Matrix String Partition Functions

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We evaluate quasiclassically the Ramond partition function of Euclidean $D = 10$ $U(N)$ super Yang-Mills theory reduced to a two-dimensional torus. The result can be interpreted in terms of free strings wrapping the space-time torus, as expected from the point of view of Matrix string theory. We demonstrate that, when extrapolated to the ultraviolet limit (small area of the torus), the quasiclassical expressions reproduce exactly the recently obtained expression for the partition of the completely reduced SYM theory, including the overall numerical factor. This is an evidence that our quasiclassical calculation might be exact.

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1. Introduction

The interpretation [1] of the dimensional reductions of ten-dimensional supersymmetric Yang-Mills (SYM) theory as effective theories for the dynamics of \( p \)-dimensional extended objects (D\( p \)-branes) initiated a new wave of interest in these theories. It culminated in the BFSS conjecture [2] that a system of interacting D0-branes, described by the \( D = 10 \) SYM theory reduced to one dimension, provides, in the large \( N \) limit, a constructive definition of M-theory, the hypothetical theory encompassing all known string theories and 11-dimensional supergravity. It also lead to Matrix string theory [3,4,5], which describes non-perturbatively type IIA string theory by \( D = 10 \) SYM theory reduced to two dimensions. Finally, an interpretation of the completely reduced SYM theory as the full perturbative and non-perturbative type IIB string theory have been advanced in [6]. All three reduced SYM theories are closely related, and in the large \( N \) limit each one contains, in a certain sense, the other two. We will refer to the SYM theory reduced to 0, 1, 2 dimensions as the IKKT, BFSS, DVV model, correspondingly.

The most basic information about these theories, namely, their vacuum structure, can be obtained by studying their partition functions\(^1\). In particular, the partition function of the DVV model is expected to describe, at least in the IR limit, a gas of extended objects – bound states of F- and D-strings.

The only partition function computed at present is that of the completely reduced theory, which we will denote by \( Z_{\text{IKKT}} \). This partition function was studied by several groups [7,8,9,10] in order to prove the existence of bound states in the BFSS model [7,8,11]. It was conjectured by Green and Gutperle [11] that

\[
Z_{\text{IKKT}} = \mathcal{F}_N \sum_{m \mid N} \frac{1}{m^2},
\]

\(^1\) By partition function of a supersymmetric matrix model we understand the volume form in the sector with a minimal number of fermionic zero-modes.
where the numerical factor $F_N$ was computed later by Krauth, Nicolai, and Staudacher [9]. Recently, this conjecture was rigorously proved by Moore, Nekrasov and Shatashvili [10]. The partition function of the IKKT model is related to the dynamics of the type IIB D-instantons, or Euclidean wrapped D0-branes in a T-dual picture, and the result (1) is coherent with the existence of bound state of D0-branes.

In this paper we present the quasi-classical calculation of the partition function of the matrix string theory (the DVV model) compactified on a rectangular torus $T^2$ with periods $R$ and $T$, with Ramond-Ramond boundary conditions. We find

$$Z_{DVV} = \sum_{m|N} \frac{1}{m} \sum_{p \in \mathbb{Z}} e^{-\frac{R^2}{2}Np^2}. \quad (2)$$

The partition functions of the DVV and the IKKT models are related to the half-BPS saturated amplitudes like the $t_8t_8R^4$ term in type IIB theory [12,13]. Our computation confirms the rules for counting wrapped D-branes used in [14,15,12,13]. In particular, we see that the relevant excitations in the IR limit can be interpreted as strings whose world sheet defines a multiple cover of the space-time torus, and that the the sum over all coverings becomes, in the large $N$ limit, an integral over the complex structures of the world-sheet tori, with the standard modular-invariant integration measure.

Being derived using the saddle-point method, this expression can be trusted in principle only in the IR limit (large area $RT$ of the torus). However, there are some reasons to believe that the result is actually exact. Indeed, if we modify the integration measure by subtracting the constant mode of one of the components of the gauge field and then calculate the quasi-classical partition function, the result will reproduce exactly, after being extrapolated to the limit $R \to 0$, $T \to 0$, the expression (1) for the partition function of the completely reduced theory, including the numerical factor computed in [9,16].

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2 The gauge coupling constant can be absorbed in the area of the torus and therefore is considered to be equal to one.
is therefore very plausible that our results are valid everywhere, i.e., that these models possess the property of having exact quasi-classics.

