performed for the heterotic prepotential with several Calabi-Yau IIA duals. In most of the tests, the weak coupling limit expansion of the heterotic prepotential has been matched with the corresponding prepotential of the type IIA side using\(^{191}\) its mirror in type IIB. Mirror symmetry supports the existence of for every Calabi-Yau manifold \( \tilde{E} \), the mirror partner \( E \) such that \( h^{(1,1)}(\tilde{E}) = h^{(2,1)}(E) \) and \( h^{(1,1)}(E) = h^{(2,1)}(\tilde{E}) \). Orbifold compactifications of the heterotic string\(^{80}\), in four dimensions on a six dimensional torus \( T^{2} \oplus T^{4} \), produces a \( T^{2} \) subspace. The classical moduli space, of \( r \) vector multiplets in the \( T^{2} \) subspace, is the group

\[
\frac{SU(1,1)}{U(1)} \bigg|_{\text{dilaton}} \times \frac{O(2,r)}{O(2) \times O(r)} \bigg/ O(2,r;Z).
\]  

(4.22)
For our case, the classical duality group comes with $r = 2$. Here, $O(2, 2; \mathbb{Z})$ is the target space duality group. The theory enjoys the non-trivial global invariance i.e identifications under target space duality symmetries $213$, $33$, $216$ the $PSL(2, \mathbb{Z})_T \times PSL(2, \mathbb{Z})_U$ dualities acting as

$$T \rightarrow \frac{aT - ib}{icT + d}, \quad U \rightarrow \frac{a'U - ib'}{ic'U + d'},$$

(4.23)

At special points in the moduli space of the torus the associated gauge group becomes enhanced to $SO(4)$ or $SU(3)$ as we have already mentioned in the introduction. In the presence of Wilson line moduli, associated with the internal torus $T_2$ the modular group acts as

$$T \rightarrow \frac{aT - ib}{icT + d}, \quad U \rightarrow U - ic \frac{BC}{icT + d'},$$

$$B \rightarrow \frac{B}{icT + d'}, \quad C \rightarrow \frac{C}{icT + d'},$$

(4.24)

with $a, b, c, d, a', b', c', d'$ are integers and $ad - bc = 1, a'b' - c'd' = 1$. The same transformation law appears also to the transformation law of the matter fields for $(2, 2)$ compactifications.

Let us discuss now Calabi-Yau manifolds. In general a Calabi-Yau manifold refer to Ricci-flat Kähler manifolds of vanishing Chern class. The condition of vanishing Chern class on a compact manifold has as a consequence the existence of a Ricci flat Kähler metric. The condition of vanishing Chern class$^{117}$ is associated with the existence of a two form $\rho$ equal to $\rho = R_{i\bar{j}} dz^i \wedge d\bar{z}^\bar{j}$, with $R_{i\bar{j}}$ the Ricci tensor. Vanishing Chern class means, Ricci flat metric, that $R_{i\bar{j}} = 0$.

The massless modes coming from the compactification on a Calabi-Yau manifold are associated$^3$ to the zero modes of the Laplace operator on the compactification manifold. The inner product is defined as $\langle \gamma, \delta \rangle = \int \gamma \wedge \ast \delta$. For our purpose it is enough to know that action of the Poincare duality operator $\ast$, in an n-dimensional manifold, transforms a $p$-form to an $n-p$ form. The number of the linearly independent $p$-forms associated to the zero modes in the action is now the number of linearly independent $p$-forms that are closed but not exact. This is defined as the Betti number $b_p$, namely $b_p = \sum_{p+q=r} h^{p,q}$. The vector space of the closed $p$-forms modulo the exact forms is the cohomology group $H^p(M, R)$ on the manifold $M$, with dimension equal to the Betti number and $H^p(M, R)$. In general, we can define a $(p, q)$ from the wedge product. So if we have a $p$-form $A$ and a $q$-form $B$, we define their product to be a $p + q$ form $(a \wedge B)_{i_1 \ldots i_p \ldots i_q} = \frac{p!q!}{(p+q)!}(A_{i_1 \ldots i_p} B_{i_1 \ldots i_q}) \pm \text{permutations},$ where

$^{117}$ By definition the 1st Chern class $c_1(X)$ is defined as $c_1 = trR$, where $R$ is the Ricci tensor.
the permutations are completely antisymmetric in all \(p + q\) indices. Forms with \((p, q)\) indices which are closed but not exact generate the Dolbeault cohomology groups \(H^{(p,q)}(X)\). On Calabi-Yau manifolds, and in general in Kähler manifolds, the Hodge-de Rham Laplacian 
\[
\Delta = \bar{\partial} \partial^* + \partial^* \bar{\partial}
\]
annihilates the \((p, q)\) forms. The associated cohomology groups \(H^{(p,q)}(X)\) decompose as \(H^k(X) = \oplus_{p+q=k} H^{(p,q)}(X)\). Here, \(H^k(X)\) is the De Rham cohomology group\(^\text{178}\) which annihilates \((p, 0)\) forms with Laplacian \(\Delta = dd^* + d^*d\). The dimensions of \(H^{(p,q)}(X)\) are the Hodge numbers \(h_{(p,q)}\) and satisfy \(h_{(p,q)} = h_{(q,p)}\) and from Poincare duality \(h_{(p,q)} = h_{(n-p,n-q)}\). The Euler number is given by \(\chi = \sum_{p,q} (-1)^{p+q} h_{(p,q)}\). For the Calabi-Yau three folds, \(h_{(1,0)} = h_{(2,0)} = 0\) and \(h_{(0,3)} = h_{(3,0)} = 1\) and the Euler number is \(\chi = 2(h^{(1,1)} - h^{(2,1)})\). Remember that for a Calabi-Yau threefold the Euler number is is two times the net number of chiral generations. For the \(K_3\) surface, the Hodge numbers are \(h_{0,0} = h_{2,0} = h_{2,2} = 1\) and \(h_{1,1} = 20\), so \(\chi(K_3) = 24\). A choice of complex coordinates\(^\text{153}, 154, 180\) in a Calabi-Yau space defines a complex structure. Complex structure deformations are parametrized by the so called complex structure moduli which are associated with the variations of the metric \(\delta g_{ij}, \delta g_{ij}\). In addition, there is an additional form of moduli, the Kähler class moduli associated with mixed indices variations of the Ricci flat Kähler metric, i.e \(\delta g_{ij}\). Variation of the metric of the Calabi-Yau space in order to preserve Ricci flatness, associates the quantities \(idg_{ij}dz^i \wedge dz^j\) to harmonic \((1, 1)\) forms and \(i \Omega^k_{ij} dz^i \wedge dz^j \wedge dz^k\) to harmonic \((2, 1)\) forms. Here, \(z^i\) are the complex coordinates\(^\text{179}\) of the Calabi-Yau manifold and \(\Omega_{ij\rho} = g_{ik} \Omega^k_{ij}\) is the constant three form. Naturally, harmonicity means that \(idg_{ij}dz^i \wedge dz^j = \sum_{i=1}^{h^{1,1}} \epsilon_i^1 \psi_i\) and \(\psi_i \in H^{1,1}\). In addition, \(i \Omega^k_{ij} dz^i \wedge dz^j \wedge dz^k = \sum_{i=1}^{h^{2,1}} \epsilon_i^2 \delta_i\) and \(\epsilon_i^2 \in H^{2,1}\). The four dimensional fields associated to the parameters \(\psi_i\) and \(\delta_i\) are the moduli of the low energy effective action. In other words, the variations of the metric associated with \(H^{1,1}\) cohomology correspond to the Kähler class moduli and variations of the metric associated with \(H^{2,1}\) cohomology to complex structure moduli. For compactifications on \(K_3\) manifolds, the moduli space of metrics with \(SU(2)\) holonomy associated to complex and Kähler deformations is \(\mathcal{M} = \frac{SO(19) \times SO(3) \times SO(19,3; \mathbb{Z})}{SO(19)} \times R^+\), where \(R^+\) is associated\(^\text{160}\) with the volume of \(K_3\). Adding the moduli coming from deformations of the antisymmetric tensor we get the moduli space of \(K_3 \frac{SO(20,4) \times SO(20)}{xSO(4)} / SO(4, 20; \mathbb{Z})\).

The low energy \(N = 1\) supergravity of type I and heterotic string theories is subject to anomalies coming from hexagon diagrams which prevent it from describing an anomaly free string theory. In this case anomalies are cancelled\(^\text{194, 193, 144}\) by the addition of appropriate
counterterms which modify the supersymmetry structure. Similarly, in six dimensions the total anomaly is associated to the eight form

\[ I_8 = \tilde{\theta}_1 tr R^4 + \tilde{\theta}_2 (tr R^2) + \tilde{\theta}_3 tr R^2 tr F^2 + \tilde{\theta}_4 (tr F^2)^2 \]  

(4.25)

where \( \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4 \) are numbers depending on the spectrum\[192, 195\] of the theory. Cancellation of anomaly requires \( \tilde{\theta}_1 = n_H - n_V + 29 n_T - 28 = 0 \), where \( n_V, n_H, n_T \) are the numbers of vector multiplets, hypermultiplets and antiselfdual tensor multiplets respectively. Because in six dimensions we have one tensor multiplet, which incorporates the dilaton, a Weyl spinor and an antiself-dual antisymmetric tensor, the last constraint becomes \( n_H - n_V = 244 \). Now Green Schwarz mechanism factorization of anomalies is at work with \( I_8 \propto -\frac{1}{(2\pi)^3 16} G \tilde{G} \), \( G = tr R^2 - \sum a \upsilon_a (tr F^2) \) and\[118\] \( \tilde{G} = tr (R \wedge R) - \sum a \tilde{\upsilon}_a tr (F \wedge F)_a \). Cancelation of anomalies requires modification of the antisymmetric field streng H as

\[ H = dB + \omega_L - \sum a \upsilon_a \omega_a^M, \quad \omega_L = tr (\omega R - \frac{1}{3} \omega^3), \quad \omega^M = tr (AF - \frac{1}{3} \omega^3). \]  

(4.26)

Here, \( \omega_L, \omega^M \) are the Yang-Mills and Lorentz Chern-Simons three forms, A the gauge field, R the Riemann tensor and \( \omega \) the spin connection. However, because \( H \) is globally defined on \( K_3 \), \( \int_{K_3} dH = 0 \). As a result, we get that the constraint

\[ \sum a n_a = 24, \quad n_a = \sum a \int_{K_3} (tr F^2)_a, = \int_{K_3} tr R^2 = 24. \]  

(4.27)

Here, the instanton number \( n_a \) becomes equal to the Euler number of \( K_3 \). Initially, in ten dimensions the unbroken group is \( E_8 \times E_8 \times U(1)^4 \), where the \( U(1) \)'s are associated with the \( T^2 \) and the graviton and the graviphoton. The spectrum can be derived from index theory\[148, 192\]. The spectrum of the theory after compactification on \( K_3 \times T^2 \) can be calculated\[192\] using index theory. The gauge group \( G \) can be broken to a subgroup \( H \), by vacuum expectation values of \( K_3 \) gauge fields in \( G \), where \( H \times G \subset G \). The gauge group \( G \) breaks into the subgroup \( H \), which is the maximal subgroup commuting with the \( G \) subgroup, the commutant of \( G \). We perform the decomposition \( adj G = \sum_i (R_i, M_i) \), where \( R_i, M_i \) representations of the gauge groups \( H \) and \( G \) respectively. Then the number of left-handed spinor multiplets transforming in the \( R_i \) representation of \( H \) is given by

\[ N_{R_i} = \int_{K_3} -\frac{1}{2} tr R_i F^2 + \frac{1}{48} dim M_i tr R^2 = dim M_i - \frac{1}{2} \int_{K_3} c_2(V) index(M_i), \]  

(4.28)

\[118\]Here, R, F are the gravitational and gauge field strengths. The coefficients \( \upsilon_a, \tilde{\upsilon}_a \) depend on the particle content and the sum is over the gauge group \( G \) factors \( G_a \).
where $V$ is the $\mathcal{G}$ bundle parametrizing the expectation values (vev’s) of the vacuum gauge fields on $K_3$. By $c_2(V)$ we denote the second Chern class of the gauge bundle $V$ and $\dim_i$ the dimension of the representation $i$. In addition, the dimension of the moduli space of gauge bundles is $4h_a - \dim(\mathcal{G}_a)$, where $h_a$ is the Coxeter number of $\mathcal{G}_a$ and $\dim$ its rank. In a general situation we allow for the gauge group $G$ to break to the commutant of $\otimes \mathcal{G}$, by embedding the gauge connections of a number of a product of gauge bundles $V_a$ with gauge group $\mathcal{G}_a$ into $G$, resulting in the breaking of $G$ into the commutant of $\otimes_a \mathcal{G}_a$. In this way, we identify, for manifolds of $SU(2)$ holonomy, the spin connection of $K_3$ with the gauge group $\otimes_a \mathcal{G}_a$, breaking the $G$ symmetry into $H$. This is the analog of breaking the gauge group $E_8$, in manifolds of $SU(3)$ holonomy, by the standard embedding\cite{199} of the $SU(3)$ gauge connection into the spin connection, to the phenomenologically interesting $E_6$ gauge group. Embedding an $SU(2)$ gauge bundle\footnote{Here, $h(SU(2)) = 24.$} into one of the $E_8$’s, we get 45 hypemultiplet scalars plus a contribution of 20 from the gravitational multiplet, making a total of 65 hypermultiplets. In addition, we get a number of 56’s in $E_7$ giving $N_{56} = 10$.

4.2 Special Geometry and Effective Actions

In this part of the Thesis we will describe properties of the low energy effective actions of $N = 2$ effective string theories. In $N = 2$ supersymmetric Yang-Mills theory the action is described by a holomorphic prepotential $F(X)$, where $X^A$ ($A = 1, \ldots, n$) are the complex scalar components of the corresponding vector superfields. Two different functions $F(X)$ could correspond to equivalent equations of motion. In general such equivalences involve symplectic reparametrizations combined with duality transformations.

The couplings of the classical vector multiplets with supergravity are determined by a holomorphic function $F(X)$, the prepotential function which is a holomorphic function of $n + 1$ complex variables $X^I$ ($I = 0, 1, \ldots, n$) and it is a homogeneous function of degree two\footnote{Here, $h(SU(2)) = 24.$} in the fields $X^I$. The general action for vector multiplets coupled to $N = 2$ supergravity was first obtained with the superconformal tensor calculus.

In $N = 2$ supergravity theories, supersymmetry demands an additional vector superfield $X^0$ which account for the accommodation of the graviphoton. It stands for the $I = 0$ component of
the vector multiplets and it belongs to a compensating multiplet. The graviphoton is the vector component of the compensating multiplet and is the spin one gauge boson of the supergravity multiplet. The coordinate space of physical scalar fields belonging to vector multiplets of an $N = 2$ supergravity is described from special Kähler geometry [226, 238], with the Kähler metric $g_{AB} = \partial_A \partial_B K(z, \bar{z})$ resulting from a Kähler potential of the form

$$K(z, \bar{z}) = -\log \left(i \bar{X}(z) F_I(X(z) - i X^I(z)), \bar{F}_I(\bar{X}(\bar{z}))\right), F_I = \frac{\partial F}{\partial X_I}, \bar{F}_I = \frac{\partial F}{\partial \bar{X}_I}$$  (4.29)

and the Riemann curvature tensor satisfying [239]

$$R^A_{BCD} = 2 \delta^A_{(B^C)} - e^{2K} W_{BCE} \bar{W}^{EAD},$$  (4.30)

where $W_{abc}$ is a holomorphic 3-index symmetric tensor given by

$$W_{ABC} = F_{IJK}(X(z)) \frac{\partial X^I(z)}{\partial z^A} \frac{\partial X^J(z)}{\partial z^B} \frac{\partial X^K(z)}{\partial z^C}. \quad (4.31)$$

By choosing inhomogeneous coordinates $z^A$ the so called, special coordinates, defined by $z^A = X^A/X^0$, $A = 0, \ldots, N$ or by $X^0(z) = 1$, $X^A(z) = z^A$, the Kähler potential can be written as [240]

$$K(z, \bar{z}) = -\log \left(2(\mathcal{F} + \bar{\mathcal{F}}) - (z^A - \bar{z}^A)(\mathcal{F}_A - \bar{\mathcal{F}}_A)\right), \quad (4.32)$$

where $\mathcal{F}(z) = i(X^0)^{-2}F(X)$. Up to a phase, the proportionality factor between the $X^I$ and the holomorphic sections $X^I(z)$ is given by $\exp \left(\frac{i}{2}K(z, \bar{z})\right)$. The kinetic energies of the gauge fields are

$$\mathcal{L}_{\text{gauge}} = -\frac{i}{8} \left(\mathcal{N}_{IJ} F_{\mu\nu}^{+ I} F_{\mu\nu}^{+ J} - \bar{\mathcal{N}}_{IJ} F_{\mu\nu}^{- I} F_{\mu\nu}^{- J}\right), \quad (4.33)$$

where $F_{\mu\nu}^{\pm I}$ represents the selfdual and anti-selfdual $F_{\mu\nu}^{\pm} = (1/2)(F_{\mu\nu}^{I} \pm \bar{F}_{\mu\nu}^{I})$ field strengths proportional to the symmetric tensor

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \frac{\text{Im}(F_{IK}) \text{Im}(F_{IL}) X^K X^L}{\text{Im}(F_{KL}) X^K X^L}, \quad F_{IJ} = \frac{\partial^2 F}{\partial X_I \partial X_J}, \quad \bar{F}_I = \mathcal{N}_{IJ} X^J. \quad (4.34)$$

Here, $\mathcal{N}$ is the field-dependent tensor of the gauge involved in the gauge couplings $g$, $g_{IJ}^2 = \frac{i}{4}(\mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ})$. The generalized $\theta$ parameters $\theta_{IJ} = 2\pi^2(\mathcal{N}_{IJ} + \bar{\mathcal{N}}_{IJ})$. Subscripts on the $F$ variable denote derivatives and repeated indices, as usual, are summed.

The equivalence of equations of motion under different functions $F(X)$ could describe equivalences under electric-magnetic dualities of the field strengths, and not local gauge transformations.
to the vector potentials $A^I_\mu$. Because for the non-Abelian case, such a duality is meaningless since
the equations of motion cannot be made invariant under the symplectic transformations which
will be defined in \((4.37)\), we will work with abelian gauge fields. Note, that a non-abelian gauge
field have only electric charge. In this way, when all the fundamental fields are neutral, one can
freely choose any integral basis for the electric and magnetic charges.

