EMITTANCE DILUTION IN TRANSFER LINES

P.J. Bryant

CERN, Geneva, Switzerland

Abstract
Expressions for the emittance dilution following alignment errors, dipole field errors, gradient errors, betatron mismatches and thin scatterers are derived assuming that complete filamentation occurs after the error. For an error just prior to injection into a circular machine these expressions are fully valid. When combining the errors in transfer lines, it is assumed that full filamentation exists between errors, which is unrealistic, but it still provides a good way of deciding power supply tolerances.

1. INTRODUCTION

This paper is complementary to Beam Transfer Lines by P.J. Bryant in the proceedings of the first CAS General Accelerator Physics Course at Orsay [1].

The title emittance dilution refers to the effective growth in emittance, which is seen when a beam filaments and spreads itself over a larger phase space area. This occurs in a circular machine when the phase space area of the injected beam does not correspond exactly to the matched ellipse of the machine. At first, the unmatched ellipse tumbles over and over inside the matched ellipse, but after many turns small non-linearities, which introduce a variation of tune with amplitude, cause the beam to grow tails, which wrap round and round inside the matched ellipse (see Fig. 1). For all practical purposes the beam has a new and bigger emittance, but Liouville's Theorem rules out that the true emittance can be increased in this way. The true beam emittance must remain constant. This can be understood by saying that the true emittance becomes distended and inter-twined with unoccupied phase-space. In this sense, it is a dilution of the beam's phase space density, which explains the title. However, this is more usually referred to as emittance blow-up, or emittance growth.

![Filamenting beam](image1)

![Filamented beam](image2)

Fig. 1 Filamentation
2. EMITTANCE DILUTION FROM MISALIGNMENT, DIPOLE AND GRADIENT ERRORS

2.1 Dipole and misalignment errors in transfer lines [2]

The motion of a particle in a transfer line can be written as

\[ y = \alpha \beta \sin (\phi + B) \]  

(1)

where, \( y \) is the general transverse co-ordinate representing either \( x \) (horizontal) or \( z \) (vertical)
\( \beta \) is the betatron amplitude
\( \phi \) is the betatron phase advance
\( A \) and \( B \) are constants.

This motion is an ellipse in phase space with

\[ y' = \frac{A}{\sqrt{\beta}} \cos (\phi + B) - \frac{A\alpha}{\beta} \sin (\phi + B). \]  

(2)

Rearranging we have

\[ Y = \frac{y}{\sqrt{\beta}} = A \sin (\phi + B) \]  

(3)

\[ Y' = \frac{y\alpha}{\sqrt{\beta}} + y'\sqrt{\beta} = A \cos (\phi + B), \]  

(4)

where \((Y, Y')\) are known as normalised phase space co-ordinates since with these variables particles follow circular paths. Note: \( y' \) denotes \( dy/ds \) while \( Y' \) denotes \( dY/d\phi \) and \( 2\alpha = -d\beta/ds \). The transformation to \((Y, Y')\) is conveniently written in matrix form.

\[
\begin{pmatrix}
Y \\
Y'
\end{pmatrix}
= 
\begin{pmatrix}
1/\sqrt{\beta} & 0 \\
\alpha/\sqrt{\beta} & \sqrt{\beta}
\end{pmatrix}
\begin{pmatrix}
y \\
y'
\end{pmatrix}.
\]  

(5)

Consider now a beam for which the equi-density curves are circles in normalised phase space. If this beam receives an unwanted deflection, \( D \), it will appear at the time of the deflection as shown in Fig. 2(a). However, this asymmetric beam distribution will not persist. As the beam continues along the transfer line and into the following circular accelerator small non-linearities will cause the beam to filament. This means that the particles will re-distribute themselves randomly in phase, while maintaining their distance from the origin, so as to restore rotational symmetry. Thus after a sufficient time has elapsed the particles, which without the deflection \( D \) would have been at point \( P \) in Fig. 2(b), will be uniformly distributed at a radius \( D \) about the point \( P \). For one of these particles the projection onto the \( Y \)-axis will be

\[ Y_2 = Y_1 + D \cos \theta, \]  

