THERMODYNAMICS OF STRONG INTERACTIONS

R. Hagedorn

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The following pages should be considered as Section 12 of these lectures.

Equation (4.10), on p. 39, should read:

\[
\sum \left[ \ldots \right] \quad \Rightarrow \quad \sum_{N=0}^{\infty} \int \left[ \ldots \right] \prod_{i=1}^{N} \rho(w_i) dw_i
\]

Equation (6.2), on p. 61, should read:

\[
\rho(m) = \sum_{M=2}^{\infty} \left[ \frac{V_0}{(2\pi)^3} \right]^{n-1} \frac{1}{M!} \int \delta^3 \left( \sum_{i=1}^{n} p_i \right) \delta \left( m - \sum_{i=1}^{n} \sqrt{p_i^2 \pm m_i^2} \right) \prod_{i=1}^{n} \rho(w_i) dw_i d^3 p_i
\]
12. **ADDENDUM**

**Developments 1971 to end of 1972**

Since these lecture notes have been written up, the following further development has taken place:

12.1 **General structure**

The plausibility argument given in Section 6.2 for the power of $m$ in front of the mass spectrum being $-3$ has now become a proof (Nahm 1972).

The analogy to nuclear physics has been stressed and Ericson fluctuations are predicted (Frautschi 1972 a, b).

The phase-space bootstrap equation of Frautschi (Eq. 6.4 of these lectures) has been formulated with respect to invariant phase space and the asymptotic form with the power $-3$ has been reconfirmed by the explicit solution of a simplified bootstrap model (Yellin 1972).

The importance of using the microcanonical description in particle physics situations has been stressed by showing that it gives results that are different from those obtained with the canonical one (Carlitz 1972).

The decay of fireballs has been followed through all generations down to stable end products, and the general shape of the observable momentum spectra is described in realistic versions of the model (Hamer 1972, Frautschi and Hamer 1972).

Truncated bootstrap equations have been solved (Chiu 1972, Chiu and Johnson 1972).

Integral equations for the linear chain decay and for generating functions of general decay have been set up and solved. One single integral equation describes composition as well as decay (down to the end products) of fireballs. (Montvay 1972 a, b, Hagedorn and Montvay 1973, Ilgenfritz and Kripfganz 1973.)

12.2 **Application to high-energy collisions**

The model has been applied to collisions of non-identical particles ($\pi p, Kp$) along the lines described in this lecture, Section 10.1.8, with the result that the same $F(\lambda)$ as for $pp$ collisions can reproduce the experiments well (Matthäus and Ranft 1972, Buschbeck and Hödl 1972).
The strong bootstrap version (with $m^{-3}$ in front of the exponential) has been introduced into the thermodynamical model for collisions (Hagedorn and Ranft 1972, Letessier and Tounsi 1972 a, b, c).

Scaling and limiting fragmentation have been reconsidered (Ranft 1971, Hagedorn and Ranft 1972).

Arguments for abandoning the view described here in Sections 10.1.2 and 10.1.3 have been given with the result that $F(\lambda)$ and $F_0(\lambda)$ are combined in one single function which never appears squared; also the necessity of factors $N(T)$ appearing in pair production might be doubled (Letessier and Tounsi 1972 a, b, Ranft 1972).

Gamma-spectra have been computed (Htun Than et al. 1972).

Good fits to data from 20 GeV to ISR energies are now made with one set of parameters (Ranft 1972, Letessier and Tounsi 1972 b, Htun Than and Ranft 1972).

Neutral spectra can be calculated (Letessier and Tounsi 1972 c).

Introducing one additional free parameter (not needed in single-particle spectra) describing the energy dependence of the average fireball mass, the model can be used to compute two (and more) particle spectra (correlations) as a function of this free parameter. The model can reproduce the results of multiperipheral as well as diffractive models, and continuously changes from the one over a superposition of both to the other extreme. Comparison with experiments leads to a determination of the new parameter and to its interpretation in terms of an intermediate (more multiperipheral than diffractive) situation (Ranft and Ranft 1972 a, b).

On the whole it can be said that while the general spirit of the statistical bootstrap approach and its most prominent result, the exponential mass spectrum implying a limiting temperature, have not been changed, many details could be considerably improved. Parts of these lecture notes must now be considered as outdated, in particular all those in which the once-favoured solution with $m^{-3/2} \exp(m/T_0)$ is taken as a basis of discussion.
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REFERENCES
The statement on p. 105, that in astrophysics one must take $V_0$ in Eq. (9.23), is wrong. The whole expression does there not apply, while it remains true for particle physics. The reason is, that it fails -- as stressed on p. 99, 100, 105 -- if the condition $1 \gg \xi^{(1)} \gg \xi^{(2)} \gg \ldots$ is violated. This happens not only when $m \ll T$ but also if $V \gg V_0$, since

$$\sum \chi_{s_0} \approx \int p_s \left| \vec{p}, \tau \right| d^3 p \approx V_{z_{-m}} \left( \frac{\mu T}{2\pi} \right)^{3/2} \exp \left( -\frac{\mu}{T} \right)$$

is proportional to $V$. In particle physics $[V = V_0, m = m_p, z_m = 4, T \approx T_0]$ one finds $\xi_{\tau Y} \approx 6.7 \times 10^{-2}$, hence here Eq. (9.23) is valid, but for much larger $V$ the supposition $\xi_{\tau Y} \ll 1$ is not fulfilled and Eq. (9.23) is wrong. Instead, the results obtained with chemical potentials become correct, so that there the average number of e.g. antinucleons (i.e. pairs) is $\sim \exp(-m_p/T)$ and not $\exp(-2m_p/T)$.

Why does the equilibrium density of $N\bar{N}$ pairs depend on the volume? In each volume $V_0$ the number of $N\bar{N}$ pairs is correctly given by Eq. (9.23). Let us build up a large volume $V$ from little boxes $V_0$. Could anything change, if we remove the walls separating the cells $V_0$ from each other? Yes: the numerical value for $\xi_{\tau Y}$ found above, implies that only one in $\approx 200$ cells contained a pair, which, the walls taken away, escapes. For annihilation an N must meet an $\bar{N}$ and that happens (without walls) very rarely at a rate proportional to the product of the densities $n(N) \cdot n(\bar{N})$. Originally, just after removing the walls, $n(N) = n(\bar{N}) \sim \exp(-2m/T)$. Thus the rate of annihilation is $\sim \exp(-4m/T)$, much smaller than the rate of creation $[\text{Eq. (9.23)}] \sim \exp(-2m/T)$. Hence the densities increase until a new equilibrium is reached when the density-dependent rate of annihilation equals the constant rate of production:

$$n(N) \cdot n(\bar{N}) = \text{const} \exp(-2m/T)$$
$$n(N) = n(\bar{N}) = \text{const} \exp(-m/T)$$

in accordance with the result obtained by using chemical potentials (here = 0). On the other hand chemical potentials cannot be used in $V_0$ for the reasons explained in the text p. 105.
1. **INTRODUCTION**

Thermodynamics of strong interactions is an idealization\(^*\), consisting in an attempt to describe the properties of highly excited hadronic matter by means of the technical tools of statistical thermodynamics, in fact of simple-minded equilibrium thermodynamics. Nowhere shall I use irreversible thermodynamics, nor will I consider thermic interaction between systems with fast relative motions. Everything, as far as statistical thermodynamics is concerned, is considered in some sort of local rest frame. Here, however, we use fully relativistic statistical thermodynamics with unlimited creation and absorption of particles. In particular, the energy of a particle is \( E = \sqrt{p^2 + m^2} \) and the total energy of a system is the sum \( \sum_{i} E_i \) of all its particles; it includes thus all rest masses.

In order to apply equilibrium thermodynamics to highly excited hadronic matter, which we can produce only in high energy hadron collisions and which therefore, as a whole, is not in peaceful equilibrium but in a state of vehement and complicated collective motion, we have to postulate that the strong forward-backward motions can be kinematically separated from the isotropic heat motion. In fact, we even have to postulate, that the collective motions appear only in the direction of the collision axis and do not give rise to turbulence: if there were turbulence, then we would not know how to separate isotropic turbulence from isotropic heat motion, and the predictions of the thermodynamic description of highly excited hadronic matter could not be compared to the experiment. Thus we formulate:

**Postulate 1**

In high energy collisions of hadrons, collective motions have only components in the direction of the collision axis. It is possible to find a continuum of co-moving Lorentz frames (local rest frames), such that a co-moving observer will, in his neighbourhood, see only thermic motion. Turbulence is absent.

\(^*\) For the first time considered by Koppe (1948, 1949) and not, as often unjustly quoted, by Fermi (1950).
This postulate is based on experience; in particular, the absence of
turbulence is concluded a posteriori from the fact that thermic motion
is observable (transverse momentum distribution), and in close agreement
with what one expects from pure thermodynamics undisturbed by turbulence.
As we will see that the heat motion is small and nearly independent of
the collision energy, while turbulence should grow with the collision
energy, it is really surprising that even at very high energies
\( p_0 \sim 10^6 \text{ GeV/c} \), it should be negligible compared to thermic motion.

After having stated Postulate 1, we are left with a system without
collective motions: the one seen by the co-moving observer in a small
neighbourhood.

This system, if describable by statistical thermodynamics, must then
have two properties:

- many degrees of freedom,
- some sort of internal equilibrium,

and being taken out of the world of hadrons, it has a third property:

- strong internal interactions.

In view of the difficulties encountered in statistical thermodynamics
of real (=interacting) gases, one might expect the presence of strong
interactions making the task impossible. As a kind of miracle, it turns
out that just the extreme strength of strong interactions enables one to
incorporate them in a formalism which is almost as simple as that of a
non-interacting (ideal) gas. Our gas will consist of infinitely many com-
ponents; it is this infinity of components that represents the strong
interactions. In a way, this is what everybody has done all the time:
if one wishes to describe helium gas, then one does not start with protons,
neutrons, electrons, and light quanta in a box, and then apply strong
interactions to bind protons and neutrons to \( \alpha \)-particles and electromag-
netic interactions to bind the electrons to them, one just starts with
the atoms and need not care any more about strong- and a good part of
electromagnetic interactions. Similarly, in our case, the resonant states
of strong interactions will be treated as particles in their own right. While we are formally dealing with free particles, the bulk of strong interactions will be represented by the rich spectrum of hadrons.

It will turn out, that the very fact that we assume the object with the above three properties to exist and to be describable by statistical thermodynamics, fixes up the structure of this theory and forces us into a general hadronic bootstrap resulting in the statement:

Each hadron consists of all other hadrons and none is elementary; the number of different kinds of hadrons is infinite and their mass spectrum grows exponentially. The macroscopic sign of this microscopic property is the existence of a universal ultimate temperature $T_0$ for all matter in equilibrium.

So far for the strong interactions governing our gas. The other two requirements: many degrees of freedom and some sort of internal equilibrium are satisfied: the first, because particle creation and absorption makes the number of degrees of freedom virtually infinite, even if the average number of created real particles is only ten or so. The second, because unlike in ordinary thermodynamics this "some sort of equilibrium" need not be achieved by means of a large number of collisions of already existing particles, namely:

Our system, in high energy collisions, decays in about $10^{-23} \text{ s}$ and somehow equilibrium must be achieved in even a shorter time. This cannot go via many collisions among each other of already created secondaries. That would not only take too long, but also it would make it difficult to understand how and why even the masses and numbers of the particles created depend on the very equilibrium which they shall bring about by their mutual collisions. The equilibrium on whose existence we have little doubt, must therefore be something instantaneous, at least it seems to take much less than $10^{-23} \text{ s}$ to establish it.
In fact, it will be shown that the particles are created into a pre-established equilibrium.

So, let us assume that the requirements are somehow fulfilled and that a thermodynamic description is worth trying. The plan of these lectures is then to first say something about the separation of collective from thermic motions and then to go through the thermodynamic part in some detail. At the end the purely empirical collective motions will be added to the more theoretical thermodynamic description, in order to confront the model with experiments.

Our units are:

\[ \hbar = c = k \text{ (Boltzmann's constant)} = 1, \]

so that the only arbitrary unit is that of the energy, which we measure in MeV and/or GeV. As a convenient inconsequence, we shall sometimes use the words centimeter, second, etc. in the text.

2. SEPARATION OF KINEMATICS AND THERMODYNAMICS

Statistical thermodynamics describes systems with many degrees of freedom by considering the ensemble of all possible states, characterizing classes of them by macroscopic observables and taking the largest classes as representing reality. The largest classes contain states, whose typical property is uniformity. It is for this reason that one obtains isotropic distributions in momentum space and coordinate space, in particular distributions of the Maxwell-Boltzmann-Planck type.

The angular distribution of particles created in high-energy hadron collisions seems to rule out any thermodynamic description of the production process. This need not be true; it is seen by imagining two gas clouds colliding in vacuum with a relative speed exceeding by far the mean thermic velocity inside each one: seen from the collision frame the angular distribution of particles after the collision is anisotropic, but it might well be that at any place a co-moving coordinate frame can be found, seen from which all motion in a small neighbourhood is isotropic and thermic.
Seen from the collision frame, there might be a continuum of such comoving frames with different speeds. Our Postulate 1 supposes that such frames do indeed exist and that they all move in the direction of the collision axis, representing thus a continuum of collective motions. It is by no means a priori evident that Postulate 1 is true for colliding hadronic matter, but it is empirically true and will not be discussed further; we simply take it for granted.

Once we know two things:

- the local thermodynamic parameters (temperature, density, pressure, etc.),
- the distribution of collective velocities,

then we can describe the highly anisotropic global aspect in any coordinate frame. As this global picture is a superposition of Lorentz-transformed local pictures, it is clear that no coordinate frame for the global description is better than any other one. We shall formulate everything in the collision centre-of-mass frame, but also consider the laboratory frame and the projectile rest frame.

2.1 Collective velocity distribution

Our present consideration is purely classical. At a given instant, say $t = 0$, a hadron-hadron collision may look like this:

![Fig. 1](image-url)
At each point \( \mathbf{x} \), there is a velocity distribution centred around a mean value \( \mathbf{\bar{v}} \) which we call the local collective velocity. This velocity distribution is a rapidly varying function of \( \mathbf{x}, \ t \) and \( \mathbf{\bar{v}} \), and it depends also on the impact parameter \( b \). We measure collective velocities conveniently, not by \( v \) itself, but by

\[
\lambda = \frac{\text{sign}(\mathbf{v})}{\gamma - 1} \frac{\gamma - 1}{\sqrt{e-1}},
\]

(2.1)

where \( \gamma = (1 - v^2)^{-\frac{1}{2}} \) is the local Lorentz factor and \( \gamma_0 = (1 - v_0^2)^{-\frac{1}{2}} \) the centre-of-mass (c.m.s.) Lorentz factor of the incoming particles. For definiteness we assume here a proton-proton collision. One can easily see that \( |\lambda| \) is the ratio of local kinetic energy density to incoming kinetic energy density. This variable has the advantage of being restricted to an energy-independent interval. Let then

\[
u(\lambda, \mathbf{x}, t, b) \]

be the probability density to find the collective velocity \( \lambda \) at the place \( \mathbf{x} \) at time \( t \) in a collision with the impact parameter \( b \).

2.2 Local excitation energy density

We now go into one of the co-moving Lorentz frames (\( \lambda \)-frame) and find ourselves surrounded by excited hadronic matter without collective motions -- at least in a small neighbourhood. The local energy density is greater than that of the incoming proton, because part of the kinetic energy has been transformed into excitation energy. In order to calculate the local energy density belonging to \( \lambda \), we formulate:

Postulate 2

All the kinetic energy of the incoming particle (\( |\lambda| = 1 \)) that has disappeared in decelerating hadronic matter to \( |\lambda| < 1 \) is adiabatically converted into local excitation energy (heat).
This postulate is as little a priori obvious as the first, but it is plausible: locally created heat can be transported away only through emission of particles, but before that starts, it must become hot. We will see that production rates are governed by a Boltzmann factor \( \exp (-m/T) \) and that there is a maximum temperature so that always \( T < T_0 = m_\pi^2 \); this means that, whatever the temperature actually reached may be, particle production is relatively exponentially negligible at any lower temperature, and it is absolutely negligible for any much lower temperature. In other words: if a piece of hadron matter is decelerated and heated up, then, at any state of deceleration and heating, all previous heat losses are negligible compared to the actual heat content just reached. Thus it is heated up (almost) adiabatically and then decays. The very fact that we observe high mass resonant states may be interpreted in these same words.

If we now take Postulate 2 for granted, we follow a piece of hadron matter through the process of deceleration. We define

\[
\varepsilon_0 = \frac{m}{V_0} = \text{average proton rest energy density}
\]

\[
V_0 = \frac{4\pi}{3} m_\pi^{-3} = \text{natural hadron volume} .
\]

Our piece of matter (filling a unit volume) has total energy \( E_0 = \varepsilon_0 V_0 \) before the collision. When it reaches a velocity \( \lambda < 1 \), then it has been excited to an energy density \( \varepsilon > \varepsilon_0 \), and it moves with Lorentz factor \( \gamma \). Thus its total energy is now \( E = \varepsilon \gamma \). Postulate 2 says \( E = E_0 \), hence

\[
\varepsilon = \frac{E_0 \gamma}{\gamma} .
\]

2.3 Local production spectrum

By equation (2.4) we assign to any collective velocity \( \lambda \) a definite energy density. The local particle production will, of course, be a function of the local energy density only, i.e. not depend on when, where, and on what impact parameter this local situation was created. Let then
\( f_m^{(\lambda', \varepsilon)} \) be the isotropic momentum spectrum of particles \( m \) produced in the \( \lambda \)-frame at local energy density \( \varepsilon \).

We shall determine this function from thermodynamics later.

2.4 The velocity weight functions

Let

\[ L^{(R)}(\lambda, \gamma_0) \]
be the Lorentz transformation which takes the spectrum from the \( \lambda \)-frame to the arbitrary frame \( R \).