Let us define the tensors \([226, 241]\) $G^\pm_{\mu \nu I}$ as

\[
G^+_{\mu \nu I} = N_{IJ} F^J_{\mu \nu} + Z_{IJ} G^-_{\mu \nu I}, \quad G^-_{\mu \nu I} = \bar{N}_{IJ} F^J_{\mu \nu}.
\]  

(4.35)

Then the set of Bianchi identities and equations of motion for the abelian gauge fields is expressed
as

\[
\partial^\mu (F^+_{\mu \nu I} - F^-_{\mu \nu I}) = 0, \quad \partial^\mu (G^+_{\mu \nu I} - G^-_{\mu \nu I}) = 0.
\]  

(4.36)

These are invariant under the symplectic $Sp(2n + 2, \mathbb{R})$ transformations

\[
F^+_{\mu \nu I} \rightarrow \tilde{F}^+_{\mu \nu I} = U_I^J F^+_{\mu \nu J} + Z^{IJ} G^+_{\mu \nu J} \quad \text{and} \quad G^+_{\mu \nu I} \rightarrow \tilde{G}^+_{\mu \nu I} = V_J^I G^+_{\mu \nu J} + W_{IJ} F^+_{\mu \nu J},
\]  

(4.37)

where $U$, $V$, $W$ and $Z$ are constant, real, \((n + 1) \times (n + 1)\) matrices. Alternatively,

\[
\begin{pmatrix}
F^+_{\mu \nu I} \\
G^+_{\mu \nu I}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
U & Z \\
W & V
\end{pmatrix}
\begin{pmatrix}
F^+_{\mu \nu} \\
G^+_{\mu \nu}
\end{pmatrix},
\]  

(4.38)

with

\[
\mathcal{O} \overset{\text{def}}{=} \begin{pmatrix}
U & Z \\
W & V
\end{pmatrix}, \quad \in Sp(2n + 2, \mathbb{R})
\]  

(4.39)

and

\[
\mathcal{O}^{-1} = \Omega \mathcal{O}^T \Omega^{-1} \quad \text{and} \quad \Omega = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]  

(4.40)

The matrices $U$, $V$, $W$ and $Z$, satisfy

\[
U^T V - W^T Z = V^T U - Z^T W = 1, \quad U V W = W^T U \quad \text{and} \quad Z^T V = V^T Z.
\]  

(4.41)

The kinetic term of the vector fields does not preserve its form under general $Sp(2n + 2, \mathbb{R})$
transformations and only the equations of motion and Bianchi identities are in fact equivalent.
In the case of abelian gauge fields, one can always choose a coordinate basis $X^I$ for which the
prepotential $F$ does exist. On the other hand, in string theory, the dilaton dependence of the gauge couplings is explicit only in a basis where $F$ does not exist.

Target-space duality transformations can always be implemented as $Sp(2n+2, R)$ transformations of the period vectors $(X_I, F^I)$ of special geometry. For target-space duality transformations the Lagrangian is left invariant by the subgroup that satisfies $W = Z = 0$ and $V^T = U^{-1}$. The presence of $Z = 0$ and $V^T = U^{-1}$ together with the condition that $W^T U$ has to be a symmetric matrix provides the semiclassical transformations

$$
\tilde{X}^I = U^I_J X^J, \quad \tilde{F}^{\pm I} = U^I_J F^{\pm J}, \quad \tilde{F}_I = [U^{-1}]^T_I F_J + W_{IJ} X^J, \quad \tilde{N} = [U^{-1}]^T \mathcal{N} U^{-1} + W U^{-1}, \quad (4.42)
$$

which may be implemented as Lagrangian symmetries of the vector fields $A^I_\mu$. The last term in (4.42) amounts to a constant shift of the theta angles at the quantum level. Because such shifts are quantized, the symplectic group must be restricted to $Sp(2n + 2, \mathbb{Z})$. Such shifts in the $\theta$-angle do occur whenever the one-loop gauge couplings have logarithmic singularities at special points in the moduli space where massive modes become massless. It will be confirmed by our results as well later in this chapter. Constant shifts in the theta angle occur, when we encircle the singular line $T = U$ at the quantum moduli space. As a result such symmetries are associated with the semi-classical one-loop monodromies around such singular points. An other form of duality transformations interchanges the field-strength tensors $F^I_{\mu \nu}$ and $G_{\mu \nu I}$ and correspond to electric-magnetic dualities. These transformations appear as $U = V = 0$ and $W^T = -Z^{-1}$, and $\tilde{N} = -W \mathcal{N}^{-1} W^T$, so that they give rise to an inversion of the gauge couplings and hence must be non-perturbative. In the heterotic string theory, such transformations represent the $S$-dualities.

The classical rigid field theory is not associated with field dependence of the physical observables. However, by introducing a cut-off at the Planck scale, in the quantum theory, the superheavy states are integrated out leaving only the light fields. Integration of the heavy fields induce moduli dependence in the effective theory. In real terms, to properly describe the low energy theory of the physical vacuum, the field-dependent couplings of the EQFT should be written as complete analytic functions of the moduli fields and the dependence on all the other fields must be described by a truncated power series.

Dividing the scalars as $z^A = X^A / X^0$ belonging to vector multiplets into moduli $\Phi^a = -i z^a$ and “matter” scalars $\Upsilon^k = -i z^k$, we expand the prepotential $\mathcal{F}$ of the theory as a truncated
power series in the matter scalars as
\[
\mathcal{F}(\Phi, \Upsilon) = h(\Phi) + \sum_{cd} f_{cd}(\Phi) \Upsilon^c \Upsilon^d. \tag{4.43}
\]

All scalars in the non-Abelian vector multiplets may be considered as matter \[172\] and not as moduli, since their vacuum expectation values can induce a nonzero mass for some of the non-Abelian fields. For such non-Abelian matter, the gauge symmetry of the prepotential requires for the gauge kinetic function of each non-abelian gauge group factor \((a), f_{ab}(\Phi) = \delta_{ab} f_{(a)}(\Phi)\).

Scalars in vector multiplets neutral under an abelian symmetry must be considered as moduli, otherwise as matter. For hypermultiplets in the effective theory charged under an Abelian gauge symmetry, the scalar superpartner of that gauge boson should be regarded as matter since its vacuum expectation value can in principle give masses to all charged hypermultiplets. But if all the light particles are neutral with respect to some Abelian gauge field, then its scalar superpartner is a moduli. So we divide the Abelian vector multiplets into \(\Phi^a\) and \(\Upsilon^a\) such that all the light of the EQFT are exactly massless for \(\Upsilon^a = 0\) and arbitrary \(\Phi^a\). In this limit the heterotic string moduli space factorises in the product \[181\] form \(\mathcal{M}_{\text{het}} = \mathcal{M}_{\text{IIA}} \times \mathcal{M}_{\text{IIB}}\), where the moduli spaces for the type IIA and IIB represent vector multiplets. The effective quantum field theory must satisfy several constraints. In particular, the Wilsonian prepotential of an \(N = 2\) supersymmetric theory must be a holomorphic function and expanding,
\[
K(\Phi, \bar{\Phi}, U, \bar{U}) = K(\Phi, \bar{\Phi}) + \sum_{ab} Z_{ab}(\Phi, \bar{\Phi}) U^a \bar{U}^b + \ldots, \tag{4.44}
\]

with
\[
K(\Phi, \bar{\Phi}) = -\log \left( 2(h + \bar{h}) - \sum_{a} (\Phi^a + \bar{\Phi}^a)(\partial h_a + \bar{\partial h}_a) \right) \tag{4.45}
\]
and
\[
Z_{ab}(\Phi, \bar{\Phi}) = 4e^{K(\Phi, \bar{\Phi})} \operatorname{Re} f_{ab}(\Phi). \tag{4.46}
\]

The Wilsonian gauge couplings follow from eqn.\((4.34)\). In addition the vector superpartners of \(\Phi\) do not mix with the graviphoton and hence the Wilsonian gauge couplings are simply \((g_{ab}^{-2})^W = \operatorname{Re} f_{ab}(\Phi)\) and for non-Abelian gauge fields \((g_{(a)}^{-2})^W = \operatorname{Re} f_{(a)}(\Phi)\). In contrast, the vector superpartners of the moduli mix with the graviphoton and as a result the Wilsonian gauge couplings \((g_{ab}^{-2})^W, (g_{a0}^{-2})^W\) and \((g_{00}^{-2})^W\) exhibit explicit non-holomorphic function moduli dependence. The complete result for the Wilsonian prepotential of \(N = 2\) theories, gives that it
is only renormalized only up to one loop order of perturbation theory on analogy with the rigid case. Thus,

$$\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \mathcal{F}^{(NP)},$$

where $\mathcal{F}^{(0)}$ represents the tree level prepotential, $\mathcal{F}^{(1)}$ is the one loop correction while $\mathcal{F}^{(NP)}$ receives corrections from world-sheet instantons and other non-perturbative effects. The perturbative one-loop correction to the prepotential of vector multiplets in decomposable and non-decomposable orbifold constructions of the heterotic string will be calculated later. Note, that the one loop correction to the prepotential of the vector multiplets has been calculated before, indirectly via its third derivative, in [172, 173]. Analytically,

$$h(\Phi) = h^{(0)}(\Phi) + h^{(1)}(\Phi), f_{ab} = f^{(0)}_{ab}(\Phi) + f^{(1)}_{ab}(\Phi),$$

and for the non-Abelian gauge group factors involved in the theory the Wilsonian gauge couplings read

$$(g^{-2}_{(a)})^W_a = \text{Re} f^{(0)}_{ab}(\Phi) + \text{Re} f^{(1)}_{ab}(\Phi).$$

Renormalization is up to one loop order, as it happens in the $N = 1$ Wilsonian couplings of effective field theories. In Calabi-Yau manifolds, special geometry is associated with the description of their moduli spaces. We will give more details at the end of the next section.

### 4.3 Low energy Effective theory of N=2 Heterotic superstrings and related issues

In this section, we will describe the low energy theory of $N = 2$ symmetric orbifold compactified heterotic superstrings. In addition, we will describe properties of the effective theory of type II superstrings compactified on a Calabi-Yau three fold. For heterotic strings compactified on a six dimensional orbifold, we consider the case where the internal torus lattice action correspond to the topus decomposition $T^4 \oplus T^2$. The moduli space of the torus is parametrized by the usual moduli $T$ and $U$. These moduli are part of the vector multiplet moduli space. Properties concerning the moduli space of such theories have already been discussed in section 4.2. Theories, with the same structure including e.g $N=2$ orbifold compactifications of the heterotic strings and $N=2$ heterotic string compactified on the $K_3 \times T^2$. The low-energy theory describing any classical $(2,2)$ vacuum includes the gravitational sector, containing the graviton, dilaton the
axion and the superpartners, together with the $E_8 \otimes E_8$ gauge multiplets and a set of chiral superfields which constitute the $27, \bar{27}$ representations of $E_6$ matter fields. In addition, world sheet supersymmetry demands that each $27, \bar{27}$ supermultiplet of matter fields is accompanied by an $E_6$ singlet moduli superfield, representing the moduli $^{27}$. These moduli in the case of Calabi-Yau threefolds correspond to the deformation parameters of the Kähler and complex structure. Note, that $(2, 2)$ symmetric orbifold compactifications of the heterotic string flow to their Calabi-Yau counterparts, after blowing up the twisted moduli scalars$^{[22, 30]}$ associated with the fixed points of the orbifold. Twisted moduli are not neutral with respect to the gauge group of the $(2, 2)$ theory and will not considered here, as they will not be involved in our discussions. The Kähler function $K$ characterizing the general heterotic $(2, 2)$ compactifications has the following power expansion$^{[22]}$ in the matter fields $K = \Sigma + \ldots$, where $\ldots$ represent a power expansion in terms of matter fields and $\Sigma$ has a block diagonal structure in $(1, 1)$ and $(2, 1)$ moduli, i.e, $\Sigma = \Sigma^{(1,1)} + \Sigma^{(2,1)}$ The neutral moduli of heterotic string compactifications are coordinates in a manifold with real dimension $2(h^{(1,1)} + h^{(2,1)} + 1)$. The additional complex dimension refers to the dilaton axion system. In reality, in all heterotic $(2, 2)$ compactifications the moduli spaces for the $(1, 1)$ and $(2, 1)$ moduli spaces are special Kähler spaces and the Kähler potential must be treated using the language of special geometry.

The axion is subject to the discrete Peccei-Quinn symmetry to all orders of perturbation theory. Since the axion is connected through a duality transformation to the antisymmetric tensor field, whose vertex operator decouples at zero momentum, this means that every physical amplitude involving $B_{\mu\nu}$ at zero momentum is zero. As a result the effective theory of the heterotic superstring is independent of the field $B_{\mu\nu}$ at zero momentum and the coupling of field appear only through its derivative. The dilaton and the axion belong to a vector multiplet. Since the axion couples to the dilaton D via the complex scalar $S$, which we will refer next as the dilaton, we conclude that any dependence of holomorphic quantities, e.g the Wilsonian gauge couplings, will be through the combination $S + \bar{S}$. However, these arguments$^{[22]}$ are not valid non-perturbatively. The structure of the heterotic vector multiplet moduli space is given by the

\footnote{They are the highest components of chiral primary fields of the left moving superalgebra.}
\footnote{Related discussion related to the expected corrections to the holomorphic superpotential will be discussed in chapter five.}
coset manifold based on the symmetric space

\[
\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2, n-1)}{SO(2) \times SO(n-1)}.
\]

The first factor corresponds to the dilaton. The prepotential for this space reads

\[
F(X) = -\frac{X^1}{X^0} X^2 X^3 - \sum_{l=4}^{n} (X'^l)^2.
\]

while the values of the moduli are identified as

\[
S = -i \frac{X^1}{X^0}, \quad T = -i \frac{X^2}{X^0}, \quad U = -i \frac{X^3}{X^0}, \quad \phi^i = -i \frac{X^{i+3}}{X^0}, \quad (i = 1, \ldots, P),
\]

with the remaining \(X'^l, C^a = -i X^{a+P+3}/X^0, a = p+4, \ldots, n\) to correspond to matter scalars. From the values of the moduli previously given, it follows that the the Kähler potential is

\[
K = -\log \left( (S + \bar{S})(T + \bar{T})(U + \bar{U}) - \sum_i (\phi^i + \bar{\phi}^i)^2 - \sum_a (C^a + \bar{C}^a)^2 \right)
\]

while we get in terms from quantities defined previously that

\[
h(o) = -S \left( TU - \sum_i (\phi^i)^2 \right), \quad f^{(o)} = S, \quad K = -\log(S + \bar{S}) - \log((T + \bar{T})(U + \bar{U})
\]

\[
- \sum_i (\phi^i + \bar{\phi}^i)^2, \quad Z = \frac{2}{(T + \bar{T})(U + \bar{U}) - \sum_i (\phi^i + \bar{\phi}^i)^2}.
\]

Especially for the non-Abelian factors in the gauge group \(G\) (or more generally any non-moduli vector multiplets) the tree-level gauge coupling is universal. In the language of special geometry, comparing (4.54,4.49) we conclude that dilaton’s vacuum expectation value, \(g^{2}_{(a)} = \text{Re}S\).

If we examine the various couplings for the vector superpartners of the moduli we see that the couplings involving the coupling of the dilaton with itself and the moduli \(T, U\) or the graviphoton are not become weak in the large dilaton limit as they should be. This is a sign that we are using a wrong symplectic basis. By changing to an other symplectic basis, e.g replacing the \(F_{\mu}^{\nu}\) with its dual field strength, we find that the couplings are now weakly coupled in the large-dilaton limit. In this way, we are using a basis and \((X^l, F_I) \rightarrow (\tilde{X}^l, \tilde{F}_I)\) where

\[
\tilde{X}^l = X^l \text{ for } I \neq 1, \quad \tilde{X}^1 = F_1, \quad \tilde{F}_I = F_I \text{ for } I \neq 1, \quad \tilde{F}_1 = -X^1
\]
and the components of the symplectic matrix $\tilde{O}$ are defined as
\[
\begin{pmatrix}
\hat{X}^I \\
\hat{F}_J
\end{pmatrix} = \tilde{O} \begin{pmatrix}
X^K \\
F_L
\end{pmatrix}.
\] (4.56)

The elements of $\tilde{O}$ are as in (4.39) and obey
\[
U^I_J = V^J_I = \delta^I_J \text{ for } I, J \neq 1, Z^{11} = 1, W_{11} = -1
\] (4.57)

In this new basis, the prepotential does not exist, since in the new basis the matrix $S^I_J$ has zero determinant and the definition of a prepotential is meaningless [227]. In the transformed basis, the Kähler potential for the moduli and the gauge couplings are found to be [172]
\[
\hat{K}_\Phi = K_\Phi = -\log(S + \check{S}) - \log(2(\check{z}^J \eta_{JI} \check{z}^I)), \quad \hat{N}_{IJ} = -2i\check{S}\eta_{IJ} + 2(S + \check{S}) \times \frac{\eta_{IK} \eta_{JL} (\check{z}^K \check{z}^L + \check{z}^K \check{z}^L)}{\check{z}^K \eta_{KL} \check{z}^L}.
\] (4.58)

In the transformed basis the couplings behave strongly in the small dilaton limit. In this limit, the target space dualities of $N = 2$ heterotic string vacua leave the classical lagrangian invariant, under transformations when $\hat{W} = \hat{Z} = 0$ and $\hat{U}, \hat{V} \in SO(2,2 + P)$. In fact, it is clear that the Kähler potential is invariant under symplectic transformations which act on the $(X^I, F_I)$. Moreover, in the absence of the one loop correction to the prepotential, we can use the $PSL(2, Z)_T$ target space duality symmetry subgroup of the full symmetry group of toroidal compactifications to study the transformation behaviour of the period vectors of special geometry. In sum, in the symplectic basis $\hat{X}^I, \hat{X}_J$, we get that the corresponding symplectic matrices are given by
\[
\hat{U} = \begin{pmatrix}
d & 0 & c & 0 & 0 \\
0 & a & 0 & -b & 0 \\
b & 0 & a & 0 & 0 \\
0 & -c & 0 & d & 0 \\
0 & 0 & 0 & 0 & 1_P
\end{pmatrix}, \quad \hat{V} = (\hat{U}^T)^{-1} = \begin{pmatrix}
a & 0 & -b & 0 & 0 \\
0 & d & 0 & c & 0 \\
-c & 0 & d & 0 & 0 \\
0 & b & 0 & a & 0 \\
0 & 0 & 0 & 0 & 1_P
\end{pmatrix},
\] (4.59)

while $\hat{W} = \hat{Z} = 0$. Especially under the generator $g_1 : T \rightarrow \frac{T}{T + 1} \in \Gamma_3^3$, we get that $\hat{U}$ is defined as follows
\[
g_1 : \hat{U} = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}, \quad V = (\hat{U}^T)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (4.60)
In a similar way we can derive the matrices corresponding to the generators for $\Gamma^o_3$, $g_2 : T \to T+1$. Considerations involving the calculation of the full quantum duality group $\Gamma^o(3)$ will not included here.

