LEPTONIC K DECAYS

Carlo Rabbia
CERN

1. INTRODUCTION

I shall discuss some of the problems related to leptonic decays of charged and neutral K-mesons. Most of the topics of the present report have already been discussed in the review paper given by J. Cronin at the 14th International Conference on High-Energy Physics, held at Vienna in 1968. This present paper will be divided into five parts.

i) The $K^0_{e2}$ decay.
ii) The $K^0_{e3}$ decays, and the determination of the Cabibbo angle.
iii) The problem of the $K^0_{\mu3}$ form factors.
iv) The $\Delta I = \frac{1}{2}$ rule for leptonic decays.
v) Tests of the selection rule $\Delta Q/\Delta S$ for $K^0_{e2}$ and $K^0_{e4}$ decays.

2. THE DECAY $K^+ \rightarrow e^+ + \nu$

The decay of the charged kaon into an electron and a neutrino provides a sensitive test of the presence of a pseudoscalar interaction. Without radiative corrections, the branching ratio of $K^+ \rightarrow e^+ + \nu$ relative to $K^+ \rightarrow \mu^+ + \nu$ is

$$R_\circ = \left[ \frac{1 - (m_e/m_K)^2}{1 - (m_\nu/m_K)^2} \right]^2 \times \left| \frac{f_p}{f_\nu} \right|^2 \left( \frac{f_p - (m_e/m_K) f_\nu}{f_\nu} \right)^2,$$

where $f_p$ and $f_\nu$ are the pseudoscalar and axial vector coupling constants. They can be complex if T invariance is violated. For a pure axial vector coupling we get $R_\circ = 2.57 \times 10^{-5}$, and for pure pseudoscalar coupling $R_\circ = 1.10$.Muon-electron universality is assumed in the calculation. The small branching ratio predicted for an axial vector coupling poses severe experimental problems.

2.1 Radiative corrections

Radiative corrections have been evaluated by Berman (BERMAN, 1959) and Neville (NEVILLE, 1961). They are significant and therefore we shall discuss them in some detail. There are three contributions:

i) virtual photon processes;
ii) inner bremsstrahlung;
iii) $K^+ \rightarrow e^+ + \nu + \gamma$ in which the photon is emitted from the structure of the decay vertex.

For a pure axial vector interaction the correction to the branching ratio due to the first two effects is:
\[ R = R_0 \left( 1 - 0.185 + 0.0273 \ln \left( \frac{2 \epsilon}{m_e} \right) - 0.032 \ln \left( \frac{2 \epsilon'}{m_e} \right) \right) \]

The second term takes into account the suppression of the rate due to virtual photons; the third term results from inner bremsstrahlung over the energy interval accepted \( E_{\text{max}} - \epsilon \) to \( E_{\text{max}} \) of the decay positron; the last term is the corresponding term for \( K^+ \to \mu^+ + \nu + \gamma \).

The third effect, namely \( K^+ \to e^+ + \nu + \gamma \) proceeding by a direct process has been considered in detail by Neville (NEVILLE, 1961). This radiative process in some circumstances may be comparable to the \( K^+ \to e^+ + \nu \) rate in spite of the additional electromagnetic vertex since now the depression factor \( m_\nu/m_K \) is missing. So far the decay rate \( K^+ \to e^+ + \nu + \gamma \) is unknown. Both axial-vector and vector couplings are possible. The vector contribution can be approximately evaluated, assuming it is mediated by an exchange of a \( K^* \) state between \((K\gamma)\) and \((\omega\nu)\) vertices. One could guess an axial-vector contribution of about the same strength. With these assumptions the rate for electron energies in the interval \( E_{\text{max}} - \epsilon \) to \( E_{\text{max}} \) is as large as the whole non-radiative process for \( \epsilon = 25 \text{ MeV} \). However, as one can easily see with helicity considerations, events in which the electron is emitted with an energy close to \( E_{\text{max}} \) are characterized by a lepton pair and a photon, approximately collinear, going in opposite directions. Therefore the process \( K^+ \to \pi^+ + \nu + \gamma \) can be largely suppressed, detecting the photon in anticoincidence.

The weak intermediate boson, if it exists, contributes to the radiative corrections. The effect has been calculated by Shaffer (SHAFFER, 1962), who finds terms of the order of \((m_K/m_W)^2\). At present \((m_K/m_W)^2 \leq 1/16\), and the effect is far too small to become of experimental interest. Larger effects are predicted for the radiative decay \( K^+ \to e^+ + \nu + \gamma \) (LEE, 1960).

2.2 Experimental results

Two measurements of the \((K^+ \to e^+ + \nu)/(K^+ \to \mu^+ + \nu)\) branching ratio have been reported so far. The first (BOWEN, 1967) led to the observation of seven events with an expected background contamination of 2.8 events. Authors have quoted an absolute branching ratio:

\[ \frac{\Gamma(K^+ \to e^+ + \nu)}{\Gamma(K^+ \to \text{all})} = \left( 2.4^{+4.8}_{-4.3} \right) \times 10^{-5} \]

However, this value does not include a contribution of \( K^+ \to e^+ + \nu + \gamma \) events due to inner bremsstrahlung which, if subtracted, would reduce the branching ratio to:

\[ \frac{\Gamma(K^+ \to e^+ + \nu)}{\Gamma(K^+ \to \text{all})} = \left( 1.9^{+5.7}_{-4.2} \right) \times 10^{-5} \]

A second, more refined, experiment has been reported by the Oxford Group (BOTTERILL, 1967; BOTTERILL, 1968). These authors have observed ten events with an estimated background contamination of only 3%. The result (Fig. 1) has been normalized against \( K_{u3} \) events observed with alternative triggering conditions and gives:

\[ \frac{\Gamma(K^+ \to e^+ + \nu)}{\Gamma(K^+ \to \text{al})} = \left( 1.9^{+0.7}_{-0.5} \right) \times 10^{-5} \]
The contribution of the radiative decays $K^+ \rightarrow e^+ + \nu + \gamma$ is strongly suppressed (factor of 50) by a $\gamma$-counter in anticoincidence, and it is assumed to be negligible.

Both experimental results are fully consistent with the prediction

$$\frac{\Gamma(K^+ \rightarrow e^+ + \nu)}{\Gamma(K^+ \rightarrow \text{all})} = 2.4 \times 10^{-5}$$

computed for $f_p = 0$, muon-electron universality, and radiative corrections. Results can be interpreted in terms of the complex ratio $f_p/f_A$ (Fig. 2). For the most unfavourable case:

$$|f_p/f_A| < 2 \times 10^{-3} \quad (90\% \text{ confidence level}),$$

unless, of course, a larger amount of pseudoscalar coupling is exactly compensated by some degree of failure of $\mu$-e universality.

3. THE DECAYS $K^+ \rightarrow \pi^0 + e^+ + \nu$ AND $K_L^0 \rightarrow \pi^+ + e^- + \nu$

With the usual convention of an even $\pi$-K parity, scalar, vector, and tensor interactions can contribute to the decay process. The interpretation of the electron decays is
Fig. 2: Limits for the (complex) ratio between axial vector and pseudoscalar coupling which follow from experimental value of the $K^+ \rightarrow e^+ + \nu$ branching ratio.

made simpler, since terms of the order of $m_e c^2$ are generally negligible compared with energies involved. These terms can, however, be important in the case of $K \rightarrow \pi + \mu + \nu$ decays.