Then the spectrum of particles \( m \), seen in \( R \), will be

\[
W^{(R)}(\vec{p}) \, d^3p = \\
= \int d^3x \, dt \, 2\pi b \, db \, \mu(\lambda, \vec{x}, t, b) \, L^{(R)}(\lambda, \gamma_0) \left\{ f_m^{(\lambda', \varepsilon)} \, d^3p \right\} \, d\lambda
\]

We see that neither \( L^{(R)}(\lambda, \gamma_0) \) nor the spectrum depend on the coordinates \( \vec{x}, t, b \), but only on \( \lambda \) and \( \gamma_0 \). We can thus at once integrate over \( \vec{x}, t, b \) and define the velocity weight function

\[
F(\lambda, \gamma_0) \equiv \int d^3x \, dt \, 2\pi b \, db \, \mu(\lambda, \vec{x}, t, b, \gamma_0)
\]

Thus, \( F(\lambda) \) collects all contributions to a certain value of \( \lambda \) coming from the whole history and all places of a given collision and then averages it over impact parameters, i.e. over a large ensemble of "identical" collisions. All these contributions have the same \( \lambda \) and therefore the same local momentum spectrum of particles \( m \) and the same Lorentz trans-
formation. \( F(\lambda) \) attaches a weight to each \( \lambda \). Thus

\[
W(R) \, d^3 p = \int_{-1}^{1} F(\lambda) L(R, \gamma_0) \left\{ f_{m}(p', \varepsilon) \, d^3 p' \right\} \, d\lambda. \tag{2.9}
\]

We consider \( F(\lambda) \) as an empirical function, which serves to disentangle thermodynamics from the rest and which cannot be calculated within the model. It can be determined from experiments, once \( f_m(p', \varepsilon) \) is known. Without going into much detail, we remark a few properties of \( F(\lambda) \):

- One might expect that each sort of particle requires its own function \( F_i(\lambda) \). In fact, only two such functions are needed to describe rather well all production spectra: one, \( F(\lambda) \), for the newly created particles and another one, \( F_0(\lambda) \), for the throughgoing particles (i.e. the incoming protons) and their decay products.

- We normalize the functions \( F(\lambda) \) and \( F_0(\lambda) \) such that:

\[
\int_{-1}^{1} F(\lambda) \, d\lambda = \int_{0}^{1} F(\lambda) \, d\lambda = 1. \tag{2.10}
\]

This is convenient in the case of pp collisions, where, for symmetry reasons, \( F(\lambda) = F(-\lambda) \) and \( F_0(\lambda) = F_0(-\lambda) \). If target and projectile are different (\( \pi p \), etc.), the functions are no longer symmetric.

- One might expect that the functions \( F(\lambda) \) and \( F_0(\lambda) \) depend on the collision energy. Our choice of a fixed interval \(-1 < \lambda < 1\) and of a fixed normalization permits only a change of shape, and we thus hope to minimize the possible energy dependence. From experiments between 10 and 70 GeV/c no energy dependence can be concluded.

In the course of the lecture several further remarks will be made about these functions. More details about them are found in the article, where they were first introduced [Hagedorn and Ranft, 1968, from now on quoted as (II)].
One may guess how these functions might look like by considering a multi-peripheral graph drawn in a somewhat unconventional way:

![Diagram](image)

Fig. 2

On this basis one expects that $F(\lambda)$ will be centred around small $|\lambda|$ and $F_0(\lambda)$ around $|\lambda| = 1$. This is indeed what one finds experimentally. Fig. 3 shows the actual functions as determined by fitting the model to data. A calculation based on a multi-Regge description has yielded a theoretical function $\phi(\lambda)$ which is rather close to the empirical $F(\lambda)$ (Ranft and Ranft, 1970).

A very similar description of high-energy collisions was given independently by Jabs (1969). He interprets a collision as a two-fireball mechanism (not as a continuum of fireballs as we do) and his distribution functions are probability distributions for the velocities of his two fireballs. Perhaps this is more reasonable than our approach, but the results are about the same and an experimental decision is difficult.
Fig. 3 The velocity weight functions $F(\lambda)$ and $F_0(\lambda)$. The first is for newly created particles and puts more weight on low velocities = = central parts; the second belongs to throughgoing particles which can be very peripheral. In fitting these functions to experiments they had the freedom to choose more complicated shapes including several bumps, even $\delta$-like. They took the simple shapes and there is no evidence for or against $\delta$-like behaviour at $\lambda = 0$ and/or $\lambda = 1$. With these two energy independent functions all spectra from 12 to 70 GeV/c are rather well fitted. So far a systematic change of shape with primary energy of $F(\lambda)$ and $F_0(\lambda)$ could not be detected. Figure taken from (II).
2.5 Limiting fragmentation

This notion, created by Benecke et al., 1969, says that in a collision the target and the projectile break up into fragments, such that for \( E_0 \rightarrow \infty \) the distribution of fragments tends to a limiting function in the respective rest frames of the colliding particles; for the target particle this hypothesis reads

\[
\lim_{E_0 \rightarrow \infty} W_{m} \left( \frac{\vec{p}}{E_0} \right) = \rho_{m}^{(1)}(\vec{p}). \tag{2.11}
\]

Similarly for two-particle distributions, etc.

This limit is not uniform in \( |\vec{p}| \): the higher we choose the value of \( |\vec{p}| \) the higher will be the value \( E_0 \) above which the deviations from \( \rho_{m}^{(1)}(\vec{p}) \) become arbitrarily small.

I shall now show that this hypothesis, if one accepts Eq. (2.9), is equivalent to the hypothesis made in (II) that \( F(\lambda) \) and \( F_{0}(\lambda) \) become energy independent for sufficiently large \( E_0 \). Consider a fixed value of \( |\lambda| = (\gamma - 1)/(\gamma_0 - 1) \). If \( E_0 \) becomes very large, then \( \gamma_0 = E_{CM}/2m_p \) becomes very large and for fixed \( \lambda \) also \( \gamma \) becomes very large. In that case \( \lambda \approx \gamma/\gamma_0 \) and the corresponding energy density is \( \varepsilon = \varepsilon_0/\lambda \). Hence in that case in Eq. (2.9) the spectrum \( f_{m}(\vec{p}',\varepsilon) \) in the \( \lambda \)-frame becomes \( f_{m}(\vec{p}',\varepsilon_0/\lambda) \), i.e. it becomes independent of the primary energy and depends only on \( \lambda \). The same is true by assumption for \( F(\lambda) \). Therefore, if we fix a value \( \lambda_0 \) as small as we wish, then the whole contribution

\[
W_{m}^{(\lambda_0)}(\vec{p}) = \int_{-\infty}^{\infty} F(\lambda) L(\lambda,\varepsilon_0) \left\{ \rho_{m}^{(1)}(\vec{p}',\varepsilon_0/\lambda) \right\} d\lambda \tag{2.12}
\]

* We denote the primary laboratory energy by \( E_0 \) and the CMS collision energy by \( E_{CM} \).
depends on $\gamma_0$ only through the Lorentz transformation $L(\lambda, \gamma_0)$ if we chose the collision energy $E_0$ sufficiently large, namely such that $\lambda_0 \gamma_0 \gg 1$, or, with $\gamma_0 = E_{CM}/2m$: such that $E_{CM} \gg 2m/\lambda_0$. But this whole contribution (2.12) is moving backward in CMS with velocities $\lambda$ corresponding to Lorentz factors $\gamma = \lambda \gamma_0$. The CMS moves forward in lab. with Lorentz factor $\gamma_0$ and therefore, seen from the lab., the forward Lorentz transformation with $\gamma_0$ and the backward motion in CMS with $\gamma = \lambda \gamma_0$ compensate to some extent. This becomes clear if one writes down the laboratory four-velocity of a piece of excited matter (in short "fireball") moving backward with velocity $-|\lambda|$ in CMS: for $\gamma$ and $\gamma_0 \gg 1$ one finds

$$\mathbf{V}_{lab}^{(\lambda)} = \left( \gamma \lambda, \beta \lambda \gamma \lambda \right) = \left( \frac{\gamma^2 + \lambda^2}{2|\lambda|}, \frac{-\gamma^2}{2|\lambda|} \right).$$

(2.13)

This is then, for $\lambda > \lambda_0$ and $E_{CM} \gg 2m/\lambda_0$, independent of the primary energy; furthermore, particle emission from this fireball is energy independent too ($\varepsilon = \varepsilon_0/\lambda$) and therefore the fireball belonging to $\lambda$ contributes an energy independent part to the lab. spectrum. This is true for all $-1 \leq \lambda \leq -\lambda_0$. Thus -- under the assumption that $F(\lambda)$ and $F_0(\lambda)$ are energy independent -- the whole contribution (2.12), transformed to the lab. system, becomes energy independent:

$$W^{(\lambda_0)}_{m, lab} (\mathbf{p}) = \int_{-1}^{-\lambda_0} F(\lambda) L_{lab} (\lambda) \left\{ \int_{\gamma}^{1} \frac{d^4 \lambda'}{E_{CM}/\lambda} \right\} d\lambda ,$$

(2.14)
with \( L_{\text{lab}}(\lambda) \) transforming from the \( \lambda \)-frame to the lab. frame using the four velocity (2.13) which depends only on \( \lambda \) and no longer on \( \gamma_0 \). In the words of Benecke et al.: "it stems from breaking up the target particle". The next contribution, from \(-\lambda_0 \) to \( \lambda_0 \) is "at rest" in CMS and therefore in lab. has energies of the order of \( \gamma_0 \approx \sqrt{E_0} \) while the part coming from \( \lambda_0 \) to 1 goes forward with \( \gamma \approx \lambda \gamma_0 \) in CMS and therefore in the lab. has energies of order \( \gamma^2 \approx E_0 \). This contribution comes from the breakup of the projectile particle. Thus, there is, in the lab. frame, one part from \((-1 < \lambda < -\lambda_0)\) of the spectrum which for \( E_0 \to \infty \) stays unchanged, one part (from \(-\lambda_0 < \lambda < \lambda_0\)) moving on to higher and higher energies of order \( \sqrt{E_0} \) and one part \((\lambda_0 < \lambda < 1)\) moving still faster towards higher energies of order \( E_0 \). In symmetric collisions like pp there is of course a complete symmetry between these classes of contributions: by transforming from the lab. frame to the projectile frame one interchanges them in an obvious way. For non-symmetric collisions like \( pp \) there is only a qualitative or approximate symmetry because then \( F(-\lambda) \neq F(\lambda) \).

The value of \( \lambda_0 \), that is, the borderline between the contribution to the "limiting fragmentation" or "breakup of the target" and the other parts of the spectrum depends on choice: the smaller \( \lambda_0 \) is chosen, the larger is the primary energy \( E_0 \) above which the contribution (2.12), seen from lab., becomes energy-independent: \( \lambda_0 \gamma_0 \gg 1 \) implies \( E_{\text{CM}} \gg 2m/\lambda_0 \). All our previous considerations are invalid for \( \lambda \ll 2m/E_{\text{CM}} \).

Thus limiting fragmentation is equivalent to

\[
\lim_{E_0 \to \infty} F(\lambda, E_0) = F(\lambda) \]
\[
\lim_{E_0 \to \infty} F_0(\lambda, E_0) = F_0(\lambda) \]  

(2.15)
3. THE MAIN PRINCIPLES OF STATISTICAL THERMODYNAMICS IN A NUTSHELL

3.1 The statistical ensemble of Gibbs (canonical ensemble)

The technique employed in hadron thermodynamics is that of the Gibbs canonical ensemble. If one renounces on all mathematical rigour, the main ideas can be formulated in a few lines of elementary algebra. This shall be done here just as a repetition of what we once have learned in a course on thermodynamics. For further reading, I recommend the marvellously short book (100 pages!) by Erwin Schrödinger (Schrödinger 1946), which is a real piece of art. The following discussion has been inspired by this book.

Let \( M \) be the macroscopic physical system which we wish to describe by statistical thermodynamics (in our context a piece of excited hadronic matter will be considered as macroscopic). Imagine now a number \( N \to \infty \) of such systems, all equal to \( M \). As macroscopic systems they are, however, not so equal to each other that they could not be distinguished; therefore we shall label them \( M_1, M_2 \ldots M_N \ldots \). The Gibbs canonical ensemble is the whole ensemble of them, packed together and in mutual thermic contact, but as a whole isolated from the rest of the world. The thermic contact may be arbitrarily small: such that the energy levels of the individual systems are barely changed.

![Diagram of the Gibbs canonical ensemble](image)

The canonical ensemble is isolated from the rest of the world:

\[
\sum_{M} E_M = E^{(N)} = \text{constant}
\]

\( M \)

and \( N \) = constant

**Fig. 4** A Gibbs canonical ensemble
The individual systems $M$ have energy levels $E_1, E_2 \ldots E_i \ldots$ which are the same for each $M$, because the $M$'s are equal. Suppose each $M$ has a front plate with a little screen showing a number $i$ that indicates on which energy level $E_i$ this system actually is. Because of the thermic contact, each $M$ frequently changes from one level $i$ to another $k$, wandering thus through the whole level spectrum. Certain levels will be occupied more often than others; the most frequently occupied levels lie closely together and constitute the average energy of $M$. We shall now see how the $M$'s are distributed over the level spectrum.

3.2 The distribution of the ensemble

Suppose we read off at given times $t_0, t_1, \text{etc.}$, the state of our ensemble by recording the numbers $i$ shown on the front plates. We sort them into a histogram:

![Histogram](image)

Fig. 5 An instantaneous state of the ensemble; at this moment six systems are found to be on level $E_1$, none on $E_3$, etc.

This histogram can equally well be represented by a point in an infinite dimensional space $\mathbb{R}^\infty$:

$$\vec{n} = (n_1, n_2, n_3, \ldots, n_i, \ldots)$$  \hspace{1cm} (3.1)
whose components \( n_i \) count the number of systems occupying level \( E_i \).

Evidently

\[
\sum n_i = N \\
\sum n_i E_i = E^{(N)}
\]  

(3.2)

To each vector \( \vec{n} \) corresponds a distribution of our \( N \) systems over the energy level spectrum of \( M \) (not over the spectrum of the ensemble!).

Suppose we record such instantaneous states, that is: such vectors \( \vec{n} \), millions and millions of times. The vectors will then show a distribution in \( \mathbb{R}_+ \) which is densely centred somewhere, while the rest of \( \mathbb{R}_+ \) is essentially unpopulated. Why? Because a distribution \( \vec{n} \) can be achieved in many different ways \( \square \) obeying, of course, the conditions (3.2):

\[
P(\vec{n}) = \frac{N!}{n_1! n_2! \ldots n_i! \ldots}
\]  

(3.3)

is the number of possible realizations of \( \vec{n} \). The density of points \( \vec{n} \) in \( \mathbb{R}_+ \) will be proportional to \( P(\vec{n}) \) and therefore be maximal around the vector

\[
\vec{n}_{\text{max}} = (\bar{n}_1, \bar{n}_2, \bar{n}_3, \ldots, \bar{n}_i, \ldots)
\]  

(3.4)

for which \( P(\vec{n}) \) = maximum. It can be shown that for \( N \to \infty \), this maximum is extremely sharp and that all the rest can be neglected: almost all \( \vec{n} \) fall together with \( \vec{n}_{\text{max}} \).

We calculate the maximum of the logarithm of \( P(\vec{n}) \) with the supplementary conditions (3.2). We add these conditions multiplied by Lagrange parameters \( \alpha \) and \( \beta \), and find then the unconditional extremum (it will be a maximum) of

\[
\ln P(\vec{n}) - \alpha \sum n_i - \beta \sum n_i E_i
\]  

(3.5)
which is given by

\[
\frac{\partial}{\partial n_j} \left[ -\sum_i \ln(n_i!) - \alpha \sum_i n_i - \beta \sum_i n_i E_i \right] = 0 \tag{3.6}
\]

Hence

\[
-\frac{\partial}{\partial n_j} \ln(n_j!) - \alpha - \beta E_j \bigg|_{n_j = \bar{n}_j} = 0 \tag{3.7}
\]

Remember now Stirling's formula

\[
\ln(k!) = \ln 1 + \ln 2 + \ldots + \ln k \approx \int_1^k \ln x \, dx
\]

so that

\[
\frac{d}{dk} \ln(k!) \approx \ln k \quad \text{(for } k \to \infty). \tag{3.8}
\]

Now, since \( N \to \infty \), each \( n_i \to \infty \); hence we can use (3.8) without remorse. Then (3.7) gives

\[
\ln \bar{n}_j = -\alpha - \beta E_j \quad \text{or} \quad \bar{n}_j = e^{-\alpha - \beta E_j} \tag{3.9}
\]

where \( \alpha \) and \( \beta \) have to be determined such that the supplementary conditions (3.2) are fulfilled

\[
\sum_i n_i = e^{-\alpha} \sum_i e^{-\beta E_i} = N \quad (\Rightarrow \infty) \tag{3.10}
\]

\[
\sum_i n_i E_i = e^{-\alpha} \sum_i E_i e^{-\beta E_i} = E(N) \quad (\Rightarrow \infty)
\]
If the level spectrum \( \{E_1, E_2, \ldots, E_i, \ldots\} \) is known, then \( \alpha \) and \( \beta \) can be calculated by solving Eq. (3.10). But we need not even do this, because we can at once eliminate \( \alpha \) by dividing the second equation by the first, obtaining thus the average energy \( \bar{E} \) of our system \( M \):

\[
\frac{E^{(N)}}{N} \equiv \bar{E} = \frac{\sum E_i e^{-\beta E_i}}{\sum e^{-\beta E_i}} \equiv -\frac{d}{d\beta} \left( \ln \sum e^{-\beta E_i} \right) . \tag{3.11}
\]

Furthermore, select randomly a system \( M \) from the ensemble; the probability to find it on the level \( E_j \) is given by equations (3.9) and (3.10):

\[
\frac{\bar{n}_j}{N} \equiv \bar{n}_j = \frac{e^{-\beta E_j}}{\sum e^{-\beta E_i}} \equiv -\frac{1}{\beta} \frac{\partial}{\partial E_j} \left[ \ln \sum e^{-\beta E_i} \right] . \tag{3.12}
\]

Again, the parameter \( \alpha \) has disappeared and we shall see that indeed we need not worry about \( \alpha \), nor about \( \beta \); we shall find out the physical significance of \( \beta \) in a moment.

Before we do that, let me stress the central importance of the quantity \( \sum e^{-\beta E_i} \). It is so important, that it has its own name:

\[
\sum (V, T) \equiv \sum e^{-\beta E_i} = \text{"partition function" or "sum over states"} \tag{3.13}
\]

The variable \( V \) indicates that the level spectrum of the system depends on its volume, and \( T \) anticipates that \( \beta \) will turn out to be the inverse of the temperature.

The importance of \( Z(V, T) \) lies in the fact that the most probable distribution \( \bar{n}_{\text{max}} = (\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_j, \ldots) \) is obtained as the logarithmic derivative of \( Z \) with respect to the \( E_j \), and that together with this distribution all expectation values of all interesting quantities are also calculable from \( Z(V, T) \), as for instance the mean energy \( \bar{E} [\text{eq. (3.11)}] \). Thus one can say without exaggerating:
the central task of statistical thermodynamics is to calculate the partition function

\[ Z(V, T) = \sum_i e^{-\beta E_i} \]

which implies the knowledge of the level spectrum.

