Cours/Lecture Series

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1996–1997 ACADEMIC TRAINING PROGRAMME

LECTURE SERIES

SPEAKER : M. NEUBERT, CERN-TH
TITLE : Quantum Chromodynamics
TIME : 11, 12, 13, 14 & 15 November from 11.00 to 12.00 hrs
PLACE : Auditorium

ABSTRACT

Quantum chromodynamics (QCD) is the fundamental theory of the strong interactions. It is a local, non-abelian gauge theory describing the interactions between quarks and gluons, the constituents of hadrons. In these lectures, the basic concepts and phenomena of this theory will be introduced in a pedagogical way. Topics will include: asymptotically free partons, colour and confinement; non-abelian gauge invariance and quantization; the running coupling constant; deep-inelastic scattering and scaling violations; the operator product expansion; chiral and heavy-quark symmetries. Some elementary knowledge of field theory, abelian gauge invariance and Feynman diagrams will be helpful in following the course.
QUANTUM CHROMO-DYNAMICS
Q.C.D.

Matthias Neubert, CERN

1. Introduction; Geometry of Gauge Invariance

2. Renormalization and Running Coupling; Applications of pQCD

3. Deep Inelastic Scattering; Scaling Violations and Parton Evolution

4. Operator Product Expansion

5. Non-perturbative Methods

(CERN, 11-15 Nov. 96)

Books: Peskin + Schröder, Quantum Field Theory
+ many others


INTRODUCTION

MATTER: fermions, interacting primarily by vector boson exchange

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quarks: u c t
        d s b } constituents of hadrons

FORCES: electromagnetism

- gauge invariance
- massless vector boson: $\gamma$
  $\Rightarrow$ QED

weak interactions

- heavy vector bosons: $W^\pm, Z^0$
  $\Rightarrow$ correct theory requires spont. sym. breaking and unification
  with EM: $\text{SM} = SU_L(2) \times U_Y(1)$

strong interactions

- for long time obscure that caused by vector boson exchange
* Interactions are strong $\Rightarrow$ perturbation theory fails

* Quarks do not exist as free particles
  $\Rightarrow$ existence and quantum numbers inferred from hadron spectrum

**Quark Model:**
- **Flavour:** u, d, s (c, b)
- **Spin:** $\frac{1}{2}$, fract. charges
- **Colour** ($\Delta^+$) $\Rightarrow N_c = 3$
- **Mesons:** $q_i \bar{q}_i$
- **Baryons:** $\varepsilon_{ijk} q_i q_j q_k$

**Explains:**
- Electromagnetic and weak interactions of hadrons
- Isospin symmetry and $SU(3)$ multiplets

**No Explanation:**
- Confinement (non-observation of fractionally charged particles)
- Hadron structure
- Nature of the strong force
Figure 30.1: SU(4) 16-plets for the (a) pseudoscalar and (b) vector mesons made of u, d, s, and c quarks. The nonets of light mesons occupy the central planes, to which the c\bar{c} states have been added. The neutral mesons at the centers of these planes are mixtures of uu, dd, ss, and c\bar{c} states.

Figure 30.2: SU(4) multiplets of baryons made of u, d, s, and c quarks. (a) The 20-plet with an SU(3) octet. (b) The 20-plet with an SU(3) decuplet.
* strong force becomes weak at large momentum transfer

**pp collisions:** \((\sqrt{s} \geq 10 \text{ GeV})\)
- production of pions along collision axis, with small transverse momenta \((\leq 1 \text{ GeV})\)

\[ \Rightarrow \text{proton} = \text{loosely bound assemblage of components that cannot absorb large } q^2 \] (jelly)

**Deep inelastic scattering:**
- 20 GeV electrons scattered from hydrogen target; large deflection angles \((\text{large } q^2)\)
- substantial rate observed, as if proton was elementary particle with em. interaction
- but very few times proton in final state

**PARTON MODEL:**
* proton = loosely bound system of partons \((\text{quarks, antiquarks, glue})\)
* partons have electromagnetic interactions of elementary particles, but are unable to exchange large \(q^2\) by strong interactions
$e^-$ can knock a quark out of proton then soft interactions: jets of hadrons collinear with struck quark

\[ s = (\mathbf{P} + \mathbf{k})^2 \]
\[ Q^2 = -(\mathbf{k} - \mathbf{k}')^2 \]
\[ x = \frac{Q^2}{2 \mathbf{P} \cdot \mathbf{q}} \]

$\Rightarrow$ prediction:

\[
\frac{d^2 \sigma}{dx \, dQ^2} = \frac{2 \pi \alpha^2}{\alpha^4} \left[ 1 + \left(1 - \frac{Q^2}{xs} \right)^2 \right] \sum_i Q_i^2 f_i(x)
\]

$Q^2$ independent!

BJORKEN SCALING

* to em. probe, structure of proton looks the same no matter how hard proton is struck

* in proton rest frame:

\[ E_X = \frac{Q^2}{2x M_p} \gg M_p \]
\[ \tau_{\text{scatt.}} \sim \frac{1}{E_X} \]

$\Rightarrow$ on short time scales, partons are essentially free
Figure 14.2. Test of Bjorken scaling using the $e^- p$ deep inelastic scattering cross sections measured by the SLAC-MIT experiment, J. S. Poucher, et. al., Phys. Rev. Lett. 32, 118 (1974). We plot $d^2 \sigma / dx dQ^2$ divided by the factor (14.9) against $x$, for the various initial electron energies and scattering angles indicated. The data span the range $1 \text{ GeV}^2 < Q^2 < 8 \text{ GeV}^2$. 
* How to reconcile QM with DIS?
* How can same force be strong at low $q^2$ but weak at large $q^2$?

→ picture:

[Diagram showing force proportional to separation]

* Conflict with Quantum Field Theory?

**SOLUTION:**

- discovery of asymptotic freedom as property of non-abelian gauge theories

- QCD: $SU(3)$ gauge group of colour, with quarks assigned to fundamental repr.

**LARGE $q^2$:** Feynman diagrams (pQCD)

**LOW $q^2$:** QCD ⇒ stat. mechanical system on 4-d Euclidean lattice

- confinement of colour
  - asymptotic states are singlets
**GEOMETRY OF GAUGE INVARIANCE**

ABELIAN GAUGE GROUP: \( \psi(x) \rightarrow e^{i\alpha(x)} \psi(x) \)

\[ \Rightarrow \bar{\psi} \psi \text{ invariant, but problem with derivatives:} \]

\[ n^\mu \partial_\mu \psi(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \psi(x+\varepsilon n) - \psi(x) \right] \]

\[ \uparrow \quad \uparrow \]

makes no sense!

\[ \Rightarrow \quad \text{compensating factor ("link"):} \]

\[ U(y,x) \in U(1) : \quad U(x,x) = 1 \]

\[ U(y,x) \rightarrow e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)} \]

infinitesimal form:

\[ U(x+\varepsilon n, x) = 1 + i \varepsilon n^\mu \left( -e A_\mu(x) \right) + \sigma(\varepsilon^2) \]

\[ \uparrow \]

"connection"

\( \psi(y) \) and \( U(y,x) \psi(x) \) have same transf. law

\[ \Rightarrow \quad \text{covariant derivative:} \]

\[ n^\mu D_\mu \psi(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \psi(x+\varepsilon n) - U(x+\varepsilon n, x) \psi(x) \right] \]

\[ = n^\mu \left( \partial_\mu + eA_\mu(x) \right) \psi(x) \]

transformation laws:

\[ D_\mu \psi(x) \rightarrow e^{i\alpha(x)} \psi(x) \left( \text{same as } \psi(x) \right) \]

\[ A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{\varepsilon} \partial_\mu \alpha(x) \]
field strength:  (kin. term for $A_\mu$)
\[
[D_\mu, D_\nu] \psi(x) \to e^{i\alpha(x)} [D_\mu, D_\nu] \psi(x)
\]
\[
[D_\mu, D_\nu] = i e \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) = i e F_{\mu\nu}
\]
not a differential op. $\Rightarrow$ gauge invariant

Geometrical interpretation:

\[
\mathcal{U}_{12}(x) = \begin{pmatrix} x+e n_2 \\ x \end{pmatrix} = 1 - i e^2 e F_{12} + o(e^3)
\]
"plaquette"
(non-commutativity of "comparisons")

$\Rightarrow$ Lagrangian:
\[
\mathcal{L} = \bar{\psi} \gamma^\mu i D_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - c \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}
\]
+ irrelevant operators $(d \geq 6)$

Maxwell-Dirac Lagrangian
= most general, renormalizable Lagrangian
with local $U(1)$ gauge invariance
NON-ABELIAN GAUGE GROUP:

\[ \gamma = \begin{pmatrix} \gamma_r \\ \gamma_g \\ \gamma_b \end{pmatrix} \quad \text{where} \quad \psi(x) \rightarrow V(x) \psi(x) \]

with: \[ V(x) = \exp(i \alpha_a(x) t_a) \in SU(3) \]

\[ \uparrow \]
generators

\[ \Rightarrow 3 \text{ orthogonal symmetry motions} \]

that do not commute with one another

**comparator:**

\[ U(y, x) \rightarrow V(y) \ U(y, x) \ V^+(x) \quad \text{unitary} \]

\[ U(x + \varepsilon n_x, x) = 1 + i \ v \ n_x \ q \ \ A^a_{\mu} (x) \ t_a + O(\varepsilon^2) \]

\[ \uparrow \]

\( (N^2 - 1) \) vector fields

\[ A^a_{\mu} t_a \rightarrow V A^a_{\mu} t_a V^+ + \frac{1}{i g} (\partial^\mu V) V^+ \]

**covariant derivative:**

\[ D_{\mu} = \partial_{\mu} - ig A^a_{\mu} t_a \]

\[ D_{\mu} \psi(x) \rightarrow V(x) \ D_{\mu} \psi(x) = \text{same as} \ \psi(x) \]

**field strength:**

\[ [D_{\mu}, D_{\nu}] = -ig \ F_{\mu\nu} t_a \rightarrow V(x) \ [D_{\mu}, D_{\nu}] \ V^+(x) \]

with: \[ F_{\mu\nu} = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + g \ f^{abc} A_{\mu}^b A_{\nu}^c \]

\[ [t_a, t_b] = i \ f^{abc} t_c \quad \text{non-abelian}! \]
\[ \Rightarrow \quad \text{tr} \left( F^a_{\mu \nu} t_a \right)^2 = \frac{1}{2} F^a_{\mu \nu} F^{\mu \nu, a} \quad \text{gauge invariant} \]

\text{Lagrangian:}

\[ \mathcal{L}_{\text{had}} = \overline{\psi} i D \psi - m \overline{\psi} \psi \]

\[ - \frac{1}{4} F^a_{\mu \nu} F^{\mu \nu, a} - \theta \frac{g}{16\pi} \epsilon^{\mu \nu \rho \sigma} F^a_{\mu \nu} F^a_{\rho \sigma} \]

\[ \uparrow \]

violates P and T

("strong CP problem")

\underline{QUANTIZATION:}

\[ \frac{\delta \mathcal{L}}{\delta \hat{A}_0} = 0 \quad \Rightarrow \quad \text{need to fix a gauge:} \quad \partial^a A^a_\mu = \omega^a \]