In Calabi-Yau manifolds, special geometry is associated with the description of their moduli spaces. In type IIB, the $H^{2,1}$ cohomology describes the deformation of the complex structure of the Calabi-Yau space $\mathcal{M}$. Now the Kähler metric for the $(2,1)$ moduli is defined from the Weyl Peterson metric $\sigma_{ij}$, namely

$$G_{ij} = \sigma_{ij} / (i(\int_\mathcal{M} \Omega \wedge \bar{\Omega})), \quad (4.62)$$

where

$$\varphi_i = (1/2)\varphi_{ik\lambda\rho}dx^kdx^\lambda dx^\rho, \quad \sigma_{ij} = \int_\mathcal{M} \varphi_i \wedge \bar{\varphi}_j$$

and $\varphi_{ik\lambda\rho} = (\partial g_{\beta\xi}/\partial t^i)\Omega^\xi_{k\lambda}$. Here, $t_i = 1, \ldots, b_{2,1}$ and $G_{ij} = -\partial_i \partial_j (i \int_\mathcal{M} \Omega \wedge \bar{\Omega})$. The three form tensor $\Omega$ is given in terms of the holology basis $\alpha, \beta$ as $\Omega = X^I \alpha_I + i F_I \beta^I$. The complex structure is described by the periods of the holomorphic three form $\Omega$ over the canonical homology basis. Here, the periods are given by $X^I = \int_{A^I} \Omega$, $i F_I = \int_{B^I} \Omega$ the integral of the holomorphic three form over the homology basis. The Kähler potential comes from the moduli metric

$$G_{ij} = -i \partial_i \bar{\partial}_j \{i \int \Omega \wedge \bar{\Omega}\}, \quad K = -\log(X^I \bar{F}_I + \bar{X}^I F_I). \quad (4.64)$$

Now the Riemann tensor is defined as

$$R_{ijkl} = G_{ij} G_{kl} - \bar{C}_{ijk} G^{\bar{m}n} e^{2K}, \quad (4.65)$$

where the expression of the Yukawa couplings in a general coordinate system are given by $C = f \Omega \wedge \partial_i \partial_j \partial_k \Omega$, $\partial_i = \partial / \partial z^i$. The holomorphic function $F$ does not receive quantum corrections from world-sheet instantons and as a consequence neither the the Kähler potential derived from it. Calabi-Yau threefolds can be constructed among other ways as a hypersurface or as a complete intersection of hypersurfaces in a weighted projective space $P^N(\vec{w})$. Remember, that the complex projective space $CP^N$ is the space defined by the homogeneous complex coordinates $Z_1, \ldots, Z_{N+1}$ which obey $(Z_1, \ldots, Z_{N+1})^{\lambda \neq 0} \equiv (\lambda^{w_1} Z_1, \ldots, \lambda^{w_{N+1}} Z_{N+1})$ for complex $\lambda$. The threefold is obtained

\[\text{In the rest of the section the notation for the special coordinates is as follows, } Z^i = -iX^i/X^o.\]
from the $CP^4$, the quintic with the general equation $\sum_{k_i} \alpha_{k_1k_2k_3k_4k_5} x_{k_1} x_{k_2} x_{k_3} x_{k_4} x_{k_5} = 0$ while the $K_3$ can be obtained from the $\sum_{k_i} \alpha_{k_1k_2k_3k_4} x_{k_1} x_{k_2} x_{k_3} x_{k_4} = 0$, in projective $P^3$ and $P^2$ respectively(rp). They describe complex manifolds parametrized by 135 and 35 complex coefficients $a_{k_i}$ rp, which after removing an overall redundancy they give 101, 19 elements of $H^{(1,1)}$ rp.

The weighted projective space $P^N(\vec{w})$ is defined by the conditions on the homogeneous coordinates $(Z_1, \ldots, Z_{N+1}) \equiv (\lambda^d_1, \ldots, \lambda^d_{N+1})$ and $P^N(\lambda^d_1Z_1, \ldots, \lambda^d_{N+1}Z_{N+1}) \equiv \{e^*= C^{N+1}/(Z_1 = 0, \ldots, Z_{N+1} = 0)\}$. The last condition excludes the origin of the complex space. The $d_i$ are the weights and the sum of the weights is the degree of the variety.

4.4 * One loop correction to the prepotential from string amplitudes

4.4.1 One loop contribution to the Kähler metric - Preliminaries

The one-loop Kähler metric for orbifold compactifications of the heterotic string, where the internal six torus decomposes into $T^2 \oplus T^4$, was calculated in [56]. In this section, we will use the general form of the solution for the one loop Kähler metric appearing in [56, 173] to calculate the one loop correction to the prepotential of $N=2$ orbifold compactifications of the heterotic string.

While the one loop prepotential has been calculated with the use of string amplitudes in [173], in my Thesis I will provide an alternative way of calculating the one-loop correction to the prepotential of the vector multiplets of the $N=2$ orbifold compactifications of the heterotic string. Note that in the following we will change notation, following the spirit of the calculation in [173], namely all moduli fields, including the dilaton, are rescaled by a factor of $i$, $P \rightarrow iP$.

In this section, we will describe the background theory of the one-loop contribution to the Kähler metric. For this purpose, we will use not the standard supergravity lagrangian up to two derivatives in the bosonic fields, described by the superconformal action formula

$$e^{-1}\mathcal{L} = -\frac{3}{2}[S_o S_o e^{-\frac{i}{4}G(Z,\bar{Z})}]_D + \left( [S_o^3]_F + h.c \right) + \frac{1}{4} \left( [f_{ab}(Z)W^a W^b]_f + h.c \right)$$

with matter described by chiral multiplets $Z_i$ only. Instead, we will use the linear multiplet formulation [84, 217, 56]. Note that both formulations are equivalent, since the linear multiplet can always be transformed in to a chiral multiplet by a supersymmetric duality transformation. In eqn. (4.66), $S_o$ is the chiral compensator field, and $G(Z,\bar{Z}) = K(Z,\bar{Z}) + \log |w(Z)|^2$ the gauge kinetic function, where $K$ is the Kähler potential and $w$ the holomorphic superpotential.
In addition, \( W^a \) is the chiral spinor superfield of the Yang-Mills field strength \( F^a_{\mu\nu} \), and \( D, F \) subscripts refer to the vector density and chiral density in superspace.

In the superconformal formalism\cite{218}, the action for the linear multiplet is given up to one loop order by

\[
\mathcal{L} = -(S_o \bar{S}_o)^{3/2} \left( \frac{\hat{L}}{2} \right)^2 e^{-G^{(o)}} + \left( \frac{\hat{L}}{2} \right) G^{(1)} + (S^3 w)_F
\]

where now the gauge kinetic function is given by \( G^{(o)}(z, \bar{z}) + lG^{(1)}(z, \bar{z}) \). The vev of \( l \) is the four dimensional gauge coupling constant \( g^2 \).

Eqn.(4.67) does not have the gravitational kinetic energy \( \propto R \) term to its canonical form. Instead, the chiral compensator field is used to properly normalising its coefficient, procedure which fixes the value of the compensator field. The advantage of using the linear multiplet instead of the chiral multiplet in eqn.(4.66) is that it provides an easy way of calculating\cite{56} one loop corrections to the Kähler metric. An easy way to see this comes from the following equation\cite{123}, which includes the bosonic kinetic energy terms,

\[
\mathcal{L}_{\text{bosonic}} = -\frac{1}{4l^2} \partial_{\mu} l \partial^\mu l + \frac{1}{4l^2} h^\mu h_\mu - G_{ij} \partial_\mu z^i \partial^\mu \bar{z}^j \left( \frac{i}{2} \right) (G_{ij} \partial z^j - G_{ij} \partial_\mu \bar{z}^j) h^\mu.
\]

The last term in eqn.(4.68) reveals that the one loop correction to the Kähler metric \( G_{z,\bar{z}} \) will come by calculating the CP-odd part of the amplitude between the complex scalars and the antisymmetric tensor \( b^{\mu \nu} \)

\[
< z(p_1) \bar{z}(p_2) b^{\mu \nu}(p_3) >_{\text{odd}} = i \varepsilon^{\mu \nu \rho \lambda} p_{1 \rho} p_{2 \lambda} G^{(1)}_{z,\bar{z}}.
\]

Here, \( G \) is the Kähler metric and \( h^\mu = \frac{1}{2} \varepsilon^{\mu \nu \lambda \rho} \partial_{\nu} b_{\lambda \rho} \) is the dual field strength of the antisymmetric tensor field \( b_{\lambda \rho} \).

The amplitude receives contributions only from \( N=2 \) sectors. We are not considering contributions to the Kähler metric which arise from \( N = 1 \) sectors, since these contributions arise only in \( N=1 \) orbifold compactifications of the heterotic string. Here, we are only interested in the geometry underlying the \( N = 2 \) sectors.

Lets us suppose that the internal six dimensional lattice decomposes into \( T^2 \oplus T^4 \), with the \( T^2 \) inside the unrotated plane. Compactifications of the heterotic string in four dimensions with \( N = 2 \) supersymmetry involve a \( U(1) \times U(1) \) gauge group from the untwisted \( T^2 \) unrotated.
subspace. This plane is parametrized in terms of moduli T, U. For special points in the moduli space, namely the $T = U$ line the gauge group becomes enhanced to $SU(2) \times U(1)$. It can become enhanced to $SO(4)$ or $SU(3)$ along the $T = U = i$ or $T = U = e^{2\pi i/3}$ lines respectively.

In this subspace of the Narain moduli space, we will be interested mostly, to calculate the moduli dependence of the one loop correction to the prepotential. Denote the untwisted moduli from a $N = 2$ sector by P, where P can be the T or U moduli parametrizing the two dimensional unrotated plane. Then the one loop contribution to the Kähler metric is given by

$$G^{(1)}_{P \bar{P}} = \frac{1}{(P + P)^2} \mathcal{I}, \quad \mathcal{I} = \int d^2 \tau \frac{\partial \tau}{\tau^2} \partial_{\bar{\tau}} (\tau Z) F(\tau).$$

(4.70)

Here, the integral is over the fundamental domain, and the factor $\frac{1}{(P + P)^2}$ is the tree level moduli metric $G^{(0)}_{P \bar{P}}$. $Z$ is the partition function of the fixed torus

$$Z = \sum_{(P_L, P_R) \in \Gamma_{2,2}} q^{P_L^2/2} \bar{q}^{P_R^2/2}, \quad q \equiv e^{2\pi \tau}, \tau = \tau_1 + \tau_2,$$

(4.71)

and $P_L, P_R$ are the left and right moving momenta associated with this plane. $F(\tau)$ is a moduli independent meromorphic form of weight $-2$ with a single pole at infinity due to the tachyon at the bosonic sector. The function $F$ was fixed in [173] to be

$$F(\tau) = -(1/\pi) \frac{j(\tau)[j(\tau) - j(i)]}{j_{\tau}(\tau)}, \quad j_{\tau} \overset{\text{def}}{=} \frac{\partial j(\tau)}{\partial \tau},$$

(4.72)

where $j$ the modular function for the group $SL(2, Z)$.

4.4.2 * Prepotential of vector multiplets/Kähler metric

For the calculation of the prepotential of the vector multiplets we will will follow the approach of [173]. Recalling the general form of the prepotential eqn. (4.43)

$$K = -\ln(iY), \quad F = STU + f(T, U).$$

(4.73)

The lagrangian (4.67) may be related to the chiral multiplet one (4.66), by a duality transformation. We introduce the dilaton S as a Lagrange multiplier into (4.67), e.g $(\mathcal{L} - L(S + \bar{S})/4)_D$. Using the equation of motion for S we get

$$(\mathcal{L} - L \partial \mathcal{L})_D \equiv -\frac{3}{2} S_{\alpha} S_{\dot{\alpha}} e^{-\mathcal{K}}.$$ 

(4.74)

\[124\] A function f is meromorphic at a point A if the function h, $h(z) \overset{\text{def}}{=} (z - A)f(z)$ is holomorphic (differentiable) at the point A. In general, this means that the function h is allowed to have poles.
In this form the Kähler potential has an expansion as

\[ K = - \ln\{(S - \bar{S}) - 2G^{(1)}\} + G^{(o)} \]  \hspace{1cm} (4.75)

Expanding \[ K^{(1)} \]

\[ K^{(1)}_{PP} = \frac{2i}{(S - \bar{S})^{G^{(1)}_{PP}}}, \quad G^{(1)}_{TT} = \frac{i}{2(T - \bar{T})^2} \left( \partial_T - \frac{2}{T - \bar{T}} \right) \left( \partial_U - \frac{2}{U - \bar{U}} \right) f + c.c. \]  \hspace{1cm} (4.76)

Using the equations for the momenta

\[ p_L = \frac{1}{\sqrt{2ImTImU}}(m_1 + m_2 \bar{U} + n_1 \bar{T} + n_2 \bar{U} \bar{T}), \quad p_R = \frac{1}{\sqrt{2ImTImU}}(m_1 + m_2 \bar{U} + n_1 T + n_2 T \bar{U}) \]  \hspace{1cm} (4.77)

we can prove that \[ I \] satisfies\[ ] the following differential equation

\[ \left\{ \partial_T \partial_T + \frac{2}{(T - \bar{T})^2} \right\} I = - \frac{4}{(T - \bar{T})^2} \int d^2\tau F(\tau) \partial_\tau (\partial^2_\tau + \frac{i}{\tau_2} \partial_\tau)(\tau_2 \sum_{PL,PR} q^{P_L^2/2} \bar{q}^{P_R^2/2}). \]  \hspace{1cm} (4.78)

The integral representation of eqn.(4.78) is a total derivative with respect to \( \tau \) and thus zero. However, the integral can give non-vanishing contributions at the enhanced symmetry points \( T=U \). Solving (4.78) away of the enhanced symmetry points gives

\[ \left\{ \partial_T \partial_T + \frac{2}{(T - \bar{T})^2} \right\} I = \left\{ \partial_U \partial_U + \frac{2}{(U - \bar{U})^2} \right\} I = 0. \]  \hspace{1cm} (4.79)

The singularity structure of (4.79) at the enhanced symmetry point\[^{125}\] will be taken into consideration later in its integral representation. The general solution of the (4.77) is\[^{173}\]

\[ I = \frac{1}{2i} \left( \partial_T - \frac{2}{(T - \bar{T})} \right) \left\{ (\partial_U - \frac{2}{U - \bar{U}}) f(T, U) + (\partial_U + \frac{2}{U - \bar{U}}) \bar{f}(T, \bar{U}) \right\} + c.c. \]  \hspace{1cm} (4.80)

It can be shown\[^{173}\] that \( \bar{f} \) is zero. Note that \( f \) represents the one-loop correction to the prepotential of the vector multiplets \( T, U \) and determines via eqn.(4.76) the one loop correction to the Kähler metric for the \( T, U \) moduli. In\[^{173}\] it was shown function \( f(T, U) \) of (4.80) satisfies the differential equation

\[ -i(U - \bar{U})D_T \partial_U f = \partial^3_T f, \]  \hspace{1cm} (4.81)

\[^{125}\]Enhanced symmetry point behaviour at a general point in the moduli space has been examined in chapter three. Direct application to the momenta of eqn’s.(4.77), shows that they correspond to the lattice points \( m_2 = -n_1 = \pm 1, m_1 = n_2 = 0 \) and gauge group enhancement from \( U(1) \times U(1) \rightarrow U(1) \times SU(2) \).
where $D_T = \partial_T + \frac{2}{(T-T)}$ is the covariant derivative. Expansion of the l.h.s and integration by parts results in

$$f_{TTT} = 4\pi^2 \frac{U - \bar{U}}{(T - \bar{T})^2} \int d^2 \tau F(\tau) \sum_{P_L, P_R} P_L \bar{P}_R^2 q^{P_L^2/2} q^{P_R^2/2}. \tag{4.82}$$

Examination of the behaviour of the r.h.s of eqn. (4.82) under separately modular transformations $SL(2, Z)_T$, $SL(2, Z)_U$, together with examination of its singularity structure at the enhanced symmetry point $T = U$, uniquely determines the well known solution of the third derivative of the vector multiplet prepotential. Remember that we examine the behaviour of the prepotential including the region of the moduli space where we have gauge symmetry enhancement to $U(1) \times SU(2)$.

For $N=2$ heterotic strings compactified on decomposable orbifolds

$$f_{TTT} = -\frac{2i}{\pi} j_T(T) \left\{ \frac{j(U)}{j(T)} \right\} \left\{ \frac{j_T(T)}{j_U(U)} \right\} \left\{ \frac{j(U) - j(i)}{j(T) - j(i)} \right\}. \tag{4.83}$$

In [172, 173] $f_{TTT}$ was determined by the property of behaving as a meromorphic modular form of weight 4 in $T$. In addition, $f_{TTT}$ had to vanish at the order 2 fixed point $U = i$ and the order 3 fixed point $U = \rho$ of the modular group $SL(2, Z)$. Moreover, it had to transform with modular weight $-2$ in $U$ under $SL(2, Z)_U$ transformations and exhibit a singularity at the $T = U$ line. The prepotential function for the $f_{UUU}$ is obtained by the replacement $T \leftrightarrow U$.

Here, we find the the equation for $f_{UUU}$. In simple form

$$\partial_U^3 f = f_{UUU} = -i(T - \bar{T})^2 \partial_T D_U \partial_U I, \tag{4.84}$$

where $D_U = \partial_U + \frac{2}{U - \bar{U}}$, the covariant derivative with respect to $U$ variable, transforms with modular weight 2 under $SL(2, Z)_U$ modular transformations, namely

$$U^{SL(2, Z)_U} \rightarrow \frac{aU + b}{cU + d}, \quad D_U \rightarrow (cT + d)^2 D_U. \tag{4.85}$$

We should notice here, that because of the symmetry exchange $T \rightarrow U$, the result for $f_{UUU}$ comes directly from (4.83), by the replacement $T \rightarrow U$. However, this can be confirmed by the solution of (4.84). In addition, we will find the differential equation for the function $f$. The calculation of the prepotential $f$ comes from the identity

$$(f)_{proj} = 2i(T - \bar{T})^2 (U - \bar{U})^2 \partial_U \partial_T I. \tag{4.86}$$
Explicitly, 
\[
(f)_{\text{proj}} = 2i(T - \bar{T})^2(U - \bar{U})^2 \partial_U \partial_T \int F d^2 \tau \frac{d^2 \tau}{\tau_2} \partial_\tau (\tau_2 Z) F(\bar{\tau}).
\] (4.87)
As we can see the one loop correction to the holomorphic prepotential comes by taking derivatives of \(I\) with respect to the conjugate moduli variables from which the holomorphic prepotential does not have any dependence. The holomorphic prepotential is defined projectively, by taking the action of the conjugate moduli derivatives on the holomorphic part of the one loop Kähler metric integral \(I\). In this way, we always produce the differential equation for the \(f\) function from the string amplitude. In addition, the solution of this equation calculates the one loop correction to the Kähler metric. Using now, the modular transformations of the momenta
\[
(P_L, \bar{P}_R) \overset{SL(2, \mathbb{Z})_T}{\rightarrow} \left( \frac{cT + d}{cT + d} \right)^{\frac{1}{2}} (P_L, \bar{P}_R), \quad (P_L, \bar{P}_R) \overset{SL(2, \mathbb{Z})_U}{\rightarrow} \left( \frac{cU + d}{cU + d} \right)^{\frac{1}{2}} (P_L, \bar{P}_R),
\] (4.88)
we can easily see that the one loop prepotential has the correct modular properties, it transforms with modular weight \(-2\) in \(T\) and \(-2\) in \(U\). Eqn. (4.86) is the differential equation that the one loop prepotential satisfies. The solution of this equation determines the one loop correction to the Kähler metric and the Kähler potential for \(N = 2\) orbifold compactifications of the heterotic string. Compactifications of the heterotic string on \(K_3 \times T_2\), appears to have the same moduli dependence on \(T\) and \(U\) moduli, for particular classes of models\([173, 148, 184, 182]\). Formally, the same routine procedure, namely taking the derivatives with respect to the conjugate \(T\) and \(U\) moduli on \(I\), can be applied to any heterotic string amplitude between two moduli scalars and antisymmetric tensor, in order to isolate from the general solution (4.86) the term \(f(T, U)\). The solution for \(f_{TTT}\) in eqn. (4.83) was derived for \(N = 2\) compactification of the heterotic strings in \([173]\) via the modular properties of the one loop prepotential coming from the study of its integral representation (4.82). Specific application for the model based on the orbifold limit of \(K_3\), namely \(T^4/\mathbb{Z}_2\), was given in \([130]\). At the orbifold limit of \(K_3\) compactification of the heterotic string the Narain lattice was decomposed into the form \(\Gamma^{22,6} = \Gamma^{2,2} \oplus \Gamma^{4,4} \oplus \Gamma^{16,0}\). It was modded by a \(\mathbb{Z}_2\) twist on the \(T^4\) part together with a \(\mathbb{Z}_2\) shift \(\delta\) on the \(\Gamma^{(2,2)}\) lattice. For reasons of level matching \(\delta^2\) was chosen to be 1/2. By an explicit string loop calculation via the one loop gauge couplings in \([130]\), from where the one loop prepotential was extracted with an ansatz, they were able to calculate the third derivative of the prepotential. It was found to agree with the result of \([172, 173]\) which was calculated for the S-T-U subspace of the vector multiplets of the orbifold compactification of the heterotic string.
In reality, $F(\tau)$ is the trace of $F(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{c}{24}} / \eta(\tau)$ over the Ramond sector boundary conditions of the remaining superconformal blocks. For the S-T-U model with instanton embedding $(d_1, d_2) = (0, 24)$ the supersymmetric index was calculated in [130] in the form

$$\frac{1}{\eta^2} Tr_R F(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{c}{24}} = -2i \frac{E_4 E_6}{\Delta},$$

where $F$ is the right moving fermion number. Expanding $I$ we get that

$$I = \left( -i \pi \right) \int \frac{d^2 \tau}{\tau_2} \left( \frac{\tau^2}{2\pi \tau_2} \right) \bar{F}(\bar{\tau}).$$

Specific tests of dual pairs were performed, in the spirit of [130], in [200, 201, 202, 203, 204]. Particular examples of calculating the prepotential for dual pairs will not be performed here.