3.3 Relation of statistics to thermodynamics

We now link these statistical considerations to thermodynamics. In thermodynamics we consider a system (just one of our \( N = \infty \)), and ascribe to it energy, temperature, entropy, volume, pressure, and so on. If we speak of temperature \( T \), then we imagine our system coupled to a thermostat or embedded in a heat bath of temperature \( T \). Being in thermic contact with a heat bath, its energy content is not exactly sharp, but fluctuates very little around some mean value \( \bar{E} \), relative fluctuations are smaller, the larger our system is.

The first law of thermodynamics states

\[ d\bar{E} = T \, dS - P \, dV. \]  \hspace{1cm} (3.14)

We introduce the "free energy" \( F \) by

\[ F \equiv \bar{E} - T \, S. \]  \hspace{1cm} (3.15)

Hence

\[ dF = d\bar{E} - T \, dS - S \, dT = -S \, dT - P \, dV. \]  \hspace{1cm} (3.16)
While the "natural variables" of $\tilde{E}$ are $S$ and $V$ -- namely the ones of which $d\tilde{E}$ is a complete differential -- the natural variables of $F$ are seen to be $T$ and $V$.

We find then, among other relations of this sort, that

$$S = -\left(\frac{\partial F}{\partial T}\right)_V.$$  \hspace{1cm} (3.17)

Inserting this into (3.15) yields

$$\tilde{E} = F + TS = F - T\frac{\partial F}{\partial T} \equiv T^2 \frac{\partial}{\partial T} \left[\frac{F}{T}\right] \equiv -\frac{\partial}{\partial (\frac{1}{T})} \left[\frac{F}{T}\right].$$ \hspace{1cm} (3.18)

This can be compared to the corresponding formula in the statistical description [(3.11) and (3.13)]:

$$\overline{E} = -\frac{\partial}{\partial \beta} \left[\ell \nu Z(V,T)\right].$$ \hspace{1cm} (3.19)

This is the average energy of a system, which as a member of an ensemble of $N = \infty$ such systems, is in weak contact with them -- in other words: the $N - 1$ remaining systems of the Gibbs canonical ensemble play the role of a heat bath for the one system considered. Thus identifying eqs. (3.18) and (3.19) we read off:

$$\overline{E} = -\frac{\partial}{\partial (\frac{1}{T})} \left[F(V,T)/T\right] = -\frac{\partial}{\partial \beta} \left[\ell \nu Z(V,T)\right].$$
So that we can identify
\[ \beta = \frac{1}{T} \]
\[ \hat{\ln} Z = -\frac{F}{T} + G(V) \]  \hspace{1cm} (3.20)

where \( G \) is still an arbitrary function of \( V \). We determine this function by comparing the partial derivatives with respect to \( V \). We find

\[ \frac{\partial}{\partial V} \hat{\ln} Z = -\frac{1}{T} \sum \left( \frac{\partial E_i}{\partial V} \right) e^{-\frac{E_i}{T}} \sum e^{-\frac{E_i}{T}} = -\frac{1}{T} \left( \frac{\partial E_i}{\partial V} \right) \]  \hspace{1cm} (3.21)

Thus

\[ \frac{\partial}{\partial V} \left[ -T \hat{\ln} Z(V, T) \right] dV = \left( \frac{\partial E_i}{\partial V} \right) dV. \]  \hspace{1cm} (3.22)

This is the total change of the energy of \( M \), brought about by the small shifts of all individual levels \( E_i \), when the volume \( V \) is changed by \( dV \), in other words: it is the external work \( -PdV \) done by the system when \( V \) is increased by \( dV \); thus, with (3.20)

\[ \frac{\partial}{\partial V} \left[ -T \hat{\ln} Z \right] = -P = \frac{\partial F}{\partial V} - T \frac{\partial G}{\partial V}. \]  \hspace{1cm} (3.23)

On the other hand, from (3.16)

\[ \frac{\partial F}{\partial V} = -P \]  \hspace{1cm} (3.24)
so that \( G(V) \) can at most be a constant, since \( \partial G/\partial V \) must vanish. Such a constant is of no interest, because the free energy is anyway only defined up to a constant. Therefore, we have finally

\[
\beta = \frac{4}{T} \\
\ln Z(V, T) = -\frac{F(V, T)}{T} \\
\bar{E} = T^2 \frac{\partial}{\partial T} \left[ \ln Z(V, T) \right] \\
P = T \frac{\partial}{\partial V} \left[ \ln Z(V, T) \right] \\
\omega_j = -T \frac{\partial}{\partial \bar{E}_j} \left[ \ln Z(V, T) \right].
\]

(3.25)

In this notation the Lagrange parameter \( \beta \) has obtained the physical significance of the inverse temperature and the relation to thermodynamics is fully established. \( P \) and \( \bar{E} \) appear in a parametric representation which may be rewritten as

\[
P = f(\bar{E}, V) = g(T, V) = \mathcal{H}(\bar{E}, T),
\]

(3.26)

which is the equation of state of our system. Hence literally everything is calculable from \( Z(V, T) \).

3.4 The importance of the level density \( \sigma(E, V) \)

Let us illustrate these results. A macroscopic system has "continuous energy"; that is to say, its energy levels lie extremely dense. Therefore, we write

\[
Z(V, T) = \sum_i e^{-\frac{E_i}{T}} = \int_0^\infty \sigma(E, V) e^{-\frac{E}{T}} dE
\]

(3.27)
with \( \sigma(E, V) dE = \text{number of energy levels in } \{E, dE\} \).

We see here that for calculating \( Z(V, T) \), we do not need to know the levels \( E_1, E_2, \ldots E_i \) individually; if they lie close enough, their density \( \sigma(E, V) \) suffices to calculate \( Z(V, T) \):

\[
Z(V, T) \quad \text{and} \quad \sigma(E, T)
\]

are Laplace transforms of each other and the knowledge of one implies that of the other.

What can we conclude about \( \sigma(E, V) \) from our daily-life knowledge about thermodynamic systems?

- The energy of a macroscopic body is "continues", hence \( \sigma(E, V) \) is "large" (in some sense or other);

- The energy of a macroscopic body is "sharp"; this means that \( \sigma(E, V) \) is fast increasing with \( E \). This can be seen as follows: Eq. (3.25) says

\[
\bar{E} = T^2 \frac{\partial}{\partial T} \left[ \bar{\nu} \int \sigma(E, \nu) e^{-E/T} dE \right]
\]

\[= \frac{\int E \sigma(E, \nu) e^{-E/T} dE}{\int \sigma(E, \nu) e^{-E/T} dE} \tag{3.28} \]

Thus, \( \bar{E} \) is the mean value of \( E \) with respect to the distribution function \( \sigma(E, V) e^{-E/T} \) which, for sharp \( E \), must have a sharp maximum at \( E = \bar{E} \).

This is the case when \( \sigma(E, V) \) is a rapidly increasing function of \( E \), which indeed it is, as we remember from the textbooks [see also (3.35a), (5.34)].

If \( \sigma(E, V) e^{-E/T} \) has a sharp maximum at \( E \approx \bar{E} \), then with some \( \Delta E \) around \( \bar{E} \)

\[
Z(V,T) \approx \sigma(\bar{E}, \nu) e^{-\frac{\bar{E}}{T}} \Delta E. \tag{3.29}
\]
Then indeed, we obtain back from (3.25)

$$\bar{E} = T^2 \frac{\partial}{\partial T} [\ell \omega \bar{Z}] = T^2 \frac{\partial}{\partial T} [\ell \omega \bar{S}(\bar{E}, \nu) - \frac{\bar{E}}{T} + \ell \omega \Delta \bar{E}] = \bar{E}$$

as we should; but furthermore, we find with (3.17) for the entropy

$$S(\bar{E}, \nu) \equiv - \frac{\mathcal{O} F}{\ell T} = \frac{\partial}{\partial T} [T \ell \omega \bar{Z}] = \ell \omega \bar{Z} + T \frac{\mathcal{O}}{\ell T} [\ell \omega \bar{Z}]$$

$$\approx \ell \omega \bar{S}(\bar{E}, \nu) - \frac{\bar{E}}{T} + \ell \omega \Delta \bar{E} + \frac{\bar{E}}{T} = \ell \omega [\bar{S}(\bar{E}, \nu) \Delta \bar{E}]$$

(3.30)

We thus obtain the key relation

$$\ell \omega [\bar{S}(\bar{E}, \nu) \Delta \bar{E}] \equiv \bar{S}^e(\bar{E}, \nu)$$

(3.31)

which once more illustrates the central role played by the level density \(\sigma(E, V)\). We learn still another important lesson by comparing eqs. (3.27) and (3.31)

$$Z = \int \sigma(E, V) e^{-E/T} dE$$

has taught us that the detailed level spectrum \(\{E_1, E_2, \ldots, E_i \ldots\}\) need not be known and that the knowledge of \(\sigma(E, V)\) suffices to calculate everything; now from \(\ln [\sigma(E, V) \Delta E] \approx S(E, V)\), we see that the thermodynamically (macroscopically) relevant quantity is not even \(\sigma(E, V)\) but its logarithm.

This last remark seems perhaps senseless, because mathematically \(f(x)\) and \(\ln f(x)\) contain exactly the same information. Mathematically, yes, but not physically in a thermodynamical sense:
two systems with level densities $\sigma_1(E, V)$ and $\sigma_2(E, V)$ become thermodynamically equivalent for $E \to \infty$, when

$$\lim_{E \to \infty} \frac{\ln \sigma_1(E, V)}{\ln \sigma_2(E, V)} = 1$$

no matter to what limit $\sigma_1/\sigma_2$ or $\sigma_1 - \sigma_2$ tend.

Example: for an ideal gas of $N$ particles, we have [see (3.34) and (3.35a)]

$$\sigma_N(E, V) = \frac{1}{\mathcal{V}} \cdot \left(\frac{V}{N}\right)^N \cdot \left(\frac{mE}{3\pi N}\right)^{3N/2}$$

Thus, comparing two gases with $N_1$ and $N_2 = N_1 - n$ particles respectively, we have for $T \to \infty$ and $n << N_1$

$$\frac{\sigma_{N_1}(E, V)}{\sigma_{N_2}(E, V)} \sim \left(\frac{V}{N}\right)^n \cdot \left(\frac{mE}{3\pi N}\right)^{3n/2} \quad \Rightarrow \quad \infty \quad \text{for } E \to \infty, \text{ N fixed}$$

$$\ln \frac{\sigma_{N_1}(E, V)}{\sigma_{N_2}(E, V)} \sim 1 + \frac{n}{N_1} \quad \Rightarrow \quad 0 \quad \text{for } N \to \infty, \text{ E fixed}$$

In other words: if from a macroscopic gas, I remove a few molecules, then the level density becomes, for $E \to \infty$, infinitely different from what it was before, but the thermodynamics is not changed at all.

The thermodynamic equivalence of two systems with logarithmically equal level densities (i.e. equal entropies) will be used in our treatment of strong interactions; a first application will be seen now.
3.5 Relation of \( \sigma(E, V) \) to phase space

Consider a gas of \( N \) free particles of mass \( m \) and spin zero in a cubical box of volume \( V = L^3 \). To calculate the number of levels in \( \{E, dE\} \), we start by writing down a single particle wave function. As the particle is enclosed in a box, not all momenta \( \vec{p} \) are permitted. The simplest way to obtain the permitted ones (\( \vec{p}_a \)), is to imagine the box repeated periodically in all three space directions, and then to ask for the set of plane wave states having the same periodicity. Thus if

\[
\vec{n} = (n_1, n_2, n_3) \quad \text{with} \quad n_i = 0, \pm 1, \pm 2 \ldots
\]

then

\[
e^{i \vec{p}_a \cdot \vec{x}} = e^{i \left[ \vec{p}_a \cdot (\vec{x} + \vec{n} \cdot \vec{L}) \right]} \tag{3.32}
\]

Hence

\[
V \vec{p}_a \cdot \vec{n} = \vec{k} \cdot 2\pi \quad ; \quad \vec{k} = 0_i \pm 1_i, \pm 2 \ldots
\]

therefore the vector \( \vec{a} = \left( L/2\pi \right) \vec{p}_a \) can only have integral positive and negative components:

\[
\vec{a} = (\alpha_1, \alpha_2, \alpha_3) \quad ; \quad \alpha_i = 0, \pm 1, \pm 2 \ldots
\]

The number of permitted states equals the number of lattice points \( \vec{a} \).

As the \( \vec{a} \)'s sit on a unit lattice, the number of \( \vec{a} \)-points equals the volume of \( \vec{a} \)-space:

\[
\left[ \text{number of states in} \quad d^3\vec{a} \right] = d^3 \vec{a} = \left( \frac{L}{2\pi} \right)^3 d^3 \vec{p} ;
\]

in other words: the number of states in \( d^3 \vec{p} \) is

\[
2(\vec{p}) d^3 \vec{p} = \frac{V}{(2\pi)^3} d^3 \vec{p} = \frac{V}{2\pi^2} p^2 dp . \tag{3.33}
\]
The same result is often "derived" from the uncertainty relation $\Delta x \cdot \Delta p \geq \hbar$ which says that for given $\Delta x = L$, only states with $\Delta p \geq \hbar/L$ can be distinguished. Therefore, in $dp$, at most $dp/\Delta p = Ldp/\hbar$ different states can be counted. Extending this argument to three dimensions yields

\[ \frac{2}{(\hbar)}^3 d^3 \rho \]

which is identical to (3.33), because of our convention $\hbar = h/2\pi = 1$.

We now derive $\sigma_N(E, V)$ for $N$ particles. Obviously, since the particles do not interact, the differential number of states for $N$ particles is

\[ \left[ \frac{2}{(2\hbar)^3} d^3 \rho \right]^N = \left[ \frac{V}{(2\hbar)^3} \right]^N \prod_{i=1}^N d^3 \rho_i \]

and the number of states $\sigma(E, V)$ is obtained by integrating over all momenta with the condition that $\sqrt{\sum_{i=1}^N \frac{p_i^2}{m^2}} = E; \ \text{hence for identical particles}$

\[ \sigma_N(E, V) = \frac{V^N(2\pi)^{-3N}}{N!} \int d^3 \rho \left( E - \sum_{i=1}^N \sqrt{\frac{p_i^2}{m^2}} \right) \prod_{i=1}^N d^3 \rho_i \cdot \quad (3.34) \]

This is a familiar expression, which -- apart from a minor difference -- looks like the probability for an $N$-particle state in the Fermi statistical model (Fermi, 1950): there is an "interaction volume" $(V/\hbar^3)$ taken to the $N^{th}$ power, there is the $N!$ for equal particles and there is the phase space (more precisely: momentum space) integral which is so dear to us in particle physics.

The minor difference is that the $\delta$-function for momentum conservation, which we are used to in particle physics, is missing. If it were not a minor difference, thermodynamics of strong interactions would be more complicated. That it is a minor difference is exactly due to the circumstance mentioned at the end of the last section, namely that $\ln \sigma(E, V)$ and not $\sigma(E, V)$ itself is the physically interesting quantity. As this
point is important, we illustrate it by calculating both densities, with
and without momentum conservation, for the case of non-relativistic particles,
where the phase space integrals can be solved exactly. As the factor
\( V^N \cdot [N!(2\pi)^{3N}]^{-1} \) is the same in both cases, we compare the momentum space
integrals. In the case of momentum conservation we should not require the
total momentum to be exactly zero (then the integral vanishes), but only
that the total momentum \( \vec{P} = \sum \vec{p}_i \) be limited to \( |\vec{P}| < \Delta P \). Then we compare	two integrals of equal dimension (we put \( E - N \cdot m = E_{\text{kin}} \)):

\[
I(E,N) = \int \delta \left( E_{\text{kin}} - \frac{4}{2m} \sum \rho_i^2 \right) \prod_i d^3 \rho_i
\]

(3.35)

\[
I(E,N,\Delta P) = \left[ \frac{4\pi}{3} \Delta P^3 \right] \int \delta \left( E_{\text{kin}} - \frac{4}{2m} \sum \rho_i^2 \right) \frac{\delta^3(\sum \vec{p}_i)}{\prod_i d^3 \rho_i}.
\]

(3.36)

The result of some lengthy algebra is (without approximations)

\[
I(E,N) = \frac{(2\pi m E_{\text{kin}})^{3N}}{E_{\text{kin}}^{2N} \Gamma \left( \frac{3N}{2} \right)}
\]

(3.35a)

\[
I(E,N,\Delta P) = \frac{(2\pi m E_{\text{kin}})^{3N}}{E_{\text{kin}}^{5/2} \Gamma \left( \frac{3N-3}{2} \right)} \frac{1}{(2\pi m N)^{3/2}} \left[ \frac{4\pi}{3} \Delta P^3 \right]
\]

(3.36a)

Hence, with some slight approximation (Sterling formula)

\[
\frac{\sigma_N(E,V)}{\sigma_N(E,V,\Delta P)} = \left[ \frac{E_{\text{kin}}}{\Delta P^2/2m} \right]^{3/2}
\]

(3.37)

which diverges for \( E_{\text{kin}} \to \infty \). On the other hand,

\[
\lim_{\Delta P \to 0} \frac{\ln \sigma_N(E,V)}{\ln \sigma_N(E,V,\Delta P)} \Rightarrow 1
\]

(3.38)
no matter how \( N \) and/or \( E \) _kin_ tend to \( \infty \). Thus, thermodynamically it is an unnecessary luxury to keep the total momentum fixed (within \( \Delta P \)). In fact, eq. (3.37) tells us clearly why. Let us ask: how large a \( \Delta P \) must we allow in order to make not only (3.38) but even (3.37) equal to one? The answer is [with \( \overline{p^2} = \text{mean square momentum per particle} \)]

\[
\frac{\Delta P^2}{2N} = \overline{E_{\text{kin}}} = N \cdot \frac{\overline{p^2}}{2N} \]

hence

\[
\Delta P = \sqrt{N} \cdot \sqrt{\overline{p^2}}. \tag{3.39}
\]

This is an obvious result: if we add up \( N \) random vectors with mean length \( \sqrt{\overline{p^2}} \), then the result is just given by the r.h.s. of this equation. In other words:

\( \sigma(E, V, \Delta P) \) and \( \sigma(E, V) \) differ only then from each other, if we require the total momentum \( \Delta P \) to be much smaller than its statistical expectation value \( \sqrt{\overline{p^2}} \cdot N \).