"cov. gauge"

\underline{SUBLTLETY:}

\[ \text{gauge-fixing condition } G(A) = 0 \text{ leads to} \]

\[ \text{functional determinant:} \]

\[ \det \left( \frac{\delta G(A)}{\delta a} \right) = \det \left( \frac{1}{g} \partial^a D_\mu \right) = \text{const.} \]

\[ \frac{1}{g} \partial^a \partial b + f_{abc} \partial^a A^c_\mu \]

\[ \Rightarrow \quad \text{additional, } A\text{-dependent term in} \]

\[ \text{functional integral!} \]
**FADDEEV–POPOV TRICK:**

\[
\det \left( \frac{\delta G(A)}{\delta a} \right) = \int \mathcal{D}[c] \mathcal{D}[\bar{c}] \ e^{i \int dx \ \bar{c} (-\partial^2 + \xi) c}
\]

\(c(x)\): anti-commuting, scalar fields in adjoint repr.

\[\Rightarrow\] wrong relation between spin and statistics
to be physical states, but elegant mathematical trick

**ghost fields**

**Ghost Lagrangian:**

\[
\mathcal{L}_{\text{ghost}} = \bar{c}^a \left( -\partial^2 \delta^{ac} - g f^{abc} \partial^\mu A^b_\mu \right) c^c
\]

**Feynman rules:**

\[
\begin{array}{ccc}
\text{a} & \rightarrow & \text{b} \\
\hline
\text{p} & \rightarrow & \text{b} \\
\hline
\text{p} & \rightarrow & \text{b} \\
\end{array}
\]

\[
\begin{align*}
\text{p} & \rightarrow \text{b} = \frac{i}{p^2} \delta^{ab} \\
\text{p} & \rightarrow \text{b} = g f^{abc} p^c \\
\end{align*}
\]
**WILSON LINES AND LOOPS**

* abelian comparator for large separation:

\[ U_p(y, x) = \exp \left( -i e \int_{\gamma} A_\mu(z) \right) \]

satisfies

\[ U_p(y, x) \rightarrow e^{i\alpha(y)} U_p(y, x) e^{-i\alpha(x)} \]

since:

\[ A_\mu(z) \rightarrow A_\mu(z) - \frac{1}{e} \partial_\mu \alpha(z) \]

\[ \Rightarrow \text{Wilson line depends on path } P : \]

\[ U_p(x, x) = \exp \left( -i e \int_{\gamma} A_\mu(z) \right) \neq 1 \]

gauge-invariant Wilson loop

* all locally gauge-invariant functions of \( A_\mu \) can be related to Wilson loops!

**Stoke's Theorem:**

\[ U_p(x, x) = \exp \left( - \frac{i e}{2} \int_{\Sigma} d\sigma^{\nu\rho} F_{\mu\nu} \right) \]

flux integral of field strength
* non-abelian generalization:

\[
U_p(y,x) = \mathcal{P} \exp \left( i g \int dz^\mu A^a_\mu(z) t_a \right)
\]

"path ordering"

\[\Rightarrow\] Wilson loop:

\[
\text{tr} \ U_p(x,x) = \text{tr} \ \mathcal{P} \exp \left( i g \int dz^\mu A^a_\mu(z) t_a \right)
\]

**PATH ORDERING:**

path \( \mathcal{P}(s) : \quad z(0) = x \ , \quad z(1) = y \ ; \quad s \in [0,1] \)

\[\Rightarrow\] ordering such that matrices corresponding to higher values of \( s \) stand to the left

\[
U_p(y,x) = \sum_{n=0}^{\infty} (ig)^n \int ds_1 \int ds_2 \cdots \int ds_n
\]

\[
x \ \frac{dz^\mu}{ds} \ A^a_\mu(s_i) t_a \cdots \frac{dz^\eta}{ds_n} \ A^b_\eta(s_n) t_b
\]
QCD FEYNMAN RULES:

Quark propagator:
\[
\frac{i}{p - m + i\eta}
\]

Gluon propagator:
\[
\frac{-i\delta^{ab}}{p^2 + i\eta} \left( g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right)
\]
gauge parameter

Fermionic vertex:
\[
ig \gamma^\mu t_a
\]

3-boson vertex:
\[
g_{abc} \left[ g^{\mu\nu}(k - p)^\xi + g^{\nu\xi} (p - q)^\mu \\
+ g^{\chi\rho} (q - k)^\rho \right]
\]

4-boson vertex:
\[
-ig^2 \left[ f^{abe} f^{cde} \left[ g_{rs} g^{\nu\xi} - g_{\mu\xi} g^{\nu s} \right] \\
+ 2 \text{ perm.} \right]
\]
Ghost propagator:

\[ a \rightarrow b \quad \frac{i\delta^{ab}}{p^2 + i\eta} \]

Ghost vertex:

\[ c^\mu, b \rightarrow a \quad = g f^{abc} p^c \]

Lagrangian:

\[ \mathcal{L}_{\text{act}} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - \frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi} (\partial_\mu A^a_\mu)^2 \]
\[ + g A^a_\mu \bar{\psi} \gamma^\mu \tau_a \psi - g f^{abc} (\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c} \]
\[ - g^2 (f^{abc} A^a_\mu A^b_\nu) (f^{cde} A^{\mu c} A^{\nu d}) \]
\[ - \bar{c}^a \sigma^2 c^a - g f^{abc} \bar{c}^a \partial^\mu A^b_\mu c^c \]

Covariant gauges:

\[ \xi = 1 \quad \text{'t Hooft - Feynman gauge} \leftarrow \]
\[ \xi = 0 \quad \text{Landau gauge} \]
\[ \ldots \]
RENORMALIZATION AND RUNNING

COUPLING:

* gauge-invariant regularization scheme:
  \[ d = 4 - 2 \varepsilon \]
  "dimensional regularization"
  \[ \Rightarrow \dim [g_{\text{bare}}] = \varepsilon \]

* replace bare fields and parameters in \( \mathcal{L}_{\text{QCD}} \) by renormalized quantities:

  \[
  \begin{align*}
  \psi_{\text{bare}} &= Z_2^{1/2} \psi_{\text{ren}} \\
  A_{\text{bare}}^\mu &= Z_3^{1/2} A_{\text{ren}}^\mu \\
  g_{\text{bare}} &= Z_g \mu^\varepsilon g_{\text{ren}} \\
  \end{align*}
  \]
  \( \mu : \text{renorm. scale} \)
  \( \uparrow \)
  \( \text{dimensionless} \)

  \[ \Rightarrow \text{quark-gluon vertex:} \]

  \[
  (g \bar{\psi} \gamma^\mu \psi)_{\text{bare}} = Z_1 \mu^\varepsilon (g \bar{\psi} \gamma^\mu \psi)_{\text{ren}}
  \]

  \text{with:} \quad Z_1 = Z_g Z_2 Z_3^{1/2}
* rewrite Lagrangian: \((m=0)\)

\[
\mathcal{L}_{\text{QCD}} = \overline{\psi} i \gamma \cdot \partial \psi - \frac{1}{4} (F_{\mu \nu}^a)^2 + \left[ \text{gauge-fixing} \right] \\
+ \delta_2 \overline{\psi} i \gamma \cdot \partial \psi - \delta_3 \frac{1}{4} (F_{\mu \nu}^a)^2 \\
+ \delta_1 \mu^g \overline{\psi} A^a \psi + \ldots \right) \text{ counter terms}
\]

with: \(Z_n = 1 + \delta_n\)

**COUNTER TERMS:**

\[
P \quad = \quad i \gamma^\mu \delta_2
\]

\[
k_{\mu \nu} = -i \left( k^2 g_{\mu \nu} - k^\mu k^\nu \right) \delta_{ab} \delta_3
\]

\[
\xi_{\mu \nu} \quad \quad = i g \gamma^\mu \tau^a \delta_4
\]

\[
\Rightarrow \text{ to be adjusted so as to cancel divergences of Green functions}
\]

"subtraction prescription" \((\text{MS}, \overline{\text{MS}}, \ldots)\)
Why dimensional regularization?

⇒ only known regularization scheme that preserves Ward identities (i.e. gauge invariance) in non-abelian gauge theories!

How does it work?

* typical loop integral: (after Wick rotation: \( k_0 \rightarrow i k_0 \))

\[
I_4 = \int d^4k \frac{1}{(k^2 + m^2)^2} \quad \text{logarithmically divergent as } k \rightarrow \infty
\]

but \( I_n = \int d^n k \frac{1}{(k^2 + m^2)^2} \) is finite if \( n < 4 \)

* more generally, can compute loop integrals in an \( n \)-dimensional Euclidean space as function of \( n \) using:

\[
d^n k \rightarrow 1 |k|^{n-1} \, dl kl \, d\Omega_{n-1}
\]

⇒ by means of analytic continuation in \( n \rightarrow d \) can uniquely define result for arbitrary \( d \)
Dimensional regularization:

\[ d = 4 - 2\epsilon < 4 \]

\[ \Rightarrow \] logarithmically divergent integrals become finite for \( \epsilon > 0 \), but are singular as \( \epsilon \to 0 \)

typically:

\[ \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \Theta(\epsilon) \]

\* in dimensional regularization, mass dimensions of fields and couplings change:

\[ S = \int d^d x \ L(x) \text{ dimensionless} \]

\[ \Leftrightarrow \ [L] = d \]

\[ [\Psi] = \frac{d-1}{2} = \frac{3}{2} - \epsilon \]

\[ [A_\mu] = \frac{d-2}{2} = 1 - \epsilon \]

\[ [g] = \frac{4-d}{2} = \epsilon \]

\[ \Rightarrow \] gauge coupling no longer dimensionless!
**QUARK SELF-ENERGY**: \( m = 0 \); Feynman gauge

\[
\begin{align*}
\int \frac{d^d k}{(2\pi)^d} & \ i \mu \, g \, t_a \, \gamma^\mu \ i \frac{i}{p+k} \ i \mu \, g \, t_a \, \gamma^\mu \ \frac{-i}{k^2} \quad (d = 4 - 2\varepsilon) \\
= & \ (i g)^2 \ \frac{t_a \, t_a}{2-d} \ \mu^{2\varepsilon} \ \int \frac{d^d k}{(2\pi)^d} \ \frac{p+k}{(p+k)^2} \ \frac{k^2}{k^2} \\
= & \ + \ i g \ \frac{g^2}{(4\pi)^2} \ \left( \frac{-p^2}{4\pi \mu^2} \right)^{-\varepsilon} \ \Gamma(\varepsilon) \ \frac{\Gamma(2-\varepsilon) \ \Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \\
= & \ \frac{\alpha_s}{4\pi} \ \text{divergent as } \varepsilon \to 0 \\
\therefore \quad \varepsilon \to 0 \quad & \quad \rightarrow \quad i g \ \frac{\alpha_s}{4\pi \varepsilon} \ + \ \text{finite terms} \\
\Rightarrow \quad \text{adjust counter term } \delta_2 \ \text{to cancel the} \ \frac{1}{\varepsilon} \ \text{pole (MS prescription):} \\
\delta_2 & = \ - \ C_F \ \frac{\alpha_s}{4\pi \varepsilon} \quad (\text{Feynman gauge})
\end{align*}
\]
GLUON SELF-ENERGY:  (Feynman gauge)