The paper is organised as follows. In Section 2, we define the DVV partition function and give a quantitative description of the dimensional reductions giving the BFSS and IKKT partition functions. In Section 3, we present the derivation of (2). In Section 4 we calculate the DVV partition function modified by a projector eliminating the constant mode of the spatial component of the gauge field and check that when extrapolated to the the UV limit, it coincides with (1). Our conclusions are presented in Section 5.

2. The partition functions of the DVV, BFSS and IKKT matrix models

1. The functional integral of Matrix string theory (the DVV model) compactified on a torus

The Euclidean DVV model is 10-dimensional $U(N)$ SYM theory dimensionally reduced to a two-dimensional cylinder [5]. The field content of the theory includes the two-dimensional $U(N)$ gauge field $\{A_\alpha\}_{\alpha=1}^2$, the bosonic Higgs fields $\{X_I\}_{I=1}^8$ and the Majorana-Weyl fermions $\{\Psi_\alpha\}_{\alpha=1}^{16}$. In order to study the partition function, we compactify the theory on a rectangular\(^3\) two-torus $T^2$ with periods $R$ and $R$,

$$T^2 = \{\vec{\sigma} = (\sigma, \tau) \mid \sigma \in [0, R], \tau \in [0, T]\}.$$ 

The DVV partition function is obtained by inserting in the functional integration measure a product of delta-functions compensating a minimal set of pairs of fermionic and bosonic zero-modes

$$\Psi_\alpha^{(0)} = \frac{\text{Tr} \int_{T^2} d^2\sigma \Psi_\alpha}{\sqrt{NRT}} , \quad X_I^{(0)} = \frac{\text{Tr} \int_{T^2} d^2\sigma X_I}{\sqrt{NRT}} .$$ (3)

The DVV partition function is is given by the functional integral

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\(^3\) The matrix string action assumes that the metric on the two-dimensional space-time is $g_{ab} = \delta_{ab}$. 

3
\[
Z_{\text{DVV}} = \int \frac{DA_a}{\text{Vol}(G)} \, DXD\Psi \prod_{I=1}^{8} \delta \left( \frac{X^{(0)}_I}{\sqrt{2\pi}} \right) \prod_{\alpha=1}^{16} \Psi^{(0)}_{\alpha} \exp \left( -S_{\text{DVV}}[A,X,\Psi] \right) \tag{4}
\]

with action
\[
S_{\text{DVV}} = \int \frac{d^2\sigma}{T^2} \Tr \left[ \frac{F_{ab}}{4} + \frac{1}{2} [D_a, X_I]^2 + \frac{i}{2} \Psi^T[\Psi, \Psi] \right. \\
\left. + \frac{1}{2} \Psi^T \Gamma_I [X_I, \Psi] - \frac{1}{4} \sum_{I,J} [X_I, X_J]^2 \right] \tag{5}
\]

where \(D_a = \partial_a - iA_a\) is the covariant derivative. The integration measure over the gauge fields is normalised as usual by the volume \(\text{Vol}(G)\) of the gauge group.

2. Relation to the partition function of the BFSS matrix model compactified on a circle

After shrinking one of the periods of the torus, the theory degenerates to a SYM theory reduced to one dimension. In the limit \(R \to 0\), the component \(A_\sigma\) of the gauge field enters in the action in the same way as the eight Higgs fields, and the \(O(8)\) symmetry of the action is enhanced to \(O(9)\). The functional integral (4) describes, in this limit, the partition function of a one-dimensional reduction of the SYM theory, with \(A_\sigma\) playing the role of \(X_9\). However, this is not yet the partition function of the BFSS model because the integration measure of the field \(A_\sigma\) is not identical to that of the rest eight Higgs fields. In order to obtain the same integration measure, we have to introduce in the original functional integral (4) a \(\delta\)-function of the zero mode
\[
A_\sigma^{(0)} = \frac{\Tr \int_{T^2} d^2\sigma A_\sigma}{\sqrt{NRT}} \tag{6}
\]