For a sufficiently large number of particles, this value is, however, ridiculously small. The mean collective velocity \( \Delta v \) and kinetic energy \( \Delta E \) connected to it is given by \( \Delta P = M \cdot \Delta v = N \cdot m \cdot \Delta v \), and from (3.39) it then follows

\[
\Delta N = \sqrt{\frac{\Delta P^2}{2N} / N} \quad ; \quad \Delta E_{\text{kin}} = \frac{\Delta P^2}{2N^2} = \frac{\overline{E_{\text{kin}}}}{N} = \frac{3}{2} T. \tag{3.40}
\]

In other words: if we neglect momentum conservation in the momentum space integral, then we permit random collective movements of the system which are by a factor \( \sqrt{N} \) smaller than the mean movement per particle, and the total kinetic energy contained in this collective motion is of the same order as the kinetic energy per particle and equals the thermic energy at temperature \( T \) of the whole system considered as a mass point.
Of course, the thermodynamics of a macroscopic system is insensitive to whether we fix it by bolts and screws to the floor or just put it there unfastened\(^*\). This is the physical meaning of eq. (3.38). Needless to show that a similar result is obtained if one considers conservation of angular momentum (Cerulus, 1961; Ericson, 1961; Jocs and Satz, 1964; Koba, 1961; Kotanski and Zalewski, 1965; Zalewski, 1965). These considerations lead us to a few interesting observations for the application in particle physics.

3.6 The bridge between particle physics and thermodynamics

- The bridge from particle physics to thermodynamics is the phase space integral. As far as a thermodynamic approach is justified, momentum conservation is unimportant. If necessary, we may reinforce it on final results.

- With \( S = \ln \sigma(E, V) \) and \( \partial S/\partial E = 1/T \) [see (3.14)], we can assign a temperature to any phase space integral with the result (non-relativistic particles \( m \gg T)\)

\[
\frac{3}{2 N} T = \begin{cases} 
\frac{E}{(N-2/2)} & \approx \frac{E}{N} \quad \text{(no momentum conservation)} \\
\frac{E}{(N-5/3)} & \approx \frac{E}{N} \quad \text{(with momentum conservation)}
\end{cases}
\] (3.41)

Again, this does not depend significantly on momentum conservation.

Of course; the assignment of a temperature to a given \( N \)-particle state with given energy \( E \) makes sense only if we know that the phase space is isotropically populated -- in other words: if collective motions have been separated off, as we did in section 1.

\(^*\) Total momentum is conserved for the unfastened system!
Our way from the single particle wave function to the phase space integral has led us to the old, non-covariant momentum space integral. Of course, the whole expression (3.34) for \( \sigma_N(E, V) \) is covariant. If one would insist on making the momentum space itself covariant by replacing \( d^3p_i \) by \( d^3p_i/2E_i \), then one has, at the same time, to replace \( V \) by \( V \cdot 2E_i \) (and leave it under the integral). This is equivalent to normalizing the wave function of particle \( i \) in \( V \cdot 2E_i \) instead of in \( V \). In particle physics this has the welcome consequence that the S-matrix elements become invariant, but for a thermodynamic description it is inconvenient to give to each particle another volume that depends on the particle's energy.

In order to give a thermodynamic description of hadronic processes, we shall thus start from an S-matrix formulation with non-covariant phase space. As the particle number is not fixed (we do not describe single events but averages over many), we have to sum over \( N = 0 \ldots \infty \). As furthermore we shall take into account all kinds of hadrons, we have to sum over the hadronic mass spectrum. Thus our density of states will be something like

\[
\mathcal{S}(E, V) = \sum_N \sum_{\text{masses}} \mathcal{S}_N(E, \text{masses}, V)
\]

where \( V^N \) will correspond to an averaged squared matrix element [compare (3.34)] and \( \sigma_N \) to an N-particle phase space integral. Such integrals are difficult to calculate, and the sum over all \( N \) and all kinds of particles would require the evaluations of millions of them. Remembering eq. (3.27):

\[
Z(V, T) = \sum_i e^{-E_i/T} = \int \sigma(E, V) e^{-E/T} dE,
\]

we see that we have the choice between the calculation of millions of phase space integrals (microcanonical description, \( E \) fixed) or of the sum over \( \exp(-E_i/T) \) (canonical description, \( T \) fixed). It will turn out that while the former is literally impossible, the latter is straightforward.

Note again that the whole makes sense only after we have separated off collective motions (Chapter 1).
4. **MOTIVATION OF THE PARTICULAR STRUCTURE OF HADRON THERMODYNAMICS**

In this section, we motivate (we do not prove) the particular hadron thermodynamics which claims to be able to take into account strong interactions — maybe fully, certainly the main part of them — in spite of the well-known difficulties in the theory of real gases.

4.1 **We start from S-matrix**

In chapter 1, we have separated off collective motions. We therefore deal with excited hadronic matter in the rest frame of a co-moving observer who, according to Postulate 1, sees only isotropic motion in his immediate neighbourhood. Such a situation arises e.g. in $p\bar{p}$ annihilation at rest. Suppose we could somehow prepare an initial state $|i\rangle$ of high energy with zero baryon number, isospin and strangeness and with negligible momentum and angular momentum. The probability to observe $N$ final particles is

$$P(E_i, N) = \int |g\langle S | S' \rangle|^2 \delta(E - \sum E_i) \delta^3(\sum \vec{p}_i) \prod_i d^3 \vec{p}_i,$$

$$= \int |S'|^2 dR_N(E_i, m_1, m_2, \ldots, m_N). \quad (4.1)$$

(The second expression defines a shorthand notation.) We use the noncovariant $d^3p$ for the reasons explained in the last section; states, between which the reduced S-matrix (into which we absorb all factors like $1/N!$) is taken, are then normalized over a fixed volume $V$. By hypothesis, the initial state is isotropic, hence, the phase space is isotropically populated. If we knew the squared matrix element, then we could take its mean value in front of the integral and write

$$P(E_i, N) = \sqrt{|S'|^2 \cdot \sigma(E_i, N)}. \quad (4.2)$$
Unfortunately, this would not yet permit us to go over to thermodynamics, because $|<S'>|^2$ itself is a function of E and N. We do not wish to make the naive assumptions of the statistical model; we rather wish to take interaction seriously.

4.2 The method of Beth and Uhlenbeck and of Belenkij

The idea of Beth and Uhlenbeck is simply this: $\sigma_N(E, V, m_1 \ldots m_N)$ is the level density of N non-interacting particles of masses $m_1 \ldots m_N$.

Suppose now two of them, 1 and 2, rigidly bound together, then we have $N-1$ non-interacting particles with level density $\sigma_{N-1}(E, V, m_{12}, m_3 \ldots m_N)$, where $m_{12}$ is the mass of the bound state: the interaction that is responsible for the rigid binding manifests itself in a change of the level density.

Take a realistic example: a dilute He gas is perfectly described by assuming N free, non-interacting He atoms, instead of 6 N protons, neutrons, electrons with complicated, unmanageable interactions. Thus, interaction may be represented by a change of the level density. After all, in eq. (4.2), both factors are functions of E and N, and nobody tells us how exactly they factorize, and with what N. Though interaction need not lead to bound states, it is tempting to ask: how does it influence the level density? This is what Beth and Uhlenbeck do.

Consider N particles in a box. This time, we choose it spherically with radius R (the shape of the normalization box does not matter). The single particle wave functions are now spherical waves. From the condition that they vanish on the walls ($r = R$), one then derives (in a more complicated way) exactly the same level density as in (3.34). Now we single out particles 1 and 2 and ask for the effect of their interaction. If $r$ and $p'$ denote their relative distance and momentum, then their internal asymptotic 1th partial wave will be

$$
\Psi_{\ell}(r, p') \sim \frac{1}{r} \sinh \left( p' r - \frac{\ell \pi}{2} + \delta_\ell \right).
$$

(4.3)
The boundary condition at $R$ implies

$$p' R - \ell \frac{\pi}{2} + \delta_{\ell} (\theta') = n_{\ell} \pi$$

$$n_{\ell} = 0, 1, 2, \ldots$$

and the number of permitted partial wave states in $\{p', dp'\}$ becomes

$$\frac{dn}{dp'} dp' \left( \frac{R}{\pi} + i \frac{d \delta_{\ell}}{d p'} \right) dp'.$$

Thus, the level density of the unperturbed 2-particle state ($dp' \cdot R/\pi$) acquires an additional term, due to the $l$th partial wave phase shift. We have, therefore, the choice between two descriptions:

- **either** we use (4.1) with unperturbed phase space and describe interaction by $|S'|^2$,
- **or** we calculate phase space with distorted waves and eliminate the corresponding part of the interaction from $|S'|^2$.

[The most modern formulation of this old idea is due to Dashen, Ma and Bernstein (1969).]

Since the knowledge of all phase shifts in all channels -- including, of course, 2, 3, ... n-body phase shifts -- is equivalent to the full S-matrix, we might attempt to push the method of Beth and Uhlenbeck to the limit, by extending the previous consideration to all phase shifts (between any number of particles and in all partial waves) and to unlimited creation and absorption of particles -- in other words: by calculating all possible phase space integrals with fully distorted wave functions, thereby reducing the S-matrix to an uninteresting constant and expressing the effect of strong interactions entirely by the change of level density.
they bring about by phase shifts and particle creation. (As an analogy, remember the improvements in the theory of nuclear beta-decay, if, for the outgoing electron, one uses Coulomb wave functions instead of plane waves).

Unfortunately, in spite of the effort of many physicists, the full $S$-matrix and with it the set of all phase shifts of strong interactions are unknown. How then can we include the effect of the set of all phase shifts? We do it by a circular syllogism which amounts to a complete and universal bootstrap of all hadrons. A circular reasoning may start anywhere; let us start from the fact that strong interactions produce resonances.

If there exists, in the $l$th partial wave at $p' = p^*$, a resonance between particles 1 and 2, then $\delta_1$ increases by $\pi$ in a small interval around $p^*$. Let this resonance have zero width -- we shall relax on this later on -- hence, $(1/\pi) \, d\delta_1/dp' = \delta(p' - p^*)$. Thus, if we insert (4.5) back into (4.1) and sum over partial waves, the first term $R/\pi$ yields the old integral (4.1), while the second one, $\delta(p' - p^*)$, yields an integral with particles 1 and 2 appearing as a single object with mass $m^* = \text{invariant mass} \, M_{12}(p^*)$. Hence,

$$
P(E|N) = \int |S|^2 \, dR_N \left( E, \, \mu_1, \mu_2, \ldots, \mu_N \right)$$

$$+ \int |S'|^2 \, dR_{N-1} \left( E, \, \mu_1, \mu_3, \ldots, \mu_N \right) \tag{4.6}$$

where $S'$ is replaced by $S''$, indicating that part of the interaction is eliminated from $S'$. This transition from (4.1) to (4.6) with the help of (4.5), when there is a resonance, has been done explicitly for the first time by Belenkij (1956). Since then, it has become a standard technique in the statistical model. However, in all subsequent statistical model applications, only a small number of resonances was permitted, so that the beautiful method of Beth and Uhlenbeck was not really exploited.
The reason was simply, that if one takes into account many resonances, the number of phase space integrals becomes astronomical: at 30 GeV it is about $10^{10}$, if all presently known particles and resonances are taken, but there are many more waiting to be discovered. We, however, shall use thermodynamics, where astronomical numbers are a virtue.

4.3 The momentum spectrum $f^\rightarrow_m(p, E)$

We wish to calculate the momentum distribution of a given kind of particle (mass $m$), no matter how many other particles there are, and no matter what they do. For the $N$-particle state this distribution follows from $P(E, N)$ by suppressing the $d^3p$-integration of the particle considered. For the phase space element with $d^3p$ suppressed, we write $dR_N/d\vec{p}$. Hence, from (4.6)

$$f^\rightarrow_m(p, E) = \int |S|^2 \frac{dR_N}{d\vec{p}} (E, \mu_1, \ldots)$$

(4.7)

$$+ \int |S''|^2 \frac{dR_{N-i}}{d\vec{p}} (E, \mu_1^+, \mu_3, \ldots).$$

Let us continue to eliminate interaction from $|S''|^2$ by introducing resonances between all particles: 2-body, 3-body, ..., n-body resonances and even resonances between particles and resonances, between resonances and resonances and so on.

Suppose we knew all resonances and suppose that strong interactions were fully described by the set of all possible production processes, together with the set of all hadrons including all resonances ("resonance saturation", we shall relax on this later on), then the final result will be

$$f^\rightarrow_m(p, E) d^3p = K \cdot d^3p \cdot \sum_c \int \frac{dR_c}{d\vec{p}} (E, \text{masses}).$$

(4.8)
Here $K$ is what remains of $|S|^2$ after shifting all interactions into phase space: a simple constant. $\int_C$ goes over all possible configurations: namely all possible particles and resonances and all possible numbers of them. Nobody can actually calculate this by evaluating individual phase space integrals. But just because it is too complex as it stands, thermodynamics can take over: in terms of statistical thermodynamics (4.8) is the microcanonical description of the momentum distribution in a gas, consisting of undetermined numbers of all kinds of freely created particles; in other words: of hadronic black body radiation with fixed energy and with total momentum $\vec{p}$, where $\vec{p}$ is the momentum at which $f_m(\vec{p}, E)$ is evaluated. As $|\vec{p}|$ is of the order of the average momentum per particle, (it does not matter if it is even ten times bigger), the phase space integrals are to be evaluated at values of $|\vec{p}|$ which essentially lie in the region $\Delta P$, discussed in section 3.5, eq. (3.39). This enables us to neglect momentum conservation; thus we have to deal with hadronic black body radiation in the usual sense. [Once $f_m(\vec{p}, E)$ has been calculated, a momentum cut-off can be imposed on the result in order to repair the leak.]

We now go over from the microcanonical ($E$-fixed) to the canonical ($T$-fixed) description and can at once conclude that $f_m(\vec{p}, E)$ must have the form of a Planck distribution (we shall later derive this result from $\ln Z$):

$$f_m(\vec{p}, T(E))d^3p = \text{const.} \left[ \exp \frac{\sqrt{\vec{p}^2 + m^2}}{T(E)} + 1 \right]^{-1} d^3p$$

(4.9)

+ for fermions, - for bosons.

4.4 Definition of the hadronic mass spectrum

The task may seem to be settled now -- but it is not. Before this formula can be applied, we have to know what value of $T$ belongs to a given energy density. Thus all our arguments, so far, amount only to having shifted the problem; it now reads:

- calculate the relation between energy density $E/V$ and temperature $T$ for hadronic black body radiation.
This new problem is equivalent to calculating either \( \sigma(2, V) \), or the partition function; and that we can do if we know which particles and resonances make up our hadron gas: we need the full hadronic mass spectrum.

To count all particles and resonances, sharp or large, including even a smooth background, we introduce the hadronic mass spectrum \( \rho(m) \), such that \( \rho(m) dm \) is the number of thermodynamically distinct "particles" in the mass interval \( \{m, dm\} \).

The \( 2J + 1 \) orientations of a spin \( J \) particle correspond thermodynamically to \( 2J + 1 \) distinct particles; thus \( \rho(m) \) contains a spin-isospin multiplicity factor \( (2J + 1)(2I + 1) \) and a further factor 2, if particle \( \neq \) anti-particle. Resonances contribute to \( \rho(m) \) via \( (1/\pi) \int \frac{d\omega}{d\omega} \), and therefore need not be sharp any longer; as \( \rho(m) \) is anyway conceived as a continuous function (containing a few \( \delta \)-bumps for the stable particles), it may also accommodate a smooth background. Let "everything necessary" (whatever that means), be contained in this unknown function: the problem has now become to find out what this \( \rho(m) \) is. But assume we knew it; then

\[
\sum_{\text{all configurations}} \Rightarrow \sum_{N=0}^{\infty} \sum_{i=1}^{N} \int \cdots \rho(m_i) dm_i \quad (4.10)
\]

Thus our piece of highly excited hadronic matter -- from now on called a fireball -- is thermodynamically described as hadronic black body radiation, enclosed in a box of volume \( V \), containing undetermined numbers of \textit{all} kinds of hadrons. The number of different sorts of hadrons with mass in \( \{m, dm\} \) is given by \( \rho(m) dm \). The most important property of this "gas" is:

Although this gas consists \textit{formally} of free, non-interacting particles, (almost) all strong interactions are properly taken into account by the hadronic mass spectrum \( \rho(m) \) and by the freedom to create and annihilate the particles described by \( \rho(m) \).
Note the *fundamental difference* between \( \rho(m) \), which counts the possible mass-states of the hadrons making up our fireball, and \( \sigma(E, V) \), which counts the number of states of the fireball as a whole. Example: in ordinary electromagnetic (em) black body radiation we have \( \rho_{\text{em}}(m) = 2\delta(m) \), while \( \sigma_{\text{em}}(E, V) \sim \exp \frac{E}{\hbar} \) [see Section 5.2]. Thus \( \rho_{\text{em}} \) is zero, except at \( m = 0 \), while \( \sigma_{\text{em}} \) grows almost exponentially. The hadronic \( \rho(m) \) is of course much richer than \( \rho_{\text{em}} \), since there are so many particles and resonances; but the real surprise will be that for strong interactions the two "fundamentally different" functions \( \rho(m) \) and \( \sigma(E, V) \) will become the same!
4.5 Duality and bootstrap

Let us now look at a fireball graphically: its particles interact strongly, two colliding ones may be just deflected or they may go into a resonant state, in either case any number of other particles of any sort may be created, which interact in the same way, -- on the whole, it looks like this:

![Diagram of a fireball under the microscope]

Fig. 6 A fireball under the microscope

Now, if in this picture we draw all possible lines, allow any number of lines emerging from a vertex and allow the lines to carry any of the masses of strong interactions: we claim, that then this is strong interactions. One might object that all this are S-channel resonances, and that the typical high-energy behaviour of the virtual subcollisions are not taken into account. However, if our \( \rho(m) \) is going to contain the contributions of all phase shifts, then nothing has been forgotten.

Also duality (Dolen, Horn, Schmid, 1967, 1968; Schmid 1968) tells us that even the high-energy Regge-behaviour of the subcollisions is
properly taken into account by a sum over s-channel resonances plus a
background [which we assume to be included in $\rho(m)$].

Thus, supposing the complete $\rho(m)$ to be known, the thermodynamics of
our gas will amount to a complete (and dual) picture of strong interactions,
as far as a thermodynamic description is possible at all.

Now we shall see that the simple assumption that this thermodynamical
picture corresponds to something real, forces us to accept a bootstrap
mechanism which fixes the mass spectrum (almost uniquely) and thereby
the whole theory. Thus, either this theory works or it does not -- but
it is essentially fixed up by the assumption that it exists at all. That
goes as follows (here comes the promised circular syllogism):

1) Suppose that a thermodynamic description of strong interactions is
possible; in other words: that its objects, the fireballs, exist.

2) If fireballs exist, they may have any total energy $E = m$; their
level density $\sigma(m, V)$ will be a function [Laplace transform of $Z(V, T)$]
which is calculable inside the theory, once the mass spectrum $\rho(m)$
is given.