Ward identity: \( g^\mu \left( \frac{\partial}{\partial k^\mu} \right)_\nu = 0 \)

\[ \Rightarrow k^\mu \frac{\partial}{\partial k^\mu} = i \left( k^2 g^{\mu\nu} - k^\mu k^\nu \right) \pi(k^2) \]

but individual diagrams are not transverse

* abelian contribution:

\[ \sum_f \frac{\alpha_s}{4\pi} \left( -\frac{4}{3} g^f T_F \right) \left( k^2 g^{\mu\nu} - k^\mu k^\nu \right) + \ldots \]

\[ \text{tr}(t_a t_b) = T_F \delta_{ab} \]

* non-abelian contributions:

\[ \frac{\alpha_s}{4\pi} \sum_b C_A \left( \frac{19}{12} k^2 g^{\mu\nu} - \frac{11}{6} k^\mu k^\nu \right) + \ldots \]

\[ = 0 + \ldots \]

\[ \frac{\alpha_s}{4\pi} \sum_b C_A \left( \frac{1}{12} k^2 g^{\mu\nu} + \frac{1}{6} k^\mu k^\nu \right) + \ldots \]

\[ \Rightarrow \text{sum} = \frac{\alpha_s}{4\pi} \frac{5}{3} C_A \left( k^2 g^{\mu\nu} - k^\mu k^\nu \right) + \ldots \]

with:

\[ \sum_{facd} f_{bcd} = C_A \delta_{ab} \]
\( \delta_3 = \left( \frac{5}{3} C_A - \frac{4}{3} n_f T_F \right) \frac{\alpha_s}{4\pi\varepsilon} \quad \text{(Feynman gauge)} \)

**VERTEX CORRECTION:** (Feynman gauge)

\[
\begin{align*}
\gamma^{\mu a} & \quad = \quad i \frac{\alpha_s}{4\pi\varepsilon} \, g \, \gamma^\mu \gamma^a \, \left[ (C_F - \frac{1}{2} C_A) + \frac{3}{2} C_A \right] \\
& \quad + \ldots
\end{align*}
\]

\( \Rightarrow \) counter term:

\( \delta_1 = - (C_F + C_A) \frac{\alpha_s}{4\pi\varepsilon} \quad \text{(Feynman gauge)} \)

**SU(N) GROUP FACTORS:**

\[
C_F = \frac{N^2 - 1}{2N} \quad , \quad C_A = N \quad , \quad T_F = \frac{1}{2}
\]

**CHARGE RENORMALIZATION:**

\[
Z_q = \frac{Z_1}{Z_2 \, Z_3^{1/2}} = 1 + \delta_1 - \delta_2 - \frac{1}{2} \delta_3
\]

\[
= 1 + \frac{\alpha_s}{4\pi\varepsilon} \left( - \frac{11}{6} C_A + \frac{2}{3} n_f T_F \right)
\]

(gauge independent !)
QCD $\beta$ FUNCTION

Scale dependence of renormalized coupling:

$$\mu \frac{d}{d\mu} g_{\text{bare}} = 0$$

$$\Rightarrow \mu \frac{d}{d\mu} g(\mu) + \epsilon g(\mu) = -g(\mu) \frac{1}{Z_2} \mu \frac{d}{d\mu} Z_2$$

$$(-2) \times \text{coefficient of } \frac{1}{\epsilon}$$

* $\beta$ function:

$$\beta(\alpha_s) = \mu \frac{d}{d\mu} \alpha_s = -\frac{\alpha_s^2}{2\pi} \left( \frac{11}{3} C_A - \frac{4}{3} n_f T_F \right) + \mathcal{O}(\alpha_s^3)$$

$$\Rightarrow \text{crucial!}$$

$$\Rightarrow \text{for SU}(N) \text{ with } n_f < \frac{11}{2} N \text{ flavours, } \beta(\alpha_s) < 0$$

**ASYMPTOTIC FREEDOM**

* "running coupling":

$$\alpha_s(\mu^2) = \frac{\alpha_s^2}{\mu^2}$$

$$\int \frac{d\mu}{\alpha_s} = -\int d\ln \mu^2 \frac{\beta_0}{4\pi}$$

$$\Rightarrow \alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\beta_0}{4\pi} \alpha_s(\mu^2) \ln Q^2/\mu^2} \equiv \frac{4\pi}{\beta_0 \ln Q^2/\lambda^2}$$

$$\alpha_s(Q^2) \rightarrow 0 \text{ as } Q^2 \rightarrow \infty$$
Screening (QED):

\[ \nabla \cdot \mathbf{E}^a = g g^a + g f^{abc} \mathbf{A}^b \cdot \mathbf{E}^c \] (Coulomb gauge)

\[ \Rightarrow SU(2) : \]

(a)

(b)

(c)

Figure 12: Electromagnetic charge screening in a dipolar medium.

Figure 16.11. The effect of vacuum fluctuations on the Coulomb field of an SU(2) gauge theory. In (a), a fluctuation \( A^2 \) occurs on top of the \( 1/r^2 \) field \( E^1 \). These combined fields generate a sink of the field \( E^3 \), as shown in (b). The \( E^3 \) field, in turn, combines with \( A^2 \) to create an effective \( E^1 \) dipole, shown in (c). The dipole points toward the original charge, enhancing its field at large distances.
**APPLICATIONS OF pQCD**

**$e^+e^-$ ANNIHILATION:**

\[
R_{e^+e^-} (s) = \frac{\sigma (e^+e^- \rightarrow \text{hadrons})}{\sigma (e^+e^- \rightarrow \mu^+\mu^-)} \quad j \quad s \ll M_Z^2 
\]

* for free quarks:

\[
\begin{align*}
\text{same as } & \quad e^- \quad \gamma \quad q_f^- \\
\text{same as } & \quad e^- \quad \gamma \quad \mu^-
\end{align*}
\]

with replacement:

\[
(-e)^2 \rightarrow N_c Q_f^2 \, e^2 \quad (s \gg m_f^2)
\]

* QCD corrections:

virtual gluon

real gluon emission

$\Rightarrow$ infrared singularities cancel between virtual and real photons (KLN theorem)
soft-gluon emission happens on large time scales and thus does not affect probability that a $q\bar{q}$-pair was produced, but only properties of final state in which $q\bar{q}$-pair evolves

$\Rightarrow$ IR insensitivity of inclusive processes

* Result:

$$ R_{e^+e^-}^{\text{QCD}}(s) = N_c \sum_f Q_f^2 \left\{ 1 + \frac{\alpha_s(\mu^2)}{\pi} \right. $$

$$ + \left[ 1.986 - 0.115 n_f - \frac{\beta_0}{4} \ln \frac{s}{\mu^2} \right] \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 + \ldots \right\} $$

$\uparrow$

log. dependence on $s$

RG improvement:

$$ \mu \frac{d}{d\mu} R \left( \frac{s}{\mu^2}, \alpha_s(\mu^2) \right) = 0 \quad \Rightarrow \quad R \left( \frac{s}{\mu^2}, \alpha_s(\mu^2) \right) = R \left( 1, \alpha_s(s) \right) $$

hence:

$$ R_{e^+e^-}^{\text{QCD}}(s) = N_c \sum_f Q_f^2 \left\{ 1 + \frac{\alpha_s(s)}{\pi} + (1.986 - 0.115 n_f) \left( \frac{\alpha_s(s)}{\pi} \right)^2 + \ldots \right\} $$

$\Rightarrow$ resums all large logarithms $\sim (\alpha_s \ln s/\mu^2)^n$

into running coupling constant $\alpha_s(s)$
Figure 18.6. Experimental measurements of the total cross section for the reaction $e^+e^- \rightarrow \mu^+\mu^-$ at energies below 3 GeV, compared to the prediction of perturbative QCD for 3 quark flavors. The data are taken from the compilation of M. Swartz, *Phys. Rev.* D (to appear). Complete references to the various results are given there.
HADRONIC $\tau$ DECAYS:

$$R_{\tau} = \frac{\Gamma(\tau^{-} \rightarrow \nu_{\tau} + \text{hadrons})}{\Gamma(\tau^{-} \rightarrow \nu_{\tau} e^{-}\bar{\nu}_{e})}$$

* for free quarks:

\[ \begin{array}{c}
\tau^{-} \rightarrow \nu_{\tau} \\
W^{-} \rightarrow d, s \\
\bar{u}
\end{array} \]

same as

\[ \begin{array}{c}
\tau^{-} \rightarrow \nu_{\tau} \\
W^{-} \rightarrow e^{-} \\
\bar{\nu}_{e}
\end{array} \]

with replacement:

$$G_{F} \rightarrow N_{c} \left( |V_{ud}|^2 + |V_{us}|^2 \right) G_{F}$$

$$= 1$$

* QCD corrections similar as for $R_{e^+e^-}$

* result:

$$R_{\tau}^{\text{QCD}} = N_{c} \left\{ 1 + \frac{\alpha_{s}(m_{\tau}^2)}{\pi} + 5.202 \left( \frac{\alpha_{s}(m_{\tau}^2)}{\pi} \right)^2 + \ldots \right\}$$

(after RG improvement)
Experimental evidence for the running of $\alpha_s(Q^2)$:

$\alpha_s(M_2^2) = 0.116 \pm 0.005$
$e^+e^- \rightarrow \text{JETS}$:

$$\sigma(e^+e^- \rightarrow q\bar{q}) = \frac{4\pi\alpha^2}{3s} N_c \sum_f Q_f^2 \left\{ \frac{1 + \sigma(\alpha_s)}{\sigma_0} \right\}$$

$\Rightarrow$ 2-jet events; dominant process

3-jet events:

$\Rightarrow$ additional gluon; suppressed by $\alpha_s$:

$$\frac{d^2\sigma(e^+e^- \rightarrow q\bar{q}g)}{dx_1 \, dx_2} = \frac{2\alpha_s}{3\pi} \, \sigma_0 \, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

with:

$$x_i = \frac{2E_i^{\text{CMS}}}{s} \quad 0 < x_i < 1 \quad x_1 + x_2 + x_3 = 2$$

2-jet invariant mass:

$$S_{13} = (p_1 + p_3)^2 = 2p_1 \cdot p_3 = s \, (1-x_2)$$

$$S_{23} = (p_2 + p_3)^2 = 2p_2 \cdot p_3 = s \, (1-x_1)$$
cross section diverges if \( x_1 \rightarrow 1 \) or \( x_2 \rightarrow 1 \)

\[ x_1 \rightarrow 1: \quad \text{either } p_3 \rightarrow 0 \quad \text{"soft gluon"} \]
\[ \text{or } p_3 \parallel p_2 \quad \text{"collinear divergence"} \quad \text{("mass singularity")} \]

* but in both cases \( q(\bar{q}g) \) is indistinguishable from a 2-jet event!

\[ \Rightarrow \text{soft and collinear divergences in } e^+e^- \rightarrow q\bar{q}g \]
\[ \text{cancel against infrared divergence in } e^+e^- \rightarrow q\bar{q} \text{ arising from} \]

\[ \text{CAREFUL DEFINITION OF "JETS": (e.g. JADE alg.)} \]

\[ 3\text{-jet} \iff s_{ij} > y s \]
\[ \iff x_i < 1 - y \]
\[ 2\text{-jet otherwise} \]