The matrix element for $K \rightarrow \pi + e + \nu$ decay, neglecting terms of the order $m_e$, is of the general form:

\[ M \sim m_K \left( \frac{f_\pi}{m_K} \right) \bar{u}_e (1 + Y_\pi) u_\nu + \frac{1}{2} \frac{f_\pi}{m_K} \left( p_{K^0} + p_{\bar{K}^0} \right) \bar{u}_\nu \gamma^\mu \left( 1 + y_\gamma \right) u_e \]

\[ + \frac{1}{m_K} \frac{f_T}{m_K} \left( p_{K^0} - p_{\bar{K}^0} \right) \bar{u}_\nu \sigma_{\mu\nu} \left( 1 + Y_\gamma \right) u_e \]

where $p(K)$ and $p(\pi)$ are the four-momenta for the $K$ and $\pi$, respectively. All form factors are functions of the four-momentum transfer:

\[ q^2 = (p^{(K)} - p^{(\pi)})^2 = (m_K - m_\pi)^2 - 2m_K T_\pi \]

which can take values in the interval $m_\pi^2 - 7.1 m_K^2$.

The variation of the form factor with $q^2$ is determined by the strong interactions. If a single $K^*$ state of mass $M$ dominates, the $q^2$ dependence is determined by a single propagator. In particular, the $K^*(890)$ has quantum numbers $1^-$, suitable for a decay by means of vector interaction. Then:

\[ f(q^2) = f(0) \frac{M^2}{M^2 + q^2} \]
If several states are exchanged a multi-parameter expansion is required. Since the \( f(q^2) \) dependence on \( q^2 \) is weak, only the first two terms of a power expansion are retained:

\[
\tilde{f}(q^2) = \tilde{f}(0) \left[ 1 + \lambda \frac{q}{m_\pi} \right]
\]

An additional four-momentum transfer dependence would be generated by the weak intermediate boson \( W \), if it exists:

\[
\tilde{f}_\omega(q^2) = \tilde{f}(0) \frac{m_W^2}{m_W^2 + q^2} = \tilde{f}(0) \left[ 1 + \lambda_W \frac{q}{m_\pi} \right]
\]

where \( \lambda_W = \left( \frac{m_\pi}{m_W} \right)^2 \). Since the mass of the \( W \) meson is now \( > 2.0 \) GeV, these effects are expected to be smaller than the non-locality due to strong interactions. Unfortunately, the two effects are experimentally indistinguishable. In particular, since the electron and neutrino are still emitted at the same point, the energy spectrum of the electron at a fixed \( q^2 \) is unchanged.

3.1 Form of the decay interaction

All available data are consistent with the absence of scalar and tensor couplings. Experimentally one must be able to separate the effects due to momentum dependence of a single dominant vector form factor from small additional contributions of other types of couplings. Very fortunately, vector-tensor and vector-scalar interference terms are of order \( \left( \frac{m_e}{m_K} \right) \) and can be neglected. Thus the analysis involves only two unknown parameters \( |f_S/f_V| \) and \( |f_T/f_V| \). Experimental upper limits on \( |f_S/f_V| \) and \( |f_T/f_V| \) have been summarized in Table 1.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Assumption</th>
<th>( f_S/f_T )</th>
<th>( f_T/f_V )</th>
<th>Confidence level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kulyukina, 1967</td>
<td>Constant form factors</td>
<td>&lt; 0.15</td>
<td>&lt; 1.0</td>
<td>68%</td>
</tr>
<tr>
<td>Jensen, 1964</td>
<td>Constant form factors or same energy dependence</td>
<td>&lt; 0.13</td>
<td>&lt; 1.20</td>
<td>83%</td>
</tr>
<tr>
<td>Cester, 1966</td>
<td>Constant form factors</td>
<td>&lt; 0.26</td>
<td>&lt; 0.80</td>
<td>90%</td>
</tr>
<tr>
<td>Kalamus, 1967</td>
<td>( V, ) variable S,T constant form factors</td>
<td>&lt; 0.30</td>
<td>&lt; 1.10</td>
<td>95%</td>
</tr>
<tr>
<td>Bellotti, 1968</td>
<td>( V, ) variable S,T constant form factors</td>
<td>&lt; 0.18</td>
<td>&lt; 1.0</td>
<td>90%</td>
</tr>
<tr>
<td>Botterill, 1968</td>
<td>( V, ) variable S,T constant form factors</td>
<td>&lt; 0.23</td>
<td>&lt; 0.58</td>
<td>90%</td>
</tr>
</tbody>
</table>

\[
M = m_K f_S(q^2) \bar{u}_\nu (1 + \gamma_s) u_e + \frac{1}{2} \bar{f}_s(q^2) (p^{(K)} + p^{(\pi)})_\alpha \bar{\nu}_\nu \gamma_\alpha (1 + \gamma_s) u_e + \frac{1}{m_K} f_T(q^2) p_\alpha \bar{p}_B \bar{\nu}_\nu \sigma_\alpha \beta (1 + \gamma_s) u_e
\]
In one class of experiments, only the electron spectrum is observed. In others, the full reconstruction of the event is exploited. One can conclude that

\[
\left| \frac{f_S}{f_V} \right| < 0.2 \quad \text{and} \quad \left| \frac{f_T}{f_V} \right| < 0.58,
\]

corresponding to intensities of about \( |I_S/I_V| < 9 \times 10^{-2} \) and \( |I_T/I_V| < 2.0 \times 10^{-3} \), respectively. On the basis of these results we shall assume a strict validity of the V-A form of the interaction, and set \( f_S = f_T = 0 \). For consistency of notation with universal conventions we shall replace \( f_V(q^2) \) with \( f_+(q^2) \) and denote with \( \lambda_+ \) the linear four-momentum coefficient in the form factor.

3.2 Experiments on the energy dependence of the form factor \( f_+(q^2) \)

Several experiments have looked at effects of the four-momentum transfer dependence of the vector form factor

\[
f_+(q^2) = f_+(0) \left[ 1 + \lambda_+ \frac{q^2}{m_T^2} \right].
\]

The relevant information is presented in Tables 2 and 3 for \( K^+ \) and \( K_L \) decays respectively. Some results are consistent with a constant form factor; others observe a small energy variation. Applying to all results the standard procedure of weighted averages, the combined results are

\[
\lambda_+^K = + 0.029 \pm 0.010 \quad \text{(for} \quad K^+ \rightarrow \pi^+ + \nu \text{)}
\]

\[
\lambda_+^\bar{K} = + 0.049 \pm 0.008 \quad \text{(for} \quad K_L \rightarrow \pi^0 + e^+ + \nu \text{)}
\]

The \( \Delta I = \frac{1}{2} \) rule for leptonic decays predicts the same energy dependence of the form factors for \( K^+ \) and \( K_L \). A value \( \lambda_+ = 0.025 \) corresponds to a characteristic mass \( M \) for the 1\(^-\) intermediate state equal to 880 MeV. An error of \( \pm 0.005 \) corresponds to an uncertainty in the mass of about \( \pm 90 \) MeV.

### Table 2

<table>
<thead>
<tr>
<th>Authors</th>
<th>Method</th>
<th>Radiative corrections</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jensen, 1964</td>
<td>Pion spectrum</td>
<td>No</td>
<td>(-0.01 \pm 0.029)</td>
</tr>
<tr>
<td>Bellotti, 1967</td>
<td>Dalitz plot</td>
<td>Yes</td>
<td>(0.045 \pm 0.017)</td>
</tr>
<tr>
<td>Imlay, 1967</td>
<td>Dalitz plot</td>
<td>No</td>
<td>(0.016 \pm 0.016)</td>
</tr>
<tr>
<td>Kalmus, 1967</td>
<td>Pion spectrum</td>
<td>No</td>
<td>(0.028 \pm 0.013)</td>
</tr>
<tr>
<td>Butterill, 1968</td>
<td>Electron spectrum</td>
<td>Yes</td>
<td>(0.080 \pm 0.040)</td>
</tr>
<tr>
<td>Eisler, 1968</td>
<td>Pion spectrum</td>
<td>No</td>
<td>(-0.020 \pm 0.080)</td>
</tr>
</tbody>
</table>

Weighted average \( \lambda_+ = 0.029 \pm 0.010 \)
3.3 Test of $\nu$-e locality

It is assumed that even in the presence of a $W$ particle, the neutrino and electrons are still emitted at a point. To test such an assumption experimentally, one has to look for dependence of the form factor on the Lorentz invariant associated to the $(\nu$-$e)$ pair, namely:

$$ q_{\nu}^2 = \left[ p_{\nu}^{(\mu)} - p_{e}^{(\mu)} \right]^2 $$

A possible diagram which could produce a $(\nu$-$e)$ non-locality is shown in Fig. 3. It is not evident how such a diagram could give origin to a $V$-$A$ coupling (WILLIAMSON, 1965).