of the \(\sigma\)-component of the gauge field. Let us denote by \(\langle \delta(A_\sigma^{(0)}/\sqrt{2\pi}) \rangle_{\text{DVV}}\) the DVV functional integral (4) with the inserted \(\delta\)-function. In the limit \(R \to 0\) it indeed reduces to the partition function \(Z_{\text{BFSS}}\) of the BFSS model with action
\[
S_{\text{BFSS}} = R \int_0^T d\tau \Tr \left( \frac{1}{2} [D_\tau, X_I]^2 + \frac{i}{2} \Psi^T[D_\tau, \Psi] - \frac{1}{4} \sum_{I,J} [X_I, X_J]^2 + \frac{1}{2} \Psi^T \Gamma^I [X_I, \Psi] \right) \tag{7}
\]
defined on the circle of perimeter \( T \):

\[
\left\langle \delta \left( A_\sigma^{(0)} / \sqrt{2\pi} \right) \right\rangle_{DVV} \to_{R \to 0} Z_{BFSS},
\]

3. The IKKT model as the high-temperature limit of the BFFS model

In the limit \( T \to 0 \) the BFSS functional integral reduces to an integral over the constant modes of the bosonic and fermionic matrix fields. The resulting finite-dimensional integral over traceless matrix variables gives the partition function of the IKKT model. The \( T \to 0 \) limit of the BFSS model has been already analysed by Sethi and Stern \[8,11\] in connection with the computation of the Witten index. Below we repeat briefly their argument, adjusting it to our notations, which are slightly different.

The partition function of the completely reduced theory is defined as the finite matrix integral

\[
Z_{IKKT}(g) = \prod_{\mu=1}^{10} [dX^\mu] \prod_{\alpha=1}^{16} [d\Psi_\alpha] \exp \left( -\frac{1}{g} S_{IKKT}[X,\Psi] \right)
\]

with the action

\[
S_{IKKT} = -\frac{1}{4} \sum_{\mu,\nu} \text{Tr} [X_\mu, X_\nu]^2 + \frac{1}{2} \sum_{\mu} \text{Tr} \Psi^T [\Gamma^\mu X_\mu, \Psi].
\]

In refs. \[9,10\] the integral was understood as an integral over traceless matrices. Here we use the same normalisations as in \[9\] but, in order to facilitate the comparison with the previous integrals, we define the integration measures by

\[
[dX^\mu] = dX^\mu \delta \left( \frac{\text{Tr} X^\mu}{\sqrt{2\pi N}} \right), \quad [d\Psi_\alpha] = d\Psi_\alpha \frac{\text{Tr} \Psi_\alpha}{\sqrt{N}}
\]

where \( dX \) and \( d\Psi \) are the flat measures with normalization

\[
\int dX e^{-\frac{1}{2} \text{Tr} X^2} = 1, \quad \int d\Psi_\alpha d\Psi_\beta e^{-\text{Tr} \Psi_\alpha^T \Psi_\beta} = 1.
\]
The partition function of the IKKT model, with the above normalisation of the measure, is equal to [9,10]

\[ Z_{\text{IKKT}}(g) = g^{-\frac{7}{2}(N^2-1)} \mathcal{F}_N \sum_{m|N} \frac{1}{m^2} \]  

(13)

where \( \mathcal{F}_N \) is the "group factor" [9]

\[ \mathcal{F}_N = \frac{\sqrt{N} N!}{(\sqrt{\pi})^{N^2-1}} \frac{1}{2\pi N} \prod_{k=1}^{N} \left( \frac{2\pi}{k!} \right)^{2k}. \]  

(14)

The BFSS action (7) reduces, in the limit \( T \to 0 \) and after identifying the constant mode of the gauge field with the Higgs field \( X_{10} \), to the \( O(10) \)-symmetric IKKT action (10) multiplied by the area \( RT \) of the torus.

The limit for the measure is less trivial. Let us first consider the measure of the gauge field. The one-dimensional gauge field \( A \) corresponds, by the exponential map, to a generic element of the local gauge group \( U(\tau) = \hat{T} e^{\int_0^\tau A(t)dt} \). It is therefore convenient to parametrise the constant mode \( A \) of the gauge field in terms of the group element \( U = e^{iTA} \) and integrate over the Haar measure on \( SU(N) \) (normalised as \( \int_{SU(N)} dU = 1 \)). In the vicinity of any of the \( N \) central elements of \( SU(N) \) the group element can be parametrised by \( U = e^{2\pi ik/N} e^{iTA} \) and the Haar measure becomes

\[ dU \to \frac{T^{N^2-1}}{N\mathcal{F}_N} [dA] \]  