\( \Rightarrow \) finite 2- and 3-jet cross sections, which depend on the y-cut
ratio of 3-jet to 2-jet events:

\[
R_3(y) = \frac{2\alpha_s(k_s)}{3\pi} \left\{ 3(1-2y) \ln \frac{y}{1-2y} + 2 \ln^2 \frac{y}{1-y}
+ \frac{5}{2} - 6y - \frac{9}{2} y^2 + 4 \ln \left( \frac{y}{1-y} \right) - \frac{\pi^2}{3} \right\}
\]

with:

\[y < k < 1\] (scale ambiguity)

\[\Rightarrow \text{result unreliable for } y \to 0 \quad (\sim \ln^2 y)\]

---

**Figure 20:** Energy dependence of 3-jet event production rates \(R_3(y = 0.8)\), compared with predictions of analytic \(O(\alpha_s^2)\) QCD calculations, with the hypothesis of an energy independent \(\alpha_s\), and with the abelian vector theory in \(O(\alpha_s^4)\) (taken from Ref. [43]).
most elementary description:

- for large $Q^2$, electron knocks a quark out of the proton

- much later, soft processes cause quark to materialize as a jet of hadrons

- in reference frame where proton is moving fast, partons are almost collinear with proton momentum and light-like:

\[ p = \xi P \quad ; \quad p^2 = 0 \quad (|p^2| \ll s) \]

\[ 0 \leq \xi \leq 1 \]

- to leading order in $\alpha_s(Q^2)$ gluon emission during the collision, as well as large parton transverse momenta, can be ignored
PARTON MODEL:

* DIS cross section = sum over parton cross sections weighted by parton distribution functions:

\[ d\sigma(\nu p \rightarrow eX) = \int d\xi \sum_i f_i(\xi) \ d\sigma(\nu f_i \rightarrow eX) \]

with:

\[ f_i(\xi) \ d\xi = \text{probability of finding constituent } f_i \text{ with momentum fraction } \xi \]

⇒ PDF: non-perturbative quantities describing internal structure of proton

* relate parton kinematics with DIS observables:

\[ \hat{t} = q^2 = -Q^2 \]

\[ \hat{s} = 2p \cdot k = 2\xi P \cdot k = \xi s \]

and:

\[ (p+q)^2 = 2p \cdot q + q^2 = 2\xi P \cdot q - Q^2 = 0 \]

\[ \xi = \frac{Q^2}{2E_q} = x \]

another useful variable:

\[ y = \frac{2P \cdot q}{s} = \frac{P \cdot q}{P \cdot k} = \frac{E_e - E_e'}{E_e} = \frac{Q^2}{xs} \]

in proton rest frame

\[ 0 \leq x, y \leq 1 \]
* result:

\[
\frac{d^2 \sigma}{dx \, dy} (ep \rightarrow eX) = \sum_i x f_i(x) Q_i^2 \frac{2\pi \alpha^2 \mathcal{F}}{Q^4} \left[ 1 + (1-y)^2 \right]\]

\[\mathcal{F}^e_{xe}(x)\]

* rederive this from more general formalism:

\[M(ep \rightarrow eX) = -e^2 \frac{\bar{u}(k') \gamma_\mu u(k)}{q^2} \int d^4x e^{i q \cdot x} \frac{1}{q^2} \left\langle x | J^\mu(x) | P \right\rangle\]

electrons

\[\text{hadronic matrix element of current:}\]

\[J^\mu = \sum_q Q_q^2 \bar{q} \gamma_\mu q\]

\[\text{photon propagator}\]

\[\Rightarrow \text{amplitude must be squared and summed over final states}\]

**OPTICAL THEOREM:**

\[6 \sim \text{imaginary part of forward Compton amplitude}\]

(= matrix element of current product)

\[
\frac{d^2 \sigma}{dx \, dy} = \frac{2\alpha^2 y}{Q^4} \left( k_\mu k_\nu' + k_\nu k_\mu' - g_{\mu\nu} k \cdot k' \right) \text{Im} W^{\mu\nu}(P, q)
\]

with:

\[W^{\mu\nu}(P, q) = i \int d^4x e^{i q \cdot x} \left\langle P | T \{ J^\mu(x), J'^\nu(0) \} | P \right\rangle\]
Figure 18.9. Computation of the cross section for deep inelastic electron scattering: (a) general structure of the amplitudes; (b) application of the optical theorem.

Figure 18.10. Evaluation of $W^{\mu\nu}$ in the parton model.
most general Lorentz-invariant decomposition allowed by current conservation \((q_v W^\nu = 0)\):

\[
W^\nu(x,q) = i \int dx e^{iq \cdot x} \langle \mathcal{P} \mathcal{T} \{ J^\mu(x), J^\nu(0) \} \mathcal{P} \rangle 
\]

\[
= \left( \frac{q^\mu q^\nu}{q^2} - g^{\mu \nu} \right) W_1 + \left( P^\nu - q^\nu \frac{P \cdot q}{q^2} \right) \left( P^\mu - q^\mu \frac{P \cdot q}{q^2} \right) W_2
\]

where \(W_1, W_2\) depend on \(q^2, P \cdot q\) or, equivalently, \(Q^2, x\)

general result for DIS cross section:

\[
\frac{d^2 \sigma}{dx dy} = \frac{\alpha^2 s}{Q^4} \left[ y(1-y) s \text{ Im } W_2 + 2xy^2 \text{ Im } W_1 \right]
\]

other common notation:

\[
\text{Im } W_2 = \frac{4\pi x}{Q^2} F_2 , \quad \text{Im } W_1 = \pi F_1
\]

RESULT:

\[
\frac{d^2 \sigma}{dx dy} = \frac{2 \pi \alpha^2 s}{Q^4} F_2(x,Q^2) \left[ 1 + (1-y)^2 + y^2 \left( \frac{x F_1(x,Q^2)}{F_2(x,Q^2)} - 1 \right) \right]
\]

(exact formula)
PARTON MODEL:

⇒ replace proton matrix element by sum over quark matrix elements weighted with parton distribution functions

⇒ result:

\[ F_2(x, Q^2) = \sum_i x f_i(x) Q_i^2 \]

\[ F_4(x, Q^2) = \sum_i f_i(x) Q_i^2 \]

* Callan-Gross relation:

\[ F_2 = x F_4 \]

NOTE:

at large \( Q^2 = -q^2 \) the bilocal object

\[ W^{\mu\nu} = i \int dx \; e^{i q \cdot x} \langle T \{ J^{\mu}(x), J^{\nu}(0) \} \rangle \]

is short-distance dominated and can be evaluated using the Operator Product Expansion

⇒ rigorous derivation of parton-model results and corrections to it!
DIS WITH NEUTRINOS:

* electron scattering not sufficient to determine individual parton distribution functions
* neutrinos interact through weak interactions and thus probe different quantum numbers

\[
\frac{d^2\sigma}{dx\,dy} (\bar{\nu}_e p \rightarrow \mu^- X) = \frac{G_F^2 \, s}{\pi} \left[ x \, d(x) + (1-y)^2 \times \bar{u}(x) \right]
\]

\[
\frac{d^2\sigma}{dx\,dy} (\bar{\nu}_e p \rightarrow \mu^+ X) = \frac{G_F^2 \, s}{\pi} \left[ (1-y)^2 \times u(x) + x \, \bar{d}(x) \right]
\]

\[
\uparrow \quad \text{valence quark distributions} \quad \uparrow \quad \text{"sea" quark distrib.}
\]

⇒ individual mapping of quark distribution functions possible
Figure 17.3. The distribution in $y$ of neutrino and anti-neutrino deep inelastic scattering from an iron target, as measured by the CDHS experiment. J. G. H. de Groot, et. al., Z. Phys. C1, 143 (1979). The solid curves are fits to the form $A + B(1-y)^2$. 
PARTON DISTRIBUTIONS:

* non-perturbative quantities describing internal structure of nucleon
  ⇒ extract from data

* scaling violations: weak logarithmic $Q^2$ dependence due to higher-order QCD effects
  ⇒ allow to determine gluon distribution $g(x)$

* because of probabilistic interpretation, distributions are normalized in a way that reflects proton quantum numbers:

\[
\int_0^1 \, dx \, [ \, u(x) - \bar{u}(x) \, ] \, = \, 2
\]

\[
\int_0^1 \, dx \, [ \, d(x) - \bar{d}(x) \, ] \, = \, 1
\]

\[
\int_0^1 \, dx \, [ \, s(x) - \bar{s}(x) \, ] \, = \, 0
\]

etc.

momentum sum rule:

\[
\int_0^1 \, dx \, x \cdot [ \, u(x) + d(x) + s(x) + \bar{u}(x) + \bar{d}(x) + \bar{s}(x) + g(x) \, ] \, = \, 1
\]

EXPERIMENTALLY:

⇒ gluons carry ~ half of proton momentum!
Figure 17.6. Parton distribution functions $x f_j(x)$ for quarks, antiquarks, and gluons in the proton, at $Q^2 = 4$ GeV$^2$. These distributions are obtained from a fit to deep inelastic scattering data performed by the CTEQ collaboration (CTEQ2L), described in J. Betti, et al., Phys. Lett. B304, 159 (1993).

Figure 17.21. The $u$ quark parton distribution function $x f_u(x, Q)$ at $Q = 2, 20$, and $200$ GeV, showing the effects of parton evolution according the Altarelli-Parisi equations. These curves are taken from the CTEQ fit to deep inelastic scattering data described in Fig. 17.6.
SCALING VIOLATIONS AND PARTON EVOLUTION

- Corrections to parton model come from
  - Gluon emission during scattering process
  - Transverse parton momenta
  - ...

- Corrections are of order:
  - \([\alpha_s(Q^2)]^n\), \(\left(\frac{\Lambda^2}{Q^2}\right)^n\) small
  - \((\alpha_s(Q^2) \ln \frac{Q^2}{\Lambda^2})^n\) large logs. \(\Rightarrow\) can be summed using RG eqns.
    (Altarelli - Parisi eqn.)