(6)
(a) Beam directly after deflection, $D$  
(b) Particle distribution after phase randomization (Filamentation)

Fig. 2. Effect of an unwanted deflection

where the subscripts 1 and 2 denote the unperturbed and perturbed positions respectively. Taking the square of this amplitude

$$Y_2^2 = Y_1^2 + 2Y_1 D \cos \theta + D^2 \cos^2 \theta$$

(7)

and then averaging over the particles around the point $P$ after filamentation has randomized the kick gives

$$\langle Y_2^2 \rangle_p = \langle Y_1^2 \rangle_p + 2 \langle Y_1 D \cos \theta \rangle_p + \langle D^2 \cos^2 \theta \rangle_p .$$

(8)

Since $Y_1$ and $D$ are uncorrelated (i.e. $D$ does not depend on $Y_1$), the second term can be written as

$$2 \langle Y_1 D \cos \theta \rangle_p = 2 \langle Y_1 \rangle_p \langle D \cos \theta \rangle_p .$$

(9)

The second factor is zero, since $D$ is a constant [Fig. 2(a)], which gives

$$\langle Y_2^2 \rangle_p = \langle Y_1^2 \rangle_p + \frac{1}{2} \langle D^2 \rangle_p = \langle Y_1^2 \rangle_p + \frac{1}{2} D^2 .$$

(10)
However, this result is true for any $P$ at any radius $A$ and hence it is true for the whole beam and

$$\langle \gamma^2 \rangle = \langle \gamma^2 \rangle_1 + \frac{1}{2} D^2 .$$  \hspace{1cm} (11)

Equation (11) is clearly related to an effective increase in emittance, but since there is a frequent confusion over the definition of emittance, this must be defined before continuing.

**Definition.** Emittance, $\varepsilon = \pi \sigma^2 / \beta = \pi \langle \gamma^2 \rangle$  \hspace{1cm} (12)

where, $\sigma$ is the r.m.s. amplitude of the particle distribution when the phase space population is projected onto the $y$-axis.

Emittance is associated with the width of the beam and this is characterised by the r.m.s. amplitude of the distribution. For a Gaussian distribution everyone has a clear understanding of the beam size knowing the r.m.s. value, but for other distributions it is perhaps not so obvious. This point is discussed in reference 2 with particular reference to the ISR. It turns out that except for rather strange distributions the luminosity in the final machine is predictable to a few percent knowing only the $\sigma$ of the beam.

Returning to Eq. (11), the emittance increase will be,

$$\varepsilon_2 = \varepsilon_1 + \frac{\pi}{2} D^2 ,$$  \hspace{1cm} (13)

where, by definition, $\varepsilon = \pi \langle \gamma^2 \rangle$. The subscripts 1 and 2 refer to the unperturbed and perturbed emittances respectively, and remember that $Y = y / \sqrt{\beta}$. Expanding the deflection, $D$,

$$D^2 = (\Delta Y)^2 + (\Delta Y')^2 = (\Delta y)^2 \frac{(1 + a^2)}{\beta} + (\Delta y')^2 \beta$$  \hspace{1cm} (14)

and substituting into (13) gives the emittance blow-up, in terms of the basic errors. Thus

$$\varepsilon_2 = \varepsilon_1 + \frac{\pi}{2} [(\Delta y)^2 \frac{(1 + a^2)}{\beta} + (\Delta y')^2 \beta] ,$$  \hspace{1cm} (15)

where, $\Delta y$ is an alignment error

$$\Delta y' = \frac{\lambda \Delta B}{B \rho}$$ an angle error from a field error $\Delta B$ of length $\lambda$. 

2.2 Gradient errors in transfer lines [2]

Consider once again a beam for which the equi-density curves are circles in normalised phase space. If this beam sees a gradient error, $k$, the equi-density curves directly after the perturbation will be ellipses as shown in Fig. 3(a). Since the object of this analysis is to evaluate the effects of small errors, it is sufficient to regard this gradient error as a thin lens with the transfer matrix

$$
\begin{pmatrix}
  y_2 \\
y'_2
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  k & 1
\end{pmatrix}
\begin{pmatrix}
  y_1 \\
y'_1
\end{pmatrix}
$$

(16)

where, $k = \frac{\Delta B}{B_0}$, an amplitude dependent kick arising from the gradient error $\Delta B$ of length $\ell$.