(15)

where \( \mathcal{F}_N \) is given by (14) and the measure \([dA]\) is identical to the measure \([dX]\) defined in (11). (The details of the calculation can be found in [16].) As was explained by Sethi and Stern [8], in the limit \( T \to 0 \) the integral over the gauge field is saturated by the vicinity of the \( N \) central elements of \( SU(N) \) and therefore by eq. (15)

\[ \mathcal{D}A \to \left( \frac{T^{N^2-1}}{\mathcal{F}_N} \right) [dX_{10}]. \]  

(16)

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4 The overall power of \( g \) can be determined by dimensional arguments. The integration over the fermions gives a pfaffian which is a homogenous polynomial in \( X/g \) of degree \( 8(N^2 - 1) \). The rescaling \( X \to g^{1/4} X \) makes the action \( g \)-independent and produces a factor \( g^{8(N^2 - 1)(8(1-1/4)-10/4)} = g^{7/2(N^2-1)}. \)
For the measures of the matter fields we find, from the kinetic part of the action (7) and the normalization (12),
\[ DX \rightarrow \left( \frac{T}{R} \right)^{-\frac{9}{2}N^2} \prod_{I=1}^{9} dX_I, \]
\[ D\Psi \rightarrow R^{-\frac{16}{7}N^2} \prod_{\alpha} d\Psi_\alpha. \] (17)

Taking into account the rescaling of the zero-modes
\[ \prod_I \delta \left( \frac{X_I^{(0)}}{\sqrt{2\pi}} \right) \rightarrow (RT)^{-\frac{2}{7}} \prod_I \delta \left( \frac{\text{Tr}X_I}{\sqrt{2\pi N}} \right), \]
\[ \prod_\alpha \delta \left( \frac{\Psi_\alpha^{(0)}}{\sqrt{2\pi}} \right) \rightarrow (RT)^{\frac{16}{7}} \prod_\alpha \delta \left( \frac{\text{Tr}\Psi_\alpha}{\sqrt{N}} \right), \] (18)
and combining all factors of $T$ and $R$, we finally obtain
\[
\left\langle \delta \left( \frac{A^{(0)}}{\sqrt{2\pi g}} \right) \right\rangle_{\text{DVV}} \xrightarrow{r,T \to 0} \frac{(RT)^{-\frac{2}{7}(N^2-1)}}{T \mathcal{Z}_N} \mathcal{Z}_{\text{KKKT}}(1/RT) = \frac{1}{T} \sum_{m|N} \frac{1}{m^2}.
\] (19)

3. Quasiclassical calculation of the partition function of the DVV model

As argued by Dijkgraaf, Verlinde, and Verlinde [5], in the infrared limit $RT \to \infty$ the non-diagonal components of the gauge and matter fields become infinitely massive and the bosonic and fermionic potentials turn into constraints. Under the constraint that all matrices $\Phi = \{X_I, \Psi_\alpha, iD_a = i\partial_a + A_a\}$ are simultaneously diagonalisable, for each field configuration there exists a unitary matrix $V(\sigma, \tau)$ such that
\[ \Phi(\sigma, \tau) = V^{-1}(\sigma, \tau)\Phi^D(\sigma, \tau)V(\sigma, \tau), \]
where $\Phi^D = \text{diag}\{\Phi_1, ..., \Phi_N\}$. We have therefore
\[ \Phi^D(R, \tau) = \hat{S}^{-1}\Phi^D(0, \tau)\hat{S}, \]
\[ \Phi^D(\sigma, T) = \hat{T}^{-1}\Phi^D(\sigma, 0)\hat{T}. \] (20)
where \( \hat{S} = V(0, \tau)V^{-1}(R, \tau) \) and \( \hat{T} = V(\sigma, 0)V^{-1}(\sigma, T) \). By construction, \( \hat{S}\hat{T} = \hat{T}\hat{S} \).

Assuming that all the eigenvalues are distinct, the only unitary transformations relating two diagonal matrices represent permutations of their diagonal elements. Therefore the matrices \( \hat{S} \) and \( \hat{T} \) act as two commuting permutations \( \hat{s} : i \to s_i \) and \( \hat{t} : i \to t_i \) of the symmetric group \( S_N \).

\[
\begin{bmatrix}
\hat{T}^{-1}\Phi\hat{T}
\end{bmatrix}
_i = \Phi_{t_i},
\begin{bmatrix}
\hat{S}^{-1}\Phi\hat{S}
\end{bmatrix}
_i = \Phi_{s_i}
\tag{21}
\]

and, in particular, do not depend on the coordinates \( \sigma \) and \( \tau \).

