ABSTRACT

Computers are an ideal tool for increasing the accuracy of algebraic calculations, and taking the tedium out of pen and paper manipulations. The current generation of algebra systems are drawing on recent advances in mathematics to extend their capabilities beyond human ability. These lectures will describe the methods used to implement algebra on a computer, some of the underlying mathematical theory and the techniques available to users to make the most of the algebraic facilities in the solution of physical problems. The lectures will be illustrated by references to standard available algebra systems.
The Science Loop

IDEA

Think

Calculation

COMPARE

PREDICT

Experiment
#MACSYMA

THIS IS MACSYMA 255

FIX 255 DSK MACSYM BEING LOADED
LOADING DONE

(C1) \text{DIFF((x^2-1)^5,x,5)}/(2^5*\text{FACT(5)});
\begin{align*}
\frac{7200 \times (x - 1)^2 + 3840 \times x + 19200 \times (x - 1)^3}{3840}
\end{align*}

(C2) \text{EXPAND(%)};
\begin{align*}
\frac{63 \times x^5}{8} - \frac{35 \times x^3}{4} + \frac{15 \times x}{8}
\end{align*}

(C3)
\[
\frac{d^5}{dx^5} \left( (x^2-1)^5 \right)
\]
\[
2^5 \quad 15
\]
\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \]

```plaintext
TIME=0.01 SECS
store in Use=11, Ceiling=71 out of 14839 words
A[2] = -( (1/2) - (3/2) h*2 )

TIME=0.02 SECS
store in Use=31, Ceiling=55 out of 14839 words
A[4] = -( (3/2) h - (5/2) h*3 )

TIME=0.11 SECS
store in Use=31, Ceiling=135 out of 14839 words
A[5] = $ (3/4) - (15/4) h*2 + (15/8) h*4 )$

TIME=0.55 SECS
store in Use=31, Ceiling=179 out of 14839 words
A[5] = ( (15/8) h - (35/4) h*3 + (35/8) h*4 )

TIME=1.77 SECS
store in Use=177, Ceiling=237 out of 14839 words
A[5] = -( (5/15) - (125/15) h*2 + (315/16) h*4 - (231/16) h*5 )

TIME=3.12 SECS
store in Use=143, Ceiling=283 out of 14839 words
A[7] = -( (35/16) h - (315/16) h*3 + (543/13) h*5 - (329/16) h*7 )

TIME=4.14 SECS
store in Use=179, Ceiling=343 out of 14839 words
A[8] = ( (35/129) - (315/32) h*2 + (3455/32) h*4 - (3553/32) h*6 + (1935/128) h*8 )
```
Types of Algebra System

- Big but "straightforward" problems
  (REDUCE) CANAL SHEEP TRIGMAN ETC
- Making harder things "straightforward"
  REDUCE, MACSYMA
- Casual, Day to Day, calculations
  MACSYMA
  SCRATCH PAD
  mMATH
  REDUCE
MAJOR ALGEBRA SYSTEMS

MACSYMA

Multics, Symbolics, VAX

Large; very broad

REDUCE

VAX, HP, DEC 20, IBM, 68000 based
CDC, CRAY, HLH Orion, IBM PC

Not so complete; still developing

MAPLE

VAX

Fast growing; innovative

SCRATCHPAD

IBM only

Advanced; unavailable
Equations in General Relativity

Covariant metric

\[ ds^2 = g_{ij} dx^i dx^j \]

Contravariant metric

\[ g_{ij} g^{jp} = \delta^j_i \]

Christoffel Symbols

\[ [ij, k] = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \]

\[ \left\{ \frac{k}{ij} \right\} = g^{kp} [ij, p] \]

Riemann Tensor

\[ R_{ijkl} = \frac{\partial}{\partial x^k} [j,l,i] - \frac{\partial}{\partial x^l} [j,k,i] + [i,\ell,p] \left\{ \frac{\ell}{jk} \right\} - [ik,p] \left\{ \frac{p}{ji} \right\} \]