![Diagram showing effect of a gradient error](image)

**Fig. 3** Effect of a gradient error

Denoting the matrix in (5) as $T$, it is easy to show that

$$
\begin{pmatrix}
  y_2 \\
y'_2
\end{pmatrix} = T \begin{pmatrix}
  1 & 0 \\
  k & 1
\end{pmatrix}^{-1} \begin{pmatrix}
  y_1 \\
y'_1
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  k\beta & 1
\end{pmatrix} \begin{pmatrix}
  y_1 \\
y'_1
\end{pmatrix}.
$$

(17)

It is now convenient to find a new co-ordinate system $(YY, YY')$, which is at an angle $\theta$ to the $(Y, Y')$ system, and in which the perturbed ellipse is a right ellipse [see Fig. 3(b)].
\[
\begin{array}{c}
\begin{pmatrix}
Y Y_2
\end{pmatrix}
= 
\begin{pmatrix}
C & -S \\
S & C
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
K\beta & 1
\end{pmatrix}
\begin{pmatrix}
Y_1
\end{pmatrix}
\end{array}
\] (18)

where \(s\) and \(c\) denote \(\sin \theta\) and \(\cos \theta\) respectively.

Introducing the initial distribution \(Y_1 = A \sin (\phi + B)\), \(Y_1' = A \cos (\phi + B)\), from (3) and (4) in the above expression for the new distribution, gives

\[
\begin{align*}
Y Y_2 &= A \sqrt{1 + s^2 k^2 \beta^2 - 2sc k\beta \sin (\phi + B)} \\
Y Y_2' &= A \sqrt{1 + c^2 k^2 \beta^2 + 2sc k\beta \sin (\phi + B + \phi')} \\
\end{align*}
\] (19)

where

\[
\begin{align*}
\phi &= \tan^{-1} \left( \frac{-s}{c - sk\beta} \right) \quad \text{and} \quad \phi' &= \tan^{-1} \left( \frac{c}{s + ck\beta} \right).
\end{align*}
\] (20)

The \((Y Y_2, Y Y_2')\) ellipse will be a right ellipse when \((\phi, \phi') = \pi/2\), which gives the condition

\[
\tan (2\phi) = 2/k\beta.
\] (21)

Equations (19) can be simplified using (21) and the relationship \((\phi, \phi') = \pi/2\) and rewritten as

\[
\begin{pmatrix}
Y Y_2
\end{pmatrix}
= 
\begin{pmatrix}
\tan \theta & 0 \\
0 & 1/\tan \theta
\end{pmatrix}
\begin{pmatrix}
Y Y_1
\end{pmatrix}
\] (22)

where, \(Y Y_1 = A \sin (\phi + B)\)

\(Y Y_1' = A \cos (\phi + B')\)

i.e. \(Y_1\) and \(Y_1'\) with a phase shift

\(B' = (B + \phi) = [B + \tan^{-1} (1/\tan \theta)]\).

Thus it has been possible to diagonalise Eq. (18) by introducing a phase shift \(\phi\) into the initial distribution. Equation (22) is therefore not a true point-to-point transformation, as is (18), but since the initial distribution is rotationally symmetric the introduction of this phase shift has no effect.

The distance from the origin of a perturbed particle is obtained from Eq. (22) as

\[
(Y Y_2^2 + Y Y_2'^2) = [(\tan \theta)^2 (A^2 \sin^2 (\phi + B')) + \left(\frac{1}{\tan \theta}\right) (A^2 \cos^2 (\phi + B'))]
\] (23)
Averaging over $2\pi$ in $\phi$ gives
\[
\langle YY^2 + YY'^2 \rangle = \frac{1}{2} \left( \tan^2 \theta + \frac{1}{\tan^2 \theta} \right) \langle A^2 \rangle ,
\]  \hspace{1cm} (24)

but
\[
\langle A^2 \rangle = \langle YY^2 + YY'^2 \rangle = \langle Y^2 + Y' \rangle
\] \hspace{1cm} (25)

and from (21)
\[
(\tan^2 \theta + \frac{1}{\tan^2 \theta}) = k^2 \beta^2 + 2.
\]