We have said that one important aspect of the expected duality is that the vector moduli space of the heterotic string must coincide at the non-perturbative level with the tree level exact vector moduli space of the type IIA theory. For the type IIA superstring compactified on a Calabi-Yau space $X$ the internal $(2, 2)$ moduli space, $N = 2$ world-sheet supersymmetry for the left and the right movers, is described, at the large complex structure limit of $X$, by the Kähler moduli, namely $B + iJ \in H^2(X, C)$, where $B + iJ = \sum_{i=1}^{h} (1, 1)(B + iJ)_a e_a$ with $B_a$, $J_a$ real numbers and $t_a = (B + iJ)_a$ representing the special coordinates and $e_a$ a basis of $H^2(X, C)$. In the content of the moduli of the Calabi-Yau space of $X$, the holomorphic prepotential at the large radius limit takes the form

$$F = -i \sum_{\alpha, \beta, \gamma} (D_\alpha \cdot D_\alpha \cdot D_\gamma) t_\alpha t_\beta t_\gamma - \frac{\zeta(3)}{2(2\pi)^3} \sum_{(d_i)_{i=1,...,n}} n_{d_1,...,d_n} L i_3(\Pi_{i=1}^{n} q_{d_i}^3),$$

where $L i_3(x) = \sum_{j \geq 1} \frac{x^j}{j^3}$. The first term in eqn. (4.91) is a product of the the Calabi-Yau divisors $D$, associated to the basis $e_a$, and the second term represents world-sheet instanton contributions. The $n_{d_1,...,d_n}$ are the world sheet instanton numbers. Performing duality tests between a heterotic model and its possible type IIA dual is then equivalent to comparing the weak coupling limit of the prepotential of the vector multiplets for the heterotic string with the large radius limit of (4.91). After identifying the heterotic dilaton with one of the vector moduli of the

\textsuperscript{126}Let us consider the target space of a complex manifold $M$ with dimension $n$. Choose coordinates on $M$, $\phi_m$ and $\bar{\phi}_m$. Then $M$ admits a Kähler structure if we can define a $(1, 1)$ form $J$ with the property $J = i G_{i\bar{m}} \partial \phi_m \wedge \partial \bar{\phi}_l$ where for a Kähler manifold the metric is $G_{i\bar{m}} = \partial \phi_m \partial \phi_l K = \frac{\partial}{\partial \phi_m} \frac{\partial}{\partial \phi_l} K$, and the Kähler potential is $K$. 
type IIA model in the form \( t_s = (B + iJ)_s = 4\pi iS \), the type IIA prepotential takes the general form\[^{154}\]

\[
\mathcal{F}_{IIA} = -\frac{1}{6} C_{ABC} t^A t^B t^C - \frac{\chi(3)}{2(2\pi)^3} \sum_{d_1,\ldots,d_n} n_{d_1,\ldots,d_n} \mathcal{L}_{id} e^{2\pi i \sum_k d_k t^k},
\]

where we are working inside the Kähler cone\[^{45, 186}\] \( \sigma \overset{\text{def}}{=} \{ \sum \rho J_\rho | t_\rho > 0 \} \), where \( J_\rho \) generate the cohomology group \( H^2(X, \mathbb{R}) \) of the Calabi-Yau three fold \( X \). In a particular symplectic basis eqn. (4.92) can be brought in the form\[^{183}\]

\[
\mathcal{F}_{IIA} = -\frac{1}{6} C_{ABC} t^A t^B t^C + \sum_{A=1}^{h_{1,1}} \frac{c_2 \cdot J_A t^A}{24} + \ldots,
\]

where \( c_2 \cdot J_A = \int_X c_2 \wedge J_A \) is the expansion of the second Chern class of the Calabi-Yau three fold in terms of the basis \( J_A^* \) of the cohomology group \( H^4(X, \mathbb{R}) \). The cohomology group \( H^4(X, \mathbb{R}) \) is dual to the \( H^2(X, \mathbb{R}) \), namely \( \int_X J_A^* \wedge J_B = \delta_{AB} \). In \[^{184}\] it was noticed that the nature of type II-heterotic string duality test has to come from the \( K_3 \) fiber structure over \( P^1 \) of the type IIA side. The form of the \( K_3 \) fibration can be found\[^{184, 182}\] by taking for example the CY in \( P^4(1,1,2k_2,2k_3,2k_4) \) and then set \( t_o = \lambda x_1 \). After rescaling \( x_1 \rightarrow x_1^{1/2} \) we arrive at the equation for the hypersurface

\[
F(\lambda) Z_1^d + Z_2^{d/k_2} + \ldots = 0.
\]

The degree \( d = 1 + k_2 + k_3 + k_4 \). For generic values of \( \lambda \) eq. (4.94) is a \( K_3 \) surface in weighted \( P^3 \). So \( P^4(1,1,2k_2,2k_3,2k_4) \) is a \( K_3 \) fibration fibered over the \( P^1 \) base which is parametrized by \( \lambda \). At the large radius limit of \( X \), in (4.93), the heterotic dilaton \( S \) is identified as one of the vector multiplet variables as \( t_s = 4\pi i S \). Confirmation of duality between dual pairs is then equivalent to the identification\[^{146}\]

\[
\mathcal{F}_{IIA} = \mathcal{F}_{IIA}(t^s, t^i) + \mathcal{F}_{IIA}(t^i) = \mathcal{F}_\text{het}(S, \phi^I) + \mathcal{F}_\text{het}^{(1)}(\phi^I).
\]

Here, we have expand the prepotential of the type IIA in its large radius limit, namely large \( t_s \).

In the heterotic side, we have the tree level classical contribution as a function of the dilaton \( S \) and the vector multiplet moduli \( \Phi^I \), in addition to the one loop correction as a function of only the \( \Phi^I \). The general differential equation for the one loop correction to the heterotic prepotential \( \mathcal{F}_\text{het}^{(1)} \) was given before by eqn. (4.86). Summarizing, the existence of a type II dual to the weak coupling phase of the heterotic string is exactly the existence of the conditions\[^{189}\]

\[
D_{ass} = 0, \quad D_{ssi} = 0 \text{ for every } i, \ldots
\]
Moreover, from eqn.(4.54) we see that the tree level heterotic prepotential can be expanded in the form
\[ F^{(o)} = -S(\eta_{ij}M_i M_j - \delta_{ij}Q^i Q^j), \quad \eta_{ij} = \text{diag}(1, -1, \ldots, -1), \] (4.97)
from which we can infer that
\[ \text{sign}(D_{sij}) = (+, -, \ldots, -) = \text{sign}(\eta_{ij}). \] (4.98)

However, there is another condition which will be usefull. It originates from the higher derivative gravitational couplings of the heterotic vector multiplets and the Weyl multiplet of conformal \( N = 2 \) supergravity. The relevant couplings originate from terms in the form \( g_n^{-2} R^2 G^{2n-2} \), where \( R \) is the Riemann tensor, \( G \) the field strength of the graviphoton. The \( g_n \) couplings obey \( g_n^{-2} = \text{Re} \tilde{F}_n (S, M^i) + \ldots \). The same of couplings appear in type II superstring. In the heterotic side they take the form
\[ \tilde{F}_n = \tilde{F}^{(0)}(S, M_i) + \tilde{F}^1(M^i) + \tilde{F}^{NP}(e^{-8\pi^2 S}, M^i), \quad \tilde{F}_1 = 24S, \quad \tilde{F}^o_{n \geq 1} = \text{const}, \] (4.99)
where \( S \) is the heterotic dilaton and \( M^i \) the rest of the vector multiplets moduli. Such terms appear as well in the effective action of type II vacua and they have to match with heterotic one’s due to duality. In the large radius limit the higher derivative couplings satisfy (the lowest order coupling) \( \tilde{F}_1 \rightarrow -\frac{4\pi i}{18} \sum_a (D_a \cdot c_2) t_a \), where \( c_2 \) is the second Chern class of the three fold \( X \). Because at the tree level, \( g_1^2 = \text{Re} \tilde{F}_1 \) we can infer the result that
\[ D_a \cdot c_2 (X) = 24. \] (4.100)

The last condition represents the mathematical fact that the Calabi-Yau threefold \( X \) admits a fibration \( \Phi \) such as there is a map \( X \rightarrow W \), with the base being \( P^1 \) and generic fiber the \( K_3 \) surface. Furthermore, the area of the base of the fibration gives the heterotic four dimensional dilaton.

4.5 * One loop prepotential - perturbative aspects

Since we have finished our discussion of the general properties of the \( N = 2 \) heterotic strings, we will now discuss the calculation of the perturbative corrections to the one loop prepotential.
Let us expand at the moment the expression of eqn. (4.43) around small values of the non-moduli scalars $C_a$ as in (4.53) and (4.54)

$$F = -S(TU - \sum_i \phi^i \phi^i) + h^{(1)}(T, U, \phi^i), \quad f_{(a)} = S + f_{(a)}^{(1)}(T, U, \phi^i),$$

(4.101)

where the functions $h^{(1)}$ and $f_{(a)}^{(1)}$ enjoy a non-renormalization theorem, namely they receive perturbative corrections only up to one loop order. Its higher loop corrections in terms of the $1/(S + \bar{S})$ vanish, due to the surviving of the discrete Peccei-Quinn symmetry to all orders of perturbation theory as a quantum symmetry. For the same reason, $f_{(a)}^{(1)}$ receives corrections only up to one loop level. The one loop prepotential, if we expand it in the general form $F(X) = H^{(0)}(X) + H^{(1)}(X)$ with $H^{(1)}(X) = -i(X^0)^2 \Omega^{(1)}$ where the superscripts denote tree level and one loop corrections respectively gives us through relations related to the basis $X^I, F_J$ that the following relations are valid

$$\hat{F}_I = -2iS_{IJ} \hat{X}^J + H^{(1)}_I, H^{(1)}(X) = \frac{1}{2} \hat{F}_I \hat{X}^I$$

(4.102)

with

$$H^{(1)}_I = \partial H^{(1)}/\partial X^I$$

(4.103)

The loop corrections to the prepotential have to take into account the generation of the discrete shifts in the theta angles due to monodromies around semi-classical singularities in the quantum moduli space where previously massive states become massless. In this way, the classical transformation rules of (4.59) become modified to

$$\hat{X}^I \rightarrow \hat{U}^I_J \hat{F}_I \rightarrow \hat{V}^I_J + \hat{W}_{IJ} P^J$$

(4.104)

with

$$\hat{V} = \hat{U}^T, \quad \hat{W} = \hat{V} \Lambda, \quad \Lambda = \Lambda^T.$$  

(4.105)

and $\hat{U} \in SO(2, P + 2, Z)$. In the classical theory $\Lambda = 0$ but in the full quantum theory around a singularity, the closed monodromy gives rise to $\hat{F}_I \rightarrow \hat{F}_I + \Lambda_{IJ} \hat{F}_I$ and the transformation rule of the one loop prepotential becomes

$$H^{(1)}(\hat{X}) = H^{(1)}(X) + 1/2 \Lambda_{IJ} \hat{X}^I \hat{X}^J.$$  

(4.106)

As a consequence, the one loop prepotential changes by a quadratic polynomial in $T$ and $U$ when moving around a semi-semiclassical singularity. In the language of special geometry this
f(T, U) \rightarrow (i c T + d)^{-2} f(T, U) + \Pi(T, U).

(4.107)

A special aspect of the theory is related to the transformation rule of the dilaton. At the level of string theory the dilaton vertex has a fixed relation to the vector tensor multiplet and it is invariant under any symmetries of the string theory. However when we are discussing the vector multiplet which is dual to the vector tensor multiplet the dilaton is no longer invariant under the perturbative symmetries of string theory and is receiving perturbative corrections. It follows via the relations (4.103), (4.104) and the relation $X^1 = -\hat{F}_1 = -iS\hat{X}^a$ that the dilaton transforms as

$$S \rightarrow \tilde{S} = S + \frac{iV^J_I (H_J^{(1)} + \Lambda_{JK}\hat{X}^K)}{U^I_0\hat{X}^I}.$$  

(4.108)

But if we insist in keeping the dilaton invariant then we can define an 'invariant' dilaton as

$$S_{inv} = S + \frac{1}{2(P + 4)} \left[ i\eta^{IJ}H_{IJ}^{(1)} + L \right],$$

(4.109)

where $L$ obeys $L \rightarrow L - i\eta^{IJ}\Lambda_{IJ}$. We will find now the transformation properties of the non-moduli gauge couplings $f_{(a)}(\Phi)$. When we are discussing about the physical properties of a low energy theory, we have to distinguish about the momentum dependent physical gauge couplings and the Wilsonian gauge couplings. The effective gauge couplings account for all the quantum effects both at high and at low energy. As a result the low energy effects due to massless particles give rise to non-holomorphic moduli dependence of the effective gauge couplings and to all orders of perturbation theory it has been found [86] that

$$g_{a}^{-2}(\Phi, \bar{\Phi}, p^2) = Re f_{(a)}(\Phi) + \frac{b_a}{16\pi^2} (\log \frac{M_P^2}{p^2} + K_{a}(\Phi , \bar{\Phi})) + constant.$$  

(4.110)

Finally the Wilsonian couplings $f_{(a)}$ transform as

$$f(\Phi) \rightarrow f(\Phi) - \frac{b_{(a)}}{8\pi^2} \log (\hat{U}^J_0 \hat{X}^J / \hat{X}^0).$$

(4.111)

In that case, under target space duality

$$T \rightarrow \frac{aT - ib}{icT + d}, \quad U \rightarrow U,$$

$$h(T, U) \rightarrow \frac{h(T, U) + \Xi(T, U)}{(icT + d)^2}, \quad f_a(S, T, U) \rightarrow f_a(S, T, U) - \frac{b_a}{8\pi^2} \log(icT + d)$$  

(4.112)
and a similar set of transformations under $PSL(2, Z)_U$. The net result is that $\partial^3 h^{(1)}(T, U)$ is a single valued function of weight $-2$ under U-duality and $4$ under T-duality.