QUALITATIVE PICTURE:

- Small \(Q^2\) \(\rightarrow\) large \(Q^2\)
Leading-order contribution of quark with momentum fraction $y$ to quark distribution function $q(x)$:

\[ q(x) = \int_0^1 dy \, q(y) \, \delta(y-x) \]

If quark emits gluon before probed by photon, its momentum fraction is degraded to $yz$ ($0 \leq z \leq 1$):

\[ \Delta q(x) = \frac{\alpha_s}{2\pi} \int_0^1 dy \, q(y) \int_0^1 dz \, \delta(yz-x) \left[ P_{qq}(z) \, \frac{\ln \frac{Q^2}{\mu^2}}{y^2} + C(z) \right] \]

with:

\[ P_{qq}(z) = C_F \left[ \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \, \delta(1-z) \right] \]

"splitting function (q → q)"

\[ \left\{ \begin{array}{l}
0 \int dz \, f(z) \frac{1}{(1-z)_+} = \int_0^1 dz \, \frac{f(z) - f(1)}{1-z} \\
\end{array} \right. \]

**NOTE:**

\[ \ln \frac{Q^2}{\mu^2} = \text{phase space for gluon emission} \]

(with quarks \(\propto\) on-shell)

collinear singularity
* with increasing $Q^2$ can resolve shorter distances inside proton

$\Rightarrow$ e.g., can resolve a quark with momentum fraction $x$ as a pair of quark + gluon with smaller momentum fractions

* thus, with increasing $Q^2$ expect qualitative changes in distribution functions:

- gluon emission shifts valence-quark and sea-quark distributions towards smaller $x$

- $g \rightarrow q\bar{q}$ increases amount of sea quarks (mainly at low $x$)

**SCALING VIOLATIONS:** $f_i(x, Q^2)$
multiple gluon radiation:

\[ \sim \frac{1}{n!} \left( \frac{a_s}{\pi} \right)^n \ln^n \frac{Q^2}{\mu^2} \]

from region where gluon transverse momenta are strongly ordered:

\[ p_{1T} \ll p_{2T} \ll p_{3T} \quad \text{etc.} \]

(otherwise no collinear singularity)

* collinear divergence can be absorbed into bare distribution function:

\[ q(x, Q^2) = q(x, \mu^2) + \frac{a_s}{2\pi} \ln \frac{Q^2}{\mu^2} \int \frac{dy}{y} q(y) P_{qq} \left( \frac{x}{y} \right) \]

\[ \text{cancel!} \]

physical distr. fct.

**EVOLUTION EQUATION**: (non-singlet distr.)

\[ Q^2 \frac{d}{dQ^2} q(x, Q^2) = \frac{a_s(Q^2)}{2\pi} \int \frac{dy}{y} q(y, Q^2) P_{qq} \left( \frac{x}{y} \right) \]

RG improvement resums all large logarithms of multiple emissions!
above equation only correct for flavour non-singlet distributions such as \([u(x) - d(x)]\)
where gluon contributions \(g \rightarrow q \bar{q}\) cancel

\[ \Rightarrow \text{ in general, must also include:} \]

\[
\begin{align*}
q \rightarrow q: & \quad y \quad zy \\
q \rightarrow g: & \quad y \quad zy \\
g \rightarrow q: & \quad y \quad zy \\
g \rightarrow g: & \quad y \quad zy
\end{align*}
\]

\[ P_{qq}(z): \text{ given above} \]
\[ P_{gq}(z) = P_{qq}(1 - z) \]
\[ P_{qg}(z) = T_F \left[ z^2 + (1 - z)^2 \right] \]
\[ P_{gg}(z) = 2 C_A \left[ \frac{z}{\langle 1 - z \rangle^2} + \frac{1 - z}{z} + z(1 - z) \right] + \frac{\alpha_s}{2} \delta(1 - z) \]
\( Q^2 \frac{d}{dQ^2} \left( \frac{q(x, Q^2)}{g(x, Q^2)} \right) = \frac{\alpha_s(Q^2)}{2\pi} \int \frac{dy}{x} \begin{bmatrix} P_{qq}(\frac{x}{y}) & P_{qg}(\frac{x}{y}) \\ P_{gq}(\frac{x}{y}) & P_{gg}(\frac{x}{y}) \end{bmatrix} \begin{bmatrix} q(x, Q^2) \\ g(x, Q^2) \end{bmatrix} \)

* matrix integro-differential equation

⇒ can be solved numerically, starting from input set of distributions at some value \( Q_0^2 \)

* evolution equation simplifies for moments:

\[
M_n^q(Q^2) = \int_0^1 dx \, x^{n-1} \, q(x, Q^2)
\]

\[
M_n^g(Q^2) = \int_0^1 dx \, x^{n-1} \, g(x, Q^2)
\]

then:

\[
Q^2 \frac{d}{dQ^2} \begin{bmatrix} M_n^q(Q^2) \\ M_n^g(Q^2) \end{bmatrix} = \frac{\alpha_s(Q^2)}{2\pi} \begin{bmatrix} \delta_{qq} & \delta_{qg} \\ \delta_{gq} & \delta_{gg} \end{bmatrix} \begin{bmatrix} M_n^q(Q^2) \\ M_n^g(Q^2) \end{bmatrix}
\]

ANOMALOUS DIMENSIONS:

\[
\delta_{qq} = C_F \left( -\frac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{k=2}^{n} \frac{1}{k} \right)
\]

\[
\delta_{qg} = T_F \frac{2+n+n^2}{n(n+1)(n+2)}
\]

\[
\delta_{gq} = C_F \frac{2+n+n^2}{n(n^2-1)}
\]

\[
\delta_{gg} = 2C_A \left( -\frac{1}{12} + \frac{1}{n(n-1)} + \frac{1}{(n+1)(n+2)} - \sum_{k=2}^{n} \frac{1}{k} \right) - \frac{2}{3} T_F n_f
\]

\[
\gamma_{ij}^n = \int_0^1 dz \, z^{n-1} \, P_{ij}(z)
\]
Figure 17.22. Dependence on $Q^2$ of the combination of quark distribution functions $F_2 = \sum_f xQ^2 f(x, Q^2)$ measured in deep inelastic electron-proton scattering. The various curves show the variation of $F_2$ for fixed values of $x$ and the comparison of this variation to a model evolved with the Altarelli-Parisi equations. The upper six data sets have been multiplied by the indicated factors to separate them on the plot. The data were compiled by M. Virchaux and R. Voss for the Particle Data Group, Phys. Rev. D80, 1173 (1994), Fig. 33.2. The complete references to the original experiments are given there.
for non-singlet distributions:

\[ M_n^{q,NS}(Q^2) = M_n^{q,NS}(Q_0^2) \left( \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right)^{d_n} \]

\[ d_n = -\frac{2 \gamma_{q_n}^n}{\beta_0} > 0 \]

\[ \rightarrow \text{decrease as } Q^2 \text{ increases} \]

flavour-singlet distributions for \( n = 2 \)

\( \approx \text{average parton momenta} \):

\[ \sum (x_i Q^2) = \sum \left[ q_f(x, Q^2) + \bar{q}_f(x, Q^2) \right] \]

then:

\[ Q^2 \frac{d}{dQ^2} \left( \begin{array}{c} M^S_2(Q^2) \\ M^g_2(Q^2) \end{array} \right) = \frac{\alpha_s(Q^2)}{2\pi} \left( \begin{array}{cc} -\frac{4}{3} C_F & 2n_f \cdot \frac{1}{3} T_F \\ \frac{4}{3} C_F & -\frac{2}{3} n_f T_F \end{array} \right) \left( \begin{array}{c} M^S_2(Q^2) \\ M^g_2(Q^2) \end{array} \right) \]

eigenvalues:

\[ \gamma_{(1)} = 0 \quad \gamma_{(2)} = -\frac{4}{3} C_F - \frac{2}{3} n_f T_F \]

\[ \rightarrow \quad M^S_2(Q^2) + M^g_2(Q^2) \]

\[ = \int_{\Xi} x \left[ \Sigma(x, Q^2) + g(x, Q^2) \right] = 1 \]

\( Q^2 \) independent!

(momentum sum rule)
moreover:

$$\frac{M_2^\Sigma(Q^2) - r M_2^g(Q^2)}{M_2^\Sigma(Q_0^2) - r M_2^g(Q_0^2)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)^{D_2}$$

with:

$$r = \frac{n_F T_F}{2 C_F} \rightarrow \frac{3}{4}$$

$$(\text{for } n_F = \frac{3}{2})$$

$$D_2 = -\frac{2 \gamma_{(Q)}}{\beta_0} \rightarrow \frac{56}{75}$$

\Rightarrow \text{ using that } M_2^\Sigma + M_2^g = 1 \text{ gives:}

$$\frac{M_2^\Sigma(Q^2) - \frac{r}{1+r}}{M_2^\Sigma(Q_0^2) - \frac{r}{1+r}} = \frac{M_2^g(Q^2) - \frac{1}{1+r}}{M_2^g(Q_0^2) - \frac{1}{1+r}} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)^{D_2}$$

**FIXED POINTS:**

$$M_2^\Sigma = \frac{r}{1+r} \rightarrow \frac{3}{7} \quad \text{total quark momentum fraction}$$

$$M_2^g = \frac{1}{1+r} \rightarrow \frac{4}{7} \quad \text{total gluon momentum fraction}$$

\Rightarrow \text{ close to experimental data}
processes with large momentum transfer involve interaction vertices (operators) separated by small distances $x^2 \sim 1/Q^2$

\[ \text{DIS:} \]
\[ W_{\mu
u}(P,q) = i \int d^4 x \, e^{i q \cdot x} \left< P | T \{ \bar{J}_\mu(x), J_\nu(0) \} | P \right> \]

with:
\[ J_\mu = \sum_f Q_f^2 \bar{q}_f \gamma_\mu q_f \quad \text{em. current} \]

\[ \frac{d^2 \sigma}{dx \, dy} (e^+ e^- \rightarrow e^+ e^- X) \sim \text{Im} W_{\mu\nu} \]

\[ e^+ e^- \rightarrow \text{hadrons:} \]

\[ \text{hadronic part of vacuum polarization} \]

\[ \text{Re} e^+ e^- (s) = 12 \pi \, \text{Im} \, \Pi_h (-s) \]

with:
\[ \Pi_h^{\mu\nu}(q) = i \int d^4 x \, e^{i q \cdot x} \left< 0 | T \{ \bar{J}_\mu(x), J_\nu(0) \} | 0 \right> \]

\[ = (-q^2 g^{\mu\nu} + q^\mu q^\nu) \Pi_h (-q^2) \]

(optical theorem)
large $q$ probe small distances $x$

$\Rightarrow$ expand bilocal operator product $J(x)J(0)$

around $x=0$

**BUT:** products of fields at same point may be singular in QFT

$\Rightarrow$ singularities absorbed into short-distance (Wilson) coefficient functions

**OPE:**

$$T\{J(x), J(0)\} = \sum_n c_n(x, \mu) \ O_n(0, \mu)$$

renorm. scale

where:

$$\{O_n\} = \text{complete basis of local operators with same global quantum numbers as original operator product (otherwise unrestricted)}$$

* dimensional analysis:

$$c_n(x, \mu) \sim \left(\frac{1}{|x|}\right)^{6-d_n} \tilde{c}_n(x^2\mu^2) \Rightarrow c$-number fcts.

$$\text{dimensionless}$$

$\Rightarrow$ singular as $x \to 0$ if $d_n < 6$

**NOTE:**

- coefficient $c_n$ carry all dependence on $x$ ($x \xrightarrow{FT} q$)