If the dependence on $q_e^2$ is not too strong, one can parametrize:

$$ p_+(q^2_h, q^2_e) = f(0,0) \left[ 1 + \lambda_+ \frac{q^2_h}{m_h^2} + \lambda_e \frac{q^2_e}{m_e^2} \right] $$

Fig. 3: Diagram of an interaction which could produce a deviation from $\nu$-$e$ locality.

where the suffixes $h$ and $e$ indicate hadrons and lepton Lorentz invariants, respectively.

A fit of the data of Bellotti et al. (BELLOTTI, 1967) has given $\lambda_e = 0.011^{+0.013}_{-0.011}$, independently of the value of $\lambda_+$ in the interval $-0.1 < \lambda_+ < 0.1$. Note that the presence of a tensor term would also simulate experimentally $\lambda_e \neq 0$.

3.4 Radiative corrections

The radiative corrections have been evaluated by Ginsberg (GINSBERG, 1967), assuming a phenomenological $K$-$\pi$ vertex and using perturbation theory. The result depends logarithmically on a cut-off $\Lambda$, just as in the case of the ordinary beta decay. The correction is considerable, averaging about 5% in absolute magnitude over the larger part of the Dalitz plot (Fig. 4). The dependence on the cut-off is weak for "reasonable" choices of $\Lambda$, i.e. around the proton mass. The most apparent effect of the radiative corrections is that they simulate an additional energy dependence of the form factor (Fig. 5). It is interesting to remark that in most of the experiments on $\lambda_+$ (see Tables 3 and 4), radiative corrections have not been accounted for in the analysis.
Fig. 4 : Fractional radiative corrections in per cent at various points of the $K_{e3}^+$ Dalitz plot, for $\Lambda = m_p$ (GINSBERG, 1967).

Fig. 5 : Zero-order pion spectrum of $K^+ \rightarrow \pi^0 + e^+ + \nu$ decay and radiative corrections for $\Lambda + m_p$ (GINSBERG, 1967).
Table 3

Linear energy dependence parameter $\lambda_\ast$ for $K_L$ decays

<table>
<thead>
<tr>
<th>Authors</th>
<th>Method</th>
<th>Radiative corrections</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Luers, 1964</td>
<td>Dalitz plot</td>
<td>No</td>
<td>$+0.07 \pm 0.06$</td>
</tr>
<tr>
<td>Fisher, 1965</td>
<td>Dalitz plot</td>
<td>No</td>
<td>$+0.15 \pm 0.08$</td>
</tr>
<tr>
<td>Firestone, 1967</td>
<td>Dalitz plot</td>
<td>No</td>
<td>$-0.01 \pm 0.02$</td>
</tr>
<tr>
<td>Kadyk, 1967</td>
<td>Pion spectrum</td>
<td>No</td>
<td>$+0.01 \pm 0.015$</td>
</tr>
<tr>
<td>Lowys, 1967</td>
<td>Pion spectrum</td>
<td>No</td>
<td>$+0.08 \pm 0.100$</td>
</tr>
<tr>
<td>Aronson, 1967</td>
<td>Pion spectrum</td>
<td>No</td>
<td>$+0.020 \pm 0.013$</td>
</tr>
<tr>
<td>Basile, 1968</td>
<td>Dalitz plot</td>
<td>No</td>
<td>$+0.023 \pm 0.012$</td>
</tr>
</tbody>
</table>

Weighted average $\lambda_\ast = +0.019 \pm 0.008$

Table 4

Determinations of the transverse muon polarization in $K_{\mu 3}$ decays

<table>
<thead>
<tr>
<th>Authors</th>
<th>Initial state</th>
<th>Method</th>
<th>$&lt; p_\perp &gt;$</th>
<th>Im ($\xi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young, 1967</td>
<td>$K_L$</td>
<td>Counters</td>
<td>$(0.25 \pm 1.25) \times 10^{-2}$</td>
<td>$-(1.25 \pm 6.6) \times 10^{-2}$</td>
</tr>
<tr>
<td>Bartlett, 1966</td>
<td>$K_L$</td>
<td>Counters</td>
<td>$(2 \pm 7) \times 10^{-2}$</td>
<td>$+(11 \pm 35) \times 10^{-2}$</td>
</tr>
<tr>
<td>Bettels, 1968</td>
<td>$K^+$</td>
<td>Heavy-liquid bubble chamber</td>
<td>-</td>
<td>$0.1 \pm 0.4$</td>
</tr>
</tbody>
</table>

3.5 The $K^+ \to \pi^0 + e^+ + \nu$ branching ratio

The experimental results are summarized in Fig. 6. The experiment of Callahan (CALLAHAN, 1966) performed in a heavy-liquid bubble chamber, appears in substantial disagreement with the bulk of the other results, especially with the magnetic spectrometer results of Auerbach et al. (AUERBACH, 1967), Botterill et al. (BOTTERILL, 1968), and the heavy-liquid bubble chamber experiment of the $X_2$ collaboration (EIGHTEN, 1968).

Background corrections are normally quite small, and the major problem in the determination of the $K^+ \to e^+ + \pi^0 + \nu$ branching ratio is in estimating the detection efficiency. Callahan has evaluated the detection probability for positrons by means of a Monte Carlo program in which the history of each particle was followed in the heavy-liquid bubble chamber. The detection efficiency was found to vary between about 80% at 50 MeV and 40% at 200 MeV. Uncertainties are, of course, quite difficult to estimate. On the basis of "results of similar programs and general programs of this type", an error of 2% was attributed by Callahan to the efficiency calculations.
Fig. 6: Summary of experimental results on the $K^+ \rightarrow \pi^0 + e^+ + \nu$ branching ratio.

In the more recent experiments with a magnetic spectrometer (Auerbach, 1967; Botterill, 1968) positrons are identified with the help of a gas Čerenkov counter. The counter has been extensively tested and calibrated, and is practically 100% efficient.

In the experiment of the $X_2$ collaboration (Eichten, 1968) in the 1.1 m³ CERN heavy-liquid bubble chamber, a fiducial volume cut on the decay point and a potential path-cut were applied, so as to ensure 100% detection efficiency to each secondary. The potential path-cut is intended to ensure the observation of the entire track to a hypothetical decay positron of maximum energy produced in the same point and emitted in the same direction as the retained track.

Averaging all results, except the one of Callahan, we find

$$\frac{\Gamma(K^+ \rightarrow \pi^0 + e^+ + \nu)}{\Gamma(K^+ \rightarrow \text{all})} = (4.99 \pm 0.14) \times 10^{-2}$$

and a $\chi^2 = 3.8$ for 9 degrees of freedom (if the result of Callahan is included, the $\chi^2$ increases to 30 for 10 degrees of freedom). For a $K^+$ lifetime $\tau_{K^+} = (1.236 \pm 0.003) \times 10^{-8}$ sec, we compute $\Gamma(K^+ \rightarrow \pi + e^+ + \nu) = (4.00 \pm 0.09) \times 10^6$ sec⁻¹.