Ricci Tensor

\[ R_{ij} = g^{pq} R_{ijpq} \]

Ricci Scalar

\[ R = g^{pq} R_{pp} \]
La métrique de Bondi

Bondi’s Metric

\[
\frac{1}{g} = \begin{pmatrix}
-\frac{r^2 U e^{2\gamma}}{r} + \frac{V e^{2\beta}}{r} & e^{2\beta} & r^2 U e^{2\gamma} & 0 \\
e^{2\beta} & 0 & 0 & 0 \\
r^2 U e^{2\gamma} & 0 & -r^2 e^{2\gamma} & 0 \\
0 & 0 & 0 & -r^2 \sin^2 \theta e^2
\end{pmatrix}
\]

\(U, V, \beta, \gamma\) functions of \\
\(u, r\) and \(\theta\)

La métrique de Sachs est impossible (???)
General Relativity

- Calculation of Riemann, Ricci, Einstein & Weyl tensors

- Calculation of Petrov types

- Solution of the field equations by approximation

- Establishing a consistent set of equations for a type of universe

"Kacbek & Kman's thing."
The Equivalence Problem in G.R.

Does there exist a real transformation

\[ g_{ij} \rightarrow g_{ij}^* \]

Are they the same space?

Classical Answer:
Tenth Deriv. Rijke ; 14 rank tensor
\[ \sim 20 \times 4^{10} \] expressions

Kaschede & Sman answer:
Up to 4th deriv Rijke , incremental
Figure 22: Definition of the coordinate system used to calculate the disturbing function.
Method of the Lunar Theory

Determine total energy,

Newtonian energy + disturbing energy

Use Hamilton's equations;

Delaunay made ~300 contact transformations by hand in 20 yrs
Derive's method ~ Lie transform - ~2460 cpu

This was done ~1970; but the technique is very important

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A Basic Algebraic Technique - Repeated Approximation

Simple Example: Solve \( y^2 = 1 - e \) for small \( e \)

as a series \( y = f(e) \)

We will ignore the binomial theory

Step 1: Assume \( e \) is zero

\( y_0 = \pm 1 \)
Step 2: Assume \[ y = y_n + \eta + O(n+2) \]
with \( y_n \) correct to order \( n \) in \( e \), and \( \eta \) is of order \( n+1 \).

Substituting \[ y_n^2 + 2y_n \eta + \eta^2 = 1 - e + O(n+2) \]
\[ y_n^2 + 2y_n \eta = 1 - e - \eta^2 + O(n+2) \]
\[ \Rightarrow \eta = \left[ \frac{1 - e - y_n^2}{2y_n} \right]_{n+1} \]

This technique is very general:
\[ E = \varpi + esm E \quad \text{(Kepler's Equation)} \]
Picard's Method
\[ \ddot{y} + y = \varepsilon y^3 \quad \text{(Duffing's Equation)} \]

etc
Repeated Approximation

To solve \( y^2 = 1 - e \)
where \( e \) is small.

Suppose \( e \) is zero; then zero approximation is \( y_0 = 1 \)

Let \( y_n \) be correct to order \( n \) in \( e \)

Let \( y_n + \eta \) be true solution

\[
y_n^2 + 2y_n \eta + \eta^2 = 1 + e
\]

\[
2y_n \eta = 1 + e - y_n^2 + O(n+2)
\]

\[
\eta = \frac{1}{2} \left( 1 + e - y_n^2 \right)_{n+1} + O(n+2)
\]
Lindstedt–Poincaré

\[ \ddot{y} + \lambda^2 y = \varepsilon f(y, \dot{y}) + g(\tau) \]

Initial approximation

\[ y_0 = a \cos(\omega_0 t) \]
\[ \omega_0 = \lambda \]

Changing variables to \( \tau = \omega_0 t \)

\[ y'' + y = \frac{1}{\lambda} \left\{ \varepsilon f(y, \dot{y}) - (\omega_0 - \lambda) y'' \right\} \]