- do not depend on external states but only on original operators and their separation
form of coefficients \( c_n(x, \mu) \) determined by renormalization group:

\[
\text{RGE:} \quad \mu \frac{d}{d\mu} T \{ J(x), J(o) \} = 0 \quad \text{(conserved currents)}
\]

\[
\Leftrightarrow \quad \sum_n (\mu \frac{d}{d\mu} c_n(x, \mu)) O_n(\mu) + \sum_n c_n(x, \mu) \mu \frac{d}{d\mu} O_n(\mu) = 0
\]

\[
\equiv - \sum_m \delta_{nm} O_m(\mu)
\]

\[
\Rightarrow \quad \mu \frac{d}{d\mu} c_n(x, \mu) - \sum_k \delta_{kn} c_k(x, \mu) = 0
\]

without operator mixing:

\[
\mu \frac{d}{d\mu} O_n(\mu) = - \delta_n \quad O_n(\mu)
\]

\[
\mu \frac{d}{d\mu} c_n(x, \mu) = \delta_n \quad c_n(x, \mu)
\]

\[\text{NOTE:} \quad \delta_{nm} \text{ are determined by operator renormalization:}\]

\[
O_{n, \text{bare}} = \sum_m Z_{nm} \quad O_{m, \text{ren}}(\mu)
\]

\[\Rightarrow \quad \delta_{nm} = -2 \times \left[ \text{coefficient of } \frac{1}{\varepsilon} \text{-pole of } Z_{nm} \right. \text{in dimensional regularization}\]
SOLUTION OF RGE: (no operator mixing)

\[ \mu \frac{d}{d\mu} \, c_n(x,\mu) = \gamma_n \, c_n(x,\mu) \quad ; \quad c_n(x,\mu) = \left( \frac{1}{|x|} \right)^{6-d_n} \, \bar{c}_n(x^2 \mu^2) \]

a) if no running coupling (not QCD!):

\[ \bar{c}_n(x^2 \mu^2) = \bar{c}_n(1) \, \exp \left( \gamma_n \int \frac{d\mu'}{\mu} \right) \sim \left( \frac{1}{|x|} \right)^{6-(d_n+\delta_n)} \]

\[ \Rightarrow \quad c_n(x,\mu) \sim \left( \frac{1}{|x|} \right)^{6-(d_n+\delta_n)} \]

\[ \delta_n : \text{"anomalous dimension"} \]

\[ \Rightarrow \text{canonical dimension } d_n \text{ of operator } P_n \]

is changed by quantum corrections

b) with running coupling (QCD):

\[ \bar{c}_n [x^2 \mu^2, \alpha_s(\mu^2)] = \bar{c}_n [1, \alpha_s(\frac{1}{x^2})] \exp \int d\alpha \frac{\gamma_n(\alpha)}{\beta(\alpha)} \]

\[ = \bar{c}_n [1, \alpha_s(\frac{1}{x^2})] \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\frac{1}{x^2})} \right)^{-\delta_n / \beta_0} \]

where:

\[ \gamma_n(\alpha) = \frac{\alpha_s}{4\pi} \gamma^{(n)}_0 \]

\[ \beta(\alpha) = \mu \frac{d}{d\mu} \alpha_s(\mu^2) = -\frac{\alpha_s^2}{2\pi} \beta_0 \]
in detail:

\[
\exp \int \frac{\gamma_n(\alpha)}{\alpha_s(\gamma_{\alpha})} \, d\alpha = \exp \left(-\frac{\gamma_{\alpha}^{(n)}}{2\beta_0} \int \frac{d\alpha}{\alpha s(\gamma_{\alpha})} \right) = \exp \left(-\frac{\gamma_{\alpha}^{(n)}}{2\beta_0} \ln \frac{\alpha_s(\mu^2)}{\alpha_s(\gamma_{\alpha})} \right) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\gamma_{\alpha})} \right)^{-\gamma_{\alpha}^{(n)}/2\beta_0}
\]

**To prove that solution is correct, we check that:**

\[
\mu \frac{d}{d\mu} \exp \int \frac{\gamma_n(\alpha)}{\beta(\alpha)} \, d\alpha = \gamma_n(\alpha) \beta(\alpha) \left( \mu \frac{d}{d\mu} \alpha_s(\mu^2) \right) \exp \int \frac{\gamma_n(\alpha)}{\beta(\alpha)} \, d\alpha = \gamma_n(\alpha) \exp \int \frac{\gamma_n(\alpha)}{\beta(\alpha)} \, d\alpha \]

\[
\checkmark
\]
Why you should stay until the end ...

level of abstraction

\[ \text{MO} \quad \text{TU} \quad \text{WE} \quad \text{TH} \quad \text{FR} \]

day

TODAY

\[ \text{SMILE} \]
OPE ANALYSIS OF $e^+e^- \to$ HADRONS:

* OPE provides new viewpoint from which to understand hard QCD process, since large $Q^2 \leftrightarrow$ short distances

* simplest application: $e^+e^- \to$ hadrons

Polarization tensor:

$$\Pi^{\mu\nu}_h(q) = i \int dx e^{iq \cdot x} \langle 0 | T \{ \bar{J}^\mu(x), J^\nu(0) \} | 0 \rangle$$

OPE:

$$T \{ \bar{J}^\mu(x), J^\nu(0) \} = c_1(x) 1 + c_{qq}(x) \bar{q}^\nu q(0) + c_{\pi^2}(x) (\tau^a \phi^a)(0)$$

$$+ ...$$

local, gauge-invariant operators
with vacuum quantum numbers

⇒ dimensional analysis:

$$c_1 \sim \frac{1}{x^6}$$

$$c_{qq} \sim \frac{1}{x^2} \cdot m_q$$ (chiral sym. breaking) $\xrightarrow{m_Q \to 0}$

$$(\text{neglect})$$

$$c_{\pi^2} \sim \frac{1}{x^2} \cdot \alpha_s(\mu^2)$$

$C_{\text{other}}$: less singular as $x \to 0$
* after Fourier transform: \( \frac{1}{x^2} \rightarrow -q^2 \equiv Q^2 \)

\[ \pi_h^\mu\nu(q) = (-q^2 g^{\mu\nu} + q^\mu q^\nu) \pi_h(Q^2) \]

with:

\[
\pi_h(Q^2) = \tilde{c_4}(Q_s^2, \mu^2) \cdot 1 + \tilde{c_\pi^{\mu}}(Q_s^2, \mu^2) \left( \frac{\alpha_s(F^{a}_s(\mu) 10)}{Q^4} \right) + \mathcal{O}(Q^{-6}) \]

\[
\tilde{c_4}(Q_s^2, \mu^2) = -N_c \sum_{f} Q_f^2 \frac{1}{12 \pi^2} \ln \frac{Q_s^2}{\mu^2} \left\{ 1 + \frac{\alpha_s(\mu^2)}{\pi} + \mathcal{O}(\alpha_s^2) \right\}
\]

\[
\tilde{c_\pi^{\mu}}(Q_s^2, \mu^2) = N_c \sum_{f} Q_f^2 \frac{1}{48 \pi} \left\{ 1 + \mathcal{O}(\alpha_s) \right\}
\]
using $\Re e^{-}(s) = + 12 \pi \Im m \pi_{h}(-s-i\epsilon)$ we can

TRY to compute cross section:

$$\Re e^{-}(s) = N_{c} \sum_{f} Q_{f}^{2} \left\{ 1 + \frac{\alpha_{s}(\mu^{2})}{\pi} + \ldots + \frac{\pi}{4} \delta'(s) \langle \alpha_{s} F^{2} \rangle \ldots \right\}$$

standard result of pQCD

non-perturbative correction

$
\Rightarrow$ non-perturbative terms VERY SINGULAR

at $s=0$!

$
\Rightarrow$ we are trying to compute something that
we CANNOT compute:

OPE can be used in space-like region ($Q^{2}>>0$)
but not in time-like region ($s=-Q^{2} > 0$),
where $\pi_{h}(-s)$ is saturated by intermediate
hadron states

* fortunately, there is a way out:

**dispersion relations**

$\Rightarrow$ relate space-like region, where OPE works
with time-like region, where experiments are
performed
\[ \Pi_h(-q^2) \] is analytic function of \( q^2 \) with a branch cut on positive \( q^2 \) axis and no other singularities in complex plane

- discontinuity across cut given by \( 2i \text{Im} \, \Pi(-s) \) and thus proportional to \( \text{Re} e^{-}(s) \)

\[ \frac{\text{d}^2}{\text{d}q^2} \Rightarrow \Pi(-q^2) = \frac{1}{\pi} \int_{s_0}^{\infty} \text{d}s \, \frac{\text{Im} \, \Pi(-s)}{s-q^2-i\varepsilon} + \text{const.} \]

\[ \Rightarrow \text{thus, with } Q^2 = -q^2: \]

\[ M_n(Q^2) = \frac{1}{n!} \left( -\frac{\text{d}}{\text{d}Q^2} \right)^n \Pi_h(Q^2) = \frac{1}{12 \pi^2 n} \int_{s_0}^{\infty} \text{d}s \, \frac{\text{Re} e^{-}(s)}{(s+Q^2)^n} \]

\[ \text{calculable using OPE measurable} \]

**ITEP QCD sum rules (SVZ)**

QCD prediction:

\[ M_n(Q^2) = N_c \sum_f Q_f^2 \frac{1}{12 \pi^2 n} \frac{1}{Q^{2n}} \times \left\{ 1 + \frac{\alpha_s(Q^2)}{\pi} + \ldots + \frac{\pi}{4} n(n+1) \frac{\alpha_s F^2}{Q^4} + \ldots \right\} \]

\[ \Rightarrow \text{non-pert. terms increasingly important for large } n! \]
**INTERPRETATION:**

* using OPE, can compute weighted integrals of 
  \( e^+e^- \rightarrow \text{hadrons} \) cross section in QCD!

WHY? \( \rightarrow \) Cauchy theorem

\[
M_n(Q^2) = \frac{1}{\pi} \int ds \frac{\text{Im} \, \Pi_n(-s)}{(s+Q^2)^{n+1}}
\]

\[
= \frac{1}{2\pi i} \oint ds \frac{\Pi_n(-s)}{(s+Q^2)^{n+1}}
\]

\[
= \frac{1}{n!} \left( -\frac{d}{dQ^2} \right)^n \Pi_n(Q^2)
\]

---

quark-hadron duality

---

* if non-perturbative corrections to \( M_n(Q^2) \) would converge to zero uniformly in \( n \), we could invert sum rules to obtain \( \text{Re} e^{-s} \)

however: non-perturbative terms grow with \( n \)

\( \Rightarrow \) most important deviations of cross section from QCD prediction are oscillations that average out in sum rules for low \( n \) \( \Rightarrow \) resonance structure at low \( s \) values!
Figure 18.6. Experimental measurements of the total cross section for the reaction $e^+e^- \rightarrow$ hadrons at energies below 3 GeV, compared to the prediction of perturbative QCD for 3 quark flavors. The data are taken from the compilation of M. Swartz. Phys. Rev. D (to appear). Complete references to the various results are given there.
QCD EFFECTS ON WEAK INTERACTIONS:

* $b \rightarrow c$ transitions in SM:

\[ b \xrightarrow{\Delta q} c, \bar{t}, d \rightarrow W^\pm, t, \bar{u} \rightarrow x^\pm, d, \bar{u} \]

\[ x^2 \sim \frac{1}{M_W^2} \quad \text{short distance} \]

\[ \sim (ig_W)^2 \int \frac{d^4 x}{i} \exp (i q \cdot x) \quad T \{ j_{cc}^\mu (x), j_{cc}^\nu (0) \} \quad \frac{-ig_{\mu \nu}}{q^2 - M_W^2} \]

\[ = \frac{g_W^2}{4\sqrt{2} \cdot G_F} j_{cc}^\mu \cdot j_{cc}^\nu (0) + O \left( \frac{1}{M_W^2} \right) \]

\[ 4\sqrt{2} \cdot G_F \]