3.6 Determination of the Cabibbo angle $\theta_V^{(0)}$

The Cabibbo angle relative to the vector transition for meson decays, $\theta_V^{(0)}$, is determined by the ratio between the $K^+ \rightarrow \pi^0 + e^+ + \nu$ and $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ decay rates. The interest of this determination is related to a theorem by Ademollo and Gatto (Ademollo, 1964) which states that for vanishing four-momentum transfer there are no first-order corrections in the symmetry-
breaking interactions to the vector coupling constants for $\Delta S = 0$ and $|\Delta S| = 1$ currents. Hence the phenomenological angle $\theta_v^{(M)}$ is predicted to deviate from the bare universal angle $\theta$ only for the presumably much smaller second- and higher-order terms. The experimental value for the rate $\Gamma(\pi^- + \pi^0 + e^- + \nu)$ is 0.376 ± 0.026 sec$^{-1}$ is unfortunately still not as accurate as one would like it to be. Therefore $\Gamma(\pi^- + \pi^0 + e^+ + \nu)$ is usually derived from the fit values of $0^- + 0^-$ super-allowed nuclear beta-decays (FREMAN, 1964) and the further assumption of a conserved vector current. The $K^+ + \pi^0 + e^+ + \nu$ decay amplitude at zero four-momentum transfer can be determined from the observed decay rate and the $q^2$ dependence of the form factor $f_+(q^2)$, specified by the parameter $\lambda_+$ (ONEDA, 1965). The angle $\theta_v^{(M)}$ is then calculated from the expression (CARIBBO, 1966) $\Gamma(K_{e3}) = \sin^2 \theta_v^{(M)} (1 + 3.707 \lambda_+) \times 7.42 \times 10^7$ sec$^{-1}$.

The results are:
\[
\sin^2 \theta_v^{(M)} = 0.230 \pm 0.003 \quad \text{for} \quad \lambda_+ = 0
\]
\[
\sin^2 \theta_v^{(M)} = 0.242 \pm 0.008 \quad \text{for} \quad \lambda_+ = 0.020 \pm 0.005
\]

3.7 Comparison with other determinations of the Cabibbo angle

The ratio of the $K^+ + \mu^+ + \nu$ and $\pi^+ + \mu^+ + \nu$ decay rates, corrected for phase-space differences, can be used to determine the corresponding axial vector angle $\theta_A^{(M)}$. The relevant experimental results are:
\[
\Gamma(K^+ + \mu^+ + \nu) = (51.8 \pm 0.8) \times 10^6 \text{ sec}^{-1} \quad (\text{Ford 1967})
\]
\[
\Gamma(\pi^+ + \mu^+ + \nu) = (38.4 \pm 0.5) \times 10^6 \text{ sec}^{-1} \quad (\text{world average})
\]

Ignoring the possibility that the form factor might take different values at $q^2 = -m^2$ and $q^2 = -m^2_{\pi}$, one gets (BRENE, 1966):
\[
\sin^2 \theta_A^{(M)} = 0.2655 \pm 0.006
\]
where the error is only statistical. The result differs somewhat from the one for $\sin^2 \theta_v^{(M)}$:
\[
\sin^2 \theta_A^{(M)} - \sin^2 \theta_v^{(M)} = 0.092 \pm 0.008
\]

The conclusion that they are indeed different rests on the assumption that the energy dependence of the axial form factor is not significant.

Hyperon decay experiments are well accounted for with a single Cabibbo angle (FILTMITH, 1969):
\[
\sin \theta_v^{(B)} = \sin \theta_A^{(B)} = 0.227 \pm 0.006
\]

However, the first-order SU$\_3$-breaking correction factors do not need to contribute necessarily in the same way to $\theta_A^{(B)}$ and $\theta_v^{(M)}$. Therefore there is no immediate contradiction between the results $\theta_A^{(B)} = \theta_v^{(B)}$ and $\theta_A^{(M)} \neq \theta_v^{(M)}$.

Finally, the vector coupling constant $\xi_v^{(B)}$ for super-allowed $0^+ + 0^+$ beta-decay transitions and the muon decay coupling constant $\xi_{\mu}^{(B)}$ must fulfill the relation:
\[
\frac{\xi_v^{(B)}}{\xi_{\mu}^{(B)}} = \sin \theta_v^{(B)}
\]
From the value of \( G_{V}^{(E)} \) (FREEMAN, 1964) and the muon lifetime, BRENE et al. (BRENE, 1966) have derived:

\[
\sin \theta_{\nu}^{(E)} = 0.210 \pm 0.016
\]

The result includes some allowance for radiative correction on \( G_{V}^{(E)} \) (±0.5%). In spite of the considerable uncertainties that still persist on this point, the agreement between \( \theta_{\nu}^{(E)} \) and \( \theta_{\nu}^{(M)} \), i.e.

\[
\sin \theta_{\nu}^{(E)} - 5\sin \theta_{\nu}^{(M)} = (1 \pm 1) \times 10^{-3}
\]

is probably significant.

4. THE DECAYS \( K^{+} \to \pi^{0} + \mu^{+} + \nu \) AND \( K_{L}^{+} \to \pi^{+} + \mu^{+} + \nu \)

If \( K^{+} \to \pi^{0} + e + \nu \) and \( K^{+} \to \pi^{+} + \mu + \nu \) decays would not be affected by the presence of strong interactions, both processes could be completely described by the same local vector interaction. If, furthermore, (\( \mu-e \)) universality is assumed, no new information could be extracted from \( K^{+} \to \pi^{0} + \mu + \nu \) experiments which in principle is not already contained in the \( K^{+} \to \pi^{0} + e + \nu \) results. Due to the presence of the strong interactions additional terms are generated, functions of the four-momenta of the hadrons, \( p_{\mu}^{(K)} \) and \( p_{\mu}^{(\pi)} \):

\[
M \sim \frac{1}{m_{e}} \left[ f_{+}(q^{2})\left(p_{\mu}^{(K)} + p_{\mu}^{(\pi)}\right) + f_{+}(q^{2})\left(p_{\mu}^{(e)} - p_{\mu}^{(\pi)}\right)\right] \bar{u}_{\mu}^{(1+q_{5})} u_{\nu}
\]

where \( q^{2} = \left[p_{\mu}^{(K)} - p_{\mu}^{(\pi)}\right]^{2} \)

and \( \bar{e} \) stands for \( e, \mu \). This expression can be easily transformed into the following one:

\[
M \sim \frac{1}{m_{e}} \left[ f_{-}(q^{2})\left(p_{\mu}^{(K)} + p_{\mu}^{(\pi)}\right) \right] \bar{u}_{\mu}^{(1+q_{5})} u_{\nu} + m_{e} f_{-}^{(q^{2})} \bar{u}_{\mu}^{(1+q_{5})} u_{\nu}
\]

where \( m_{e} \) is the lepton mass.

Thus even if one starts with a pure vector interaction between "bare" particles, strong interaction effects generate an induced scalar term of an effective coupling constant \( m_{e}f_{-}(q^{2}) \). Obviously only the term \( f_{-}(q^{2}) \) has to be retained in the case of electron decays. However, for muonic decays, \( (m_{\mu}/m_{e}) \sim 5 \) and both terms \( f_{-}(q^{2}) \) and \( f_{-}(q^{2}) \) could give detectable effects.

In analogy to the \( K^{+} \to \pi^{0} + e + \nu \) decays\(^{*}\) the energy dependence of the form factors are parametrized as in a linear expansion:

\[
f_{\pm}(q^{2}) = f_{\pm}(0) \left( 1 + \lambda_{\pm} \frac{q^{2}}{m_{\pi}^{2}} \right)
\]

Since \( f_{-}(q^{2}) \) may be small\(^{+}\), we parametrize the ratio:

\[
\frac{\bar{f}_{-}(q^{2})}{f_{-}(q^{2})} = \frac{\bar{f}_{+}(q^{2})}{f_{+}(q^{2})}
\]

\(^{*}\) There is, however, no known \( K^{0} \to \pi^{0} \) resonance which could dominate the intermediate states relative to \( f_{-}(q^{2}) \).