Thus,
\[
\langle YY^2 + YY'^2 \rangle = \frac{1}{2} (k^2 \beta^2 + 2) \langle YY^2 + YY'^2 \rangle .
\] \hspace{1cm} (26)

As in the previous case for dipole errors, the asymmetric beam distribution will not persist. The beam will regain its rotational symmetry by filamentation or phase randomisation. Each particle, however, will maintain its distance from the origin constant. Once filamentation has occurred, the distribution will not distinguish between the YY and YY' axes and (26) can be rewritten as
\[
\langle YY^2 \rangle = \frac{1}{2} (k^2 \beta^2 + 2) \langle YY^2 \rangle
\] \hspace{1cm} (27)

and hence the effective increase in emittance will be
\[
\epsilon_2 = \frac{1}{2} (k^2 \beta^2 + 2) \epsilon_1 .
\] \hspace{1cm} (28)

2.3 Combining errors

Thus we have the expressions (15) and (28) for what happens after an alignment error, a dipole error and a quadrupole error, assuming that the beam is observed after phase randomization has taken place.

If a circular machine is at the end of the transfer line, filamentation will take place there, typically in the order of milliseconds, and the above then gives a rather exact way of calculating the effective emittance increase due to a single error in the preceeding transfer line.
It is suggested that series of errors be treated by taking them in beam order and assuming complete phase randomisation between each error. This is clearly a simplifying assumption since transfer lines are usually too short to even show the effects of filamentation and certainly there is never complete randomisation between elements in a line. In the real world adjacent magnets are often on the same transformer, which also gives correlated errors. Having pointed out these deficiencies, the above method does give a basis upon which to fix tolerances. The assumption that full randomisation takes place between elements will give a pessimistic result for the usual case of tens of elements spread over several betatron wavelengths, which errs on the correct side for fixing tolerances. Small numbers of elements, very close in phase with possible correlated errors can be underestimated by this analysis.

In general it is the kicker magnets and their supplies which cause the most concern. Pulsed power supplies with conventional magnets can usually be made sufficiently reproducible while d.c. supplies can easily reach the required tolerances.

3. **EMITTANCE DILUTION FROM BETATRON MISMATCH**

This is basically the gradient error problem of the previous section seen from a slightly different view point.

It often happens that the constraints on the linear optics are such that an analytically perfect match cannot be found between the end of the transfer line and the accelerator it serves. It also may be that measurements of the beam ellipse reveal a mismatch of unknown original. These situations pose the problem of what error in $\beta$ and $\alpha$ can be tolerated? Again for the designer this is essentially converting the mismatch into an effective emittance increase, which then has to be judged against a criterion set by the ultimate user such as loss in luminosity in a collider.

The transformation to normalised co-ordinates $\left(\gamma, \gamma'\right)$ of the phase space motion of the betatron oscillation (1) was given by (5),

$$
\begin{pmatrix}
\gamma \\
\gamma'
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{\beta_1} & 0 \\
\alpha_1/\sqrt{\beta_1} & \sqrt{\beta_1}
\end{pmatrix} \begin{pmatrix}
\gamma \\
\gamma'
\end{pmatrix}
$$

(5)

where the subscript 1 now denotes the ideal or matched values. If we use this transformation to observe a beam with a mismatch characterised by $\left(\alpha_2, \beta_2\right)$ we will see an ellipse in normalised phase space (see Fig. 4) with the equation,

$$
\gamma_2^2 \left[ \frac{\beta_1}{\beta_2} + \frac{\alpha_1}{\alpha_2} - \frac{\beta_1}{\beta_2} \frac{\beta_2}{\alpha_2} \right] + \gamma_2^2 \frac{\beta_2}{\beta_1} - 2\gamma_2 \frac{\beta_2}{\beta_1} \gamma_1 \left( \frac{\alpha_1}{\alpha_2} - \frac{\beta_1}{\beta_2} \right) \gamma_1 = A^2
$$