Fermi's 4-fermion interactions

* $b \rightarrow c e^- \bar{\nu}$:

\[ \mathcal{L}^{SL}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{cb} \bar{c}_L \gamma^\mu b_L \bar{e}_L \gamma^\nu \nu_L (0) \quad \text{"semileptonic"} \]

$b \rightarrow c \bar{u} d$:

\[ \mathcal{L}^{NL}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \bar{c}_L \gamma^\mu b_L \bar{d}_L \gamma^\nu u_L (0) \quad \text{"non-leptonic"} \]

⇒ this is an OPE!
\[ \mathcal{L}_{\text{eff}}^{\text{SL}} = \frac{4G_F}{\sqrt{2}} V_{cb} c_{SL} (M_W, \mu) O_{SL} (\mu) + o \left( \frac{1}{M_W^2} \right) \]

\[ \mathcal{L}_{\text{eff}}^{\text{NL}} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \left[ c_4 (M_W, \mu) O_4 (\mu) + c_2 (M_W, \mu) O_2 (\mu) + ... \right] + o \left( \frac{1}{M_W^4} \right) \]

where:

\[ O_{SL} = \bar{c}_L \gamma^\mu b_L \bar{L}_L \gamma_\mu \nu_L \]

\[ O_4 = \bar{c}_L \gamma^\mu b_L \bar{L}_L \gamma_\mu u_L \]

\[ O_2 = \bar{c}_L \gamma^\mu u_L \bar{L}_L \gamma_\mu b_L \]

* vector and axial-vector currents are not renormalized (current conservation)

\[ \Rightarrow O_{SL} \text{ is not renormalized} \]

\[ \Rightarrow \text{hence:} \]

\[ c_{SL} (M_W/\mu, \frac{1}{2}) = c_{SL} (1, \Delta s(M_W^2)) \]

\[ = 1 + O [\Delta s(M_W^2)] \]

not affected by large logarithms!
* renormalization of $O_1$ and $O_2$:

\[
\begin{array}{ccccccc}
D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \\
\end{array}
\]

vanish by current conservation

⇒ for the matrix element of the bare operator $O_1$, one finds:

\[
D_3 = D_4 = \frac{\alpha_s}{4\pi\varepsilon} \left[ \frac{2}{3} \langle O_1 \rangle - 2 \langle O_2 \rangle \right] + \text{finite terms}
\]

\[
D_5 = D_6 = \frac{\alpha_s}{4\pi\varepsilon} \left[ -\frac{1}{6} \langle O_1 \rangle + \frac{1}{2} \langle O_2 \rangle \right] + \text{finite terms}
\]

hence:

\[
O_{1,\text{bare}} = \left(1 + \frac{\alpha_s}{4\pi\varepsilon}\right) O_{1,\text{ren}} - 3 \cdot \frac{\alpha_s}{4\pi\varepsilon} O_{2,\text{ren}} + \ldots
\]

Similarly:

\[
O_{2,\text{bare}} = \left(1 + \frac{\alpha_s}{4\pi\varepsilon}\right) O_{2,\text{ren}} - 3 \cdot \frac{\alpha_s}{4\pi\varepsilon} O_{1,\text{ren}} + \ldots
\]

⇒ anomalous dimension matrix:

\[
\hat{\delta} = \frac{\alpha_s}{4\pi} \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}
\]
RGE:

$$\mu \frac{d}{d\mu} \begin{pmatrix} c_1(M_W, \mu) \\ c_2(M_W, \mu) \end{pmatrix} = \frac{\alpha_s(\mu^2)}{4\pi} \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} c_1(M_W, \mu) \\ c_2(M_W, \mu) \end{pmatrix}$$

⇒ solution:

$$c_{1,2}(M_W, \mu) = \frac{1}{2} \left[ c_+(M_W, \mu) \pm c_-(M_W, \mu) \right]$$

with:

$$c_\pm(M_W, \mu) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(M_W^2)} \right)^{-\chi_{\pm}/2}$$

$$\chi_+ = -2 \pm 6$$ eigenvalues

* with $n_f = 5$ flavours ($m_b < \mu < M_W$):

$$c_{1,2}(M_W, \mu) = \frac{1}{2} \left[ \left( \frac{\alpha_s(\mu^2)}{\alpha_s(M_W^2)} \right)^{-6/23} \pm \left( \frac{\alpha_s(\mu^2)}{\alpha_s(M_W^2)} \right)^{12/23} \right]$$

numerically:

$$c_1 = 1 \ , \ c_2 = 0 \quad \Rightarrow \quad c_1 = 1.11 \ , \ c_2 = -0.25$$

($\mu = M_W$)

($\mu = m_b$)

⇒ sizable effects
HEAVY-QUARK SYMMETRY

* strong interactions of hadrons containing a heavy quark $Q$ simplify in limit $m_Q \to \infty$

$\Rightarrow (Q\bar{Q}), (Qq\bar{q}) \ldots$ hadrons resemble hydrogen atom:

$\Rightarrow$ wave function of light degrees of freedom then becomes independent of spin and flavour of the heavy quark

SU($2N_c$) spin-flavour symmetry

(note: magn. moment $\sim \frac{1}{m_Q} \to 0$)

* heavy-quark symmetry = approximate global symmetry realized when:

$\ m_Q \gg \Lambda_{QCD} \ \Leftrightarrow \ \lambda_a \ll R_{had}$

Compton wave length

size of hadrons

$\Rightarrow$ construct effective Lagrangian that is explicitly invariant under this symmetry

systematic expansion in powers of $1/m_Q$
BUT: How to take limit $m_q \to \infty$ in QCD Lagrangian?

$$\mathcal{L}_a = \overline{\psi} (i \not\!\partial - m_q) \psi \quad \to \quad ?$$

TRICK: use that heavy hadron defines a rest frame, where:

$$v^\mu = (1, \vec{v}) \quad \text{hadron 4-velocity}$$

(1) Field redefinition:

$$\psi \to e^{-i m_a x^0} \begin{pmatrix} h \\ H \end{pmatrix} \quad \text{upper two components}$$

$$\psi \to e^{i m_a x^0} \begin{pmatrix} H \\ h \end{pmatrix} \quad \text{lower two components}$$

this gives:

$$\mathcal{L}_a = \overline{h} i \not\!\partial h + \overline{H} (-i \not\!\partial - 2 m_a) H$$

$$- \overline{h} i \not\!\sigma \cdot \vec{D} H - \overline{H} i \not\!\sigma \cdot \vec{D} h$$

\[
\begin{cases}
  h: \text{looks like massless field} \\
  H: \text{looks like field with mass} = 2 m_a
\end{cases}
\]

Equations of motion:

$$\frac{\delta \mathcal{L}_a}{\delta H} = 0 \quad \Rightarrow \quad (2 m_a + i \not\!\partial) H = - i \not\!\sigma \cdot \vec{D} h$$
(2) inside hadron, momentum transferred to heavy quark $\sim \Lambda_{\text{QCD}} \ll m_Q$

$$\Rightarrow \quad h \sim \frac{\Lambda_{\text{QCD}}}{m_Q} \cdot h$$

"small components" \quad "large components"

$$\Rightarrow$$ eliminate "heavy degrees of freedom" represented by $H$ from the Lagrangian:

$$L_{\text{eff}}[h] = \bar{h} i D^0 h + \bar{h} i \vec{\gamma} \cdot \vec{D} \left( 1 \mp \frac{i \delta}{2 m_Q + i D^0} \right) \vec{\gamma} \cdot \vec{D} h$$

$\sim \Lambda_{\text{QCD}}$

* using that

$$\left( i \vec{\gamma} \cdot \vec{D} \right)^2 = -(i \vec{D})^2 + \frac{g}{2} 6 \tilde{u} \tilde{F}_a \vec{t}_a$$

$$= -(i \vec{D})^2 - g \vec{c} \cdot \vec{B}_a \vec{t}_a$$

we obtain:

\[ L_{\text{eff}} = \bar{h} i D^0 h \left\downarrow \begin{array}{c} \text{SU(2N_c)} \text{ invariant} \\ \text{chromo-magnetic field} \end{array} \right\] \[ \downarrow \]

$$- \frac{1}{2 m_Q} \left[ \bar{h} (i \vec{D})^2 h + g \bar{h} \vec{c} \cdot \vec{B}_a \vec{t}_a h \right] + \sigma \left( \frac{1}{m_Q^2} \right)$$

$$\Rightarrow$$ HEAVY-QUARK EFFECTIVE THEORY
* leading term:

\[ \mathcal{L}_{m \to \infty} = \bar{h} i \sigma^\mu \gamma_5 h = \bar{h} i \gamma_\mu D^\mu h \]

⇒ explicitly invariant under spin-flavour symmetry!

**Feynman Rules:**

\[ v \quad \frac{k}{\rightarrow} = \frac{i}{v \cdot k} \quad \text{[} v^\mu : \text{hadron velocity}] \]

\[ v \quad \frac{\sigma_{\mu,\nu}}{\rightarrow} = i g v^\mu t^\nu \]

⇒ much simpler than QCD Feynman rules

⇒ \( 1/m_Q \) corrections break heavy-quark symmetry

**Applications of HQET:**

- spectroscopy of heavy hadrons
- hadronic matrix elements
- weak decays
- …
CHIRAL SYMMETRY

* strong interactions of hadrons containing light quarks simplify in limit $m_q \rightarrow 0$, but in an intricate way

* observed isospin $(u,d)$ and $SU(3)$ flavour symmetry $(u,d,s)$ are, at first sight, a MYSTERY:

$$L_{\text{light}} = \sum_{f=u,d,s} \bar{q}_f (i\gamma^\mu - m_f) q_f$$

⇒ global $U(1)$ symmetries:

$$q_f \rightarrow e^{i\phi} q_f \rightarrow \text{conservation of baryon number}$$

$$q_f \rightarrow e^{i\phi_f} q_f \rightarrow \text{flavour conservation}$$

⇒ global $SU(3)$ flavour rotations:

$$q_f \rightarrow U_{ff'} q_{f'} \quad ; \quad U_{ff'} \in SU(3)$$

⇒ Lagrangian invariant if $m_u = m_d = m_s$!

GIVEN THAT $m_d/m_u \approx 2$, $m_s/m_u \approx 40$ WHY DOES THIS SYMMETRY WORK SO WELL IN NATURE?

(⇒ quark model based on flavour symmetry)
consider "chiral limit":  \[ m_q \rightarrow 0 \]

\[ q_L = \frac{1 - \gamma_5}{2} q , \quad q_R = \frac{1 + \gamma_5}{2} q \]

then:

\[ L_{\text{light}} = \sum_{q = u, d, s} \left[ \bar{q}_L \gamma^\mu D^\mu q_L + \bar{q}_R \gamma^\mu D^\mu q_R - m_q (\bar{q}_L q_R + \bar{q}_R q_L) \right] \]

\[ \Rightarrow \] left- and right-handed fields decouple in chiral limit

\[ \Downarrow \]

global chiral symmetry:

\[ U_L(3) \times U_R(3) = U_V(1) \times U_A(1) \times SU_L(3) \times SU_R(3) \]

\[ \uparrow \]

separate flavour rotations of left- and right-hd. fields

baryon number

What about \( U_A(1) \)?

\[ q \rightarrow e^{i\phi} \gamma \cdot q \quad \text{i.e.} \quad q_L \rightarrow e^{-i\phi} q_L \]
\[ q_R \rightarrow e^{i\phi} q_R \]

\[ \Rightarrow \] symmetry of classical (tree-level) Lagrangian, but broken by quantum effects!

\[ \text{ANOMALY} \]

\[ \Rightarrow \] not a symmetry of QCD
thus, apart from a $U(1)$ symmetry associated with baryon number conservation, QCD in the chiral limit $m_q \to 0$ ($q = u, d, s$) has a global $SU_L(3) \times SU_R(3)$ chiral symmetry

this symmetry can be realized in two ways:

WIGNER–WEYL REALIZATION:

- ground-state (= vacuum) invariant
  - physical hadron states fill irreducible representations of symmetry group
  \[\Downarrow\]
  degenerate states with opposite parity!
  
