ANALYTICAL BETA FUNCTION MATCHING WITH THE THIN-LENS DOUBLET*

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The matching of the beta functions with the thin-lens doublet was discussed by P.J. Bryant in the proceedings of the CERN Accelerator School 1986 [1]. This is a revised version of his note to correct one of his formulas and, additionally, to add an important boundary condition, which restricts the free choice of the Twiss parameters. This boundary condition is necessary because the doublet has one free parameter less than the two transfer matrices for the horizontal and vertical plane. From the transfer matrices we get six free parameters for both planes, whereas the doublet is characterized by three lengths and two kick strengths (see Fig. 1). Only with this additional boundary condition is it possible to reproduce the matrix generated by the Twiss parameters.

The transfer matrix \( m_{ij} \) is defined by the Twiss parameters in the following way (if we skip the indices \( h,v \) we mean no specific plane):

\[
\begin{align*}
  m_{11} &= \sqrt{\beta_2/\beta_1} (\cos \Delta\phi + a_1 \sin \Delta\phi) \\
  m_{12} &= \sqrt{\beta_1 \beta_2} \sin \Delta\phi \\
  m_{21} &= -(1 + a_1 a_2) \sin \Delta\phi - (a_2 - a_1) \cos \Delta\phi)/\sqrt{\beta_1 \beta_2} \\
  m_{22} &= \sqrt{\beta_1/\beta_2} (\cos \Delta\phi - a_2 \sin \Delta\phi).
\end{align*}
\]

For the doublet notation we will mainly follow the original note but change a little the way of computation. The doublet is described by the multiplication of five matrices and the resulting matrix has to be identified with the matrix of the Twiss parameters:

\[
\begin{pmatrix} 1 & 1 \beta_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \beta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \beta_1 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.
\]

By multiplying this equation from the left with the inverse matrix of the drift \( l_3 \), and from the right with the inverse matrix of the drift \( l_1 \) we get:

\[
\begin{pmatrix} 1 + l_2 \beta_1 \\ \beta_1 + l_2 \beta_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \beta_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \beta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \beta_1 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} m_{11}^{-1} l_3 m_{21} & m_{12}^{-1} l_3 m_{22} + l_1 \beta_1 \beta_2 m_{21} \\ m_{21} & m_{22}^{-1} l_1 m_{21} \end{pmatrix}.
\]

* Addendum to P.J. Bryant, Betatron Matching, Proc. CAS, General Accelerator Physics, CERN 87-10 (1987).
Now we apply the result explicitly to the two planes. First we get the condition for the two matrix elements $1,1$ in both planes:

\[ 1 + l_{1h} \delta_{1h} = m_{11h} - l_{3h} m_{21h} \]
\[ 1 + l_{1v} \delta_{1v} = m_{11v} - l_{3v} m_{21v} \]

where $\delta_{1h}$ and $\delta_{1v}$ are the kick strengths in each plane, respectively. We make use of the relations $\delta_{1v} + \delta_{1h} = 0$, $l_{2v} = l_{2h} (= l_2)$, $l_{3v} = l_{3h} (= l_3)$ and solve for $l_3$ by applying the abbreviations $(m_{1jh} + m_{1jv})/2 = a_{ij}$ and $(m_{1jh} - m_{1jv})/2 = s_{ij}$:

\[ l_3 = (a_{11} - 1)/a_{21} . \]

Repeating the same reasoning for the matrix elements $2,2$ we can solve for $l_1$:

\[ l_1 = (a_{22} - 1)/a_{21} . \]

Both lengths are uniquely defined by the Twiss parameters.

Further manipulations of the equations yield the overall length and the two kick strengths:

\[ l = l_1 + l_2 + l_3 = (s_{11} + s_{22})/s_{21} \]
\[ \delta_{1h} = (s_{11} - l_3 s_{21})/l_2 \]
\[ \delta_{2h} = (s_{22} - l_1 s_{21})/l_2 . \]

By comparing the matrix elements $1,2$, we find expressions for the length $l_{2v}$ and $l_{2h}$ which must be equal:

\[ l_{2h} = m_{12h} - l_{3h} m_{22h} - l_{1h} m_{11h} + l_{1h} l_{3h} m_{21h} \]
\[ l_{2v} = m_{12v} - l_{3v} m_{22v} - l_{1v} m_{11v} + l_{1v} l_{3v} m_{21v} . \]

Equalizing both sides we get the desired boundary condition for the Twiss parameters:

\[ s_{12} a_{21}^2 - s_{22} a_{21} (a_{11} - 1) - s_{11} a_{21} (a_{22} - 1) + s_{21} (a_{11} - 1) (a_{22} - 1) = 0 . \]

In this expression we have Twiss parameters which have to be matched to the original task of the doublet and some which might be free to be defined by the boundary condition. We
can solve the equation for one Twiss parameter of the last type and calculate it by the others.

Sometimes a variation of the boundary condition is more useful, especially if one wants to keep the overall length $l$ of the doublet constant. Therefore we express the length $l$ in two different ways:

\[
l = \frac{(s_{11} + s_{22})}{s_{21}}
\]

\[
l = a_{12} - (a_{11} - 1)(a_{22} - 1)/a_{21}.
\]

These two equations can be solved for two of those Twiss parameters which are not subject to the matching condition. Sometimes it could be rather difficult to solve these equations analytically to get the Twiss parameters explicitly.

This tool and its application to the design of the lattice for the electron storage ring BESSY II were presented at the European Particle Accelerator Conference in Rome 1988 [2,3].

**Fig. 1** Thin lens doublet

**REFERENCES**