3) Consider a continuum of fireballs with different masses: somewhere
when the mass becomes smaller and smaller, selection rules begin to
make a thermodynamic description less and less applicable, and fire-
balls gradually obtain, according to their quantum numbers, individ-
ual personalities: they become what we call resonances. On the
other hand, we know that with increasing mass, resonances begin to
decay statistically into more and more decay channels, some of which
contain resonances themselves. In other words: resonances with
greater and greater mass gradually become what we call fireballs.
Thus, fireballs are resonances and vice versa. We call all of them
fireballs.

4) A thermodynamic description of fireballs which aims to include all
internal strong interactions as completely as possible, ought to do
so by using the most complete $\sigma(m)$, which counts all particles and
all resonances and, since resonances are fireballs, also all fireballs.
5) As \( \sigma(E, V_0) \) is the number of different fireballs of \textit{hadronic size} \( V_0 \) in \( \{E, dE\} \), the mass spectrum \( \rho(m) \) must be continued beyond the experimentally known region to include \( \sigma(m, V_0) \). In other words: \( \rho(m) \) and \( \sigma(m, V_0) \) are thermodynamically equivalent:

\[
\lim_{m \to \infty} \frac{\ln \rho(m)}{\ln \sigma(m, V_0)} = 1 \tag{4.11}
\]

"Logarithmic bootstrap condition".

We shall show that this condition can be fulfilled.

6) A thermodynamical system is \textit{completely} described by its partition function or, equally well, by its density of states \( \sigma(E, V) \), which can be calculated once \( \rho(m) \) is given. The condition of the equivalence of \( \rho(m) \) and \( \sigma(m, V_0) \), determines essentially their asymptotic behaviour (as we shall prove) and, since \( \rho(m) \) is known experimentally for small \( m \), it is known completely; hence, our thermodynamics is essentially (we shall see to what extent) determined; it \textit{i}s a thermodynamics of strong interactions.

Thus our supposition No. 1 of above is shown to be fulfilled by construction, the circular reasoning is consistent. In order to show that it corresponds to physical reality, we have to find out its consequences and check them with experiments.

Before we work out the whole formalism, a few comments:

- Our above circular reasoning is not only a logical bootstrap, it indeed corresponds to the idea of hadronic bootstrap: \( \text{fireballs} \equiv \text{resonances are at the same time} \)
  i) object of the description,
  ii) elementary constituents of this object,
  iii) generating the forces creating this object.

Thus a complete and universal hadronic bootstrap is solved, if the condition (4.11) is fulfilled. This bootstrap, because of its generality and its thermodynamic formulation, cannot be expected to give any details
on the bootstrapped particles (as limited bootstraps can do), because, apart from a few low-lying resonances, these particles are fireballs, and for the birds-eye of thermodynamics all fireballs are equal (apart from their mass).

- The bootstrap condition (4.11) can be posed only for \( V = V_0 \), because for any larger \( V \) strong interactions do not reach over the whole extension and then such a larger fireball is no longer a single hadron.

- We shall not try to achieve \( \rho(m) \rightarrow \sigma(m, V_0) \), because \( \sigma(m, V_0) \) contains more states than \( \rho(m) \): the hadronic mass spectrum counts each resonance only once [apart from the factor \((2J + 1)(2I + 1)\)] disregarding the states due to its possible momenta -- they appear in \( \sigma(E, V) \) --, while \( \sigma(E, V) \) is calculated under neglect of momentum conservation and therefore contains the states corresponding to the possible motions of the fireball as a whole. Comparing eqs. (3.35a) and (3.36a), we see that for non-relativistic particles \( \sigma(E, V) \) is by a factor \( E_{\text{kin}}^{\frac{3}{2}}(mN)^{\frac{3}{2}} \) greater than \( \sigma(E, V, \Delta P) \); our fireballs will turn out to contain mostly heavy, non-relativistic particles, whose total rest mass is \( \sum m = E \). Thus, for a fireball of energy \( E \), the level densities, with and without momentum conservation, will differ roughly by \( E^{\frac{3}{2}} \). Consequently, we expect \( \sigma(m, V_0) \approx m^{\frac{3}{2}} \rho(m) \), when \( m \) becomes large. The power need not be exactly \( 3/2 \); in the first paper on hadron thermodynamics (Hagedorn, 1965), a similar argument, yielding a further factor \( m^2 \), was based on the assumption that the resonances in \( \rho(m) \) might not have arbitrarily high spin; if Regge trajectories rise linearly to infinity, then that argument fails; if not, the power may become \( > 3/2 \).

Our logarithmic condition (4.11), which is sufficient for thermodynamics, makes questions about the precise differences between \( \rho \) and \( \sigma \) meaningless.

If, however, we would try to describe everything in the language of phase space (microcanonical ensemble with full energy momentum conservation), then we should require that \( \rho(m) \) and \( \sigma(m, V_0, \dot{V} = 0) \) themselves (and not only their logarithms) become equal for \( m \rightarrow \infty \). That this is indeed possible,
and leads to a more detailed picture (though to the same thermodynamics), has been shown by Frautschi (1970) in a paper "A statistical bootstrap model of hadrons".

- Once our bootstrap condition is fulfilled (by an exponential spectrum, as we shall see), we know that \( \rho(m) \) counts everything it should. However, what exactly it counts: how many stable particles, how many sharp or diffuse resonances, how much background?—these are at present academic questions; their answers may take long and may come from an unexpected direction*). In the low-mass region, where particles and resonances are individually known, \( \rho(m) \) should be identified with the observed hadronic mass spectrum.

- Our bootstrap reasoning has led to a precise prescription for building up thermodynamics of strong interactions. If one puts the condition (4.11) in words, it reads:

\[
\text{A fire-ball is}
\]

\[
\text{a statistical equilibrium (hadronic black body radiation) of undetermined numbers of all kinds of fireballs, each of which, in turn, is considered to be}
\]

- We have gone a long way from a formula, containing a squared matrix element times momentum space [eq. (4.1)], over shifting the interaction from the matrix element via distorted waves into momentum space [eq. (4.6)], to making this procedure as complete as possible by requiring it to be self-consistent [bootstrap condition (4.11)] and thus arriving at an (almost) completely fixed thermodynamics of strong interactions:

*) Repulsive forces will not create resonances, but they may still be present in \( \rho(m) \) as negative contributions which counteract the positive ones of the attractive forces. Nobody knows.
this long way should not obscure the fact that it is only a formal manipulation and generalization (according to known rules of quantum theory) of the starting formula (4.1), namely

\[ \text{Probability} = \int |\text{matrix element}|^2 \times \text{Phase space}. \]

It so happens that the result of this generalization fulfills all formal requirements for its description by the formulae of statistical thermodynamics. But the "thermodynamical equilibrium", which is thus described, is not brought about by millions of mutual collisions of given particles; no: the particles whose equilibrium is described are born into this "pre-established" equilibrium which is nothing else than the quantum mechanical probability distribution of all possible final states.

Indeed: Frautschi's (1970) paper, mentioned above, carries through the program resulting from our bootstrap argument, without ever leaving phase space and without ever speaking of temperature or equilibrium. He says: "the reason we choose to work directly with the density of states, at the cost of some awkwardness with the Pauli exclusion principle, is mainly pedagogical: phase space is more familiar to particle physicists, and the independence of the derivation from any assumptions concerning thermal equilibrium is most convincingly demonstrated by not introducing temperature at all".

We shall proceed here with thermodynamics for two reasons: firstly, because by using the notions of thermodynamics, everything becomes surprisingly simple (e.g. the Pauli principle is fulfilled with ease) and physically transparent; secondly, because the applications follow so easily without much formal (though numerical) calculations -- one almost can copy them from elementary textbooks on statistical thermodynamics.

Thus we shall use the words temperature and equilibrium; and these words have their usual physical meaning, but the mechanism, which creates equilibrium and temperature, is entirely different from that of classical statistical thermodynamics. The question, how long does it take to
establish equilibrium?, is answered by: no time at all*); but what the final state (the one to be observed in a single event) will be, that only turns out when the strong interactions cease to act between the outgoing particles, namely after \( \gtrsim 10^{-22} \) sec. Before that, any final state has a chance to emerge from the equilibrium. Thus, our "equilibrium" is equivalent to what in field theory are the interpolating fields at \( t = 0 \), when the particle number operator is not diagonal.

Therefore, the question, whether a resonance lives long enough to participate in this equilibrium, makes no sense; it helps to constitute this equilibrium by its *virtual*, not by its *actual* presence: it is contained in \( \rho(m) \) via \( (1/\tau)\delta \chi/d\rho \) and the width of this contribution may be great or small (the lifetime may be small or great): it does not matter.

However, it turns out that in this theory resonances do live long enough; namely: our bootstrap condition will lead to an exponentially increasing hadronic mass spectrum \( \rho(m) \). A fireball decays into other fireballs (rather few; in one version of the model only two -- see Section 8.1) with the same mass spectrum. The decay width of the parent fireball is proportional to the density of states of the daughter fireballs and inversely proportional to its own (it may decay into any final state but must start from a particular initial one). The result of a closer analysis is surprising and satisfying: if \( \rho(m) \) increases exponentially -- as in our theory -- lifetimes tend, for \( m \to \infty \), to a limit (corresponding to a width of \( \approx 300 \) MeV); on the other hand, if \( \rho(m) \) is either cut off above 2 to 3 GeV or continued by a power law -- both contradicting bootstrap -- then widths tend rapidly to infinity, lifetimes to zero [Matthiae, 1968].

Thus, while the actual lifetime of a resonance may in fact be irrelevant in our formalism -- extremely large "resonances" can be present in \( \rho(m) \) -- it is reassuring that everything is consistent even if one requires that the lifetimes should not be shorter than the interaction time: if there are infinitely many resonances -- as bootstrap requires -- then they are narrow enough to merit the name!

*) This "instantaneous equilibrium" applies only to spatial extensions of order \( \leq V_0 \)!
5. THE PARTITION FUNCTION AND THE
SOLUTION OF THE BOOTSTRAP CONDITION

We start from eq. (3.27) which we repeat here:

\[ Z(V,T) = \sum_i e^{-E_i/T} = \int_0^\infty \sigma(E,V) e^{-E/T} dE. \]  

(5.1)

It is extremely difficult to calculate \( \sigma(E,V) \) directly from phase space; surprisingly enough, the first expression, the sum, can be calculated easily. We shall thus do this and then use the identity with the second formulation to impose the bootstrap condition.

In order to compute \( \sum_i \exp(-E_i/T) \), we have to calculate the energies of all possible states, containing any number of any sort of hadrons. However, since all interaction is contained in the mass spectrum \( \rho(m) \), formally we have free, non-interacting particles. Therefore, a general state \( \psi_i \) is the direct product of single particle plane wave states, and its energy \( E_i \) is simply the sum of the energies of the individual free particles. Interaction energies do not occur, they are hidden in \( \rho(m) \).

5.1 The partition function

In a box of volume \( V \) only a discrete set \( \{p_\alpha^+\} \) of plane waves is allowed for each particle. Assume for the moment also that the set of all hadrons is discrete, call it \( \{m_\gamma\} \). Then the energy of particle \( \gamma \) in the momentum state \( \alpha \) is

\[ E_{\alpha\gamma} = \sqrt{p_\alpha^2 + m_\gamma^2}. \]  

(5.2)

Let further

\[ \nu_{\alpha\gamma} = \begin{cases} 0, 1 & \text{for fermions} \\ 0, 1, 2, \ldots \infty & \text{for bosons} \end{cases} \]  

(5.3)
be the "occupation numbers", indicating how many particles of the kind \( \gamma \) have momentum \( \vec{p}_\alpha \). Note that here the Pauli exclusion principle -- or: the difference between Fermi-Dirac and Bose-Einstein statistics -- is fully contained in (5.3), whereas in phase space it is difficult to accommodate.

In this notation each state \( \psi_i \) is determined (up to a phase) by one complete set of occupation numbers \( \{ \nu_{\alpha \gamma} \}_i \), which is an \( \infty \times \infty \) matrix with elements \( \nu_{\alpha \gamma} \), hence \( \psi_i \equiv \{ \nu_{\alpha \gamma} \}_i \). Thus

\[
E_i = \sum_{\alpha \gamma}^\infty \nu_{\alpha \gamma} \epsilon_{\alpha \gamma}
\]

(5.4)

and

\[
Z(\nu, T) = \sum_{\{ \nu \}} \exp \left[ -\frac{1}{T} \sum_{\alpha \gamma}^\infty \nu_{\alpha \gamma} \epsilon_{\alpha \gamma} \right]
\]

(5.5)

where this sum goes over all matrices \( \{ \nu \} \), each representing one state.

We put

\[
X_{\alpha \gamma} \equiv e^{-\frac{\epsilon_{\alpha \gamma}}{T}}
\]

(always < 1) (5.6)

and proceed:

\[
Z(\nu, T) = \prod_{\{ \nu \}} \prod_{\alpha \gamma} X_{\alpha \gamma} \equiv \prod_{\alpha \gamma} \left[ \sum_{\nu_{\alpha \gamma}} \nu_{\alpha \gamma} X_{\alpha \gamma} \right].
\]

(5.7)

[The reader can convince himself that these two expressions are identical, by considering the simplest possible example, namely where \( \{ \nu \} \) is a 2 \( \times \) 2 matrix.] We split the label \( \gamma \) into two: \( \beta \) for bosons and \( \phi \) for
fermions. Thus \( \nu_{\alpha \beta} = 0, 1, 2 \ldots \infty \), while \( \nu_{\alpha \phi} = 0,1 \) only. Then

\[
Z(\nu, T) = \prod_{\alpha \beta} \frac{1}{1 - x_{\alpha \beta}} \prod_{\alpha \phi} (1 + x_{\alpha \phi})
\]

(5.8)

\[
\ln Z = - \sum_{\alpha \beta} \ln (1 - x_{\alpha \beta}) + \sum_{\alpha \phi} \ln (1 + x_{\alpha \phi}).
\]

(5.9)

Now we go over to the continuum [see (3.33)]

\[
\sum_\alpha [... ] \Rightarrow \frac{V}{2\pi^2} \int_0^\infty [... ] p^2 dp
\]

(5.10)

\[
\sum_{\beta, \phi} [... ] \Rightarrow \int_0^\infty \rho_B F(m) [... ] dm.
\]

(5.11)

Thus

\[
\ln Z = \frac{V}{2\pi^2} \int_0^\infty p^2 dp \left[ \int_0^\infty \rho_B F(m) \ln (1 + x_{\beta m}) dm \\
- \int_0^\infty \rho_B F(m) \ln (1 - x_{\phi m}) dm \right].
\]

(5.12)

We expand the logarithms and obtain

\[
\ln Z = \frac{V}{2\pi^2} \sum_{m=1}^\infty \frac{1}{n} \int_0^\infty dm dp p^2 \rho(m,n) x_{\beta m}^n,
\]

(5.13)

where

\[
x_{\beta m}^n \equiv \exp \left( - \frac{m}{n} \sqrt{p^2 + m^2} \right)
\]

(5.14)
and

\[ p_n(m, n) = p_B(m) - (-1)^n p_F(m). \] (5.15)

In order to carry out the \( p \)-integration, we use the formulae

\[ \int_0^\infty \int_0^\infty \exp\left(-\frac{p^2 + m^2}{T}ight) d\rho \, dm = (2n-1)!! \, \rho^{2n+1} \left(\frac{T}{m}\right)^n \, K_{n+1}\left(\frac{\rho}{T}\right) \]

\[ \int_0^\infty \rho \exp\left(-\frac{p^2 + m^2}{T}ight) d\rho = 2^n \, n! \, \rho^2 \left(\frac{T}{m}\right)^{n+\frac{1}{2}} \, K_{n+\frac{3}{2}}\left(\frac{\rho}{T}\right) \]

\[ (2k-1)!! = 1, 3, 5, \ldots, (2k-1); \quad (-1)!! = 1 \] (5.16)

where \( K \) is the second modified Hankel function. For half-integer index one has

\[ K_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \, e^{-x} \sum_{k=0}^n \frac{(n+k)!}{k! \,(n-k)!} \left(\frac{1}{2x}\right)^k, \] (5.17)

which is an elementary function, while for integer \( n > 1 \), it is non-elementary with the limiting \( (x \to 0, \infty) \) behaviour

\[ \begin{align*}
K_n(x) & \to \sqrt{\frac{\pi}{2x}} \, e^{-x} \left[ 1 + \frac{(n+\frac{1}{2})(n-\frac{1}{2})}{2x} + \ldots \right] \\
K_n(x) & \begin{cases} \\
0 & x \to 0 \\
0 & x \to \infty \end{cases} \end{align*} \] (5.18)
We note these results for later use. Presently we obtain

\[
Z(V, T) = \exp \left\{ \frac{V T}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^\infty \rho(m;n) m^2 K_2 \left( \frac{m}{T} \right) \, dm \right\}
\]  

(5.19)

which shows that indeed \( Z \) is completely known if \( \rho(m; n) \) is given.

5.2 One solution of the bootstrap condition

We write --- replacing \( E \) by \( m \) as integration variable ---

\[
Z(V, T) = \int_0^\infty \sigma(m, V) e^{-\frac{m}{T}} \, dm
\]  

(5.20)

and require that (5.19) \( \equiv \) (5.20) and that for \( V = V_0 \)

\[
\frac{\ln \rho(m)}{\ln \sigma(m, V_0)} \xrightarrow{m \to \infty} 1.
\]  

(5.21)

Now the difficulty is that the bootstrap condition implies that \( \rho \) and \( \sigma \) are "almost equal", and that, since \( K_2(x) \sim e^{-x} \), both functions appear in an integral multiplied by \( \exp (-m/T) \). Therefore, both integrals are "equal" up to some power of \( m \), but in (5.19) the whole is exponentiated, in (5.19) not. How can these so very different expressions become equal? They can, roughly speaking, if we achieve to find a \( \rho(m; n) \) which makes the integral in (5.19) behave as a logarithm, so that the exponentiation is compensated.

The details are somewhat complicated. They have been discussed in the first paper on this thermodynamics [Hagedorn, 1965, henceforward quoted as (I)] and shall not be repeated here.