--- small ---

\[ = \sum k_j \cos(j \tau) + m_j \sin(j \tau) \]

so

\[ y_n = \sum_{j \neq 1} \frac{k_j \cos(j \tau)}{l^2 - j^2} + \sum_{j \neq 1} \frac{m_j \sin(j \tau)}{l^2 - j^2} + c \cos \tau + d \sin \tau \]

\[ \omega_n = \omega_1 \cdots + \varepsilon e^{\lambda \tau} \]
The Perturbed Harmonic Oscillator

\[ y'' + y = ey^3 \]

with solution having no secular terms.

Stretch time

Phase constant \( y'(0) = 0 \)

Initial solution \( \alpha \cos[t] \)

Transform to \( \tau = cT \)

\[ c_0 = 1 \]

\[ y'' + y = ey^3 + (c-1)^2 y'' + 2(c-1)y' \]

Write this as

\[ (D^2 + 1)y = G(y, c-1) \]
- Extend c by $te^n$
- Calculate the right hand side
- Remove any resonance terms by solving the coefficients of $\sin[t]$ and $\cos[t]$ for any introduced constant
- Calculate $y = \frac{1}{D^2+1} (\text{rhs})$
- Apply phase constant

$c_0 = 1 \quad y_0 = a \cos[t]$

\[
\text{rhs} = (-2eak + \frac{3}{4} ea^3) \cos[t] + \frac{1}{4} ea^3 \cos[3t]
\]

to remove resonance $k = \frac{3}{8} a^2$

\[
c_1 = 1 + \frac{3}{8}e^6 \quad y_1 = a \cos[t] - \frac{1}{32} ea^3 \cos[3t]
\]

\[
\text{rhs} = -(2e^2ak + \frac{11}{128} e^8 a^5) \cos[t] + (\frac{1}{4} ea^3 + \frac{21}{128} e^8 a^6) \cos[3t] - \frac{3}{128} e^8 a^5 \cos[5t]
\]

to remove resonance $k = -\frac{21}{256} a^4$

\[
c_2 = 1 + \frac{3}{8}e^6 - \frac{21}{256} a^6 e^4 \quad u_2 = a \cos[t] - (\frac{1}{32} ea^3 + \frac{21}{128} e^8 a^6) \cos[3t]
\]
Celestial Mechanics

- Expansion of lunar disturbing function
- Performing Delaunay Operations
- Discovering approximate contact transformations
- Production of literal form of the variational orbit
- Solution of the main problem of the lunar theory by Hill's technique
Theoretical Physics

- Study of the breaking effect on stars caused by a gas cloud
- Study of the temperature distribution inside a planet
- Perturbation theory in Quantum mechanics
- Elastic Waves in a sheared medium
Engineering

- Solution of boundary value problems for the elastic stress in a pipe junction
- Design and simulation of control systems
- Design of non-linear filters
etc

- Application of operators to wave equations
- Transformation of pictures
- Solution of differential equations near a singularity
- Solution of stiff differential equations
- Process flow in a computer system
- Number theory
- Spread of epidemics
- Random walks
- Flow of fluid through a membrane
Trouver la solution de

\[ \frac{d^2 f}{dx^2} + k^2(x) f = 0 \]

avec \( f = \frac{1}{\sqrt{2}} \exp \left[ i \int q \, dx \right] \)

et \( q = k(x) \sum_{n=0} Y_{2n}(x) \)

C'est suffisant trouver \( Y_{2n} \)

Laisse \( \Theta = \sum Y_{2n} \) et \( Y_0 = 1 \)

Définis \( \hat{A} = \frac{1}{k} \frac{dA}{dx} \)