(29)

which can be compared to the circle of the matched beam given by

$$
\gamma_1^2 + \gamma_1^2 = A^2.
$$

(30)
Equation (29) is exactly similar in form to a general phase space ellipse in normal space and if we apply the equivalents

\[ \gamma = \frac{\beta_1}{\beta_2} + \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 \frac{\beta_2}{\beta_1} \]

\[ \beta = \frac{\beta_2}{\beta_1} \]

and \[ \alpha = -\left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right) \frac{\beta_2}{\beta_1} \]

all the standard formulae [3] can be used. One can easily check for example that, \[ \gamma = (1 + \alpha^2)/\beta \] still holds.

Thus we can avoid a lot of tedious algebra and quote directly the major and minor axes, \( a \) and \( b \), as,

\[ a = \frac{A}{\sqrt{2}} \left( \sqrt{H + 1} + \sqrt{H - 1} \right) \]

\[ b = \frac{A}{\sqrt{2}} \left( \sqrt{H + 1} - \sqrt{H - 1} \right) \]

where

\[ H = \frac{1}{2} \left[ \beta_1 \beta_2 + \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 \beta_2 + \beta_2 \right]. \]
Without going through the full analysis, as in Section 2.2, we can see that the transformation from the matched circle to the ellipse can be expressed as a diagonal matrix of the form

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}
\]

(35)

providing we make a suitable rotation to align the axes of the ellipse to the coordinate axes. Furthermore from (32) and (33) we see that,

\[
\lambda = \frac{1}{\sqrt{2}} \left( \sqrt{\bar{H} + 1} + \sqrt{\bar{H} - 1} \right).
\]

(36)

A quick multiplication shows that this is consistent with

\[
\lambda^{-1} = \frac{1}{\sqrt{2}} \left( \sqrt{\bar{H} + 1} - \sqrt{\bar{H} - 1} \right).
\]

(36a)

Thus the square of the distance from the origin of a perturbed particle is,

\[
(Y_2^2 + Y_1^2) = \lambda^2 A^2 \sin^2 (\phi_1 + B) + \lambda^{-2} A^2 \cos^2 (\phi_1 + B).
\]

(37)

Averaging over $2\pi$ in betatron phase,

\[
\langle Y_2^2 + Y_1^2 \rangle = \frac{1}{2} (\lambda^2 + \lambda^{-2}) \langle A^2 \rangle.
\]

(38)

Since the factor $\langle \lambda^2 + \lambda^{-2} \rangle$ is independent of $R$ and $\phi$, (38) applies to the whole beam independent of the kind of distribution. Thus we can express the effective increase in emittance as,

\[
\epsilon_2 = \frac{1}{2} (\lambda^2 + \lambda^{-2}) \epsilon_1.
\]

(39)

First we note that even for quite large values of $\lambda$, the effect on the emittance is less than one might intuitively expect. For example if $\lambda = 1.4$ the circle in normalised phase space is deformed in the ratio of 2:1 yet the emittance increases by only 23%.

Evaluating $(\lambda^2 + \lambda^{-2})/2$ from (36) and (36a), we find

\[
\frac{1}{2} (\lambda^2 + \lambda^{-2}) = H.
\]

(40)

Thus

\[
\epsilon_2 = H \epsilon_1
\]

(41)
where

\[ H = \frac{1}{2} \left[ \frac{\beta_1}{\beta_2} + (\alpha_1 - \alpha_2 - \frac{\beta_1}{\beta_2} \frac{\beta_2}{\beta_1})^2 \frac{\beta_2}{\beta_1} \right] \]  

Assuming small errors \( \Delta \beta \) and \( \Delta \alpha \) we have approximate expressions,

\[ \varepsilon_2 = \left[ 1 + \frac{1}{2} (\Delta \alpha)^2 \right] \varepsilon_1 \quad \text{and} \quad \varepsilon_2 = \left[ 1 + \frac{1}{2} \left( \alpha \frac{\Delta \beta}{\beta} \right)^2 \right] \varepsilon_1 \ldots \]  

The expression for \( (\Delta \alpha) \) can be checked against (28) by calculating the \( (\Delta \alpha) \) due to a thin lens of strength \( k \). The expression for \( (\Delta \beta) \) shows the relative insensitivity of this type of mismatch. Indeed if \( \alpha = 0 \) the emittance is constant to first order.