Each pair of permutations describes a \( N \)-covering of the target-space torus consisting, in general, of several connected components. Each connected component can be interpreted as the world-sheet of a string wrapping several times the torus in both directions. A \( q \)-component covering corresponds to a decomposition

\[
\hat{s} = \prod_{k=1}^{q} \hat{s}^{(k)}, \quad \hat{t} = \prod_{k=1}^{q} \hat{t}^{(k)}
\]

where all factors commute with each other and satisfy

\[
(\hat{s}^{(k)})^{j_k} (\hat{t}^{(k)})^{m_k} = (\hat{s}^{(k)})^{n_k} (\hat{t}^{(k)})^{l_k} = 1, \quad \sum_{k=1}^{q} (n_k m_k - l_k j_k) = N.
\tag{22}
\]

The \( k \)-th world-sheet torus is the complex plane \( \omega = \sigma + i\tau \) factored by the periods \( \omega_1 = n_k R + i (l_k T) \) and \( \omega_2 = j_k R + i (m_k T) \), and covers the target torus \( N_k = n_k m_k - l_k j_k \) times. In this way the orbifold structure of the target space generates the sum over all twisted boundary conditions which becomes, in the limit \( N \to \infty \) and for the "long" strings only, the integral over all complex structures of the world-sheet tori.

Due to the periodicity condition (20) on the fermions, each connected component has 16 fermionic zero modes. In our problem we are only interested in contributions with exactly 16 fermionic zero-modes, therefore only one-sheet \( (q = 1) \) coverings will be relevant.

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5 After our manuscript has been finished, we learned about the paper [17] where the a similar interpretation of the partition function of pure \( U(N) \) YM theory on the torus is presented.
A covering with periods $\omega_1 = nR + i(lT)$ and $\omega_2 = jR + i(mT)$ wrapping the target torus $N = nm - jl$ times is defined by two permutations $\hat{s}$ and $\hat{t}$ satisfying

$$\hat{s}\hat{t} = \hat{t}\hat{s}, \quad \hat{s}^n\hat{t}^m = 1.$$ 

By a mapping class transformation we can reduce this to

$$\hat{s}\hat{t} = \hat{t}\hat{s}, \quad \hat{s}^n = \hat{s}^j\hat{t}^m = 1 \quad (mn = N, j = 0, 1, ..., n - 1). \quad (23)$$

The explicit solution of (23) is, up to an internal homomorphism,

$$\hat{s} = \{i \rightarrow i + m \mod N\}, \quad \hat{t} = \begin{cases} \{i \rightarrow i + 1 \mod m\} & \text{if } j = 0 \\ \{i \rightarrow i - j \mod N\} & \text{if } j = 1, ..., n - 1. \end{cases} \quad (24)$$

After having identified the distinct topological sectors, we can write the partition function as a sum over all pairs $m$ and $n$ such that $mn = N$ and $j = 0, 1, ..., n - 1$. The diagonal matrix variable $\Phi^D(\sigma, \tau)$ with boundary conditions belonging to the equivalence class $(m, n; j)$ can be considered as a scalar variable defined on the torus with periods $\omega_1 = nR$ and $\omega_2 = imT + jR$ in the $\omega = \sigma + i\tau$ plane.

Now let us proceed to the evaluation of the partition function in the infrared limit. In the limit $RT \to \infty$ the measure over the gauge field reduces to the integral over the diagonal components of the gauge field, normalised by the volume of the diagonal gauge group $G^D$ which is the localized $U(1)^N$. Taking into account the overall factor $1/N!$ because the eigenvalues are determined up to a permutation, we get

$$\int \frac{DA}{\text{vol}(G)} DXD\Psi \rightarrow \frac{1}{N!} \sum_{\hat{s}\hat{t}=\hat{s}\hat{t}} \int \frac{DA^D}{\text{vol}(G^D)} DXD\Psi^D. \quad (25)$$