  *NOT REALIZED IN NATURE!*

NAMBU–GOLDSTONE REALIZATION:

- vacuum not symmetric
  - spontaneous symmetry breaking!
  
  \[\Rightarrow\] 
  GOLDSTONE THEOREM:
  
  a massless scalar particle ("Goldstone boson") appears in the spectrum for each broken generator of the original symmetry group
existence of a mass gap:

\[ M^2 \xrightarrow{\Lambda_{SB}} \text{other particles} \]

\[ \Lambda_{SB} \xrightarrow{\text{GAP}} \text{Goldstone bosons} \]

\[ m_q \neq 0 \]

\[ \text{explicit sym. breaking} \]

\[ \Lambda_{SB} \]

\[ 0 \]

\[ \lambda^2 \]

\[ M^2 \]

\[ \Lambda_{SB} \]

\[ 0 \]

\[ \lambda^2 \]

\[ \Lambda_{SB} \]

\[ 0 \]

\[ \lambda^2 \]

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* explains why isospin and flavour $SU(3)$ work so well, although quark masses are very different (corrections are $\sim m_q/\Lambda_{SB}$)

**MECHANISM OF SPONTANEOUS SYM. BREAKING:**

- Electro-weak model: gauge symmetry broken by Higgs field $\rightarrow$ massive gauge bosons

- QCD simpler: global symmetry broken dynamically by means of a quark condensate generated by non-perturbative effects:

  $\sigma = \langle 0 | \bar{q} q | 0 \rangle = \langle 0 | \bar{q}_L q_R + \bar{q}_R q_L | 0 \rangle = 0$

  ("order parameter")

* Goldstone nature of pseudoscalar meson and the particular way of chiral symmetry breaking imply strong constraints on the strong interactions of these mesons

$$\downarrow$$

**effective low-energy Lagrangian**
**CHIRAL PERTURBATION THEORY**

Goldstone bosons = zero-energy excitations of the QCD vacuum

\[ \langle 0 | \bar{q}^i L_q^i | 0 \rangle = \frac{v}{2} U^i_3(\phi) \]

with:

\[ U(\phi) = \exp \left( \frac{2i}{f} t_a \phi_a(x) \right) \]

SU(3) generators

\[ f: \text{ mass parameter} \]

\[ \phi = \sqrt{2} t_a \phi_a = \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ -\frac{\pi^0}{2} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}} \eta \end{pmatrix} \]

\[ U(\phi) \text{ transform linearly under chiral group:} \]

\[ q_{L,R} \rightarrow V_{L,R} q_{L,R} \text{ \ where } V_{L,R} \in SU_{L,R}(3) \]

\[ U(\phi) \rightarrow V_R U(\phi) V_L^\dagger \]

* implies non-linear transformations for Goldstone boson fields
at low energies, where the Goldstone bosons are the only light degrees of freedom, the effective QCD Lagrangian can be constructed as the most general, chiral symmetric Lagrangian containing $U(\Phi)$, ordered according to number of derivatives ($\propto$ momentum expansion):

$$\mathcal{L}_{\text{eff}} = \frac{f^2}{4} \text{tr} \left( \partial_\mu U^\dagger \partial^\mu U \right) + \mathcal{O}(p^4)$$

$$= \frac{1}{2} \text{tr} \left( \partial_\mu \Phi \right)^2 + \frac{1}{12 f^2} \text{tr} \left( \Phi \partial_\mu \Phi \right)^2 + \ldots + \mathcal{O}(p^4)$$

$\uparrow$

4-meson interaction

$\Rightarrow$ to $\mathcal{O}(p^2)$, all meson interactions are determined by a single constant:

$$f = f_\pi = 93 \text{ MeV} \quad (\text{pion decay constant})$$

* can include explicit breaking due to $m_q \neq 0$:

$$\mathcal{L}_{\text{break}} = \frac{1}{2} \text{tr} \left( M^\dagger U + U^\dagger M \right); \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$\Rightarrow$ breaks chiral symmetry in same way as in QCD Lagrangian

$\Rightarrow$ gives masses to Goldstone bosons:

$$m_\phi^2 \sim m_q$$
* effective chiral Lagrangian:

\[ \mathcal{L}_{\text{eff}}^{\text{XPT}} = \frac{f^2}{4} \text{tr}(\partial_\mu u^\dagger \partial^\mu u) + \frac{|v|}{2} \text{tr}(M^\dagger u + u^\dagger M) \]

\( + \Theta(p^4) \)

(chiral perturbation theory)

\[ \pi \pi \text{ SCATTERING:} \]

\[ A(\pi^+ \pi^- \rightarrow \pi^+ \pi^-) = \frac{t - m_\pi^2}{f_\pi^2} \]

etc.

\( \rightarrow \) similar for: \( \pi \pi \rightarrow 4\pi, 6\pi \ldots \)

**MESON MASSES:**

\[ m_{\pi^+}^2 = \frac{|v|}{f^2} (m_u + m_d) = m_{\pi^0}^2 \]

\[ m_{K^+}^2 = \frac{|v|}{f^2} (m_u + m_s) \; ; \; \; m_{K^0}^2 = \frac{|v|}{f^2} (m_d + m_s) \]

\[ m_\eta^2 = \frac{|v|}{3f^2} (m_u + m_d + 4m_s) \]

\( \rightarrow \) Gell-Mann - Okubo relation: \( 4m_K^2 - 3m_\eta^2 - m_\pi^2 = 0 \)

\( \rightarrow \) extract ratios of quark masses:

\[ m_u : m_d : m_s = 0.55 : 1 : 19 \]
$U_A(1)$ ANOMALY

MYSTERY of the missing meson: → apparent contradiction

- hadronic physics described by QCD
- hadronic physics is approximately invariant under $SU_L(n_f) \times SU_R(n_f)$ chiral symmetry ($n_f = 2, 3$)
- BUT: in QCD chiral symmetry requires $m_q = 0$, in which limit the Lagrangian is also invariant under $U_A(1)$:

$$q \rightarrow e^{i \phi} q \quad \text{Noether} \quad j_\mu = \sum_q \bar{q} \gamma_\mu \gamma_5 q$$

conserved current

- this symmetry is not manifest in hadron spectrum (no parity doublets), so it must be spontaneously broken

\[\downarrow\]

must be an extra Goldstone boson

BUT: SUCH A LIGHT ISO-SINGLET PSEUDO-SCALAR MESON DOES NOT EXIST!

$U_A(1)$ problem
* on classical level, conservation of axial currents

\[ j_\mu^5 = \sum_q \bar{q} \gamma_\mu \gamma_5 q \quad ; \quad j_\mu^{a5} = \bar{q} \gamma^a \gamma_\mu \gamma_5 q \]

\[ u_A(1) \quad \text{SU}_A(3) \]

follows from equation of motion:

\[ \partial^\mu j_\mu^5 = \sum_q \left[ \bar{q} \gamma_5 (-\not{\Phi}) q + \bar{q} \gamma^5 \gamma_5 \right] \]

\[ = \sum_q 2i m_q \bar{q} \gamma_5 q = 0 \quad (m_q = 0) \]

\[ \partial^\mu j_\mu^{a5} = 0 \quad \text{similarly} \]

BUT: subtlety with UV regularization (\( \gamma_5 \))

⇒ equation of motion can be violated for off-shell quark states (⇒ loop!):

\[ q^\mu \left( \right) \neq 0 \quad \text{ANOMALY} \]

axial current

find:

\[ \partial^\mu j_\mu^5 = -\frac{g^2}{32 \pi^2} n_f \varepsilon^{\mu\nu\alpha\beta} F^a_{\nu\alpha} F^a_{\mu\beta} + 0 ! \]

\[ \partial^\mu j_\mu^{a5} = [\text{same}] \times \text{tr}[\gamma^a] = 0 ! \]

⇒ classical \( u_A(1) \) symmetry broken by quantum effects!
YET ANOTHER SUBTLETY:

⇒ RHS of anomaly is a total divergence:

$$\Theta^\mu_{\scriptscriptstyle J^5} = - \frac{g^2 n_f}{32 \pi^2} \varepsilon^{\mu\nu\rho\sigma} F^{\nu\rho}_{\mu\nu} F^{\rho}_{\sigma \rho} = \partial^\mu K_\mu$$

with:

$$K_\mu = - \frac{g^2 n_f}{16 \pi^2} \varepsilon^{\mu\nu\rho\sigma} A^a_{\nu} \left( F^a_{\rho\sigma} - \frac{2}{3} f^{abc} A^b_\rho A^c_\sigma \right)$$

⇒ $U_A(1)$ problem seems to reappear for the modified axial current $J^5_{\mu} - K_\mu$ since:

$$\Theta^\mu (J^5_{\mu} - K_\mu) = 0$$

BUT: crucial that $K_\mu$ is not gauge-invariant!

⇒ because of complicated vacuum structure in non-abelian gauge theories, there exist field configurations ("instantons") for which:

$$n = \frac{g^2}{32 \pi^2} \varepsilon^{\mu\nu\rho\sigma} \int d^4 x \, F^{a}_{\mu\nu}(x) F^{a}_{\rho\sigma}(x) \neq 0$$

\[ n = \text{integer!} \rightarrow \text{Pontryagin index} \]

⇒ configurations with different values of $n$ are connected by gauge transformations!

* NO CONSERVED $U_A(1)$ CURRENT *
π⁰ DECAY:

* although iso-vector currents \( j^a_μ \) have no QCD anomaly, they do have an em. anomaly:

\[
\Rightarrow \partial^\mu j^{\mu a}_\pi = - \frac{e^2}{16\pi^2} \epsilon^{\mu
u\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \text{Tr} \left[ Q^2_q \tau^a \right] \\
\]

\[
= \begin{cases} 
N_c/6 & a = 3 \\
0 & \text{otherwise}
\end{cases}
\]

⇒ precisely the current coupling to \( \pi^0 \) has an em. anomaly!

* anomaly responsible for \( \pi^0 \rightarrow 2\gamma \) decay:

\[
\Gamma(\pi^0 \rightarrow 2\gamma) = N_c \frac{\alpha^2}{576 \pi^3} \frac{m_{\pi^0}^3}{f_\pi^2} = 7.73 \text{ eV}
\]

experiment: \( \Gamma(\pi^0 \rightarrow 2\gamma) = 7.7 \pm 0.6 \text{ eV} \)

⇒ direct measurement of the coefficient of a quantum anomaly and one more piece of evidence that \( N_c = 3 \)

COLOUR IS REAL!