4. **EMITTANCE GROWTH FROM SCATTERING IN THIN WINDOWS**

Transfer lines are often built with a thin metal window separating their relatively poor vacuum from that of the accelerator or storage ring that they serve. The beam must pass through this window with as little degradation as possible. Luminescent screens are also frequently put into beams with the same hope that they will have a negligibly small effect on the beam emittance. It is therefore interesting to know how to calculate the blow-up for such cases.

The root mean square projected angle due to multiple Coulomb scattering in a window is given by [4,5]

\[ \sqrt{\langle \theta'^2 \rangle} = \frac{0.0141}{\beta_c} \frac{Z_{\text{inc}}}{Z_{\text{rad}}} \sqrt{\frac{L}{L_{\text{rad}}} \left[ 1 + \frac{1}{2} \log_{10} \frac{L}{L_{\text{rad}}} \right]} \text{[radian]}, \]  

where

- \( Z_{\text{inc}} \) is particle charge in units of electron charge
- \( \beta_c = v/c \)
- \( L \) is thickness of scatterer
- \( L_{\text{rad}} \) is radiation length in material of scatterer.

Consider a particle with a projected angular deviation of \( y_1' \) at the window due to the initial beam emittance. This particle receives a net projected kick in the window of \( \theta_5 \) and emerges with an angle \( y_2' \) given by

\[ y_2' = (y_1' + \theta_5). \]  

* A derivation of the emittance growth due to scattering appears in Ref. [1], but it was pointed out to be incorrect by C. Bovet and R. Schmidt (private communication) to whom the author is indebted.
By squaring and averaging over the whole beam this becomes

\[
\langle y_2' \rangle = \langle y_1' \rangle + \langle \theta_5^2 \rangle + 2\langle y_1' \theta_5 \rangle
\]

but, since the initial \( y_1' \) is in no way correlated to \( \theta_5 \) and \( \langle y_1' \rangle = 0 \),

\[
2\langle y_1' \theta_5 \rangle = 2\langle y_1' \rangle \langle \theta_5 \rangle = 0
\]

and the above simplifies to

\[
\langle y_2'^2 \rangle = \langle y_1'^2 \rangle + \langle \theta_5^2 \rangle.
\]

This describes the situation immediately after the scattering (see Fig. 5) when the beam is no longer matched.

![Diagram](image)

**Fig. 5** Effect of a thin scatterer in normalised phase space

Using the same arguments as in Section 2 we see that this initial distribution filaments and the average angular divergence becomes

\[
\langle y_2'^2 \rangle = \langle y_1'^2 \rangle + \frac{1}{2} \langle \theta_5^2 \rangle.
\]

From (3), (4) and (12) we find,

\[
\varepsilon = \pi \langle y^2 \rangle = \pi \langle y_1'^2 \rangle.
\]
Thus

\[ \varepsilon_2 = \pi \left( \frac{a^2}{\beta} \langle y_2^2 \rangle + 2 \alpha \langle y_2 y_2' \rangle + \beta \langle y_2^4 \rangle \right). \]  

(47)

The first term in (47) is unchanged by the scattering, so that \( \langle y_2^2 \rangle = \langle y_1^2 \rangle \). The second term directly after scattering yields,

\[ \langle y_2 (y_1' + \theta_5) \rangle = \langle y_1 y_1' \rangle + \langle y_1 \theta_5 \rangle \]

but since \( y_1 \) and \( \theta_5 \) are uncorrelated the second term can be written as \( \langle y_1 \rangle \theta_5 \rangle \) and is therefore zero. Finally the third term is given by (46) after filamentation, so that we have

\[ \varepsilon_2 = \varepsilon_1 + \frac{\pi}{2} \beta \langle \theta_5^2 \rangle. \]

(48)

REFERENCES