Let $Z_{(m,n;j)}$ be the value of the functional integral over the diagonal fields satisfying the twisted boundary conditions (20)-(21). The number of permutations corresponding to this condition is equal, for each $j$, to the number of combinations of $m$ and $n$ elements times the number of cyclic permutations of order $m$ and $n$,

$$\frac{N!}{n!m!} (m - 1)!(n - 1)! = (N - 1)!$$
We have therefore

$$Z_{DVV} = \frac{(N-1)!}{N!} \sum_{mn=N} \sum_{j=0,...,n-1} Z_{[m,n;j]}$$

(26)

where $Z_{[m,n;j]}$ is the partition function of the Abelian ($N=1$) DVV model defined on the torus of area $NRT$ with periods $\omega_1 = nR$ and $\omega_2 = jR + i(mT)$. The latter is a product of the partition function of the Abelian gauge field and that associated with the diagonal components of the matter field

$$Z_{m,n;j} = Z_{\text{gauge}}^{[m,n;j]} Z_{\text{matter}}^{[m,n;j]}.$$  

(27)

The contribution of the $U(1)$ gauge field to the partition function is given, in the gauge $A^D = 0$, to the functional integral with respect to the angular variable

$$\theta(\tau) = \int_0^R \text{Tr} A^D_\sigma(\tau, \sigma) d\sigma$$

$$= \int_0^{nR} A_\sigma(\tau, \sigma) d\sigma.$$  

(28)

Here $A_\sigma$ denotes the Abelian gauge field on the ”world-sheet” torus that corresponds to the collection of the $N$-component field of $A^D_\sigma$ on the space-time torus.

The gauge-field partition function is

$$Z_{\text{gauge}}^{[m,n;j]} = \int_{\theta(mT)=\theta(0)} \mathcal{D}\theta e^{-\frac{1}{\pi R} \int_0^{mT} d\sigma (\partial_\sigma \theta)^2} = \sum_{p \in \mathbb{Z}} e^{-\frac{RT}{2} Np^2}.$$  

(29)

The result of the integration depends only on the area $RT$ and not on the modular parameter of the torus which reflects the symmetry of the two-dimensional gauge theory with respect to area-preserving diffeomorphisms. Further, the integral of the matter fields with our conventions for the zero modes is exactly one due to supersymmetry [18]

$$Z_{\text{matter}}^{[m,n;j]} = 1.$$  

(30)

This gives the result (2) for the complete integral which can be written, after a Poisson resummation, as

$$Z_{DVV} = \sum_{m|N} \frac{1}{m} \sqrt{\frac{2\pi}{RTN}} \sum_{E \in \mathbb{Z}} e^{-\frac{1}{2} \frac{(2\pi E)^2}{RTN}}.$$  

(31)
4. The DVV partition function with subtracted constant mode of the gauge field

The evaluation of the DVV partition function with inserted delta function of the zero-mode $A_\sigma^{(0)}$ defined by eq. (6) can be done in the same way as in the previous section. In the topological sector $[m, n; j]$ this zero-mode is expressed through the angular variable (28) as

$$A_\sigma^{(0)} = \int_0^{mT} d\tau \theta(\tau) \sqrt{NRT}.$$ (32)

The combinatorics is the same as in the previous section and the only difference is in the expression of the gauge field partition function in each topological sector. The latter is given by the one-dimensional functional integral with respect to the field $\theta(\tau)$ taking its values in the unit circle

$$\tilde{Z}_{[m,n;j]} = \sqrt{NRT} \frac{1}{mT} \sum_{p \in \mathbb{Z}} e^{-\frac{1}{2RT}Np^2} = \frac{1}{mT} \sum_{E \in \mathbb{Z}} e^{-\frac{1}{2RT}E^2}.$$ (33)

which is evaluated as

$$\tilde{Z}_{[m,n;j]} = \sqrt{\frac{NRT}{2\pi}} \frac{1}{mT} \sum_{p \in \mathbb{Z}} e^{-\frac{1}{2RT}Np^2}.$$ (34)

Summing over all topological sectors we find, instead of (31),

$$\langle \delta \left( \frac{A_\sigma^{(0)}}{\sqrt{2\pi}} \right) \rangle_{DVV} = \frac{1}{T} \sum_{m \mid N} \frac{1}{m^2} \sum_{E \in \mathbb{Z}} e^{-\frac{1}{2RT}E^2}.$$ (35)

Comparing this expression, which by its derivation can be trusted only in the infrared limit, with Eq. (19), we see that it is equally true in the ultraviolet limit.