\(^{+}\) \( \bar{S}_{\mu} \), predicts \( f_{-} = 0 \).
as

$$\bar{\psi}(q^2) = \bar{\psi}(0) + \Lambda q^2 / m_{\pi}^2$$

where $\Lambda \approx \xi(0)(\lambda_- - \lambda_+)$. The introduction of the new parameter $\Lambda$ avoids the usual difficulty that $\lambda_-$ is undetermined when $f_+(0)$ is small.

The present status of the experimental situation will now be reviewed. The main interest in investigating $K^+ \rightarrow \pi^0 + \mu^+ + \nu$ decays is concentrated on:

i) determination of the effective scalar coupling constant $m_\mu f_-(q^2)$; and

ii) test of (u-e) universality by comparing the $f_+(q^2)$ term for muon and electron decays.

The form factor analysis will be carried out initially supposing $\xi$ to be constant. Successively, the $q^2$ dependence of $\xi$ will be discussed.

4.1 Polarization experiments in $K^+ \rightarrow \pi^0 + \mu^+ + \nu$ and $K_L \rightarrow \pi^0 + \mu^+ + \nu$ decays

The muon polarization is very sensitive to the ratio $f_- / f_+$. For $f_- = 0$ the decay interaction is a pure vector, and therefore right-handed $\mu^+$ are produced. The term $f_-$ is an effective scalar interaction, and therefore it generates left-handed positive muons. Interference between the two terms gives rise to a complicated dependence of the polarization on the decay configuration (Fig. 7). The muon polarization is, however, always complete and pointing to some direction, function of $T_\pi$, $T_\mu$, and $\xi$ (CABIBBO, 1964). Furthermore, if time reversal invariance is valid, the polarization vector lies in the decay plane. In fact, the expression

$$P_\mu = P_\pi \times \hat{P}_\mu - \sigma_\mu$$

Fig. 7: Expected total muon polarization in $K^+ \rightarrow \pi^0 + \mu^+ + \nu$ for the cases $\text{Im} \xi = 0$, $\text{Re} \xi = \pm 1$. As long as $\text{Im} \xi = 0$, polarization lies in the plane containing $\hat{P}_\pi$ and $\hat{P}_\mu$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7.png}
\caption{Expected total muon polarization in $K^+ \rightarrow \pi^0 + \mu^+ + \nu$ for the cases $\text{Im} \xi = 0$, $\text{Re} \xi = \pm 1$. As long as $\text{Im} \xi = 0$, polarization lies in the plane containing $\hat{P}_\pi$ and $\hat{P}_\mu$.}
\end{figure}
must vanish because of time-reversal invariance. A small apparent time-reversal violation is induced by the final state electromagnetic interactions between the muon and the charged pion in the case of \( K^0 \) decays (for the \( K^+ \rightarrow \pi^0 \nu \bar{\nu} \) decay such a term is evidently absent). The effect, proportional to \( \alpha \), generates an average transverse polarization of the order of \( 1\% \) (BYERS, 1965).

Experimental results on \( P_z \) are summarized in Table 4. They are fully consistent with \( \text{Im}(\xi) = 0 \), i.e., with time-reversal invariance.

Several experiments have investigated the muon polarization in the decay plane. Results are listed in Table 5. The two measurements (CUTTS, 1968; BETTELS, 1968) on \( K^+ \) decays are in good agreement (Fig. 8). An average between the two results gives \( \xi = -0.98 \pm 0.20 \) (for \( \xi = \text{constant} \)). The two older experiments on the decay \( K_L \rightarrow \pi^- + \mu^+ + \nu \) (ABRAMS, 1966; AUERBACH, 1966) give a combined value \( \xi = -1.15 \pm 0.35 \), in good agreement with the \( K^0 \) data and somewhat smaller than the more recent counter result (HELLAND, 1968), \( \xi = -1.81_{-0.26}^{+0.29} \).

### Table 5

<table>
<thead>
<tr>
<th>Authors</th>
<th>Initial state</th>
<th>Method</th>
<th>( \xi = \text{constant} )</th>
<th>( &lt; q^2 &gt; /m_{\pi}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutts, 1968</td>
<td>( K^+ )</td>
<td>Spark chambers</td>
<td>-0.95 \pm 0.3</td>
<td>( \sim 3 )</td>
</tr>
<tr>
<td>Bettels, 1968</td>
<td>( K^+ )</td>
<td>Heavy-liquid bubble chamber</td>
<td>-1.0 \pm 0.3</td>
<td>( \sim 4.9 )</td>
</tr>
<tr>
<td>Helland, 1968</td>
<td>( K_L )</td>
<td>Counters</td>
<td>-1.81 \pm 0.5</td>
<td>( \sim 2.5 )</td>
</tr>
<tr>
<td>Abrams, 1966</td>
<td>( K_L )</td>
<td>Spark chambers</td>
<td>-1.1 \pm 0.5</td>
<td>-</td>
</tr>
<tr>
<td>Auerbach, 1966</td>
<td>( K_L )</td>
<td>Spark chambers</td>
<td>-1.2 \pm 0.5</td>
<td>-</td>
</tr>
</tbody>
</table>

### 4.2 Analysis of the Dalitz plot

The density of events in the Dalitz plot of the decay is a function of \( f_+ (q^2) \) and of \( \xi(q^2) \):

\[
\mathcal{P}(E_\mu, E_\mu, f_+(q^2), \xi(q^2)) \ dE_\pi \ dE_\mu = \\
= \left| f_+(q^2) \right|^2 \left[ A(E_\mu, E_\pi) + B(E_\mu, E_\pi) \xi(q^2) + C(E_\pi) \xi(q^2) \right] dE_\pi \ dE_\mu
\]

where \( A, B, \) and \( C \) are defined as follows

\[
A = -2 q \xi \left[ E_\mu + E_\pi \right] + 2 \left[ \frac{m_\pi}{2} + \frac{m_\mu}{2} \right] E_\mu + \left[ \frac{m_\pi^2}{2} + \frac{m_\mu^2}{2} \right] E_\pi - \left[ E_\pi \frac{m_\mu^2}{2} \left( \frac{m_\pi}{q} - \frac{m_\mu}{q} \right) + m_\pi m_\mu \right]
\]

\[
B = - \frac{m_\pi}{2} \left[ E_\mu + \frac{q}{2} E_\pi - \frac{m_\mu}{2} \right] E_\mu - \frac{m_\mu}{2} \left[ E_\mu + \frac{q}{2} E_\pi - \frac{m_\pi}{2} \right] E_\pi
\]

\[
C = - \frac{m_\mu}{2} \left[ E_\pi - E_\pi \frac{m_\mu^2}{2} \right]
\]
Fig. 8 : Determinations of \( \xi(q^2) \) from muon polarization measurements for \( K^+ \rightarrow \pi^0 + \mu^+ + \nu \).

The sensitivity of the density of events to the parameter \( \xi \) is large only for the least populated part of the Dalitz plot, corresponding to low pion energies. An approximate estimate of the coefficients \( A, B, C \) gives:

\[
\begin{align*}
\mathcal{P}(e_K, e_\mu) &\sim 1 + \xi + 0.3 \xi^2 & \text{for } T_\pi < 20 \text{ MeV} \\
\mathcal{P}(e_K, e_\mu) &\sim 10 + \xi + 0.1 \xi^2 & \text{for } T_\pi > 100 \text{ MeV}
\end{align*}
\]

The analysis of the pion spectrum leads to a quadratic equation with two possible values for \( \xi \). The muon spectrum gives a unique solution for \( \xi \), since the coefficient \( C \) in the above formula depends only on \( E_\pi \).