We turn now our previous discussion to the case of $N = 2$ orbifold compactifications of six dimensional heterotic string vacua. The one loop correction to the prepotential of vector multiplets for the subspace of the Narain lattice corresponding to the $T, U$ moduli of the decomposable $T^2$ torus has already been calculated in \([172, 173]\). In explicit form may be derived from eqn.(4.83). In this section of the Thesis we will discuss the calculation of the prepotential for the case where the moduli subspace of the Narain lattice associated with the $T, U$ moduli exhibits a modular symmetry group $\Gamma^o(3)_T \times \Gamma^o(3)_U$. The same modular symmetry group appears in the $N = 2$ sector of the $N = 1$ $(2, 2)$ symmetric non-decomposable $Z_6$ orbifold defined on the lattice $SU(3) \times SO(8)$. In the third complex plane associated with the square of the complex twist $(2, 1, -3)/6$ the mass operator for the untwisted subspace was given to be

$$m^2 = \sum_{m_1, m_2, n_1, n_2 \in Z} \frac{1}{2T^2 U^2} |TU'n_2 + Tn_1 - U'm_1 + 3m_2|^2. \quad (4.113)$$

Any $(2, 2)$ orbifold will flow to its Calabi-Yau limit after giving vacuum expectation values to its twisted ”moduli” scalars. In this limit, the corresponding Calabi-Yau phase exist. Let us forget the $N = 1$ orbifold nature of the appearance of this $N = 2$ sector. Then its low energy supergravity theory is described by the underlying special geometry. The question now is if calculating the prepotential using its modular properties and the singularity structure, as this was calculated for decomposable orbifold compactifications of the heterotic string \([172]\), there is a type II dual realization. We believe that it is the case. In the analysis of the map between type II and heterotic dual supersymmetric string theories \([207, 184]\) it was shown that subgroups of the modular group do appear. In particular in one modulus deformations of $K_3$ fibrations the modular symmetry groups appearing are all connected to the $\Gamma_0(N)_+$, the subgroup of the $PSL(2, Z)$, the $\Gamma_0(N)$ group together with the Atkin-lehner involutions $T \to \frac{1}{N^2}$ in certain type II models, e.g the surface $X_{24}(1, 1, 2, 8, 12)_3^{480}$, the $K_3$ fiber is a two moduli system $X_{12}(1, 1, 4, 6)$. In certain complex structure limit the $K_3$ fiber degenerates to a $K_3$ elliptic fibration $X_{0}(1, 2, 3)$, it look locally as a torus, over $P^1$ with modular groups connected to e.g $\Gamma(3)$ and $\Gamma(2)$. We expect that the same prepotential, beyond describing the geometry of the $N = 2$ sector of $Z_6$ in exact analogy to the decomposable case, may come form a compactification of the heterotic string on the $K_3 \times T^2$. An argument that seems to give some support to our conjecture was given in
It was noted by Vafa and Witten that if we compactify a ten dimensional string theory on $T^2 \times X$, where $X$ any four manifold, acting with a $Z_2$ shift on the Narain lattice we get the modular symmetry group $\Gamma_o(2)_T \times \Gamma_o(2)_U$. They even describe the Narain lattice that exhibits this symmetry. In this respect it is obvious that our calculation of the prepotential which we present in this Thesis, may come from a shift in a certain Narain lattice of $T^2$. We suspect that this is a $Z_3$ shift but we were not able to prove it. From the mass operator (4.113) we deduce that at the point $T = U$ in the moduli space of the $T^2$ torus of the untwisted plane, with $n_1 m_1 = \pm 1$ and $n_2 = m_2 = 0$, its $U(1) \times U(1)$ symmetry becomes enhanced to $SU(2) \times U(1)$. Moreover, the third derivative of the prepotential has to transform, in analogy to the $SL(2, Z)$ case, with modular weights -2 under $\Gamma_o(3)_U$ and 4 under $\Gamma_o(3)_T$ dualities. The Hauptmodul function, the analog of $SL(2, Z)$ $j$-invariant, for $\Gamma_o(3)$ is the function $\omega$ described in chapter three. The function $\omega_o(3)$ has a single zero at zero and a single pole at infinity. In addition, its first derivative has a first order zero at zero, a pole at infinity and a first order zero at $i\sqrt{3}$. The modular form $F$ of weight $k$ of a given subgroup of the modular group $PSL(2, Z) = SL(2, Z)/Z_2$ is calculated from the formula

$$\sum_{p \neq 0, \infty} \nu_p + \sum_{p = 0, \infty} \frac{1}{m} \times (\text{order of the point}) = \frac{\mu k}{12}. \quad (4.114)$$

Here, $\nu_p$ the order of the function $F$, the lowest power in the Laurent expansion of $F$ at $p$ and $m$ is the order of the subgroup fixing the point. The index $\mu$ for $\Gamma_o(3)$ is calculated from the expression

$$[\Gamma : \Gamma_o(N)] = N\Pi_{p/N} \left(1 + p^{-1}\right) \quad (4.115)$$

equal to four. The sum of the widths at all cusps is equal to the index of the subgroup of $PSL(2, Z)$. The width at infinity is defined as the smallest integer such as the transformation $z \rightarrow (z + \alpha)$ is in the group, where $\alpha \in Z$. The width at zero is coming by properly transforming the width at infinity at zero. For $\Gamma_o(3)$ the width at $\infty$ is 3 and the width at zero is 1. The holomorphic prepotential can be calculated easily if we examine its seventh derivative. The seventh derivative has modular weight 12 in $T$ and 4 in $U$. In addition, it has a sixth order pole at the $T = U$ point whose coefficient $A$ has to be fixed in order to produce the logarithmic singularity of the one loop prepotential. As it was shown the one loop prepotential as
\( T \) approaches \( U_g = \frac{aU + b}{cU + d} \), where \( g \) is an \( SL(2, Z) \) element\(^{127}\)

\[
f \propto -\frac{i}{\pi} \{(cU + d)T - (aU + b)\}^2 \ln(T - U_g).
\]

The seventh derivative of the prepotential is calculated to be

\[
f^{TTTTTTT} = A \frac{\omega(U)^3 \omega(U)^5 (\omega'(U))^3}{(\omega(U) - \omega(\sqrt{3}))^2 (\omega(U) - \omega(T))^6} X(T),
\]

where \( X(T) \) a meromorphic modular form with modular weight 12 in \( T \). The complete form of the prepotential is

\[
f^{TTTTTTT} = A \left( \frac{\omega(U)^3 \omega(U)^5 (\omega'(U))^3}{(\omega(U) - \omega(\sqrt{3}))^2 (\omega(U) - \omega(T))^6} \right) \left( \frac{\omega(T)^6}{\omega^2(T) ((\omega(T) - \omega(\sqrt{3}))^4) \} \right). \quad (4.118)
\]

The two groups \( \Gamma^o(3) \) and \( \Gamma_o(3) \) are conjugate to each other. If \( S \) is the generator

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{we have } \Gamma^o(3) = S^{-1} \Gamma_o(3) S. \quad (4.119)
\]

So any statement about modular functions on one group is a statement about the other. We have just to replace everywhere \( \omega(z) \) by \( \omega(3z) \) to go from a modular function from the \( \Gamma^o(3) \) to the \( \Gamma_o(3) \). In other words, the results for the heterotic prepotential with modular symmetry group \( \Gamma^o(3) \) may well be described by the prepotential of the conjugate modular theory.

We have calculated the prepotential of a heterotic string with a \( \Gamma^o(3)_T \times \Gamma^o(3)_T \times Z_2^{T-U} \) classical duality group. The same dependence on the \( T, U \) moduli and its modular symmetry group appears in the \( N = 2 \) sector of the \( Z_6 \) orbifold defined on the six dimensional lattice \( SU(3) \times SO(8) \), namely the \( Z_6\)-IIb. The effective theory of the \( T, U \) moduli \( N = 2 \) sector of the \( Z_6\)-IIb orbifold, appears in \( N = 1 \) symmetric orbifold compactifications of the heterotic string. Recall now the discussion in (4.98). In general one expects \( \text{sign}(d_{\text{non}}) = (+, -, \ldots, -, 0, \ldots, 0) \), where the non zero entries correspond to the moduli from the generic \( K_3 \) fibre. The zero entries correspond to singular fibres, fibers which degenerate at points in the moduli space to non \( K_3 \) surfaces like a smooth manifold, and correspond to the heterotic side to strong coupling singularities\(^{203}\). Because of the maximum number of \( K_3 \) moduli 20, the number of generic

\(^{127}\)The same argument works for the subgroups of the modular group, but now there are additional restrictions on the parameters of the modular transformations.
fibers is constrained to be less than 20. The perturbative heterotic vacuua correspond to moduli of the type IIA coming from the generic fibres.

We believe that the nature of the lattice twist of non-decomposable orbifolds is such that its form when acting on the $N = 2$ planes may correspond to orbifold limits of $K_3$. In this phase, the $K_3$ surface can be written as an orbifold of $T^4$. The fixed points of $T^4$ under the orbifold action are the singular limits of $K_3$ because the metric on the fixed point develops singularities. The singularities of $K_3$ follow an ADE classification pattern. In fact, because at the adiabatic limit\[188\], we can do even the reverse, we can map the type II phase to the heterotic one. In the limit where the base of the fibration has a large area, but the volume of the $K_3$ fiber is of order one, we can replace the $K_3$ fibers with $T^4$ fibers. In this form, the heterotic $K_3 \times T^2$ compactification is replaced by a heterotic string description of the $T^4$ fibers, namely the Narain lattice $\Gamma^{20.4}$. 
CHAPTER 5
5. Superpotentials with T and S-duality and Effective $\mu$ terms.

5.1 Introduction

Superstring theory, if it is to have any chance to be consistent with real world, has to make definite predictions which will be subsequently verified by the experiment or even predict some new phenomena. However in order to accomplish such a role, the theory has first of all to solve its own problems. Beyond any doubt the biggest problem of all is the question of $N = 1$ space-time supersymmetry breaking. The breaking, due to the presence of the gravitino in the effective action, must be spontaneous and not explicit. This problem is crucial for the theory to make contact with the low energy physics and to correctly predict the particle masses. It is expected, that the breaking will correctly create the hierarchy between the light particles by predicting exactly the Yukawa couplings of the light particles with the Higgs scalars. As a result, at the level of supergravity theory the masses of the physical particles, directly connected to the soft term generation created by the supersymmetry breaking, will be predicted.

String theory as the only candidate for a theory which can consistently incorporate gravity, has still some problems. It has a huge number of consistent vacua without a associated mechanism which singles out one of them. Another problem is related to the determination of physical couplings and masses of the theory, which becomes a complicated dynamical problem, since they depend on the vacuum expectation values of the dilaton and the moduli. In the absence of a perturbative method to exactly fix their values, this problem is left to be fixed from the process of supersymmetry breaking. A third problem is associated with the calculation of physical mass and couplings of the quark and lepton superfields and Higgs doublets after supersymmetry breaking. In non-supersymmetric theories like those coming from the standard model or extensions of it, the scalar masses remain unprotected against quadratic divergencies, thus creating the gauge hierarchy problem. Supersymmetric gauge field theories solve technically the gauge hierarchy problem with the introduction of terms that explicitly break supersymmetry, the so called soft terms.

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128We are speaking of supersymmetric solutions of string theory since supersymmetric vacua don’t suffer from stability problems and furthermore they are known to provide a solution for the hierarchy problem.

129grand unified or technicolor theories
In gauge theories the Higgs sector in unprotected against large radiative corrections which can give very large masses to the Higgs particles, due to quadratic divergencies, therefore creating a hierarchy problem. In supersymmetric gauge theories the hierarchy problem is technically solved since the theory is free of quadratic divergencies. However, spontaneous breaking of global supersymmetry with the introduction of soft breaking terms does not produce very realistic models. In locally supersymmetric theories the soft breaking terms arise naturally in the low energy supergravity lagrangian of spontaneously broken supergravities coupled to matter multiplets[253, 256, 257]. Hierarchy remains stable against radiative corrections[269] only when \( m_{3/2} \leq 1 \) Tev. This means that the Higgs sector of the theory is protected against large perturbative corrections as long as the gravitino mass obey this constraint.

However, for the spontaneously broken \( N = 1 \) locally supersymmetric effective superstring theory[30] the contributions[46] to the one loop effective potential take the generic form

\[
V_1 = V_0 + \frac{1}{64\pi^2} \text{Str} \mathcal{M}^0 \times \Lambda^4 \log \frac{\Lambda^2}{\mu^2} + \frac{1}{32\pi^2} \text{Str} \mathcal{M}^2 \Lambda^2 + \frac{1}{64\pi^2} \text{Str} \mathcal{M}^4 \log \frac{\mathcal{M}^2}{\Lambda^2} + \ldots, \tag{5.1}
\]

where \( V_0 \) is the classical potential of the theory or order of the electroweak scale. The general form of

\[
\text{Str} \mathcal{M}^n \equiv \sum (-)^{2J_i} (2J_i + 1) m_i^n. \tag{5.2}
\]

It depends on the Higgs masses and represents the sum over powers of the field dependent mass eigenvalues of the different degrees of freedom. The divergencies depend on the metric of the chiral and gauge superfield content of the underling theory, are field dependent and are not guarantee to be vanishing.

The effects on the gauge hierarchy problem, after the spontaneous breaking of local supersymmetry, receive contributions related to the the quadratically divergent corrections to the effective potential

\[
\text{Str} \mathcal{M}^2(z, \bar{z}) = 2(n - 1 - G^I H_{IJ} G^J) m_{3/2}^2 \tag{5.3}
\]

with

\[
H_{IJ} = \partial_I \partial_J \log \det G_{MN} - \partial_I \partial_J \log \det \text{Re} f_{ab}. \tag{5.4}
\]

\(^{130}\) that is coming from the superstring vacuum in the limit of keeping \( m_{3/2}^2 \) fixed and \( k \to 0 \), with \( k \) the gravitational coupling.
So, (5.3) can be different from zero. However these contributions can be vanishing, if we demand that the moduli fields transform in a scale invariant way i.e under the target space duality symmetries in the large moduli limit of the underlying σ-model. Of course such a scenario puts constraints on the low energy content of the theory based on the need to stabilize the gauge hierarchy and not based on physical properties coming from an underling principle. This is a general problem of all the string models constructed up to know. Only special classes of models are compatible with the phenomenological requirements required to single out a particular vacuum. Other problems connected to the breaking of supersymmetry, is the question of the smallness of the cosmological constant problem and as a result the question of the selection mechanism which could proliferate string vacua. Non vanishing contributions to the cosmological constant may come from the part of the quadratically divergent contributions related to the gauge hierarchy problem as well as the non-perturbative moduli dependent part of the vacuum of the theory. At present we will not be concerned with the cosmological constant problem, but instead we will concentrating our efforts to the moduli of the dependent part of the potential.

String theory is a theory of only one scale, the string scale. Physical quantities in string theory are not input free parameters as in supergravity models. They depend on dynamical fields whose value depend on the vacuum expectation value of the dilaton and the moduli fields.

While a lot of work have been done at the level of the effective theory in order to solve the problem of supersymmetry breaking and possibly to predict the value of the dilaton, the majority of the scenarios in the works involved have failed to properly incorporate its value. The dilaton value is limited from the LEP measurements, giving support to extrapolations of the values of the gauge coupling constants, in consistency with the picture of grand unification idea at the scale $g_{uni}^2 \sim 4\pi/26$. In string theory the gauge coupling constants are 'unified', by connecting the value of the tree level gauge coupling constants to the Newton coupling constant. Recently, it was shown that the strong coupling limit of the $E_8 \times E_8$ heterotic string is given by an orbifold of the eleven dimensional M-theory, known to have as a low energy limit the eleven-dimensional supergravity. In this picture the unification of the coupling constants with...
gravitation happens at the grand unification scale.

The tree level value of the dilaton at the unification scale is $\text{Re} S \sim 2$ and is expected that such a value will be determined from an action which incorporates S-duality \cite{285, 251} as well as T-duality invariance \cite{138, 289, 291}.

Several mechanisms have been used to break consistently supersymmetry. The main flow of research has been concentrated in three main directions. The tree level coordinate dependent compactification \cite{49, 48, 50} mechanism - CDC, which extended the "Scherk-Schwarz" mechanism \cite{276} in string theory, the non-perturbative gaugino condensation \cite{274, 275, 289, 291} mechanism and via magnetized tori \cite{277, 278}.

In the CDC mechanism, spontaneous supersymmetry breaking is achieved by coupling the lattice momenta of toroidal compactifications to the charges of a $U(1)$ current. The latter does not commute with the gravitino vertex operator and it therefore breaks supersymmetry. The net result of the investigations so far, show that in this case CDC gives no-scale models with vanishing potential and zero cosmological constant at the tree level of string theory. The problem with in this approach is that supersymmetry is broken but the values of the moduli parameters are not fixed. The hope is, that they will be fixed from radiative corrections or from non-perturbative phenomena. Furthermore contributions to the cosmological constant can arise at the one loop level.

In the magnetized tori approach, a magnetic field associated with a $U(1)$ gauge group generates mass splittings among the hypermultiplets, which carry non-zero $U(1)$ charges. The two previous approaches are distinguished from the fact that in the latter case, at tree level the potential of the theory is different from zero and the gauge group $SO(32)$ can be broken down to standard model. Common future of the previous two mechanisms is the impossibility of fixing the value of the dilaton.

At this work, we will be mainly concerned with the gaugino condensation approach. Our primary concern, is the dynamical fixing of the value of the dilaton. This problem is a complicated one in string theory and one solution involves the use of multiple condensates \cite{298} to stabilize its vacuum expectation value. This approach, in contrast with the previous one's is not a 'stringy' one but field theoretical.

In this scenario the non-perturbative generated superpotential for the composite vector su-
permultiplets is responsible for the creation of the required hierarchy. Whenever the gauge interactions become strong, the condensate forms and breaks supersymmetry. As a result, an effective potential for the moduli is generated after the integration of the gauge degrees of freedom associated with the gaugino bound states. Then, with a typical value of the hidden SUSY breaking sector scale of order $\Lambda$ equal to $10^{13}$ Gev, the hierarchy in the low energy gauge sector of the model is stable against quantum corrections, if the gravitino mass is of order of $\frac{\Lambda^3}{M_P}$, i.e of one Tev.

Our study, uses the Hauptmodul functions of chapter three. The existence of the S-duality symmetry of string theory was conjectured in [284, 285]. In section (5.2) we will see how different parametrizations of the non-perturbative effects, which combine T-S duality, provide constraints [287] that severely constraint the form of the effective action. Moreover in section (5.3) we want to propose a possible supersymmetry breaking scenario, conjectural, which use S-duality [264, 265, 266], to fix the value of the dilaton. Furthermore in section (5.4) we will discuss the $\mu$ term generation in orbifold compactifications. We must say here these $\mu$ terms are part of the soft supersymmetry breaking terms of the effective low energy lagrangian of the orbifold compactifications of the heterotic string[80]. We will see later in section (5.4) the general form of the soft supersymmetry breaking terms. They include the ”trilinear” A-terms and the ”bilinear” B-terms. The resulting $\mu$ terms receive contributions from the non-perturbative superpotentials of chapter three. Because the relevant contributions to the $\mu$ terms arise in the one loop corrected effective action of orbifold compactifications of the heterotic string, they enjoy all the invariances of its effective theory in the linear representation of the dilaton.

5.2 *Constraints from duality invariance on the superpotential and the Kähler potential for the globally and the locally supersymmetric theory

The effective low energy theory of string compactifications with $N = 1$ supergravity up to two space time derivatives, is described from the following functions of chiral superfields, the gauge invariant Kähler potential $K$, the superpotential $W$ and the gauge kinetic function $f$, which is associated with the kinetic terms of the fields in the vector multiplets. The Kähler potential has to be a real analytic function of chiral superfields. The Kähler potential and the superpotential
are connected together via the relation
\[
G(z, \bar{z}) = K(z, \bar{z}) + (\log W(|z|))^2. \tag{5.5}
\]
Because the spectrum and interactions of the string vacuum are invariant under the appropriate T-duality group and T-duality has been proven to be a good symmetry in any order of perturbative string theory, the effective low energy theory of the orbifold compactification of the heterotic string on a torus has to be invariant under the $SL(2, Z)$, T-duality group with
\[
T \rightarrow \frac{\alpha T - ib}{icT + d}. \tag{5.6}
\]
When considering a global supersymmetric theory, the constraints from modular invariance on the Kähler potential and the superpotential of the theory gives that while K has to be invariant up to a Kähler transformation the superpotential has to be modular invariant. Accordingly, we can choose for the superpotential any polynomial of the modular function $\omega$ where $\omega$ is one of the j functions for the congruence subgroups of the $PSL(2, Z)$ which have been listed in chapter three. In the case of local supersymmetry the constraints are different. For effective low energy superstring theories with $N = 1$ supersymmetry the action contains the terms
\[
e^{-1} \mathcal{L} = e^G \left[ 3 - G_t G_{\bar{t}} G_{\bar{t}} \right] + e^{G/2} \xi_{\mu R} \sigma^{\mu \nu} \xi_{\nu R} + \ldots, \tag{5.7}
\]
where the first factor in parenthesis is the effective potential of the theory and the last term depend on $\xi_{\mu}$ the gravitino.