En substituant dans l'équation d'origine, on obtient

\[ \Theta^2 \varepsilon_o + \Theta^2 - \Theta^4 - \frac{\Theta \dot{\Theta}}{2} + \frac{3 \dot{\Theta}^2}{4} = 0 \]

\[ \varepsilon_o = \frac{1}{4k^2} \left\{ 3 \left( \frac{1}{k} \frac{dk}{dx} \right)^2 - \frac{2}{k} \frac{d^2 k}{dx^2} \right\} \]
Ecrit \[ \varepsilon_r = \dot{\varepsilon}_{n+1} \]

Ordre \((\varepsilon_0) = 2\)

Ordre \((\varepsilon_r) = n+2\)

Alors, \(\text{Ordre } (Y_{2n}) = 2n\)

\[ \Theta_{2n} = \bar{\Theta} + Y_{2n} \quad \text{ordre } (\bar{\Theta}) = 2n - 2 \]

\[ \begin{align*}
\bar{\Theta}^2 \varepsilon_0 + \bar{\Theta}^2 + 2 \bar{\Theta} Y_{2n} + Y_{2n}^2 + 2 \bar{\Theta} Y_{2n} \varepsilon_0 + Y_{2n}^2 \varepsilon_0 \\
- \bar{\Theta}^4 - 4 \bar{\Theta}^3 Y_{2n} + 6 \bar{\Theta}^2 Y_{2n}^2 + 6 \bar{\Theta} Y_{2n}^3 + Y_{2n}^4 \\
+ \frac{1}{2} (\bar{\Theta} + Y_{2n}) \dddot{\bar{\Theta}} + \ldots \\
+ \frac{3}{4} \dddot{\bar{\Theta}} ^2 + \ldots \end{align*} = 0 \]

\[ \Rightarrow \]

\[ Y_{2n} = \frac{1}{2} \left[ \varepsilon_0 \bar{\Theta}^2 + \bar{\Theta}^2 - \bar{\Theta}^4 + \frac{1}{2} \bar{\Theta} \dddot{\bar{\Theta}} + \frac{3}{4} \dddot{\bar{\Theta}} ^2 \right]^{2n} \]
OPERATOR E;
FORALL N LET DF(E(N), X) = E(N+1);
FOR N = 0.5 DO ORDER (E(N)) = N+1;
ARRAY Y(10);
Y(0) := 1; TH := 1;
FOR N = 1.5 DO <=
  WEIGHT (2*N);
Y(2*N) := ((E(0)+1)*TH**2 - TH**4
               + DF(TH, X, 2)*TH/2
               + 3*DF(TH, X)**2/4)/2;
  TH := TH + Y(2*N) >>;

---

Une méthode très importante

(voir Fitch, Norman et Moore 1981)
Cowling Stellar Model

\[ 2x^2 \frac{d}{dx}(yz) = 5wz \]

\[ \frac{dw}{dx} = x^2 z \]

\[ \begin{cases} 
  z = y^{3/2} & x < x_0 \\
  x^2 \frac{dy}{dx} = -Qz^2 y^{-13/2} & x > x_0 
\end{cases} \]

\[ x = 0 \quad w = 0 \quad y, z \text{ finite} \]
\[ x = x_0 \quad y, z, w, y', z', w' \text{ continuous} \]
\[ x = x_5 \quad y = z = 0 \]
\[ x \text{ is radius} \]
\[ w \text{ is mass} \]
\[ y, z \text{ pressure and luminosity} \]

**Numerical calculation:**

The inner zone is Emden equation \( n = \frac{3}{2} \).

At the surface approaches Emden \( n = \frac{13}{4} \).
Algebraic Calculation

Develop the solution around the surface \( x_5 \).