Two experimental investigations of the Dalitz plot have recently been reported, one on the charged \( K \) decay (BETTELS, 1969) and one on the long-lived decays (BASILE, 1968). The results are highly contradictory, and therefore both experiments will be discussed in more detail.

The \( K^+ + \pi^0 + \mu^+ + \nu \) Dalitz plot has been studied in the 1.1 m\(^3\) CERN heavy-liquid bubble chamber, filled with freon (\( X_0 = 2.5 \text{ cm} \)). Complete \( K^+ + \pi^0 + \mu^+ + \nu \) events, with stopping muon and both \( \gamma \)-rays converted, have been retained. In order to obtain an unbiased sample, only the hatched region of the Dalitz plot in Fig. 9 has been retained.

The cuts have the following justifications:
- to remove \( \tau' \) and \( K-\pi_2 \) events;
- \( \tau_{\pi^0} > 20 \text{ MeV} \) in order to ensure a very uniform \( \pi^0 \) detection efficiency.
Fig. 9: Dalitz plot for $K^+ + \pi^0 + \mu^+ + \nu$. Only the hatched region is normally retained, mainly in order to remove background events due to $\tau'$ and $K_{\pi_2}^+$ decays (from BETTELS, 1969).

Out of the 15,000 initial events only 3,240 survive all cuts. The sample is considered free of biases and of backgrounds. The only correction required is the $\pi^0$ detection efficiency. Two independent determinations of the $\gamma$ detection efficiency were made. The first one is based on $K^+ + \pi^+ + \pi^0$ decays in which the $\pi^0$ is monoenergetic and therefore the $\gamma$-spectrum is flat, and a second one is a determination of the conversion length of the $\gamma$-rays of a given energy.

A maximum likelihood analysis (Fig. 10) with $\lambda_{\mu} = 0.023$ has given $\xi = -0.3 \pm 0.25$. A second solution, $\xi = -3.5 \pm 0.3$ is about 1000 times less likely.

The experiment of Basile et al. has studied the $K_L + \pi^\pm + \mu^\mp + \nu$ Dalitz plot with a magnetic spectrometer and spark chambers (Fig. 11).

Muons and pions are separated by curvature and range measurements. A very interesting feature of the experiment is that the problem of the knowledge of $K^0$ momentum (determined only with a quadratic ambiguity) is very considerably reduced by the use of a very slow $K_L^0$ beam.

The detection efficiency, has been calculated from the geometry and the momentum spectrum by Monte Carlo methods (Fig. 12). The variations of the detection efficiency are most important for the lower bins of the Dalitz plot, (small $T_{\pi}$) where the sensitivity to $\xi$ is largest. An accurate knowledge of this efficiency is, of course, crucial for obtaining the result.
Fig. 10 : Likelihood curve versus $\xi = \text{constant of the Dalitz plot analysis for}$ $K^* \rightarrow \pi^0 + \mu^+ + \nu$ of the $X_2$ collaboration experiment (BETTELS, 1969).

Fig. 11 : The experimental arrangement of Basile et al. (BASILE, 1968). Charged decay products from a long-lived kaon beam are magnetically analysed and identified in the range spark chambers.
Fig. 12: Detection efficiency, in arbitrary units after the Dalitz plot of the experiment of Basile et al. (BASILE, 1968). The numbers inside the small sparks are the number of events actually recorded in each bin.

A fit to the data with constant $\xi$ has given $\xi = -3.9 \pm 0.1$. Figure 13 shows $\chi^2$ curves as a function of $\xi$ for these bands of the pion energy. The lower energy band carries indeed most of the statistical weight.

The same experiment has given the independent result $\xi = -0.4 \pm 0.3$ from the $K^{\mu}/K^{\nu}$ branching ratio. The two results are therefore inconsistent, unless of course some of the major assumptions are reviewed, such as, for instance, $\mu$-e universality. It is interesting to note that the branching ratio is determined from the total number of events, mostly concentrated in the top part of the Dalitz plot; instead, the result $\xi = -3.9 \pm 0.1$ is dominated by the much fewer events of the lower part.

A number of earlier experiments on individual lepton and pion energy spectra as well as Dalitz plot distributions have all given values for $\xi$ of about +1 (ROSENFIELD, 1968).

The experiment in the CERN heavy-liquid bubble chamber (BETTLES, 1969) is so far the only one that gives a measurement for $\xi$ in full agreement with polarization measurements.

4.3 Determinations of the $K \rightarrow \pi + \mu + \nu/K \rightarrow \pi + e + \nu$ branching ratio

The branching ratio $R = K \rightarrow \pi + \mu + \nu/K \rightarrow \pi + e + \nu$ can be used to determine $\xi$, provided $\mu$-e universality is assumed. In the case of electronic decays, the formula for the density of events in the Dalitz plot of Section 4.2 is considerably simplified, since all terms proportional to the lepton mass are now negligible:

$$\mathcal{P} (E_{\pi}, E_{\ell}) = \left| \mathcal{P}^+ (q^2) \right|^2 A_c (E_{\pi}, E_{\ell})$$

where:

$$A_c (E_{\pi}, E_{\ell}) = 2 M_K (E_{\ell}^2 + E_{\pi} E_{\ell}) + 2 M_K^2 E_{\ell} + M_K^2 E_{\pi} - E_{\pi}^2 - M_K^2$$
If (μ-e) universality holds, electronic and muonic decays have exactly the same form factor $f_e(q^2)$. Therefore the ratio between integrals over the Dalitz plot densities is a quadratic function of $\xi$ with known coefficients. The integration over the whole Dalitz plot gives *

$$R = 0.649 + 0.127 \xi + 0.00193 \xi^2$$

Several experiments are sensitive only to a fraction of the Dalitz plot surface. In this case the observed branching ratio $R'$ is still a quadratic expression in $\xi$ with calculable numerical coefficients.

The polarization measurements have given $\xi = -0.98 \pm 0.20$ for the $K^+$, and $\xi = -1.45 \pm 0.26$ for the $K_L$. Assuming (μ-e) universality and for constant form factors we calculate $R = 0.53 \pm 0.02$ for the $K^+$, and $R = 0.50 \pm 0.025$ for the $K_L$. The experimental results (Fig. 14) all give systematically larger values $R \sim 0.7$, except for the result of the $X_2$ collaboration, $R = 0.054 \pm 0.03$.

*) If $\xi$ is permitted to be complex, then one has to replace this expression with the following one:

$$R = 0.649 + 0.127 \Re (\xi) + 0.00193 |\xi|^2$$
Fig. 14: Summary of the experimental results of the $K_{3/8}^+/K_{3}^+$ branching ratio (from STIENING, 1968). The dashed line is the predicted branching ratio obtained from ($\mu$-e) universality and polarization experiments. Form factors have been taken as constants.

The deviation between the bulk of the results and the result of $X_2$ collaboration is not clearly understood. It is difficult to judge which experiment is the most superior, and it is clear that more experimental work is needed.

4.4 Energy dependence of the parameter $\xi$

The energy dependence of the parameter $\xi$ is parametrized as:

$$\xi(q^2) = \xi(0) + \Lambda \frac{q^2}{m_\pi^2}$$

with $\Lambda = \xi(0) [\lambda_+ - \lambda_-]$. Theoretical calculations based on the dominance of ($\pi$-K) resonances (MCDOWELL, 1959; DENNERY, 1963) predict very small values for $\lambda_+$ and $\lambda_-$, of the order of $1-2 \times 10^{-2}$.

Experimentally, very little is known about $\Lambda$. Polarization experiments that directly determine $\xi$ as a function of $q^2$ are compatible with such a very large range of values for $\Lambda^8$ (Fig. 8).