5. Conclusion

We have computed the quasiclassical partition function of the Euclidean matrix string theory compactified on a two-dimensional torus $\mathcal{T}^2$, with doubly periodic boundary conditions and a minimal set of zero-modes removed. We have shown that the relevant degrees
of freedom are described, in the limit of large area of the torus, by an abelian supersymmetric sigma model accompanied by a $U(1)$ gauge theory and defined on an orbifold space $S^N \mathcal{T}^2 = (\mathcal{T}^2)^{\otimes N}/S_N$ where $S_N$ is the symmetric group of $N$ elements. The sigma-model and the gauge field are coupled through the boundary conditions which are described by a pair of commuting permutations of $S_N$.

In general, each such configuration can be interpreted a set of noninteracting strings in a light-cone gauge, with additional gauge degrees of freedom on the world sheet. The world sheet of each of these strings defines a multiple covering of the space-time torus $\mathcal{T}^2$ characterized by a modular parameter $\omega_2/\omega_1$, with $\omega_{1,2}$ sweeping the lattice $\mathbb{RZ} + iT\mathbb{Z}$. In the large $N$ limit the sum over coverings converges to an integral with a correct modular-invariant measure, under the condition that we are far from the boundary of the moduli space. Thus the IR degrees of freedom of the matrix model correctly reproduce the functional integral over the configurations of the ”long” strings. On the other hand, near the boundary of the moduli space, that is when one of the periods become small, the sum conserves its discrete character and the corresponding excitations (”short” strings) describe particles. Note that the combinatorics simplifies drastically if one considers grand-canonical partition functions.

With the convention that a minimal set of zero-modes is deleted from the functional measure, which is the case considered in the present paper, the allowed field configurations of the orbifold theory are described by a single string whose world sheet wraps $N$ times the space-time torus $\mathcal{T}^2$. Our computation gives an intuitive understanding of the sum over the divisors of $N$ in the expression for the partition function of the IKKT model (1). It is quite analogous to the the argument presented by Green and Gutperle where this last partition function was compared with the partition function of the matrix quantum mechanics at infinite temperature. Their calculation was based on the assumption of the existence of bound states of D0-branes which lead to the sum over the divisors of $N$. In our case the sum over the divisors comes from the sum over the conjugacy classes of
permutations defining the different topological sectors of the $S_N$-orbifold theory. This is a trivial example of the Hecke operator, defined by its action on a modular-invariant form $\mathcal{A}$ (actually a constant in our case)

$$\mathcal{H}_N[\mathcal{A}](\tau) = \frac{1}{N} \sum_{N=m,n \atop 0 \leq j < n} \mathcal{A}\left(\frac{m\tau + j}{n}\right),$$

which takes care of the inequivalent $N$-fold wrappings. (For more general discussion see [19,20].) Comparing our calculation with the argument of Green and Gutperle we see that the bound states of D0-branes can be indeed interpreted as strings winding several times around the spatial ($\sigma$-) dimension of the two-torus. Moreover, we have shown that the high-temperature limit of the DVV model reproduces the D-instantons contributions predicted in [12], which confirms the dual interpretation of the DVV model as describing wrapped D1-branes.

We consider as the principal result of this paper not the computation of the IR partition function itself, which is rather trivial, but its comparison with the known result for the partition function of the completely reduced gauge theory (the IKKT model). For this purpose we have established the exact relations between the partition functions of the SYM theories reduced to 2, 1 and 0 dimensions, which we presented in Section 2. Then we were able to check that when extrapolated to the small area limit, our quasiclassical result reproduces exactly the known expression of the IKKT partition function. Therefore we conjecture that the expressions (31) and (35) are actually exact, that is, that the theory has the property of having exact quasiclassics. We expect that this can be proved by extending to the two-dimensional case the calculation of Moore, Nekrasov and Shatashvili [10] based on the Witten’s description of the two-dimensional gauge theory as cohomological field theory [21].

Finally, let us remark that it is possible to give a very simple explanation of the pre-exponential factors in the expression (31) if the partition function of the DVV model is considered as a certain limit of the partition function of a supersymmetric Schild string defined on the orbifold $S^N T^2$. We intend to report on this subject in the near future.
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