From the term involving the gravitino $\xi_{\mu}$, we can see that $G$ has to be modular invariant. This can be implemented either as a separately modular invariant superpotential and Kähler potential or by keeping the whole $G$ expression modular invariant. Here we will be interested in congruence subgroups of the modular group $\Gamma$ and especially those appearing in the non-decomposable orbifold constructions of the heterotic string. In fact by using the expression
\[
\int^T dt \Psi(t, N) = 4\pi \log \frac{\eta(NT)}{\eta(T)} + C = \pi/6 \log \frac{\Delta(NT)}{\Delta(T)} + C, \tag{5.8}
\]
we can identify the latter modular invariant expression as part of $\Delta(NT)/\Delta(T) = \omega(NT)$ the non-perturbative $G$ function for the non-decomposable orbifolds based on the subgroup $\Gamma_o(N)$. The

\[133\text{We consider for simplicity dependence on one modulus field.}\]

\[134\text{we are following the notation of} \ 258, 138.\]
result for the $\Gamma^o(N)$ group easily follows by replacing in (5.8) $N \to 1/N$. Note, that we have used the relation
\[
\Delta(T) = \eta^{24}(T),
\] (5.9)
where $\eta$ is the Dedekind function. The function $\Psi(T, N)$ is defined as
\[
\Psi(\tau, N) = NG_2(NT) - G_2(T),
\] (5.10)
where $T$ is a modulus field appearing in the low energy lagrangian, and under T-duality transformations is transforming covariantly
\[
\Psi(\tau, N) \rightarrow (c\tau + d)^2 \Psi(T, N).
\] (5.11)
The value of the $G_2(T)$ function is given by the Eisenstein series
\[
G_2(T) = \sum_{n_1, n_2 \in \mathbb{Z}} (in_1T + n_2)^2 = \frac{\pi^2}{3} - 8\pi^2 \sum_{m, m_1 \geq 1} mq^{mm_1},
\] (5.12)
\[
\tilde{G}_2(T) = G_2(T) - \frac{\pi}{ReT}.
\] (5.13)
It transforms inhomogenously under $SL(2, \mathbb{Z})$ transformations
\[
G_2 \rightarrow (icT + d)^2 G_2 - 2\pi ic(icT + d).
\] (5.14)

Take for example now, the non-decomposable orbifold $Z6$-IIb \[59\]. The tree level Kähler potential for the untwisted subsector with target space duality group $\Gamma^o(3)$ is $K(T, \bar{T}) = -\log(T + \bar{T})$. By using the expression
\[
G_T(T, \bar{T}) = -(1/(T + \bar{T})) + \frac{\partial \log W}{\partial T},
\] (5.15)
we will be able to fix the leading term in eqn. (5.16). The leading terms of the non-perturbative superpotential $W$ were calculated in chapter 3. The term which was associated with the contribution of particular point in the moduli space of the non-decomposable orbifold corresponding to symmetry enhancement was found in the leading order as follows
\[
W = \omega(T)\eta(T)^{-2}\eta(U)^{-2} + \ldots
\] (5.16)
By identifying
\[
\partial_T\log W = r \frac{3}{\pi}[\frac{1}{3}G_2(\frac{T}{3}) - G_2(T)] + \frac{3}{2\pi}G_2(\frac{T}{3}) + \frac{1}{2\pi}G_2(U)
\] (5.17)
we can recover back the result of eqn. (5.16). The previous function transforms in the proper way under modular transformations, has modular weight $-1$ and is the leading term in the expansion of eqn. (5.16).
5.3 * S- and T- dual supersymmetry breaking

While trying to solve the problems of string theory from the perturbative framework, it is the non-perturbative status of string theory which can at the moment give some definite answers. At the level of the creation of the non-perturbative superpotential that could give rise to dynamical determination of the vacuum expectation values of the dilaton and moduli fields, resulting in hierarchical supersymmetry breaking, the gaugino condensation mechanism was suggested as a mechanism for a realistic supersymmetry breaking in string theory. The conceptual difficulty in the above approach is that gaugino condensation by itself is a field theoretical phenomenon and does not provide for a consistent skeleton which would incorporate non-perturbative effects at the small radius limit in the \( \sigma \)-model sense. Furthermore, in the original approach a vacuum with vanishing vacuum energy and broken supersymmetry was only possible if a constant \( c \), coming from possible non-perturbative effects was present in the superpotential of the theory. However, this constant is quantised.

Later, we will make use of the target space modular invariance together with the assumption of the existence of S-duality for \( N = 1 \) vacua, to dynamically study a way of creating a modular superpotential with the correct modular invariance properties for the moduli fields coming from the compactification of our high energy vacuum. At this part of the thesis, we are using the principle of S-duality to examine possible dynamical mechanisms for fixing the value of the dilaton. We will not give emphasis to mechanisms which are concentrating only in the use of T-duality as it was the approach up to now. Models which are based solely on T-duality are clearly not satisfactory and the main drawback of the models existing in the literature is the difficulty to fix the value of the dilaton.

In general, there are two different approaches for the non-perturbative gaugino condensation. These are the effective lagrangian approach, where we can use a gauge singlet gaugino bilinear superfield as a dynamical degree of freedom and the effective superpotential approach, which was used with superpotentials transforming covariantly under T-duality, by replacing the condensate field by its vacuum expectation value. In the models of the value of the dilaton is fixed at a realistic value but supersymmetry is unbroken at the minimum, while at the models of the value of the dilaton is at a fairly good level but the
cosmological constant is negative. Here we will use the principle of S-dual gaugino condensation to describe models based on subgroups of the modular group. We will not consider the presence of hidden matter. The dynamics of the effective theory of gaugino condensation is described by the composite superfield $U = \delta_{ab} W^a_\gamma \epsilon_{\gamma\lambda} W^b_\lambda$, which at the lowest order contains the gaugino bilinear as its scalar component. Let us consider the superpotential of, which generalize the Veneziano-Yankielowicz superpotential incorporating both SL(2, Z) duality and SL(2, Z)$_S$ duality

$$W = \frac{\Psi^3}{\eta(S)} \times \left( \frac{1}{2\pi} \ln (j(S) + 3b \ln(\Psi^2_\eta^2(T)/\mu) + c \right),$$

(5.18)

where $\Psi^3 = W_a W^a$ the value of the condensate, $\mu$ is the scale of magnitude at which that the condensate forms. In the value of $c$ was fixed from the requirement that for $S, T$ equal to 1 the gaugino condensate gets an expectation value equal to $\mu$. In fact we will see that we can do more. We recognize $j$ as the j-invariant modular function for SL(2, Z).

The effective Kähler potential includes the chiral superfield $\Psi$ which transforms under T-duality with a modular weight $-1$ and which we choose it to be $K = -\log(S + \bar{S}) - 3\ln(T + \bar{T} - \Psi \bar{\Psi})$. At the weak coupling limit $S \to \infty$ the S-duality superpotential (5.18) must flow to the global limit of Veneziano-Yankielowicz models, namely $W \approx Y^3 S$. In this case, we have to adjust the modular prefactor in front, to correctly recover this limit.

In general, as it was suggested in working with the weak coupling limit of (5.18) is equivalent to working with the condensate integrated out of the form of the superpotential. If we integrate out the gaugino bilinear the resulting superpotential becomes

$$W = \frac{\mu^3(-c - b) e^{-\frac{j}{2\pi}}}{j(S) \eta^2(S) \eta^6(T)} = \frac{\mu^3 \alpha_o}{j\frac{1}{2\pi} \eta^2(S) \eta^6(T)},$$

(5.19)

where $\alpha_o = (-c - b) e^{-\frac{j}{2\pi}}$. Here, the constant b is equal, assuming $E_8$ gauge group, to $b = \beta_o(E_8)/96\pi^2$.

The auxiliary fields which when their vacuum expectation value is non-vanishing break local supersymmetry are given by

$$h^i = e^{\frac{i}{2}G^i} = |W| e^{\frac{i}{2}K} \left( K^i + \frac{W^i}{W} \right),$$

(5.20)

where K is the Kähler potential and W is the superpotential and $W^S$ denoted the derivative of the superpotential with respect to the i variable, either S or T moduli. The S-duality in-
variant superpotential will break supersymmetry if one of the auxiliary fields, either $S$ or $T$, gets non-vanishing vacuum expectation value. We are mostly interested if the $h^S$ will break supersymmetry. The scalar potential of the theory is given by

$$V = |h^S|^2 G_{SS}^{-1} + |h^T|^2 G_{TT}^{-1} - 3e^G. \quad (5.21)$$

At the moment there is some control on the $N=1$ non-perturbative aspects of heterotic string theory. Non-perturbative contributions can appear in $N=1$ heterotic strings in the form of higher weight interactions $\Pi^n W^g$, involving chiral projections of non-holomorphic functions of chiral superfields. A typical amplitude at genus $g$ involves $2g-2$ gauginos and 2 gauge bosons. In $N=2$ compactifications of the heterotic string\cite{148, 149} on $K_3 \times T^2$, the non-perturbative contributions to the prepotential of the heterotic side are calculated\cite{130} from the exact result of the type IIA dual pair. In this way world sheet instanton effects on type IIA are mapped on spacetime instanton effects on the heterotic side. In addition, in $N=4$ non-perturbative contributions involve comparison\cite{280, 281} of M-theory predictions with the loop dependence of $R^4$ terms in the effective action of type IIB or IIA. Here, we demand that S-duality is a good symmetry of, possibly of a formulation of string theory in a different form, string theory when the all non-perturbative corrections are taken into account. Since we assume that S-duality holds at the $N=1$ heterotic string theory, it has to hold at the level of the effective action as well. This means that the $G$ function of $N=1$ supergravity has to be S-duality invariant.

There are some comments that we want to make at this point. At the time that S-duality was claimed to be valid as a symmetry of the $N=1$ string effective action the $j$-invariants for the subgroups of the modular group $PSL(2, Z)$, which were clearly indentified in this Thesis, were completely unknown to the authors. In fact, a relevant comment of the authors in\cite{138} confirms this argument. In order to understand why S-duality could involve subgroups of the modular group, we must first understand that there is nothing special about $PSL(2, Z)_S$. All the evidence for $PSL(2, Z)_S$ duality involve $N=4$ heterotic strings. So the conjectural $PSL(2, Z)_S$ for $N=1$ is a scenario of convenience, since it gives us the dynamical mechanism to fix the value of the dilaton.

In general it is possible to discuss supersymmetry breaking in the presence of matter fields. However, we believe that the low energy potential of the theory\cite{168} which will determine the...
value of the continuous parameters of the theory, must not include matter fields in order that the spontaneous supersymmetry breaking to be model independent. We should note that in Seiberg-Witten pure $SU(2)$ theory the quantum symmetry groups $\Gamma^0(2)$ and $\Gamma_o(2)$ appear when the number of the hypermultiplets is equal to zero and two respectively [196]. So, if we imagine that this quantum symmetry group is the low energy limit of the duality group of the theory, then if there is S-duality present in pure $SU(2)$ Yang-Mills it has to be $\Gamma^0(2)$ or $\Gamma_o(2)$. This argument provides support to our claim that the associated high energy S-duality group of the string model might be $\Gamma^0(2)_S$ or $\Gamma_o(2)_S$. The $N = 2$ supersymmetric Yang-Mills appears at the $\alpha \to 0$ limit of the associated string theory vacuum.

We will now discuss the potential coming from the superpotential

$$W_I = \frac{\Psi^3}{\eta(2S)} \times \left( \frac{1}{2\pi} \ln(\omega(S) + 3b\ln(\Psi^2(T)/\mu)) \right),$$

(5.22)

where $\Psi^3 = W_a W^a$ the value of the condensate, $\mu$ is the order of magnitude that the condensate forms. The prefactor of $\eta^3(2S)$ was used to provide the correct modular weight of $W$ and not to fix its large $S$ limit of (5.18) following [265]. We should notice at this point that the value of $\omega(S)$ represents the value of j-invariant for the congruence subgroups of the modular group $\Gamma_o(2), \Gamma_o(3), \Gamma^0(3)$ and $\Gamma^0(2)$ which appear in the case of $(2, 2)$ symmetric non-factorizable orbifold models, when no continuous Wilson lines are involved.

We assumed that the superpotential has $\Gamma_0(2)$ S-duality and the gauge kinetic function $f$ is $\Gamma_o(2)_S$ duality invariant. This means that under strong–weak coupling duality, $1/g_{\text{non–pertur}}^2 \to 1/g_{\text{non–pertur}}^{S-1/S}$. This implies [251, 285] S-duality invariance of the effective actions under the $\Gamma_o(n)_S$ or $\Gamma^0(n)_S$ in general. Take for example $\Gamma_o(2)$ invariance. This means that $S \to \frac{S}{S+1}$.

Integration of the bilinear condensate gives the superpotential

$$W_I = \frac{\mu^3 \alpha_0}{\eta^6(T)\eta^2(2S)(\omega(S))^{\alpha_1}},$$

(5.23)

where $\alpha_1 = \frac{1}{2\pi b}$ and $\alpha_o \equiv -be^{-1}$. The Kähler potential is $K = -\log(S + \bar{S}) - \log(T + \bar{T} - \bar{Y}Y)$. The potential coming from (5.23) is

$$V_I = \frac{|W_I|^2}{S_R T_R^3} \left( S_R^2 \left( \frac{1}{2\pi} (G_2(2S)) - \alpha_1 2\pi i [E_2(S) - 2E_2(2S)] \right)^2 - 3 \right) + \frac{3\mu^6 \alpha_0^2}{4\pi^2 S_R T_R^3 \eta^4(T)} \times T_R^3 G_2^2(T) \frac{1}{\eta^2(2S)\omega(S)^{2\alpha_1}},$$

(5.24)

F-theory [169].
where  
\[ G_2(2S) = -4\pi \frac{\partial S \eta(2S)}{\eta(2S)} - \frac{2\pi}{S_R}; \ G_2(T) = -4\pi \frac{\partial T \eta(T)}{\eta(T)} - \frac{\pi}{T_R}, \] 
(5.25)

\[ S_R = (S + \bar{S}) \text{ and} \]

\[ T_R = T + \bar{T} - \frac{\mu^6 \alpha_\omega^2}{\omega^{2\alpha_1} \eta^8(2S) \eta_{12}(T)}. \] 
(5.26)

In the decompactification limits \( T_R \to \infty \), and its dual limit \( T_R \to 0 \), the potential diverges \( V_I \to \infty \). As a result, for gaugino condensation to happen, it is necessary that the theory is forced to be compactified. The potential at the limit \( S_R \to \infty \) goes to zero. It becomes a free theory only when \( 2\pi b < 6 \) holds. This means at the weak coupling limit the dilaton cannot be determined from gaugino condensates. We should note that the latter condition is more restrictive that the analogous condition\(^{[265]}\) for the modular group \( PSL(2, Z) \), namely \( 2\pi b < 12 \), where large gauge groups in the hidden sector were required to satisfy the constraint. Because for the \( E_8 \) gauge group we get that \( b|_{E_8} = 90/96\pi^2 \approx 0.09508 \) and \( b < 0.9554 \) we need a large gauge group to satisfy the constraint. Stringy constraints on the possible hidden sector gauge groups allowed to break supersymmetry can come by the use of higher order subgroups of the \( \Gamma_o(n) \) group. Namely, for the \( \Gamma_o(n) \) group the general form of the constraint \( b < \frac{12}{n\pi} \) single out at least one \( E_8 \) group factor, only for the modular groups \( \Gamma_o(3) \) and \( \Gamma_o(5) \). If we demand that the form of the allowed S-duality modular symmetry group at the weak coupling limit to be constrained only from modular invariance, then we could use \( \Gamma_o(7) \) or \( \Gamma_o(13) \) modular groups as well. Of course, nothing prevents us from using, instead for \( \Gamma_o(n) \), the \( \Gamma_o(n) \) subroups of \( PSL(2, Z) \) mentioned in chapter three.

The singular points of the potential can be read from the orbifold points. The latter are extrema of the potential\(^{[258, 138]}\). In complete analogy, we do expect the point \( S = \frac{1+i}{2} \), the fixed point of the modular group \( \Gamma_o(2) \), to be an extremum of the potential. The auxiliary field \( F_S = exp(\frac{1}{2}G)G_S \) at the orbifold point vanishes, since the function \( G_2(2S) \) vanishes at the same point.\(^{[22]}\) Alternatively, we could calculate the first derivative with respect to the T-variable.

We did not include in the fixing of the modular weight of the superpotential the prefactor \( \eta^2(S) \). Alternatively, if we want the prefactor in front of \( W \) to have the correct modular weight\(^{137}\), it was noted in \([302]\) that it can be shown numerically that the latter holds.
and the weakly coupled limit as in Veneziano-Yankielowicz models, we may have

\[ W_{II} = \frac{\Psi^3}{\eta^2(S)\omega(S)^{1/3}} \left( \frac{1}{2\pi} \ln(\omega(S)) + 3b \ln\left( \frac{\Psi\eta^2(T)}{\mu} \right) \right). \]  

(5.27)

In this case, integrating out the condensate we get

\[ W_{II} = \frac{\mu^3 \alpha_o}{\eta^2(2S)\omega(S)^{1/3} \eta^6(T)} \]  

(5.28)

with scalar potential

\[ V_{II} = \frac{|W_{II}|^2}{S_R T_R^3} \left\{ \left( \frac{1}{2\pi} (G_2(2S)) + \alpha_2 2\pi [E_2(S) - 2E_2(2S)] \right)^2 - 3 \right\} + \frac{3\mu^6 \alpha_o^2}{4\pi^2 S_R T_R^2 \eta^{12}(T) \eta^4(2S)} \times T_R^3 G_2^2(T) \frac{1}{\omega(S)^{1/3}} \]  

(5.29)

where \( \alpha_2 \equiv \frac{12 + 2\pi b}{24\pi b} \), and \( T_R = T + \bar{T} - \frac{\mu^6 \alpha_o^2}{2\pi^2 \eta^4(2S) \eta^{12}(T)} \).

Note that the following identities hold for the Hauptmodul of \( \Gamma_0(2), \frac{\Delta(S)}{\Delta(2S)} \):

\[ \frac{\partial_S \Delta(S)}{\Delta(S)} = (2i\pi) E_2(S), \quad \frac{\partial_S \Delta(2S)}{\Delta(2S)} = 2(2i\pi) E_2(2S). \]  

(5.30)

and

\[ E_2(S) = 1 - 24 \sum_n \frac{n e^{2i\pi z}}{(1 - e^{2i\pi z})} = 1 - 24 \sum_{i=1}^{\infty} \sigma_1(n) q^n, \quad E_S(S) = \frac{d}{dS} \log(\eta(S)), \]  

(5.31)

\[ \partial_S \omega(S) = 2i\pi (E_2(S) - 2E_2(2S)), \quad E_2(T) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \ldots. \]  

(5.32)

Here, \( \sigma_{p-1}(n) \) is the divisor \( \prod \sigma_{p-1}(n) = \sum_{d|n} d^{p-1} \). Using a numerical routine, the question of whether supersymmetry breaking can be solved completely. Since the expressions for the potentials are known, we can determine whether or not the auxiliary fields connected with the modulus S or T breaks local supersymmetry. Numerical minimization of the potentials \( V_I, V_{II} \) leads to same value \( T = T_1 + iT_2 = 1.03 + i0.54 \) and \( S = S_1 + iS_2 = 0.505 + i0.50 \). In fact, the only difference between the two potentials is the different value of the \( \alpha_1 \) coefficient.