Another way of determining $\Lambda$ is to attribute the apparent discrepancy between the polarization and branching ratio measurements to large four-momentum dependence of the form factor. A strong $q^2$ dependence of $\xi$ can be generated only by $\lambda_-$ since $\lambda_+ \approx 0.02$. 
In order to be more specific in the considerations that follow, we shall refer to the experimental results of the \( \chi_2 \) collaboration (BETTLES, 1968; BETTLES, 1969). A best fit analysis of the data with the linear four-momentum dependence of \( \xi(q^2) \) for the polarization, branching ratio, and Dalitz plot measurements has given the likelihood plots shown in Fig. 15.

The equal likelihood contours appear very elongated, since parameters \( \xi(0) \) and \( \Lambda \) are closely correlated. It is possible to replace the parameters \( \Lambda \) and \( \xi(0) \) with the two other parameters \( \xi(0) \) and \( \xi(-m_H^2/a) \) which are uncorrelated provided a is defined by the regression function of \( \Lambda \) into \( \xi(0) \):  
\[
\Lambda = \alpha \xi(0) + \delta
\]

The results of the three \( \chi^2 \) determinations of \( \xi(q^2) \) can then be expressed in terms of the following pairs of uncorrelated parameters:

- **Polarization measurement**
  \[
  \xi(0) = -0.6 \pm 1.4 \\
  \xi(-4.9 m_H^2) = -1.0 \pm 0.3
  \]

- **Branching ratio measurement**
  \[
  \xi(4 m_H^2) = -0.6 \pm 0.2
  \]

- **Dalitz plot measurement**
  \[
  \xi(0) = 0.1 \pm 0.20 \\
  \xi(6.8 m_H^2) = -0.33 \pm 0.24
  \]

Therefore separate measurements provide a relatively accurate determination of \( \xi \) at some "average" four-momentum transfer and a more uncertain determination of the slope. In order

![Fig. 15](image_url)

Fig. 15: Over-all best fit of the polarization, branching ratio and Dalitz plot measurements of the \( \chi_2 \) collaboration. The prediction of the Callen-Treiman relation is also shown.
to reconcile conflicting branching ratio results with polarization measurements, it is necessary to profit from the small differences between the "average" momenta of the two measurements. This can, of course, be done only at the expense of a huge value for $\Lambda^0$.

The three determinations of $\xi(q^2)$ of the $X_2$ experiment are in good agreement (Fig. 16). Averaging between the results gives:

$$\xi(0) = -0.96 \pm 0.35$$
$$\xi(5.4 m^2) = 0.57 \pm 0.10$$

$$\begin{cases} \lambda_+ = 0.05 \pm 0.12 \\ \rho = 0.23 \end{cases}$$

The results do not exclude $\xi$ from being constant, in which case the result is

$$\xi = -0.59 \pm 0.10$$

5. THE SELECTION RULE $|\Delta I| = \frac{1}{2}$ FOR LEPTONIC DECAYS

The $|\Delta I| = \frac{1}{2}$ rule states that the total isospin charge of the strongly interacting particles is $\frac{1}{2}$. Transitions $\Delta Q = -\Delta S$ imply isospin charges $|\Delta I| = 3/2$ as a consequence of the absolute relation $Q = I_3 + \frac{1}{2}(S + B)$. Therefore the selection rule $|\Delta I| = \frac{1}{2}$ implies the validity of the selection rule $\Delta Q = \Delta S$. As discussed in the next paragraph, there is at present no conclusive evidence for a violation of the $\Delta Q = \Delta S$ rule.

Fig. 16: Dependence on $q^2$ of $\xi(q^2)$ from an over-all fit to the results of the $X_2$ collaboration experiment.
The $|ΔI| = \frac{1}{2}$ rule is automatically verified in a number of leptonic processes, $K^± + μ^± + ν, Λ^0 + p + e^- + ν, ξ^- + Δ + e^- + ν$. At present the only valid test is provided* by the semi-leptonic decays of the charged and long-lived $K$ mesons.

The main prediction is that the ratios

$$\frac{\Gamma(K^- \to π^0 e^- \bar{ν})}{2 \Gamma(K^+ \to π^0 e^+ ν)}$$

and

$$\frac{\Gamma(K^- \to π^0 μ^- \bar{ν})}{2 \Gamma(K^+ \to π^0 μ^+ ν)}$$

should both be equal to unity. The ratios are increased by 1.1 due to radiative corrections (GINSBERG, 1968) and by 1.355 because of phase-space differences.

Experimental results are summarized in Table 6. They appear to be in good agreement with the predictions of the rule.

**Table 6**
The $|ΔI| = \frac{1}{2}$ rule in leptonic decays

<table>
<thead>
<tr>
<th>Quantity</th>
<th>New average (Cronin, 1968)</th>
<th>Old average Rev. Mod. Phys. 40, 77 (1968)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma(K^- \to π^+ e^- + ν)/2\Gamma(K^+ \to π^+ e^+ ν)$</td>
<td>0.99 ± 0.04</td>
<td>0.85 ± 0.05</td>
</tr>
<tr>
<td>$\Gamma(K^- \to π^+ μ^- + ν)/2\Gamma(K^+ \to π^+ μ^+ ν)$</td>
<td>1.05 ± 0.04</td>
<td>0.94 ± 0.07</td>
</tr>
</tbody>
</table>

In order to evaluate the sensitivity of the branching ratio to $ΔI = \frac{3}{2}$ contributions, we assume the possible existence of all three decay amplitudes $α_{11}, α_{31}, α_{33}$, corresponding to isospin changes $(ΔI, ΔI) = (1, 2, 1, 2), (3, 2, 1, 2)$ and $(3, 2, 3, 2)$. The terms $α_{11}, α_{31}$ are related to transitions $ΔQ = ΔS$, whereas $α_{33}$ is related to $ΔQ = −ΔS$.

Ignoring the small $CP$-violating parameter in the long-lived state $|K_L⟩ = (1/\sqrt{2}) [|K_0⟩ - |\bar{K}_0⟩]$, and for small deviations from the $|ΔI| = \frac{1}{2}$ rule, namely $|α_{31}/α_{11}| << 1, |α_{33}/α_{11}| << 1$, one finds

$$\left[ \frac{\Gamma(K^- \to π^0 e^- \bar{ν})}{2 \Gamma(K^+ \to π^0 e^+ ν)} - 1.023 \right] \approx 3 \sqrt{2} \ Re \left( \frac{α_{31}}{α_{11}} + \sqrt{3} \frac{α_{33}}{α_{11}} \right)$$

* Many verifications of this rule could be done with neutrino-induced reactions.
The parameter $a_{33}/a_{11}$ is related to the same approximations to the usual parameter $x$ measuring the amount of $\Delta Q = -\Delta S$ decay amplitude:

$$a_{33}/a_{11} \approx \sqrt{6} \approx \frac{\sqrt{6}}{x} = \frac{A(K_{e3} \rightarrow e^+\nu\bar{\nu})}{A(K_{e3} \rightarrow e^+\nu)}$$

Taking the average between the more recent results on muon and electron branching ratios (Cronin, 1968) we get:

$$\frac{\Gamma(K_{\mu3} \rightarrow \mu^+\bar{\nu}\nu) + \Gamma(K_{e3} \rightarrow e^+\nu\bar{\nu})}{2 \Gamma(K^+ \rightarrow e^+\nu) + 2 \Gamma(K^+ \rightarrow \mu^+\nu)} = 1.03 \pm 0.028$$

corresponding to:

$$\text{Re} \left( \frac{a_{33}}{a_{11}} + \sqrt{3} \frac{a_{39}}{a_{11}} \right) = (3.4 \pm 1.3) \times 10^{-3}$$

Experimental results on the $\Delta Q = \Delta S$ rule give a much larger indetermination. For the term $a_{33}$ alone:

$$\left| \text{Re} \left( \frac{a_{33}}{a_{11}} \right) \right| = \left| \text{Re} (\sqrt{6} x) \right| \leq 0.36$$

Therefore the upper limit for each of the two matrix elements $a_{31}$ and $a_{33}$ are considerably less accurate than the one on the specific linear combination.