We can, however, see by a qualitative argument, what happens: suppose some \( \rho(m) \) is given, then (5.19) can be evaluated and \( Z(V, T) \) is known. If \( Z(V, T) \) is known, then the Laplace transform (5.20) can be inverted,
and thus \( \sigma(m, V_0) \) is known. Therefore, \( \sigma \) is a functional of \( \rho \): for every given \( \rho \), \( \sigma \) can be calculated:

\[
\sigma(m, V_0) = F \left[ \rho(m) \right]
\]  
(5.22)

and we may try to solve the bootstrap condition by iteration. Let us start with the worst possible assumption, namely that there is only a finite number of different types of hadrons, so that

\[
\int_0^\infty \rho_0(m) \, dm = \beta < \infty
\]  
(5.23)

\[
\int_0^\infty \rho_F(m) \, dm = \phi < \infty
\]

The bootstrap condition \( \ln \rho(m) = \ln \sigma(m, V_0) \) is posed for \( \bar{E} = m \to \infty \); hence we need only consider the behaviour of \( Z \) for \( T \to \infty \). Then, if (5.23) is fulfilled, \( T \) can be pushed far beyond the largest masses and (5.18) gives

\[
K_2 \left( \frac{m u}{\bar{u}} \right) \xrightarrow{T \to \infty} 2 \frac{T^2}{m^2 u^2}.
\]  
(5.24)

This inserted into (5.19) yields

\[
Z^{(0)}(V, T \to \infty) \Rightarrow \exp \left\{ \frac{V T^3}{\pi^2} \sum_{\eta_4} \frac{1}{\eta_4^4} \int_0^\infty \rho(m, u) \, dm \right\}
\]

\[
= \exp \left\{ \frac{V T^3}{\pi^2} \left( \beta \frac{1}{\eta_4^4} - \phi \frac{t(u)^m}{\eta_4^4} \right) \right\}.
\]  
(5.25)
Using the well-known formulae for the Riemann ζ-function

\[ \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \]  

(5.26)

\[ \frac{7}{8} \zeta(4) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \frac{7}{8} \frac{\pi^4}{90} \]

we obtain

\[ Z^{(b)}(\nu, T \to \infty) \Rightarrow \exp \left[ -\mathcal{R} \frac{V \pi^2 T^3}{90} \right] \]  

(5.27)

with

\[ \mathcal{R} \equiv \int_0^{\infty} g_b(u)du + \frac{7}{8} \int_0^{\infty} g_f(u)du < \infty \]

The factor 7/8 is due to the Pauli principle (Fermi-Dirac statistics).

The total energy \( \bar{E} = m \) is given by (3.25)

\[ \bar{E} = m = T^2 \frac{\partial}{\partial T} \ln Z^{(b)} \Rightarrow \frac{\mathcal{R} V \pi^2}{30} T^4 \]  

(5.28)

and the energy density follows for \( T \to \infty \) a Stefan-Boltzmann law:

\[ \frac{\bar{E}}{\nu} = \mathcal{E} \Rightarrow \frac{\pi^2}{15} \cdot \frac{\mathcal{R}}{2} \cdot \frac{\pi^2}{15} = \frac{\mathcal{R}}{2} \sigma_{st}^{em} \cdot T^4 \]  

(5.29)

Here \( \pi^2/15 \) is the Stefan constant \( \sigma_{st}^{em} \) for ordinary electromagnetic black body radiation; indeed: for a light quantum gas \( \rho_f = 0, \rho_B = 2\delta(m) \), \( R = 2 \).

Thus, in passing by, we have the instructive result:
For any converging mass spectrum $\rho$ such that
\[
\int (\rho E + \frac{3}{8} p_F) du = R < \infty
\]
the black body radiation becomes thermodynamically similar to the electromagnetic one when $T \to \infty$; its Stefan constant is $\frac{1}{4} R \sigma_{st}^{(em)}$.

Now we calculate $\sigma^{(0)}(E, V)$ using the approximation (3.29) written as
\[
\sigma^{(0)}(E, V) \approx \frac{1}{\Delta E} Z(V, T) \frac{E}{T}.
\]

Let us, in (5.28), abbreviate $RV \pi^2 / 30 = R'$; then $T = (E/R')^{\frac{1}{2}}$ and $Z = \exp \left( \frac{1}{3} R'T^3 \right)$. Thus
\[
\sigma^{(0)}(E, V) \approx \frac{1}{\Delta E} \exp \left( \frac{4}{3} R' T^3 \right) = \frac{1}{\Delta E} \exp \left( \frac{4}{3} R' \frac{1}{4} E^{3/4} \right).
\]

We mention, in passing by, that eq. (3.31) gives for the entropy $S = \ln(\sigma \Delta E)$ thus
\[
S(E, V) = \frac{4}{3} \left( \frac{RV \pi^2}{30} \right)^{\frac{1}{4}} E^{3/4}
\]
and with the first law [eq. (3.14)]:
\[
\frac{1}{T} = \frac{\partial S}{\partial E} = \left( \frac{R \pi^2}{30} \frac{1}{E} \right)^{\frac{1}{4}}
\]
which is again the same as (5.29). Thus, for $R = 2$ (electromagnetic case) all formulae reduce to the well-known ones of the light quantum gas.
Identifying now $E$ with $m$, we see that

$$\sigma^{(0)}(m, \nu_0) \sim \exp\left(\text{const} \cdot m^{3/4}\right)$$  \hspace{1cm} (5.34)

Our bootstrap condition requires that $\rho(m)$ also should then at least grow like $\exp m^{3/4}$. Thus, our previous assumption $\int \rho(m) dm < \infty$ was wrong. In the next step of iteration we put

$$\rho^{(1)}(m) \sim \exp\left(\text{const} \cdot m^{3/4}\right)$$  \hspace{1cm} (5.35)

and now the evaluation of the integrals becomes less trivial and will not be done here. The result is that

$$\sigma^{(1)}(m, \nu_0) \sim \exp\left(\text{const} \cdot m / \ln m\right),$$  \hspace{1cm} (5.36)

which grows still much faster than $\sigma^{(0)}$ and almost exponentially. But for the next iteration, $\rho^{(2)}$ must grow at least as fast as that -- and so it goes on: $\sigma^{(i)}$ grows always faster than $\rho^{(i)}$; and since we always put, for the next step, $\rho^{(i+1)} = \sigma^{(i)}$, it is obvious that each $\rho$ grows faster than the previous one, and that we seem to approach rapidly an exponential function.

On the other hand: neither $\rho$ nor $\sigma$ can be permitted to grow faster than exponentially -- even $\exp (\text{const} \cdot m \cdot \ln m)$ is forbidden! -- since obviously the integrals (5.19) and (5.20) would diverge for any $T > 0$, that is: thermodynamics does not exist. One might object that the non-existence of the partition function is no argument against $\rho(m)$ growing faster than exponentially, because one could renounce on a thermodynamic description and go back to good, old phase space. However, Frautschi (1970)
has shown, that for $\rho$ growing faster than exponentially, $\sigma$ grows slower than $\rho$, so that an iteration would, in that case, approach an exponential function from above*). Thus, it is clear that non-exponential solutions do not exist and that iterations lead -- from both sides -- to an exponential solution. Its existence was first proved in 1965 [see (1) for details].

That $\rho (m) \sim \sigma(m) \sim \exp (m/T_0)$ is a possible solution, can be seen qualitatively as follows: for an exponential $\rho(m)$ there will always be masses $\gg T$ and therefrom will come the main contributions to the integral in (5.19). Thus, in this case we may use [see (5.18)]

$$K_2 \left( \frac{\tau m}{T} \right) \Rightarrow \sqrt{\frac{\pi T}{2\tau m}} e^{-\frac{\tau m}{T}}$$

and this will then appear under the integral multiplied by $\exp (m/T_0)$. Thus, the first integral ($n = 1$) will diverge for $T \rightarrow T_0$, the second one ($n = 2$) for $T \rightarrow 2T_0$, etc. Thus, for $T \rightarrow T_0$ we consider the divergent first integral and neglect the rest of the sum ($n = 2 \ldots \infty$) which can be shown to add up to a finite constant. We put

$$\rho(m) \Rightarrow \bar{\rho}(m) \exp \left( \frac{m}{T_0} \right)$$

$$\sigma(m) \Rightarrow \bar{\sigma}(m) \exp \left( \frac{m}{T_0} \right)$$

and obtain from (5.19) = (5.20) for $T \rightarrow T_0$ and with $\tau = (T_0 - T)/T_0$

$$Z(V_0 T) \Rightarrow \exp \left\{ V_0 \left( \frac{T_0}{2\tau} \right)^{3/2} \int_{M_0}^{\infty} \bar{\rho}(m) m^{3/2} e^{-\tau m} \, dm \right\}$$

$$+ V C \left( M_0, T_0 \right)$$

$$\Rightarrow \int_{0}^{\infty} \bar{\sigma}(m) e^{-\tau m} \, dm$$

(5.39)

*) For $\rho$ growing faster than exponentially, low-dimensional phase space with large masses is exponentially favoured against high dimensional phase space with small masses (at given total energy): resonances would not decay but coalesce; the world would be one big hadron and you would not read these lecture notes.
Here \( V_0 C(M_0, T_0) \) is the finite rest of terms. Thus we have two integrals, both containing the exponential function \( \exp(-\tau^\beta) \), which, for \( T > T_0 \), i.e. for \( \tau \to 0 \), tends to 1, and some non-exponential functions. One integral is exponentiated, the other not. Now we can choose \( f(m) \) such that the integral in the exponent becomes a logarithm which compensates the exponentiation: with

\[
f(m) = \frac{a_c}{m^{5/2}}
\]

(5.40)

\[
f(m) = b m^{\beta - 1}
\]

the first integral becomes, with \( \tau \to 0 \),

\[
a_c \int_{M_0}^{\infty} \frac{du}{u} e^{-\tau^\beta} = a_c \int_{M_0}^{\infty} \frac{dx}{x} e^{-x} \Rightarrow a_c \ln \left( \frac{T_0}{M_0} \right)^{\beta} \]

(5.41)

\[
= a_c \ln \frac{T_0}{T_0 - T} + a_c \ln \frac{T_0}{M_0}
\]

and the second

\[
b \int_{0}^{\infty} m \beta \ln u m \tau \, dm = \left( \frac{T_0}{T_0 - T} \right)^{\beta} \left[ b T_0^\beta \Gamma(\beta) \right].
\]

(5.42)

Thus

\[
Z(V_0 T) \Rightarrow \left( \frac{T_0}{T_0 - T} \right)^{\alpha_0} \exp \left[ V_c C(M_0, T_0) + \alpha_c \ln \frac{T_c}{M_0} \right]
\]

(5.43)

\[
\Rightarrow \left( \frac{T_0}{T_0 - T} \right)^{\beta} \left[ b T_0^\beta \Gamma(\beta) \right]
\]

with

\[
\alpha_0 \equiv a_c V_0 \left( \frac{T_0}{2\pi} \right)^{3/2} \text{ [pure number]}
\]
We now put

$$\beta = \alpha_0 = a_0 V_0 \left( \frac{T_0}{2\pi} \right)^{3/2}$$

and have solved the bootstrap condition, because

$$\frac{\ln \rho(m)}{\ln \sigma(m, V_0)} = \frac{\mu/T_0 + \mu \frac{f(m)}{m}}{\mu/T_0 + \mu \frac{g(m, V_0)}{m}} \quad \text{as} \quad \mu \to \infty$$

(5.45)

It should be mentioned that for sufficiently large $M_0$, the $M_0$-dependence of the argument of the exponential function in (5.44) disappears. Furthermore, (5.44) guarantees that $\sigma(m, V)$ depends exponentially on $V$ -- as it should to ensure additivity of the entropy. Thus, after the bootstrap condition is solved by an exponential spectrum $\rho(m; n)$, the partition function can be defined for any volume $V$ as in (5.19), and its logarithm is proportional to $V$. The specialization to $V_0$ was only necessary to impose and solve the bootstrap condition, which, for volumes exceeding the range of strong interactions, does not make sense.

While the formalism may thus be applied to any macroscopic system -- e.g. the early universe -- it happens that in application to particle physics the actual volume again is of order $V_0$. 
6. DISCUSSION OF THE BOOTSTRAP SOLUTION

6.1 Our solution

Obvious as our solution of the bootstrap condition may seem -- it is not unique. Indeed: if one chooses in (5.40) a more complicated expression with a leading power \( m^a \) and \( a \leq -\frac{5}{2} \), then the partition function converges for \( T \to T_0 \). It is then easy to reproduce in both representations (5.39) the same numerical value for \( T \to T_0 \) and even the same approach to this value. We could also admit \( a > -\frac{5}{2} \), if in \( \sigma(m) \) the factor \( g(m) \) is allowed to behave like \( \exp m^c \) with \( 0 < \varepsilon < 1 \), because our logarithmic bootstrap condition would still be fulfilled.

In reality, by the ansatz (5.40) we have already sharpened our logarithmic bootstrap condition to the more adequate

"weak bootstrap condition"

\[
\frac{c m^a \rho(m)}{\sigma(m V_0)} \xrightarrow{m \to \infty} 1 \quad \text{for some } \alpha, c
\]  

(6.1)

and shown that it can be fulfilled\(^*\). We shall from now on forget the logarithmic bootstrap condition, because we got a solution of the stronger condition (6.1). As long as we do not further sharpen this condition, any \( \rho(m) \sim c m^a \exp (m/T_0) \) with \( a < -\frac{5}{2} \) will yield a solution, because then in \( \sigma(m, V_0) = g(m, V_0) \exp (m/T_0) \) the function \( g(m, V_0) \) is polynomially bounded. The solution \( a = -\frac{5}{2} \) which we produced above is distinguished, however:

- \( a = -\frac{5}{2} \) is the endpoint of the continuum \( a \leq -\frac{5}{2} \) of solutions;
  it corresponds to the most rapid increase of \( \rho(m) \) compatible with the weak bootstrap condition,

- it is the simplest one, as it causes the leading term of \( g(m, V_0) \) to be a power of \( m \).

In (I) the solution \( a = -\frac{5}{2} \) was favoured as the most natural one. There was a philosophical argument behind it:

\(^*\) We shall not mention the constant \( c \) any more; it might be absorbed into \( \rho \) or \( \sigma \).
among all possible functions \( \rho(m) \) the exponentially increasing one is the most extreme; this probably means that strong interactions are as strong as they can be; then in \( \rho(m) = f(m) \exp(m/T_0) \), \( f(m) \) again should increase as fast as possible -- thus \( a = -\frac{5}{2} \) -- and \( T_0 \) should be as small as possible -- thus of the order of \( m_{\pi} \). Indeed \( T_0 \approx 160 \) MeV, which made the argument sound reasonable and suggested to take also \( a = -\frac{5}{2} \).

Therefore in all subsequent papers, \( a = -\frac{5}{2} \) was considered to be "the solution". Recent developments have, however, cast much doubt on this choice and brought back other solutions into serious consideration. We shall sketch this now.

6.2 The strong bootstrap condition of Frautschi

Frautschi (1970) observes that in the thermodynamic treatment \( \sigma(m, V_0) \) contains still the states corresponding to the motion of the fireball as a whole and secondly, that in our expression (5.19) zero- and one-particle states are contained, which in a true bootstrap of hadrons should be excluded. He prefers then to work in phase space with full momentum conservation, starting with two-particle states. As the conservation of total momentum will diminish \( \sigma(m, V_0) \) asymptotically by a factor \( m^{-\frac{3}{2}} \) (see the comments following the bootstrap reasoning in Section 4.5), one might indeed hope that a strong bootstrap condition \( \rho(m) \rightarrow m^{-\frac{3}{2}} \sigma(m, V_0) \) could be satisfied.

Frautschi uses \( \rho(m) \) as "input" function and calls it \( \rho_{in}(m) \), while the total density of states with momentum conservation is the "output" function called \( \rho_{out}(m) \) [thus \( \rho_{out}(m) \rightarrow m^{-\frac{3}{2}} \sigma(m, V_0) \)].

His equation then reads [compare our Eqs. (4.1), (4.6), (4.7), (4.8), (4.10)]

\[
\rho_{out}(m) = \sum_{n=2}^{\infty} \left[ \frac{V_0}{2\pi^3} \right]^{n-1} \frac{1}{n!} \int \prod_{i=1}^{n} d\mathbf{u}_i d^3p_i \rho_{in}(\mathbf{u}_i) \delta(m - \sum_i \sqrt{p_i^2 + m_i^2}) d^3(\sum p_i) \quad (6.2)
\]

and his "strong bootstrap condition" is
\[
\frac{\rho_{\text{in}}(m)}{\rho_{\text{out}}(m)} \xrightarrow{m \to \infty} 1
\]

which is the weak condition (6.1) sharpened by requiring \(\alpha = \frac{3}{2}\). He shows that this strong condition can only be satisfied by

\[
\rho_{\text{in}}(m) \sim c m e^{m/T_0}, \quad a < -\frac{5}{2}
\]

The difference with our result is that his power of \(m\) must be smaller than \(-\frac{5}{2}\) while we found that \(-\frac{5}{2}\) is also allowed. There is no contradiction; we differ by physical assumptions:

Frautschi's \(\rho_{\text{out}}(m)\) counts all possible compound states (in our words: fireballs) made up by any possible combination of hadrons, whose mass spectrum is \(\rho_{\text{in}}(m)\). Requiring \(\rho_{\text{out}} \to \rho_{\text{in}}\) is equivalent to the assumption that for \(m \to \infty\) there are no constraints on \(\rho_{\text{in}}(m)\) which are not also imposed on \(\rho_{\text{out}}(m)\). These latter ones are explicitly written down in (6.1): energy- and momentum conservation, that is all. But what if there are some more constraints on \(\rho_{\text{in}}(m)\), which have been forgotten in (6.2)?

One constraint, not made explicit in (6.2), is the Pauli exclusion principle. This will probably not matter, since in our formulation its neglect will only have an influence on the terms \(n \geq 2\) in Eq. (5.19), and these terms are regular at \(T = T_0\) and did not enter our discussion of what the power of \(m\) in front of the \(\exp m/T_0\) of \(\rho(m)\) should be.

Another constraint may be more important: we had to admit that \(\varrho\) and \(\sigma\) might differ by \(m^{3/2}\), because we neglected momentum conservation in computing \(Z\). Frautschi does not neglect this; therefore he need no longer admit a factor \(m^{3/2}\) between \(\rho_{\text{in}}\) and \(\rho_{\text{out}} \approx -\frac{3}{2}\sigma\). However, angular momentum conservation, if introduced into (6.2) would diminish \(\rho_{\text{out}}\) by some further power (see the literature quoted at the end of Section 3.5). Suppose now that the states counted in the hadronic mass spectrum \(\rho(m) \equiv \rho_{\text{in}}(m)\) obey some regularity with respect to their total spins. This regularity has then be neglected in (6.2) and thus \(\rho_{\text{out}}\) does not
obey it; hence $\rho_{\text{out}}$ and $\rho_{\text{in}}$ could again differ by a function of $m$. In fact to the $\rho_{\text{out}}$ of (6.2) belongs a definite statistical distribution of spin values, while the hadronic mass spectrum $\rho_{\text{in}}$ might have another one. This would depend very much on the structure of the set of all Regge trajectories and is presently not well enough understood to exclude that $\rho_{\text{in}}$ and $\rho_{\text{out}}$ could still differ by a function of $m$ which does not tend to 1 for $m \to \infty$.