We observe that the minimum of the potential along the S-direction is near the fixed point of the modular group \( \Gamma_0(2) \) group. The auxiliary S-field at the minimum breaks supersymmetry along the S-direction. S-duality invariant superpotentials can be studied alternatively from the superpotentials of (5.22), (5.28) by replacing \( T \to 2T \).
5.4 * Effective $\mu$ term in orbifold compactifications

5.4.1 Generalities

The hierarchy problem is solved technically in the case of $N = 1$ globally supersymmetric lagrangians with the addition of soft breaking terms, namely soft scalar masses and trilinear and bilinear scalar terms and soft gaugino masses. In general spontaneously broken locally supersymmetric quantum field theories, soft terms arise naturally from the expansion of the supergravity scalar potential

$$V = e^G[G_\alpha(G^{-1})_\beta^\alpha G^\beta - 3].$$

(5.33)

Supersymmetry is spontaneously broken by the vacuum expectation values of the hidden fields which are gauge singlets under the "observable" gauge group. The hidden fields interact only gravitationally with the observable sector fields and their decoupling from the effective action produces the soft terms. The real gauge invariant Kähler function $G$ is given as usual

$$G(z_\alpha, z_\alpha^*) = K(z_\alpha, z_\alpha^*) + \log |W(z_\alpha)|^2,$$

(5.34)

where $z_\alpha$ represent all scalar fields of the theory, including observable and hidden one's. We assume for the Kähler potential and the superpotential has the general form

$$K = K_0(h_i, h_i^*) + K_{ij}\phi_i^*\phi_j^* + (Z_{ijk}\phi_i^*\phi_j^*\phi_k^* + h.c) + \ldots,$$

$$W = W_0(h_i) + \mu_{ij}\phi_i^*\phi_j + Y_{ijk}\phi_i^*\phi_j^*\phi_k + \ldots,$$

(5.35)

where the fields $h_i$ and $\phi_i$ correspond to the hidden and observable sector scalar fields respectively. The ellipsis correspond to terms of higher order in the fields $\phi_i, \phi_i^*$. The terms $\mu_{ij}, Y_{ijk}, K_{ij}$ and $Z_{ij}$ depend on the hidden sector scalar fields $h_i, h_i^*$.

Soft terms involve mass terms for the gauginos $\lambda_i$ and the scalars $\phi_i$, the A term with couplings to trilinear superpotential terms and the B term with couplings to bilinear superpotential terms. The general form of the effective Lagrangian for the soft terms derived from the expansion of the potential (5.33) is given by

$$L_{soft} = \frac{1}{2} \sum_{\alpha} M_\alpha \bar{\lambda}_\alpha\lambda - \sum_i m_i^2 |\tilde{\phi}_i|^2 - (A_{ijk}\tilde{Y}_{ijk}\tilde{\phi}_i^*\tilde{\phi}_j^*\tilde{\phi}_k^* + B_{ij}\bar{\mu}_{ij}\tilde{\phi}_i\tilde{\phi}_j + h.c),$$

(5.36)
\[ \phi_i = K^{1/2}_i \tilde{\phi}_i, \quad \lambda_\alpha = (Ref_\alpha)^{1/2} \lambda_\alpha, \quad \tilde{Y}_{ijk} = Y_{ijk} \frac{W_o^*}{|W_o|} e^{\frac{i\theta}{2}(K_i K_j K_k)} \]  

(5.37)

Let us assume that our low energy theory is that of the minimal supersymmetric standard model. In that case the expansion of the Kähler potential and the superpotential reads

\[ K = K_o(h_l, h_{l^*}) + \sum K_i \phi_i \phi_i^* + (ZH_1 H_2 + h.c), \]  

(5.38)

\[ W = W_o(h_l) + \sum (\lambda^{ab}_L L^a E^b H_1 + \lambda^{ab}_D D^a Q^b H_1 + \lambda^{ab}_U U^a Q^b H_1 + \mu H_1 H_2) + \mu H_1 H_2. \]  

(5.39)

The summation is over all generations of fermions. In eqn.(5.39) we observe that there is a mixing term between the two Higgs fields. The appearance of the mass mixing term for the two Higgs fields of the standard model, which is necessary for the correct electroweak radiative breaking of the electroweak symmetry, must not happen through the mixing. \( W_{\text{tree}} = \mu H_1 H_2 \) at the superpotential \( W_o \) of the theory. If it happens this means that the low energy parameter \( \mu \), of the electroweak scale, is identified with a parameter of order of the Planck scale something unacceptable. In this case, the \( \mu \)-term introduces the hierarchy problem. On the other hand the value of the \( \mu \) term cannot be zero at the electroweak scalar potential. If \( \mu \) is zero, the lagrangian possess the Peccei-Quinn symmetry, which after spontaneous symmetry breaking leads to the unwanted axion. Take for example the potential of the supersymmetric standard model along the neutral direction after electroweak symmetry breaking. Then

\[ V(H_1, H_2) = \frac{1}{8}(g^2 + g'^2)(|H_1|^2 - |H_2|^2)^2 + \mu_1 |H_1|^2 + \mu_2 |H_2|^2 - \mu_3 (H_1 H_2 + h.c), \]  

(5.40)

where

\[ \mu_{1,2}^2 = m_3^{1/2} + V_o + \bar{\mu}^2, \quad \mu_3^2 = -Bm_3^{1/2} \bar{\mu}, \quad \bar{\mu} = e^{\frac{i\theta}{2} K_o} \mu \frac{W_o^*}{|W_o|}, \]  

(5.41)

The tilde are canonically normalized quantities appearing when passing to the low energy lagrangian, \( \lambda \) is the gaugino field.

138 The tilde are canonically normalized quantities appearing when passing to the low energy lagrangian, \( \lambda \) is the gaugino field.

139 Here, \( Q^a := (3, 2, 1/6) \) is the left handed quarks, \( U^a := (\bar{3}, 1, -2/3) \) the left handed antiquarks or right handed quarks, \( D^a_c := (\bar{3}, 1, 1/3) \) the left handed antiquarks, \( L^a := (1, 2, -1/2) \) the left handed leptons and \( E^a_c := (1, 1, 1) \) the right handed leptons. The \( \lambda^{ab}_L, \lambda^{ab}_D, \lambda^{ab}_U \) are the Yukawa coupling matrices. The masses of the quarks and the leptons will be generated by vacuum expectation values of the Higgs multiplets \( H_1 := (1, 2, -1/2), \) \( H_2 := (1, 2, +1/2), \) in the effective low energy theory. The number in the parenthesis represent the quantum number with respect the \( SU(3) \times SU(2) \times U(1)_Y \), while the last entry is the weak hypercharge.
$V_o$ is the cosmological constant. Here, we have assumed that $g_3 = g_2 = g_1 = \sqrt{5/3} g'$ at the unification scale and $\tilde{\mu}$ is the Higgsino mass. From the renormalization group equations we derive that if $\mu$ is zero then it remains zero in all energy scales.

If this is happen then such an appearance can have disastrous results since the minimum of the potential is at $H_1 = 0$. In this case, the d-type quarks and e-type leptons stay massless, which does not happen in reality. The last problem, related to the appearance of $\mu$, taken together with its other problem where its mass can be of order $M_{\text{Planck}}$, something unphysical for a Higgs potential of the order of the electroweak scale, constitutes the well known $\mu$ problem and several scenaria have appeared in previous years, providing a solution. Clearly the presence of such a term in the superpotential of the theory, is essential in order to avoid the breaking of the Peccei-Quinn [319] symmetry and the appearance of the unwanted [320] axion and to give masses to the d-type quarks and e-type leptons which otherwise will remain massless.

Here, we explore the origin of $\mu$ terms in orbifold compactifications of the heterotic string. We discuss particular solutions to the $\mu$ problem related to the generation of the mixing terms between Higgs fields and neutral scalars in the Kähler potential. We will examine the contribution of the $\mu$ terms to the effective low energy lagrangian of $N=1$ orbifold compactifications of the heterotic string. Alternative mechanisms for the generation of the $\mu$ term make use of gaugino condensation [316], to induce an effective $\mu$ term [321] or the presence of mixing terms in the Kähler potential [323], which induce after supersymmetry breaking an effective $\mu$ term given from the last two terms in eqn.(5.46) of order $O(m_{3/2})$. The similarity of the gaugino condensation with our approach will be shown later. Another solution, applicable to supergravity models, makes use of non-renormalizable terms (fourth or higher order) in the superpotential. They have the form $M_{\text{Pl}}^{1-n} A^n H_1 H_2$ and generate a contribution [322] to the $\mu$ term of order $\tilde{\mu} \sim O(M_{\text{Pl}}^{1-n} M_{\text{hidden}}^n)$ after the hidden fields A acquire a vacuum expectation value.

Here, we explore the origin of $\mu$ terms in orbifold compactifications of the heterotic string. We discuss particular solutions to the $\mu$ problem related to the generation of the mixing terms between Higgs fields and neutral scalars in the Kähler potential. We will examine the contribution of the $\mu$ terms to the effective low energy lagrangian of $N=1$ orbifold compactifications of the heterotic string. Let us explain the origin of such mixing terms in superstring theory [236]. We assume that our effective theory of the massless modes after compactification is that of the
heterotic string preserving $N = 1$ supersymmetry. The superpotential of the effective theory involves the moduli $M_i$ and the observable fields $\Pi^I$ and has the general form

$$ W = W^{tr} + W^{induced}, $$

(5.42)

where

$$ W^{tr}(M_i, \Pi^I) = \frac{1}{3} \tilde{Y}_{IJL} \Pi^I \Pi^J \Pi^K + \ldots, \text{ and } W^{in} = \hat{W}(M) + \frac{1}{2} \tilde{\mu}_{IJ}(M) \Pi^I \Pi^J + \ldots $$

(5.43)

with $W^{tr}$ the usual classical superpotential and $W^{induced}$ the superpotential describing our theory at energies below the the condensation scale. The Kähler potential, after expanding it in powers of the matter fields $\Pi^I$ and $\bar{\Pi}^I$, takes the generic form

$$ K = \kappa^{-2} \hat{K}(M, \bar{M}) + Z_{IJ} \Pi^I \Pi^J + \left( \frac{1}{2} H_{IJ}(\Pi, \bar{\Pi}) \Pi^I \Pi^J + \text{c.c} \right) + \text{higher order terms in } \Pi, $$

(5.44)

where $\kappa^{-2} = 8\pi/M_{Pl}^2$. The quantity $Z_{IJ}$ appearing in the previous equation, represents the normalization matrix of the observable superfields and is renormalized to all orders of perturbation theory. The corrections to the $\mu$ term that we are interested will appear below the scale of supersymmetry breaking. The calculation of the effective Lagrangian for the moduli fields

$$ V^{eff}(\Pi, \bar{\Pi}) = \kappa^{-2} K_{ij} F^i \bar{F}^j - 3\kappa^2 e^{\hat{K}} |\hat{W}|^2, \quad F^j = \kappa^2 e^{\hat{K}/2} \hat{K}^j_i (\partial_i \hat{W} + \hat{W} \partial_i \hat{K}), $$

(5.45)

where $F^j$ the auxiliary component of the individual modulus, gives after substituting the moduli fields with their vacuum expectation values, the following expressions for the masses of the observable matter fermions and Yukawa couplings

$$ \mu_{IJ} = \tilde{\mu}_{IJ} + m_{3/2} H_{IJ} - \bar{F}^j \partial_j H_{IJ}, $$

(5.46)

$$ Y_{IJK} = e^{\hat{K}/2} \bar{Y}_{IJK}. $$

(5.47)

The previous expressions induce the effective superpotential

$$ W^{eff} = \frac{1}{2} \mu_{IJ} \Pi^I \Pi^J + \frac{1}{3} Y_{IJK} \Pi^I \Pi^J \Pi^K. $$

(5.48)

\textsuperscript{140} neglecting the effects of electroweak symmetry breaking

\textsuperscript{141} at the flat limit $M_{Pl} \to \infty$ while keeping $M_{3/2}$ fixed.
After supersymmetry breaking the effective scalar potential for the observable superfields of the theory becomes \cite{99, 105} equal to

\[
V_{\text{eff}} = \sum g_{\alpha}^2/4 \left( \bar{\Pi}^I Z_{I\alpha} T_\alpha \Pi^J \right)^2 + \partial \bar{\Pi} W_{\text{eff}} Z^{IJ} \bar{\Pi}^I \Pi^J + h.c.,
\]

(5.49)

with the first line to represent the scalar potential of the unbroken rigid supersymmetry and the second line to represent the so called soft breaking terms

\[
m_{I\bar{J}}^2 = m_{3/2} Z_{I\bar{J}} - F_i \tilde{F}_\bar{j} R_{i\bar{j}I \bar{J}}, \quad A_{I\bar{J}L} = F^i D_i Y_{I\bar{J}L},
\]

\[
B_{I\bar{J}} = F^i D_i \mu_{I\bar{J}} - m_{3/2} \mu_{I\bar{J}}, \quad (5.50)
\]

responsible for the soft breaking of supersymmetry.

We are interested in the \( \mu \)-term generation in \((2, 2)\) orbifold compactifications of the heterotic string. Let us fix the notation \cite{52} first. We are labeling the \(27, \bar{27}\) with latters from the beginning (middle) of the Greek alphabet while moduli are associated with latin characters. The gauge group is \(E_6 \times E_8\), the matter fields are transforming under the \(27, \bar{27}\)'s representations of \(E_6\), \(27\)'s are related to the \((1, 1)\) moduli while \((\bar{27})\)'s are related to the \((2, 1)\) moduli in the usual one to one correspondence. The Kähler potential is given by

\[
K = G + A^\alpha A^\beta Z^{(1,1)}_{a\bar{a}} + B^\nu B^\rho Z^{(2,1)}_{\nu \rho} + (A^\alpha B^\nu H_{a \nu} + c.c) + \ldots
\]

(5.52)

with the A and B corresponding to the \(27\)'s and \(\bar{27}\)'s respectively. The additional contribution in the \(\mu\) term \(\mu_{I\bar{J}}\) which appears in eqn. (5.46) is generated from the presence of higher weight interactions \cite{236}, which are not appearing in the standard description of the low energy superconformal supergravity of the \((2, 2)\) heterotic string compactifications. In the superconformal tensor calculus \cite{218}, parts of the action are constructed as the F-component parts of chiral
superfields with weight $(3, 3)$. The previous notation, is understood to represent the general characterization of multiplets in the superconformal calculus, with the components of the weight to represent the conformal and chiral weights of the dilatations and the chiral $U(1)$ transformations of the respectively. In this way, the lagrangian density for the superpotential is obtained from a term $(\theta^3 W)_F$. The $\theta$ is the compensator field with weights $(1, 1)$. The interactions are created by including in the action chiral projections $\Pi$ acting on complex vector superfields $V$ of weight $(2, 0)$. In general, we demand $F$ terms in the action to have weights $(3, 3)$. The superpotential $W$ is a function of fields with weights $(0, 0)$ so the lagrangian density is obtained from the $F$-component of $\theta^3 W$. As matter as it concerns the $\mu$ term generation in $(2, 2)$ compactifications of the heterotic string corresponding to the presence of the $\bar{\mu}$ term, in eqn. (5.46), the higher weight interactions responsible for this task are generated from terms in the form

$$ (\theta^{-3} P_1 P_2)_F, \quad P_n \equiv (\theta \bar{\theta} e^{-K/3} f^n), \quad n = 1, 2. \quad (5.53) $$

Here, the subscript $F$ denotes the $F$-component and $f$ are complex functions with weights $(0, 0)$. The presence of mixing terms $H_{\alpha \nu}$ for the $27, \bar{27}$ in the Kähler potential (5.52) generates the contributions of the last two terms in (5.46). The presence of higher weight interactions gives the contribution

$$ \bar{\mu} = -h^\alpha W_{ABs} G^{ss} f_s^{(1)} f_{\bar{n}}^{(2)}, \quad (5.54) $$

where $W_{ABs}$ the Yukawa couplings between the scalars $s$ and the Higgs moduli $A, B$ and $G^{ss}$ the inverse Kähler metric for the $s$ fields. In addition, $h^n$ is the auxiliary field of the n-th modulus field. We have assumed an expansion of the superpotential in the form

$$ W = W_o + W_{AB} AB. \quad (5.55) $$

The superpotential of the theory in the form (5.55) comes from non-perturbative effects since terms in this form don’t arise in perturbation theory due to non-renormalization theorems\[301, 13\]. Furthermore, because supersymmetry cannot be broken by any continuous parameter\[213\], the origin of such terms may not come from a spontaneous breaking version of supersymmetry but necessarily its origin must be non-perturbative.

Contribution (5.54) vanishes if the low energy particle content is that of the minimal supersymmetric standard model. In this case, the fields $s$ either are superheavy as with no Yakawa

\[142\] The analog of $D^2$ of rigid supersymmetry
couplings with the Higgs scalars. If the superpotential of the theory includes the mixing term $\mathcal{W}_{AB}$ between the two Higges then the $\mu$ term receives an additional contribution in the form $\tilde{\mu} = e^{G/2} \mathcal{W}_{AB}$. In the following, we assume that the Higgs fields $A$, $B$ are coming from the same untwisted orbifold complex plane.

Let us assume that the low energy content of our theory is that of the minimal supersymmetric standard model. We want to examine possible $\mu$ term contributions coming from orbifold compactifications of the heterotic string. Let us examine for simplicity the non-decomposable orbifold $\mathbb{Z}_6 - IIb$. After taking into account the result for the expression (3.110) for the non-perturbative superpotential, the additional contribution $\bar{\mu}$ to the $\mu$ term becomes

$$ \bar{\mu} e^{-3s/2b} = [ (\eta^{-2}(T))(\frac{1}{3}\eta^{-2}(\frac{U'}{3}))((\partial_T \log \eta^2(T))(\partial_U \log \frac{1}{3}) \times $$

$$ \eta^2(\frac{U'}{3})) ] \tilde{W} + [ (\eta^{-2}(U')(\eta^{-2}(\frac{T}{3}))(\frac{1}{3})( (\partial_T \log \eta^2(T)) \times $$

$$ (\partial_U \log \frac{1}{3}\eta^2(\frac{U'}{3}))) ] \tilde{W} + \mathcal{O}((BC)^2), $$

(5.56)

while as matter as it concerns the observable fermion masses, Higgino masses are given by

$$ m = m_{3/2} + (T + \bar{T})h_T + (U + \bar{U})h_U + (T + \bar{T})(U + \bar{U})\bar{\mu}. $$

(5.57)

In the previous expression, we have used the tree level expressions for the wave function normalization factors, i.e $Z_{A\bar{A}} = Z_{B\bar{B}} = ((T + \bar{T})(U + \bar{U}))^{-1}$. The gravitino mass, which is associated with the scale of the spontaneous breaking of the local supersymmetry, is given by $m_{3/2} = e^{G/2} \mathcal{W}$.