6. THE $\Delta Q = \Delta S$ RULE IN LEPTONIC K DECAYS

6.1 Tests in $K_{e3}^0$ decays

The $\Delta Q = \Delta S$ rule for strangeness-changing decays predicts that:

$$\begin{align*}
K_{e3}^0 & \rightarrow \bar{e}^- + e^+ + \nu \\
\bar{K}_{e3}^0 & \rightarrow \bar{\kappa}^+ + e^- + \nu
\end{align*}$$

\[ \Delta Q = \Delta S \]
are allowed, whereas
\[
K_0 \rightarrow \pi^+ + e^- + \nu \\
\overline{K}_0 \rightarrow \pi^- + e^+ + \nu
\]
are forbidden. CPT invariance has been assumed, and the asterisk stands for complex conjugation. Parameters \( f \) and \( g \) are the form factors for the decay matrix elements. Only one form factor is appreciable in the electron case. For the sake of clarity, the subscript + used in Section 3.1 will be omitted in the present discussion.

Coefficients \( f \) and \( g \) are (relatively) real if CP invariance is valid. The decay rate in \( e^\pm \) for a \( \bar{K}^0 \) state at \( t = 0 \) is given by:
\[
N^\pm (K_0) = \frac{\sqrt{2}}{4} \left| 1 - x^2 \right| e^{-\frac{\Gamma e}{2}} + \frac{1}{2} \left| 1 + x^2 \right| e^{-\frac{\Gamma e}{2}}
\]
\[
\pm 2 \left( 1 - |x|^2 \right) \cos (\Delta m t) e^{-\left( \Gamma_L + \Gamma_S \right) t/2}
\]
\[
- 4 \text{ Im} (x) \sin (\Delta m t) e^{-\left( \frac{\Gamma_L + \Gamma_S}{2} \right) t/2}
\]

where \( \Gamma_L, \Gamma_S \Delta m = m_L - m_S \) are decay rates and masses of the long-lived and short-lived states. In the case of an initial \( \bar{K}^0 \) state, the sign of the two last terms has to be changed.

The latest experimental data are summarized in Table 7. If only the \( \Delta Q = \Delta S \) rule is questioned, namely if CP is not violated in the decay interaction, the combined result is (LITTEMENBERG, 1969):
\[
\chi = 0.03 \pm 0.025
\]

Therefore unless CP is violated in \( K_{e3} \) decays, the \( \Delta Q = \Delta S \) rule is found to be valid. If we allow \( x \) to be complex, then the combined averages are:
\[
\text{Im} (x) = -0.13 \pm 0.043
\]
\[
\text{Re} (x) = +0.14 \pm 0.05
\]

The recent counter experiment of Bennett et al. (BENNETT, 1968) is further constraining \( x \) to \( (1 - |x|^2)/(|1 - x|^2) = 1.06 \pm 0.06. \)
Table 7
Tests of $\Delta Q = \Delta S$ rule in $K_{e3}$ decays

<table>
<thead>
<tr>
<th>Authors</th>
<th>Initial state</th>
<th>Method</th>
<th>Re (x)</th>
<th>Im (x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baldo-Ceolin, 1965</td>
<td>$K^0$</td>
<td>$K^0$ charge exchange (HBC)</td>
<td>$0.06 + 0.18$</td>
<td>$-0.44 + 0.32$</td>
</tr>
<tr>
<td>Aubert, 1965</td>
<td>$K^0$</td>
<td>$K^0$ charge exchange (HBC)</td>
<td>$0.075 + 0.11$</td>
<td>$-0.21 + 0.15$</td>
</tr>
</tbody>
</table>
| Franzini, 1965    | $K^0, K^+$    | $\pi^+ p \rightarrow 
K^0, \text{spark chambers}$ | $0.17 + 0.16$| $0.0 + 0.25$|
| Feldman, 1967     | $K^0$         | $K^0$ charge exchange (HBC) | $0.17 + 0.16$| $-0.35 + 0.28$|
| Hill, 1967        | $K^0$         | $K^+ d \rightarrow ppK^0$, (D7 BC) | $0.17 + 0.16$| $-0.35 + 0.28$|
| Webber, 1968      | $K^0$         | $K^0 p \rightarrow K^- n$  | $0.15 + 0.13$| $0.08 + 0.08$|
| James, 1968       | $K^0, K^+$    | $\pi^+ p \rightarrow K^0, \text{spark chambers}$ | $0.17 + 0.16$| $-0.35 + 0.28$|
| Littenberg, 1969  | $K^0$         | $K^0$ charge exchange,      | $0.17 + 0.16$| $-0.35 + 0.28$|
| Bennett, 1968     | $K^0, K_L$    | $\text{Regeneration in C1 block}$ | $1 - |x|^2$| $= 1.06 + 0.06$|

Although there is evidence for a simultaneous violation of the $\Delta Q = \Delta S$ rule and CP in the combined averages, it is important to stress that no single experiment gives separately a statistically significant effect.

6.2 Tests in $K_{e3}$ decays

There are two possible decay processes

$$K^+ \rightarrow \pi^+ + e^- + \nu \quad \left[ K_{e4} \left( e^+ \right) \right], \quad \Delta Q = \Delta S$$

$$K^- \rightarrow \pi^- + \pi^+ + e^- + \nu \quad \left[ K_{e4} \left( e^- \right) \right], \quad \Delta Q = -\Delta S$$

The decay rate for the mode $K_{e3}(e^+)$ has been determined by Ely et al. (ELY, 1968) to be $(2.60 \pm 0.30) \times 10^{-4}$ sec$^{-1}$ on the basis of 264 events. The "forbidden" mode $K_{e3}(e^-)$ has not been observed to an upper limit in the branching ratio of $7 \times 10^{-7}$ (95% confidence level).

The $K_{e3}(e^+)$ involves both vector and axial-vector terms, although the latter is dominating. The decay $K_{e3}(e^-)$, if it occurs at all, proceeds almost entirely through an axial-vector current. The interpretation of the result in terms of the ratio of the amplitudes in the currents:

$$X = \frac{A_\nu \left( \Delta Q = -\Delta S \right)}{A_\nu \left( \Delta Q = +\Delta S \right)}$$
is made somewhat elaborate because of the difference in the dipion interactions for $K_{e4}^- (e^-)$ and $K_{e4}^+ (e^+)$ decays. The dipion is a pure $I = 2$ state for the $K_{e4}^- (e^-)$, whereas for the $K_{e4}^+ (e^+)$ it can have both $I = 0$ and $I = 1$. Both $s$- and $p$-waves are likely to be appreciable in the $K_{e4}^+ (e^+)$. Only $s$-waves would be present in $K_{e4}^- (e^-)$. If the final-state interactions are neglected completely, the result of Ely et al. leads to $|x| < 0.15$ (at 95% confidence level). The enhancement factor due to final-state interactions is estimated (KACSER, 1965) to be as large as 4, giving

$$|x| < 0.30 \quad (95\% \text{ confidence level}).$$

The conclusion refers specifically to the axial vector part of the weak current.

Acknowledgements

The author wishes to thank Dr. M. Roos for his support in averaging the experimental results, and Drs. V. Brisson and P. Petiau for discussions on the problem of $K_{u3}$ form factors.
BIBLIOGRAPHY


J. Bettels et al., X_v-collaboration, preliminary results to this Conference [V. Brisson and P. Pietra, private communication (1969)].


A. Filthuth, Report given at this Conference.


E.S. Ginsberg, private communication (1968).