A last difference might come from constraints on other quantum numbers, e.g. it could be that in $\rho_{\text{in}}$ exotic states are absent*), while they are included in $\rho_{\text{out}}$. This again leads to a difference between these two functions, which might not disappear for $m \to \infty$.

In all these cases, Frautschi's strong bootstrap condition might be modified with the result that the power $a$ in $\rho(m) \sim c m^a \exp (m/T_0)$ is no longer restricted to $a < \frac{5}{2}$ but that $a = -\frac{5}{2}$ becomes again allowed -- it needs very little; but how little depends on physics which we do not yet understand well enough. To sum up:

Our weak bootstrap condition $m^a \rho(m) \sim c(m) -- a$ not fixed -- does not touch on presently unknown details and allows any $\rho(m) \sim c m^a \exp (m/T_0)$ with $a \leq -\frac{5}{2}$;

any strong bootstrap condition -- $a$ fixed -- contains implicit or explicit assumptions about the constraints imposed on the hadronic mass spectrum $\rho(m) \equiv \rho_{\text{in}}(m)$.

Frautschi's version just excludes the endpoint $-\frac{5}{2}$ of the interval of allowed powers of $m$. One can go one step further. Nahm (1971) considers the thermodynamical formulation with momentum conservation which essentially means replacing $c(m)$ by $\rho_{\text{out}} \propto m^{3/2}$. Imposing then Frautschi's bootstrap condition, he writes our Eq. (5.39) in the equivalent form

$$ Z(V_0, T \to T_0) \Longrightarrow \exp \left\{ A(t) \int_{M_0}^{\infty} \frac{a+\frac{3}{2}}{M} e^{-\omega T} d\omega + B(t) \right\} $$

$$ \Longrightarrow \int_{M_1}^{\infty} C(t) \int \frac{a+\frac{3}{2}}{M} e^{-\omega T} d\omega + D(t) \right\} $$

(6.4)

*) e.g. nuclei and all their excited states should not be counted in $\rho_{\text{in}}(m)$ since they do not fit into $V_0$ (having a volume $\sim A \cdot V_0$). Nevertheless they are counted in $\rho_{\text{out}}$: the state with all $m_i = m_j$ does occur in Frautschi's formula (6.2) -- but it should not.
The strong bootstrap condition is here already built in; the functions $A(\tau)$, $B(\tau)$, $C(\tau)$ and $D(\tau)$ need not be specified; important is only that they all are regular at $\tau = 0$, i.e. can be expanded in a power series.

Now, with $m\tau = x$ we have for the integrals

\[
\int_{-\infty}^{\infty} x^{a+\frac{3}{2}} e^{-x} \, dx = \left(\frac{1}{e}\right)^{a+\frac{5}{2}} \int_{0}^{\infty} x^{a+\frac{3}{2}} e^{-x} \, dx
\]

\[
= \left(\frac{1}{e}\right)^{a+\frac{5}{2}} \left[\Gamma \left(a+\frac{5}{2}\right) + \ldots\right].
\]

(6.5)

We put

\[
a + \frac{5}{2} = \gamma
\]

(6.6)

then (6.4) becomes

\[
Z(\nu_0, \tau \rightarrow \hat{\tau}_0) \Rightarrow \exp \left\{ A_1(\tau) \left(\frac{1}{e}\right)^{\gamma} + B(\tau) \right\}
\]

\[
\Rightarrow C_\gamma(\tau) \left(\frac{1}{e}\right)^{\gamma} + D(\tau)
\]

(6.7)

where $\Gamma(\gamma) + \ldots$ is absorbed into $A_1(\tau)$ and $C_1(\tau)$. In (I) we argued that $\gamma > 0$ is excluded, because the upper line would diverge exponentially for $\tau \rightarrow 0$ and the lower not. For $\gamma = 0$ — which just is our old solution $a = -\frac{5}{2}$ of the weak bootstrap — we have in (6.5) to take the limit $a \rightarrow -\frac{5}{2}$ (from below) with the result $\sim \ln 1/\tau$. This is not acceptable as a solution in the present strong bootstrap case, because now the upper line of (6.7) gives $+ \tau^{-A}$ while the lower yields $C \ln 1/\tau$. Hence $\gamma \geq 0$ is excluded. Let now
\[ \gamma \equiv -\varepsilon \quad \varepsilon > 0. \] (6.8)

We then find

\[ Z(v_0, T \to T_0) \implies \exp \left\{ A_1(\tau) \tau^\varepsilon + B(\tau) \right\} \]
\[ \implies C_1(\tau) \tau^\varepsilon + D(\tau). \] (6.9)

Expanding the exponential function yields

\[ \left[ 1 + A_1(\tau) \tau^\varepsilon + \frac{1}{2} A_1^2(\tau) \tau^{2\varepsilon} + \ldots \right] \times \exp B(\tau) \]
\[ \implies C_1(\tau) \tau^\varepsilon + D(\tau). \] (6.10)

The lower line contains only \( \tau^\varepsilon \), the upper \( \tau^\varepsilon, \tau^{2\varepsilon}, \ldots \), thus \( \varepsilon \) must be half-integral; therefore \( \alpha = \gamma - \frac{3}{2} = -\varepsilon - \frac{3}{2} \) is limited to

\[ \alpha = -3, -4, -5, \ldots, \] (6.11)

because the values \( \alpha = -\frac{7}{2}, -\frac{9}{2}, \ldots \) would again produce logarithmic expressions which cannot be matched. This argument can be further sharpened by more close inspection of the strong bootstrap condition; the result is that only \( \alpha = -3 \) is a solution. We thus state Nahm's result (1971)

if the strong bootstrap condition \( \rho_{\text{in}} \to \rho_{\text{out}} [\rho(m) + m^{-\frac{3}{2}} \sigma(m, v_0)] \)

is posed, only one solution
\[ \rho(m) \rightarrow \frac{c}{m^3} e^{m/T_0} \]  

(6.12)

remains; the constants \( c \) and \( T_0 \) are related to \( V_0 \).

We repeat that the strong bootstrap condition may be too strong -- just a little too strong, so that our solution \( \rho(m) = c m^{-3/2} \exp(m/T_0) \) may become again allowed.

6.3 The mass spectrum in the Veneziano model

In this part I will be very short, because the relation between the thermodynamical and the dual approach is not yet understood. The central result of our thermodynamics is that a complete and general hadronic bootstrap is soluble and that the hadronic mass spectrum is necessarily exponential. Exactly this was also the outcome of the attempt to "factorize" the Veneziano n-point function; factorization is another technical approach to bootstrap. The surprise was great, when Fubini and Veneziano (1969) and Fubini, Gordon and Veneziano (1969) found this result. In the meantime it became considered one of the most significant implications of dual bootstrap models (instead of a serious disease). To this attitude the result of the thermodynamical bootstrap and some other general arguments [Brout 1969, Krzywicki 1969] based on duality have certainly contributed.

The present situation seems to be the following one [Lee and Leung 1970]: the Veneziano model predicts [Huang and Weinberg 1970]

\[ \rho_V(m) \rightarrow \sqrt{2\alpha'} \left( \frac{D}{24\alpha'} \right)^{\frac{D+1}{4}} \frac{m}{(m_0^2 + m^2)^{D+3/4}} \exp \left[ \frac{2\pi}{D+1} \sqrt{\frac{D}{6} (m_0^2 + m^2)} \right] \]  

(6.13)

where the constants mean

\[
\begin{align*}
\alpha_0 &= \text{intercept} \\
\alpha' &= \text{slope} \\
m_0 &= \sqrt{\alpha_0/\alpha'} = 1 \text{ GeV} \\
D &= \text{dimension of oscillator-operator}
\end{align*}
\]  

(6.14)
The "dimension" $D$ is at least 4; it probably is 5, less probably larger than 5. We cannot go into these questions here, it may suffice to say that presently $D = 5$ is favoured [Lovelace 1969].

Let us push $m$ so far that $m \gg m_0$. Then

$$\begin{align*}
\lim_{m \to \infty} S^V(m) & \sim \text{const.} \cdot m^{D+1} e^{-m/T_0} \\
T_0 &= \sqrt{\frac{3}{2\pi^2 D}}
\end{align*}$$

(6.15)

This formula is surprisingly near to what we found as solutions to the weak or strong bootstrap conditions: for $D = 4, 5, 6 \ldots$ we obtain the power of $m$ to be $-\frac{5}{2}, -3, -\frac{7}{2} \ldots$ and just $-\frac{5}{2}$ and $-3$ were favoured by us in the weak and strong case respectively. The Veneziano model seems not to require $D \geq 6$ and we do not like to accept a power $\leq -\frac{7}{2}$ for reasons which will be discussed below -- a satisfactory coincidence.

Furthermore, with $\alpha' \approx 1 \text{ GeV}^{-2}$ and $D = 4, 5, 6$ one obtains

$$T_0^V = \begin{cases} 
196 \text{ MeV} & D = 4 \\
174 \text{ MeV} & D = 5 \\
146 \text{ MeV} & D = 6 
\end{cases}$$

(6.16)

which all are not far from the experimental value $\approx 160 \text{ MeV}$ (see below).

The structures of the Veneziano $n$-point function with factorization and of our thermodynamics are technically very different, but similar in philosophy: both are dual and both postulate a general and universal bootstrap. In the Veneziano case fermions are not yet incorporated, but the thermodynamic approach shows that this does not matter so much (if there were only bosons, the terms $n \geq 2$ in (5.19) would be affected -- but they are regular at $T_0$ and do not enter the discussion of the solution). Thus it seems that

a general and universal hadronic bootstrap leads to an exponential mass spectrum

rather independently of the structure of the formalism; a viewpoint stressed already long before the discovery that also the dual models have exponential spectra [Hagedorn, 1967].
6.4 Comparison with the experiment, physical consequences

So far we have discussed the mathematical situation of the bootstrap solution, the exponential spectrum. We shall now compare the prediction

\[ p(m) \Rightarrow c m^a \exp \left( \frac{m}{T_0} \right) \]  \hspace{1cm} (6.17)

with the experiment. So far the spectrum of particles and resonances is known only up to about 2-3 GeV and only below 1 GeV we may assume to have a complete knowledge. The asymptotic formula (6.17) diverges for \( m \to 0 \) and must be modified. One possibility is to take the formula of Huang and Weinberg (6.13) and try to fit it to the experimental mass distribution. Long before the Huang-Weinberg formula was derived, L. Fogli proposed to continue our solution with \( a = -\frac{5}{2} \) to low masses by

\[ p(m) \approx \frac{a_0}{(m_0 + m)^{5/4}} \exp \left( \frac{m}{T_0} \right) \]  \hspace{1cm} (6.18)

Fig. 7 shows how well one can fit the mass spectrum with such a formula.

However, reasonable fits are possible with the Huang-Weinberg formula as well (e.g. \( D = 6, \alpha' = 1.2 \text{ GeV}^{-2}, \alpha_0 = 0.8 \)) is not bad, it gives \( T_0 = 146 \text{ MeV} \); with \( D = 5 \) a good fit can be achieved only at the cost of a much too high slope \( \alpha' = 1.8 \) and rather low intercept \( \alpha_0 = 0.4 \) leading to \( T_0 = 130 \text{ MeV} \) or with another guess-formula

\[ p(m) \approx \frac{a_0}{(m_0 + m)^{5/2}} \exp \left( \frac{m}{T_0} \right) \]  \hspace{1cm} (6.19)

with \( a_0 = 1.47 \times 10^5 \text{ MeV}^{3/2} \)

\( m_0 = 1000 \text{ MeV} \)
\( T_0 = 168.6 \text{ MeV} \)

This last is actually the best fit obtained so far, but in drawing hardly distinguishable from the fit of figure 7.
A particle or resonance is counted with its statistical weight \( W = (2J+1)(2I+1)2^d \)

\[
\alpha = \begin{cases} 1 & \text{if particle or antiparticle} \\ 0 & \text{if particle = antiparticle} \end{cases}
\]

and then represented by a Gaussian function normalized to \( W \) with width 200 MeV.

Fig. 7 The smoothed experimental mass spectrum as it developed from 1964 to 1967 and the function

\[
\rho(m) = a(m_0^2 + m^2)^{-\frac{3}{2}} \exp\left(\frac{m}{T_0}\right)
\]

which has the asymptotic form required by the theory. The figure is taken from (II).
The lesson we learn from these different fits is that the well-known parts of the spectrum do not yet suffice to fix $T_0$ and other constants or even the power of $m$; reasonable-to-good fits can be obtained with

$$130 \text{ MeV} \lesssim T_0 \lesssim 180 \text{ MeV} \quad (6.20)$$

and the value $T_0 = 160$ MeV, adopted in all previous papers, might be a good choice. What is important, is that the known part of the mass spectrum strongly supports an exponential spectrum.

What are the physical consequences? The most striking one is that $T_0$ is the universal maximum temperature for all matter because above $T_0$ the partition function and all thermodynamic functions do not exist. It is a universal maximum temperature, because any matter (e.g. neutrinos) brought to some temperature $T > T_0$ would, in the long run, by their mutual collisions create hadrons and then further hadron creation would rapidly use up the available kinetic energy until $T \leq T_0$ is reached. That there exists a highest temperature, is not so strange as it first seems: for an ideal gas with fixed number $N$ of particles, the temperature increases like $\frac{2}{3} E/N$ [see Eq. (3.41)]. If particle creation is permitted, then the temperature increases much more slowly, namely like $\left(\frac{E}{V}\right)^{1/4}$ [see Eq. (5.29)], provided the spectrum is finite. If an exponential spectrum is permitted, then, for large $E/V$, the temperature grows very slowly and approaches a limit $T_0$. It does so, because with an exponential mass spectrum it is much easier to create new particles than to increase the kinetic energies of those present. The whole has a striking similarity to boiling and therefore I once called $T_0$ the boiling point of hadronic matter (Hagedorn 1968a).

Now if $T_0$ is the maximum temperature, then the momentum distribution of particles emitted from highly excited fireballs, must tend to a limiting distribution, namely the Planck-distribution (4.9) with $T(E) \Rightarrow T_0$. Since in real collisions the longitudinal momentum distribution is largely determined by the collective motions [see Eq. (2.9)], we have to look for the transverse momentum distribution, which is invariant under the Lorentz-transformations in Eq. (2.9). This transverse momentum
distribution should be again of a similar type as the Planck distribution. This is in fact what one sees experimentally. What is important, however, is that for very large collision energies these distributions are indeed characterized by $T(E) \rightarrow T_0$ with the same $T_0 ≈ 160$ MeV as in the mass spectrum [see Section 10.1.6 and Fig. 14]:

- exponential mass spectrum $\rho(m) \sim \exp (m/T_0)$,
- maximum temperature $T_0$ and
- limited transverse momenta —

these three are in reality only different aspects of one and the same fundamental fact, namely of the general and universal hadron bootstrap.

6.5 Thermodynamic functions

After the bootstrap is solved and the asymptotic mass spectrum has been extrapolated down to the region where we know it experimentally, we can evaluate the partition function as given by Eq. (5.19) numerically. It turns out that for $m \geq 1$ GeV the argument $m/T$ of the Hankel function $K_2$ becomes so large — because $T < T_0 ≈ 160$ MeV — that all terms with $n \geq 2$ can be neglected there. For small $T$ this is obvious, for $T \rightarrow T_0$ it is still a very good approximation even if we choose the solution $\rho(m) \sim m^{-3} \exp (m/T_0)$; if we choose the one with $\rho(m) \sim m^{3/2} \exp (m/T_0)$, then the first term ($n = 1$) diverges and the rest can be neglected anyway. Thus, the procedure is to split the integral in one from $m_\Pi$ to $M ≈ 1$ GeV, use there the experimental mass spectrum $\rho(m; n) = \rho_B(m) - (-1)^n \rho_p(m)$ and evaluate the sum over as many terms $n = 1, 2, 3 ... N$ as one needs for a given accuracy and then to integrate from $M \approx 1$ GeV to $\infty$ with $\rho(m)$ given by (6.19) or some similar extrapolation, neglecting all terms $n \geq 2$. This has been carried out explicitly in (II) and a good numerical approximation to the result has been given in another paper (Hagedorn 1970a Section 4.1) both assuming the solution $\rho(m) \sim -m^{3/2} \exp (m/T_0)$. Instead of going into any detail, we only remark that

after having chosen one of the bootstrap solutions $\rho(m) \sim -m^{3a} \exp (m/T_0)$ and extrapolating it down to the known mass spectrum, the whole $\rho(m; n)$ is known and the partition function $Z(V, T)$ is a known function of $V$ (any $V$, not restricted to $V_0$) and $T$. As
all thermodynamic functions: energy, entropy, pressure etc., as well as all distribution functions are derivable from \(Z(V,T)\), the whole problem of a thermodynamic description of highly excited strongly interacting hadronic matter is solved (up to the ambiguity of which bootstrap solution should be taken).

The asymptotic behaviour for \(T \to T_0\) does, of course, not depend on the detailed structure of \(\rho(m)\) at low masses, here only the asymptotic \(\rho(m)\) matters.

One should think that in view of the exponential factor the power \(a\) of \(\rho(m) = cm^{-3} \exp(m/T_0)\) does not matter at all — this is not so. Let us consider the two cases \(a = -\frac{5}{2}\) and \(a = -3\). The first one has been worked out when we tried to find a solution of the bootstrap:

Eq. (5.43) gave

\[
\ln Z_{-\frac{5}{2}}(V_0, T \to T_0) \Rightarrow V_0 C(H_0, T_0) + \\
+ a_0 \ln \frac{T_0}{H_0} + a_0 \ln \left( \frac{T_0}{T_0 - T} \right)
\]

(6.21)

where with \(a_0\) and \(T_0\) taken from the fit of Fig. 7 we find

\[
a_0 = a_0 V_0 \left( \frac{T_0}{2\pi} \right)^{3/2} = 5.37 \quad \text{(pure number)}.
\]

(6.22)

On the other hand, with \(a = -3\), the partition function converges at \(T \to T_0\), because \(\rho(m) = cm^{-3} \exp(m/T_0)\) yields, instead of (5.41)

\[
c \int_{H_0}^{\infty} \frac{dw}{w^{3/2}} e^{-w/T} = c \sqrt{T} \int_{\frac{H_0}{T_0}}^{\infty} dx x^{-3/2} e^{-x} \approx c \sqrt{T} \Gamma(-\frac{1}{2})
\]

(6.23)

hence now, with \(\Gamma(-\frac{1}{2}) < 0\) and \(T = (T_0 - T)/T_0\)

\[
\ln Z_{-3}(V_0, T \to T_0) \Rightarrow -A \sqrt{T_0 - T} + B \quad ; \quad A > 0
\]

(6.24)
so that \( \ln Z_{-3} \) approaches the limit \( B \) from below with infinite slope. I have not worked out the values of \( A \) and \( B \), but both are proportional to \( V_0 \) as were \( V_0 \) and \( \alpha_0 \) in the \( a = -5/2 \) solution. The pressure is given by (3.25) as

\[
P = T \frac{\partial}{\partial V} \ln Z(V; T).
\]  

(6.25)

Thus (note that for black body radiation \( P \) is a function of \( T \) alone!)

\[
P_{-5/2} (T \rightarrow T_0) \Rightarrow A_1 + A_2 \ln \left( \frac{T_0}{T_0 - T} \right)
\]

(6.26)

\[
P_{-3} (T \rightarrow T_0) \Rightarrow \frac{B}{V_0} - \frac{A}{V_0} \sqrt{T_0 - T}.
\]

Thus for \( a = -5/2, \ln Z \) and \( P \) diverge logarithmically for \( T \rightarrow T_0 \), while for

\( a = -3 \) both converge to constant values, however approaching them with

infinite slope (analogy to boiling!).