The presence of higher weight interactions modifies the special geometry of $(2, 2)$ compactifications and incorporates now the matter fields $A$, $B$ associated with the $27$, $\bar{27}$'s. In particular, the Riemann tensor $R_{\alpha\beta\nu\mu}$ is modified as

$$ R_{\alpha\beta\nu\mu} = G_{\alpha\beta}G_{\nu\mu} - W_{\alpha\nu}G_{\beta\mu}(e^{G} \tilde{W}_{\beta\mu\bar{s}} - T_{\beta\mu\bar{s}}), $$

(5.58)

where $T$ is given by

$$ T_{\beta\mu\bar{s}} = (\nabla f^{(1)} f^{(2)}), \nabla [k T_{j\bar{s}i\bar{s}}] = \nabla [k] \left( e^{G} \tilde{W}_{j\bar{s}i\bar{s}} \right), T_{j\bar{s}i\bar{s}} = e^{G} \tilde{W}_{j\bar{s}i\bar{s}}. $$

(5.59)

The proposed non-perturbative superpotentials are consistent with the use of the corrected one-loop effective action which uses the linear representation of the dilaton. The expansion of
the superpotentials into the form \( W = W_0 + W_{AB}AB \), is consistent with the invariance of the one-loop corrected effective action under tree level \( \Gamma^0(3)_T \) transformations \(^{1.24}\), which leave the tree level Kähler potential invariant, only if \( W \to (icT + d)^{-1}W \) and

\[
W_0 \to (icT + d)^{-1}W_0, \quad W_{AB} \to (icT + d)^{-1}W_{AB} + icW_0.
\]

In the discussion so far we have tacitly identify the expression in eqn. (3.105), with the non-perturbative generated superpotential in \((2,2)\) orbifold compactifications. This follows \(^{132}\) from the viable identification of the expression of the topological free energy in \( N = 1 \) orbifold compactifications with the determinant of the square mass matrix. i.e.

\[
F = \log \left( \det(e^KK^{-2})|\det W_{ij}|^2 \right),
\]

where we are adopting the notations of eqn. (5.5). Especially in a gaugino condensation approach, the gaugino condensate is \( <\lambda\bar{\lambda}> \propto W(T) \).

We must say that the grouping of terms in the form presented in (3.105) is a matter of convinience. Specifically, grouping together the first with the third term and the second with the forth term we get the result (3.105) and the \( \mu \) term (5.56). On the other hand, regrouping the third term in (3.105), we get

\[
\sum_{n^2, m=0; \, q=0} \log \mathcal{M} = \log(\eta^{-2}(T)\eta^{-2}(U)(\frac{1}{3})\eta^{-2}(U)\eta^{-2}(\frac{T}{3})\frac{1}{3})(1 - 4BC \\
\times \partial_T \{ \log \eta(T) \} \log \eta(\frac{T}{3}) \partial_U \{ \log \eta(U) \} \log \eta(U)).
\]

(5.62)

In this case, the \( \tilde{\mu} \) term contribution is

\[
\tilde{\mu} = -\frac{4e^{G/2}\partial_T \{ \log \eta(T) \} \log \eta(T) \partial_U \{ \log \eta(U) \} \log \eta(U)}{9\eta^2(T)\eta^2(U)\eta^2(\frac{U}{3})\eta^2(\frac{T}{3})}.
\]

(5.63)

The last expression appears to give the same moduli dependence, in its numerator, up to numerical factors, as the ansatz used for the \( \tilde{\mu} \) term contribution to the \( \mu \) term in \((303)\). However, the tree level contribution to the non-perturbative superpotential coming from (5.63) does not have modular weight \(-1\), since in this case

\[
W_0 = \eta^{-2}(T)\eta^{-2}(U)(\frac{1}{3})\eta^{-2}(U)\eta^{-2}(\frac{T}{3})\frac{1}{3}.
\]

(5.64)

\(^{143}\)Remember, that we have changed the notation from \( U' \) to \( U \).
In [303] the square root of the denominator of the expression (5.63) was used as an ansatz for the $\mu$ term. However, here we can see that the term which could produce the same numerator dependence arise with the wrong modular weight, in its denominator. Our results saw that the ansatz used in [303] does not arise, from the calculation of the topological free energy of (2, 2) compactifications, up to $O(AB)$ terms.

Our previous work on candidate non-perturbative superpotentials can be further generalized to other classes of non-decomposable Coxeter orbifolds. For instance, in the case of the $Z(4)$ orbifold [53] with Coxeter twist defined on the lattice $SU(4)^2$ and exhibiting $\Gamma^0(2)_T \times \Gamma_U$ modular symmetry group, the non-perturbative superpotential is

$$W_{\text{non-pert}} = \left( \frac{1}{2} \eta^{-2}(T^2) \eta^{-2}(U) (1 - AB(\partial_T \log \frac{1}{2} \eta^2(T^2))(\partial_U \log \eta^2(U)) \right).$$

(5.65)

The corresponding $\tilde{\mu}$ term is

$$\tilde{\mu} = e^{G/2} \frac{1}{2} \eta^{-2}(T^2) \eta^{-2}(U) \partial_T \log(\frac{1}{2} \eta(T^2)) (\partial_U \log \eta^2(U)).$$

(5.66)

The complete list of non-perturbative superpotentials for non-decomposable orbifolds will appear in a preprint version of the Thesis results, related to generalized solutions of the $\mu$ problem.

In recent popular phenomenological studies [324] of soft breaking terms in string theories study of soft supersymmetry breaking terms in the case of of $\mu$ term from Kähler mixing reveals that the effective parameter space of the theory is non-universal in the general case, while use of tree level physical quantities in dilaton dominated scenaria constraints effectively the parameter space in terms of two independent parameters. The presence of Kähler mixing is necessary, if we want to avoid the appearance of a large $\mu$-term which makes the Higgs heavy.

The form of the $\mu$ term that we have proposed can be used to test observable CP violation effects in on-decomposable orbifold compactifications of the heterotic string in the spirit suggested in [304, 303]. We should notice that we have calculated the non-perturbative superpotential with the correct properties in the one loop corrected effective action in the linear representation of the dilaton for exactly the orbifold $Z_6 - IIb$ used there.

\textsuperscript{144} With effective low energy theory spectrum that gives rise to the particle spectrum of the minimal supersymmetric standard model.

\textsuperscript{145} This scenario guarantee the smallness of flavour-changing neutral currents.
In conclusion, in this chapter we have examined ansatz superpotentials invariant under a strong-weak coupling duality based on subgroups of the modular group $PSL(2, Z)$. The values of the dilaton coming from minimization, appears to have the same problem with superpotentials invariant under $SL(2, Z)_S$ appeared before in the literature \cite{264, 265, 266, 284}. The exact determination of the vacuum expectation value for the dilaton remains an unsolved problem. Its final solution may come when we will be able to perform the sum over all possible non-perturbative effects. In addition, we examined contributions to the $\mu$ terms in $(2, 2)$ orbifold compactifications coming from the presence of non-perturbative contributions to the superpotential of $N = 1$ non-decomposable orbifolds.
CHAPTER 6
6. Conclusions and Future Directions

In string theory, the threshold corrections are always dependent on some untwisted moduli of vector multiplets, which have the interpretation as parametrizing the size and shape of the underlying torus. This dependence comes from the integration of the heavy modes involved in the compactification process. In this Thesis, we calculated this dependence in a number of quantities of physical interest.

In chapter three, we used modular orbits in target space free energies, in $N = 1$ $(2,2)$ symmetric non-decomposable orbifold compactifications [80, 83, 66] of the heterotic string, to calculate the moduli dependence in non-perturbative superpotentials in $(2,2)$ symmetric orbifolds, threshold corrections to gauge couplings in $(2,2)$ symmetric orbifolds and threshold corrections to gravitational couplings in $(0,2)$ $N = 1$ orbifolds. We discuss the regions of moduli space, where additional massives stated become massless, signaling gauge symmetry enhancement. The same method, using modular orbits, has been appeared before in [120] in a different content, where the calculation involved decomposition of the internal lattice in the form $T^4 \oplus T^2$. In addition, we calculated the moduli dependence of threshold corrections in a class of generalized Coxeter $(2,2)$ symmetric $N = 1$ orbifolds. Similar calculations have been appeared before in [59]. Our calculation completes the calculation of threshold corrections in non-decomposable orbifolds, from the classification list of $(2,2)$ symmetric $N = 1$ orbifolds in [66]. The NPS’s calculated are of major importance in the phenomenology of superstring derived models. They may be used, for future research, in supersymmetry breaking to determine the values of the moduli involved. Note, that determining the exact values of the moduli is of particular importance since it eliminates the moduli dependence in threshold corrections to gauge couplings. Once this has been done, renormalization group equations can be used to determine whether or not the undelying string model has any relation with the real world at energies of the order of the electroweak scale. Furthermore, the identical invariance properties of the superpotential with the invariances of the one-loop corrected effective action ([236]) in the linear formulation, indicates that the topological nature of the superpotential is well inside the perturbative regime of the low energy supergravity. In addition, it appeals very interesting to apply the methodology of chapter three, to the calculation of the NPS in specific models, with dual pairs, coming from heterotic strings [148] compactified on $K_3 \times T_2$. Furthermore, it appears to us quite interesting to calculate NPS’s
using the method suggested in [305]. NPS’s were calculated [305] using M-theory compactified on Calabi-Yau four-folds, which gives \( N = 1 \) supersymmetry in four dimensions. Using F-theory (in twelve dimensions)

In chapter four, we have discussed the one loop correction to the one loop prepotential of the vector multiplets for \( N = 2 \) heterotic strings compactified on six dimensional orbifolds. The importance of our result comes from the fact, that in \( N = 2 \) supergravity theories the Wilsonian gauge couplings and the Kähler potential are determined from the holomorphic prepotential. We have established a general procedure for calculating the one loop corrections to the prepotential of the vector multiplets for \( N = 2 \) heterotic strings compactified on six dimensional orbifolds and/or for any compactification of the heterotic string on the \( K_3 \times T^2 \). The difference now is that the index in the Ramond sector, of the internal system with \((0,4)\) superconformal symmetry, counts the embedding of the instantons on the gauge bundle [130, 200] of \( K_3 \). Our solution provides for an alternative solution to the one appearing in [130] where the one loop prepotential was calculated indirectly, with an ansatz, from its relation with the Green-Schwarz term. It should be noticed that the interesting relation between \( K_3 \times T_2 \) heterotic string and the dual type \( II \), can be explored further. The most important result to our opinion at this chapter, equation 4.86 can be applied to various dual pairs [148, 210] and at present an ongoing investigation is well under way.

In [173] only the differential equation for the third derivative of the prepotential with respect to the T moduli was given. The result for the U moduli was derived by use of the mirror symmetry, in the solution of the equation for the third derivative, \( T \leftrightarrow U \). In chapter four we have provided an alternative differential equation, from [173] for calculating the third derivative of the prepotential with respect to the U moduli. ACertainly, it will give the same result. Its integral representation and the analysis of its properties will be given elsewhere.

In addition, in chapter four, we calculated the heterotic prepotential of the \( N = 2 \) heterotic string compactified on in the \((2,2)\) symmetric [29] non-decomposable orbifold \( Z_6 \) with torus lattice \( SU(3) \times SO(8) \). This model has the modular symmetry \( \Gamma^0(3)_T \times \Gamma^0(3) \). Let me call it A model. The calculation was based on the modular symmetries and the singularity structure of the prepotential following [172]. Alternatively, even though there is no heterotic in \( K_3 \times T^2 \) model, known, exhibiting the same modular symmetry, it is not out of the question that it
will not be found. Various modular symmetry groups have appeared\cite{184} in the literature with their Seiberg-Witten theory known, but with their heterotic string limit unknown. For example, take Seiberg-Witten theory. For pure $SU(2)$ theory with the number of hypermultiplets equal to $N_f = 0$, the quantum duality group\cite{132,196} leaving the dyon spectrum invariant is $\Gamma_o(2)$. In the case that the theory has $N_f = 1$ hypermultiplets the quantum symmetry group is $\Gamma_o(2)$. However, there is no corresponding string theory model known where these group appear. However, the Hauptmodul for $\Gamma_o(2)$ and $\Gamma_o(2)$ was presented\cite{146} in chapter three of the Thesis. In our case the situation, is exactly the opposite with the Seiberg theory model. We know the modular group of the $N = 2$ sector of $Z_6$ or the $K_3 \times T^2$ heterotic model, to which the duality group corresponds, but we don’t know the the Seiberg-Witten theory (SWT) with the same quantum symmetry group. However, it would be interesting to understand the way that the $K_3 \times T^2$ models could be classified, so that the exact string theory analog could possible found.

Summarizing, in view of the result of the\cite{189}, and assuming that a Calabi-Yau dual model exists, calculating the perturbative one loop prepotential at its weakly phase is equivalent to the existence of a IIA dual on its large radius limit, or the large complex structure limit for IIB, defined on a bundle with base $P^1$ and generic fibre the $K_3$ surface. In general terms, there is not concrete evidence that the duality between the heterotic string and the type II holds everywhere in the moduli space or in specific regions for a number of models. On $N = 2$ heterotic strings the gauge group can certainly be non-abelian and is bounded by its central charge to be less than twenty four, where the contribution of twenty two units comes from its internal left moving sector. The two units left come from the superpartner of the dilaton and the graviphoton. On the type II side the gauge group is abelian, and non-abelian gauge symmetry enhancement can happen at specific points in the moduli space. There is no bound on the gauge group due to the central charge. The last property makes it difficult to imagine a way such that the maximum admisible rank on type II models match the dual heterotic ones.

In chapter five, we began by examining the way that the modular functions of chapter three can help us to the building of a theoretical model which incorporates S-duality in its structure. The model had to obey a number of constraints, involving the correct modular transformations and the correct weak coupling limit. In fact, this model is supposed to be S-duality invariant,\cite{139} We have calculated the heterotic string one loop correction to the prepotential of vector multiplets corresponding to SWT $N_f = 0, 1$. The results will appear elsewhere.
for example, under the $\Gamma_o(2)_S$ congruence subgroup of the modular group. This group, appear in Seiberg-Witten theory [196] for pure $SU(2)$ Yang-Mills with $N_f$ equal to zero. Our purpose in the first part of chapter five was only zto determine whether or not is it possible for the dilaton to break supersymmetry, or even fix its value, but it was concetrated as well on the number of consistency requirements required to build the particular superpotential. Furthermore, we saw how the modular functions presented in chapter three can be used to construct superpotentials able to possible make a prediction for the value of the dilaton. Note, that predicting the dilaton value is of particular phenomenological importance since its value determines the value of the string unification scale. Because, experimental data predict that the values of the gauge couplings in the standard model seem to be unified at an energy $M_{\text{gut}} = 10^{16}$ GeV, and $\alpha_{\text{gut}} = g_{\text{gut}}^2/4\pi \approx 1/26$ is an open question or not whether a realistic superstring model can be build which simultaneously can break local supersymmetry and fix the value of the dilaton $S$ at the value predicted by the LEP data, namely $\text{Re}S \approx 2$.

In addition, we examined how the calculation of the topological free energy in chapter three can affect calculations involving $\mu$ terms coming from contributions of the higher weight F-terms in the effective theory of orbifold compactifications [80]. We provide two different examples of calculating the $\mu$ terms. However, very easily the number of examples may be increased to cover the whole list of classification of $N = 1$ Coxeter twists in symmetric orbifold compactifications. Remember, that $N = 1$ orbifolds are the more phenomenologically interesting since give chiral models in four dimensions. In addition, another way of exploring the consequences of the $\mu$ term contributions may be in determining the effects on CP violation [303, 304] in specific models coming from non-decomposable orbifolds. In particular, the $Z_6 - IIB$ orbifold which has been discussed already in the literature [303].
CHAPTER 7
Appendix A

The homogeneous modular group $\Gamma' \equiv SL(2, Z)$ is defined as the group of two by two matrices whose entries are all integers and the determinant is one. It is called the "full modular group and we symbolize it by $\Gamma'$. If the above action is accompanied with the quotient $\Gamma \equiv PSL(2, Z) \equiv \Gamma'/{\pm 1}$ then this is called the 'inhomogeneous modular group' and we symbolize it by $\Gamma$. The fundamental domain of $\Gamma$ is defined [140] as the set of points which are related through linear transformations $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$. If we denote $\tau = \tau_1 + i\tau_2$ then the fundamental domain of $\Gamma$ is defined through the relation $\mathcal{F} = \{\tau \in C | \tau_2 > 0, |\tau_1| \leq \frac{1}{2}, |\tau| \geq 1\}$. One of the congruence subgroup of the modular group $\Gamma$ is the group $\Gamma_0(n)$. The group $\Gamma_0(n)$ can be represented by the following set of matrices acting on $\tau$ as $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$:

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ ad - bc = 1, \ (c = 0 \ mod \ n) \right\}$$

(6.1)

However, we are interested on the group $\Gamma_0(2)$. It is generated by the elements $T$ and $ST^2S$ of $\Gamma$. It’s fundamental domain is different from the group $\Gamma$ and is represented from the coset decomposition $\tilde{\mathcal{F}} = \{1, S, ST\}\mathcal{F}$. In addition the group has cusps at the set of points $\{\infty, 0\}$.

We will give now some details about the integration of the integral that we used so far. The integration of eqn.(3.6) is over a $\Gamma_0(2)$ subgroup of the modular group $\Gamma$ since (3.6) is invariant under a $\Gamma_0(2)$ transformation $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$ (with $ad - bc = 1, c = 0 \ mod \ 2$). Under a $\Gamma_0(2)$ transformation (3.6) remains invariant if at the same time we redefine our integers $n_1, n_2, l_1$ and $l_2$ in the following way:

$$\begin{pmatrix} n'_1 & n'_2 \\ l'_1 & l'_2 \end{pmatrix} = \begin{pmatrix} a & c/2 \\ 2b & d \end{pmatrix} \begin{pmatrix} n_1 & n_2 \\ l_1 & l_2 \end{pmatrix}$$

(6.2)

Clearly, $c \equiv 0 \ mod \ 2$. The integral can be calculated based on the method of decomposition into modular orbits.

There are three sets of inequivalent orbits under the $\Gamma_0(2)$

1. The degenerate orbit of zero matrices, where after integration over $\tilde{\mathcal{F}} = \{1, S, ST\}\mathcal{F}$ gives as a total contribution $I_0 = \pi T_2/4$. 
2. The orbit of matrices with non-zero determinants. The following representatives give a contribution to the integral:

\[
\begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \begin{pmatrix} 0 & -p \\ k & j \end{pmatrix}, \begin{pmatrix} 0 & -p \\ k & j + p \end{pmatrix}, \quad 0 \leq j < k, \quad p \neq 0.
\]  

(6.3)

3. The orbits of matrices with zero determinant

\[
\begin{pmatrix} 0 & 0 \\ j & p \end{pmatrix}, \begin{pmatrix} j & p \\ 0 & 0 \end{pmatrix}, \quad j, p \in \mathbb{Z}, \quad (j, p) \neq (0, 0).
\]  

(6.4)

The total contribution from the modular orbits gives,

\[
I_3 = -4 \Re \ln \eta(U) - \ln \left( \frac{T_2}{4} U_2 \right) + \left( \gamma_E - 1 - \ln \frac{8\pi}{3\sqrt{3}} \right)
- \frac{1}{2} \times 4 \Re \ln \eta(U) - \frac{1}{2} \times \ln (T_2 U_2) + \frac{1}{2} \times (\gamma_E - 1 - \ln \frac{8\pi}{3\sqrt{3}})
\]

The first matrix in eqn. A.1 has to be integrated over the half–band \( \{ \tau \in \mathbb{C} \tau_2 > 0, \ |	au_1| < h \} \) as explained in ref. [71]. In contrast the second matrix has to be integrated over a half–band with the double width in \( \tau_1 \).
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