The solution has the form

\[
y = \sum_{i=0} y_i \ (x_5 - x)^i
\]

\[
z = (x_5 - x)^{9/4} \sum_{i=1} z_i \ (x_5 - x)^i
\]

\[
w = \mu - (x_5 - x)^{13/4} \sum_{i=1} w_i \ (x_5 - x)^i
\]
$$x(1-yy')(1-kx) = y(l-y)$$
References on Applications

**General:**

- Barton & Fitch
  Reports on Progress in Physics 35 1972
- Brown & Hearn
  Comp. Phy. Comm 1980
- Fitch
  Lecture Notes in Computer Scienc 22 1979

**Relativity:**

- Cohen, Leringe & Sundblatt
  GRG I 1976
- Fitch & Cohen
  GRG 1980

**Optics:**

- Hawkes
  Optik 48 29 1977
- Goto & Soma
  Optik 48 255 1977

**Fluids:**

- Graham & Moore
  MNRAS 253 617 1978
REDUCE Facilities

Interactive ALGOL (PASCAL) like language
FORTRAN Output
General system
LISP based
includes
Simplification
Indefinite Integration
Factorization
SOLVE
Big integers & big floats
MACSYMA Facilities

Interactive Language - BASIC like?

Fortran Output

Very General

Lisp Based

Includes

User Controlled Simplification

Definite & Indefinite Integration

Factorization

Limits

Power Series

Solve

Big Integers and Floats

Plotting

Factorization over finite fields
The Construction of Algebra Systems

Language:

LISP is commonest

MACSYMA, REDUCE, SCRATCHPAD, SHEE...

C

MAPLE, SMP

BCPL

CAMAL, (MAPLE)

FORTRAN

TRIGMAN, ALDES

etc

Assembler: many
Representation of Algebraic Expressions

More important than language

Three main groups

Strings, Parse Trees

Recursive Polynomial Structure

Tables
Parse Trees.

No serious system has ever used strings but starting point is Polish notation

\[ a^2 + 2ab + c \rightarrow \text{++} \Rightarrow \text{a} \ 2 \ 2 \Rightarrow \text{a} \ 2 \ 	ext{abc} \]

No ambiguity, but inconvenient

\[ \Rightarrow \text{A tree} \]

or variant
The Recursive Data Structure

Consider a polynomial in $n$ variables

$$\sum C_{i_1 \ldots i_n} x^{i_1} y^{i_2} \ldots z^{i_n}$$

or

$$\sum C_i x^i$$

with $C_i$ polynomial in $n-1$

This second form gives the representation

Variants: Can $M=0$?

Is mention of $x$ necessary?
Tabular Form

ALTRAN representation: 3 tables

\[ 3 + z + (-2)x + 5xyz + 15yz^5 + x^2t \]
Figure 3 Representation of $\frac{p}{q} a^\beta b^\gamma \cdots m^\delta$
Ordering

The entries in any of the structures need to have an order to avoid unnecessary expressions

\[ x + y + x + 2y + 3x - 3y - 4x \]

In recursive structure, order variables
& then increasing or decreasing powers

REDUCE uses decreasing powers
& "random" order of variables

Perturbation systems tend to use increasing powers
Basic Algorithms

ADD

SUBTRACT

MULTIPLE

DIFFERENTIATE

Harder Algorithms

DIVISION

FACTORIZATION

INTEGRATION
**ADDITION: REDUCE Style**

\[ X = C_i x^i + \sum C_j x^j \] or number

\[ Y = C_k y^k + \sum C_l y^l \] or number

**PROCEDURE ADD(X,Y);**

IF NumberP(X) THEN
  IF NumberP(Y) THEN X+Y
  ELSE LT Y .+ ADD(X,RED Y)
ELSE IF NumberP(X) THEN LT X .+ ADD(RED X, Y)
ELSE
  IF LVAR X = LVAR Y THEN
    IF LPow X = LPow Y THEN
      ADD(LC X, LC Y) .+ (LVAR X .LPow X) .+ ADD(RED X, RED Y)
    ELSE IF LPow X > LPow Y THEN
      LT X .+ ADD(RED X, Y)
    ELSE LT Y .+ ADD(X, RED Y)
  ELSE IF ORDER(X,Y) THEN
    LT X .+ ADD(RED X, Y)
  ELSE LT Y .+ ADD(X, RED Y)