The energy diverges in both cases; Eq. (3.25):

\[
E = T^2 \frac{\partial}{\partial T} \ln Z(V; T)
\]

(6.27)

\[
E_{-5/2} (V_0, T \rightarrow T_0) \Rightarrow T_0 \alpha_0 \frac{T_0}{T_0 - T}
\]

(6.28)

\[
E_{-3} (V_0, T \rightarrow T_0) \Rightarrow \frac{A}{2} T_0^2 \frac{1}{\sqrt{T_0 - T}}
\]

These examples may suffice to show that the physical differences of the
two favoured solutions \( a = -5/2, -3 \) are not completely negligible. We
shall encounter others.

Here is also the place to show why we would not like to accept

\( \rho(m) \rightarrow c m^a \exp (m/T_0) \) with \( a \leq -7/2 \). Namely, for \( a < -7/2 \) we obtain for
the leading integral in \( \ln Z \) [Eq. (5.41)].
\[ c \int \frac{du}{u^{\alpha}} e^{-\frac{u}{T_0}} \approx c T^{-\alpha} \Gamma(-\alpha+1); \quad \alpha > 2. \quad (6.29) \]

Thus

\[ \ln Z \Rightarrow -A (T_0 - T)\beta + B; \quad \beta > 1 \quad (6.30) \]

and

\[ E = \frac{\partial^2}{\partial T^2} \ln Z \Rightarrow \beta A T_0^{-2} (T_0 - T)^{\beta-1} + C. \quad (6.31) \]

We see that now for \( T \to T_0 \) also \( E \to \text{const.} \). That is: above a certain energy \( E_0 \) no thermodynamics is possible if in \( \rho(m) \) the power of \( m \) is smaller than \(-\frac{7}{2}\); actually the case \( a = -\frac{7}{2} \) leads to the same behaviour, \( E_0 \) is then approached with infinite slope. As long as the solutions \( a = -\frac{7}{2} \) and \( a = -3 \) are acceptable, we may discard \( a \leq -\frac{7}{2} \) as unphysical (see also Huang and Weinberg, 1970).

Conclusion

The question: which power \( a \) of \( m \) stands in front of the exponential function in the hadronic mass spectrum is all but trivial. Its answer depends on presently unknown but important details of strong interaction physics and it influences non-negligibly the asymptotic behaviour for \( E \to \infty \) not only in particle physics but also in cosmic physics (Hagedorn 1970a, Huang and Weinberg 1970).

Good numerical approximations of all thermodynamic functions have been given for the solution \( \rho(m) \to cm^{-\frac{5}{2}} \exp(m/T_0) \) in the paper (Hagedorn 1970a). There also the analytical expressions for all charac-
teristic quantities of a hypothetical fireball consisting only of particles with mass $m >> T_0$ have been derived (for $T \rightarrow T_0$ real fireballs approach this hypothetical fireball; see Sections 8.2 and 8.3).
7. **HOW TO DERIVE DISTRIBUTIONS FROM Z(V,T)**

In the section "Thermodynamics in a nutshell", it was shown that all thermodynamic functions can be derived from Z(V,T). Here, however, as a preparation to applying the theory to particle physics, we face another problem, namely that of deriving distribution functions -- momentum spectra, multiplicites, etc. -- from Z(V,T); we shall do this now.

7.1 **The momentum spectrum of freely created particles**

Freely created means: not limited (to pair- or associated creation) by some conservation law. Thus, the following consideration is strictly true only for particles with baryon number, strangeness and isospin zero; but it is nevertheless typical and may easily be modified later to obey the conservation laws.

What is the momentum spectrum \( f_m(\vec{p},T) \)? Before we derive it, let us say what it means:

\[
f_m(\vec{p},T)d^3p \text{ is the average number of particles } m \text{ emitted (without constraints) into the momentum cell } \{\vec{p},d^3p\}, \text{ without regard to whatever else happens -- i.e. no matter how many other particles there are, and no matter what they do.}
\]

Thus, in fashionable words: \( f_m(\vec{p},T) \) is the momentum spectrum of particle \( m \) in inclusive experiments.

Again, in other words: \( f_m(\vec{p},T) \) is the mean number of particles of kind \( m \) and momentum \( \vec{p} \). Let us go back for a moment to the discrete set of momenta \( \vec{p}_\alpha \) and masses \( m_\gamma \); then we can say that \( f_m(\vec{p}_i,T) \) is the mean number of particles of kind \( k \), having momentum \( \vec{p}_i \) -- but that is just the mean value of \( v_{ik} \) [the occupation number defined in (5.3)] taken over the ensemble:

\[
f_{m_k}(\vec{p}_i,T) = \overline{v}_{i,k} \quad (7.1)
\]
Thus, these mean values have to be calculated. Remaining at discrete masses but using the "continuous" counting of momentum states [see (3.33)], we see that

$$ f_m (\vec{p}, T) \, d^3 p = \frac{z_m}{(2\pi)^3} \, \overline{V} (m, \vec{p}) \, d^3 p $$

(7.2)

where

$$ z_m = \begin{cases} 
(2J + 1)(2I + 1) & \text{if spin and charge are not observed} \\
(2J + 1) & \text{if charge but not spin is observed} \\
(2I + 1) & \text{if spin but not charge is observed} \\
1 & \text{if spin and charge are observed}
\end{cases} $$

To calculate $\overline{\nu}_{ik}$ we go back to the partition function (5.7)

$$ Z(V, T) = \sum_{\{V\}} \prod_{\alpha \beta} x_{\alpha \beta}^{V_{\alpha \beta}} $$

(7.3)

Obviously

$$ \overline{\nu}_{ik} = \frac{\sum_{\{V\}} V_{ir} \prod_{\alpha \beta} x_{\alpha \beta}^{V_{\alpha \beta}}}{\sum_{\{\bar{V}\}} \prod_{\alpha \beta} \bar{x}_{\alpha \beta}^{V_{\alpha \beta}}} = \frac{x_{ir}}{\partial x_{ir}} \ln Z(V, T). $$

(7.4)

Looking at the final form (5.19) of $Z(V, T)$, we do not see any $x_{ik}$ with respect to which we could differentiate -- in fact, finally $Z$ is just a function of two variables, $V$ and $T$. So, what does (7.4) mean? It means indeed -- as we have emphasized in section 3 -- that the partition function is much more than "a thermodynamic function of $V$ and $T"$: it contains all information about probability distributions.

Eq. (7.4) makes sense if we go back to eq. (5.9)

$$ \ln Z = -\sum_{\alpha \beta} \ln (1 - x_{\alpha \beta}) + \sum_{\alpha \phi} \ln (1 + x_{\alpha \phi}) $$

(7.5)
from which (7.4) gives immediately $x_{ik} = \exp \left( -\sqrt{p_i^2 + m_k^2} / T \right)$, see (5.6)]

$$\bar{v}_{ik} = \frac{x_{ik}}{1 + x_{ik}} = \frac{1}{\exp \left( \sqrt{p_i^2 + m_k^2} / T \right) + 1}$$  \hspace{1cm} (7.6)

(- if $k$ is a boson, + if $k$ is a fermion) and thus with (7.2)

$$\rho_{mn}(\vec{p}, T) d^3 \vec{p} = \frac{\mathcal{Z}_V \, v}{2 \pi^2} \frac{p^2 d\rho}{\exp \left( \sqrt{p_i^2 + m^2} / T \right) + 1}$$  \hspace{1cm} (7.7)

(- bosons, + fermions).

This is the Planck distribution which we guessed in section 4.3, but now $T(E)$ is a calculable function of the local energy density (see section 6.5).

7.2 The general rule for deriving probabilities

The general rule, which was employed in writing down (7.4), is this: if $\{\psi'\}$ is a definite state [remember the discussion in section 5.1: $\psi_i \equiv \{\psi\}_i$], then the probability $W(\psi')$ to find the system in this state is

$$W(\psi') = \frac{\prod_{\alpha \neq \beta} x_{\alpha \beta}^{\psi'} \nu_{\alpha \beta}^{\psi'}}{\mathcal{Z}}.$$  \hspace{1cm} (7.8)

Now let me make a few speculations: if a quantum mechanical state $\psi(T, V)$ could be assigned to our system in a heat bath -- it cannot! -- then $W(\psi') \equiv |\langle \psi' \rangle \phi|^2$ would be true. This remark, while applying to a non-existent situation, is useful so far to make obvious that the trivial consequence of (7.8), namely

$$\sum_{\{v\}} W(\psi) = 1 \hspace{1cm} (7.9)$$
is the analogon to unitarity. Thus, as far as the analogy with quantum theory goes, our thermodynamic formulation is automatically "unitary". Indeed, whatever we calculate — e.g. momentum spectra, eq. (7.7) — depends on the whole set of all channels and all possible reactions (via the relation between T and E coming from the bootstrap-solving partition function) as it should in any unitary theory. Even if we would calculate the relative probabilities of two states \( \{v'\} \) and \( \{v''\} \), namely the branching ratio:

\[
\frac{W(v')}{W(v'')} = \prod_{\alpha_x} \frac{v'_\alpha_x}{v''_\alpha_x} = \exp \left( -\frac{1}{T} \sum_{\alpha_x} \varepsilon_{\alpha_x} (v'_\alpha - v''_\alpha) \right) \tag{7.10}
\]

— even then the virtual presence of all other possible states and their peculiar interrelation by bootstrap is exhibited by the explicit appearance of the temperature T(E) in this ratio. Thus "unitarity" does not only consist in the trivial normalization property of eq. (7.8). Of course, if one would try to pursue this analogy to unitarity further — which might be interesting — then the partition function must first be recalculated under strict observance of all conservation laws. Difficult as this may be, it might be rewarding in view of the long-known and striking analogy between the partition function and the S-matrix on the one hand and inverse temperature 1/T and imaginary time i.t on the other [see any modern book on the many-body problem, where this analogy is exploited; e.g. the elementary book by R. Mills (1969)]. In our language there is a singularity in the complex T-plane at \( T_0 \), from where a branch cut to \( \infty \) starts. This cut is due to the exponential mass spectrum. There is another branch cut from \( T = -\infty \) to 0, which comes from the \( K_2 \)-functions in (5.19). Therefore, our T-plane is cut from \( -\infty \) to 0 and from \( T_0 \) to \( +\infty \); the complex 1/T-plane will be cut from \( -\infty \) to \( 0^- \) and from \( 0^+ \) to \( 1/T_0 \). In the above-mentioned analogy, between S-matrix and partition function, the imaginary axis \( \pm i/T \) corresponds to real time, and to
\( \frac{i}{T_0} \) would correspond a time \( t_0 = (160 \text{ MeV})^{-1} = 10^{-23} \text{ sec} \), which always has been conjectured to be something like a "shortest time". Of course, if one considers the imaginary \( 1/T \) axis as real time, then at \( t_0 \) there is no singularity located on this axis, and one may even go smoothly through zero to negative \( t \), because the branch cuts to the left and to the right of zero have different origins and zero remains a branch point or better: a branch dipole over which one might pass without entering another sheet. After this excursion into science fiction, we return to serious things.

From (7.8) it follows: if \( S \) is any set of states \( S = \{ \{ v' \}, \{ v'' \}, \ldots \} \) then the probability to find the system in any one state \( \{ v \} \in S \), is given by

\[
\mathcal{W}(S) = \frac{i}{Z} \sum_{\{ v \}} \prod_{\alpha \gamma} v_{\alpha \gamma}
\]

(7.11)

where \( \sum_{\{ v \}} \) goes over all \( \{ v \} \in S \).

Clearly, all probability distributions can be calculated by specializing \( S \).

Two examples follow.

7.3 \textbf{Occupation number distribution}

In section 7.2 we just calculated \( v_{ik} \) in order to express by them the particle spectra. We shall derive the same result once more as an illustration to the general rule (7.11).

We start by calculating \( \mathcal{W}(v_{ik}) \), i.e. the probability to find \( v_{ik} \) particles of momentum \( p_i \) and kind \( k \); thus the set \( S \) is:

\[
S = \left\{ \{ v \} : v_{ik} \text{ fixed } ; \text{ all other } v_{\alpha \gamma} \text{ free} \right\}.
\]

(7.12)

Hence the sum in (7.11) goes over all matrices \( \{ v \} \) that have one single element \( v_{ik} \) fixed, while all other \( v_{\alpha \gamma} \) run through all possible values. We remember that sum and product can be interchanged [see (5.7)].
\[
\sum_{\{v\}} \prod_{\alpha S} x_{\alpha S}^{v_{\alpha S}} = \left( \sum_{\nu_{11}} x_{\nu_{11}} \right) \left( \sum_{\nu_{12}} x_{\nu_{12}} \right) \cdots \left( \sum_{\nu_{iR}} x_{\nu_{iR}} \right) \cdots
\]  
(7.13)

Thus the prescription to sum over all \(\nu_{\alpha S}\) with the exception of \(\nu_{ik}\) can be simply obeyed, dividing (7.13) by \(\sum_{\nu} x_{\nu_{ik}}\) and multiplying by \(x_{\nu_{ik}}\):

\[
\sum_{\{v\}} \prod_{\alpha S} x_{\alpha S}^{v_{\alpha S}} = \mathcal{Z} \cdot \frac{x_{\nu_{ik}}^{\nu_{ik}}}{\sum_{\nu} x_{\nu_{ik}}}
\]  
(7.14)

Thus, with (7.11) and \(\sum_{\nu} x_{\nu_{ik}} = (1 - x_{\nu_{ik}})^{-1}\) or \((1 + x_{\nu_{ik}})\) for \(k = \text{boson or fermion}\) respectively:

\[
\overline{W}(\nu_{ik}) = x_{\nu_{ik}}^{\nu_{ik}} \begin{cases} 
1 - x_{\nu_{ik}} & \text{(bosons)} \\
1 & \text{(fermions)}
\end{cases}
\]  
(7.15)

(= probability to find \(\nu_{ik}\) particles of kind \(k\) with momentum \(p_i\)).

Before we proceed, let us remark that this is much more than \(\nu_{ik}\): it is the whole distribution of which \(\nu_{ik}\) is only the mean value. And another important remark: \(x_{\nu_{ik}} \equiv \exp \left[ -\left( \frac{1}{T} \sqrt{p_i^2 + m_k^2} \right) \right] \) is, for most hadrons, \(<< 1\), because \(T < T_0 = m_\pi\). Now, with \(x_{\nu_{ik}} \ll 1\), eq. (7.15) tells us:

\[
\overline{W}(\nu_{ik}) \approx x_{\nu_{ik}} \approx \left[ \overline{W}(\nu_{ik} = 1) \right]^{\nu_{ik}}
\]  
(7.16)

and then

\[
\overline{\nu_{ik}} = \sum \nu_{ik} x_{\nu_{ik}} \nu_{ik} \approx x_{\nu_{ik}} \approx \overline{W}(\nu_{ik} = 1)
\]
In words: for \( m >> T \), there is no difference between bosons and fermions; the probability to find one particle \( k \) with momentum \( \vec{p}_k \) is very small \( (x_{ik}) \); the probability to find \( n \) such particles is the \( n^{th} \) power of that for finding one; the average number equals the probability to find just one. This approximation corresponds exactly to neglecting the \( \pm 1 \) in eq. (7.7).

[Of course, in a volume \( V \) there are \( (V/2\pi^2)p^2dp \) different \( \vec{p}_k \) near \( \vec{p} \) and \( (V/2\pi^2) \exp \left[ -\sqrt{p^2 + m^2}/T \right] p^2dp \) increases proportional to \( V \) as it should; it does so because with increasing volume the levels become denser, while the probability to find a given level occupied remains constant and small.]

The result (7.16) is significant, as it shows already now that if particles must be created in pairs, e.g. nucleon –antinucleon, then the probability to see one created antinucleon -- which equals the probability to see one created pair -- is proportional not to \( x_{ik} \) but to \( x_{ik}^2 = \exp \left[ -(2/T) \sqrt{p^2 + m^2} \right] \approx \exp (-2m/T) \). This is very important for heavy pair creation, where the factor 2 in the exponent may be deadly for counting rates (quarks!). We will come back to this in Section 9.

Now we proceed to calculate \( \overline{\nu}_{ik} \). From (7.15)

\[
\overline{\nu}_{ik} = \sum_{\nu} \nu_{ik} \mathcal{W}(\nu_{ik}) = \left\{ \begin{array}{l}
\left( 1 - x_{ik} \right) \sum_{\nu} \nu x_{ik} \nu \\
\frac{i}{1 + x_{ik}} \sum_{\nu} \nu x_{ik} \nu
\end{array} \right.
\]

(7.17)

\[
\sum_{\nu} \nu x_{ik} \nu = x_{ik} \frac{d}{dx_{ik}} \sum_{\nu} x_{ik} \nu = \frac{x_{ik}}{(1 - x_{ik})^2}
\]

(7.18)

\[
\sum_{\nu} \nu x_{ik} \nu = x_{ik}.
\]

(7.19)
This, inserted into (7.17), gives

\[ \frac{1}{V_{ik}} = \frac{X_{ik}}{1 \mp X_{ik}} (\pm \text{bosons}) \]

\[ \pm n \text{ fermions} \]  \hspace{1cm} (7.20)

which is, of course, identical with our previous result (7.6).

7.4 Multiplicity distribution of fireballs in a fireball

A fireball consists of fireballs, which consist of fireballs which ...

-- if that is so (and this is the whole essence of our approach), then
we cannot calculate the expected total number of observed metastable
and stable particles (\(\pi, K, p, n, \Lambda, \Sigma \ldots\)) in the final state without
following up the whole decay cascade:

```
fireball  =>  pions
fireball  =>  kaons
fireball  =>  nucleons
fireball  =>  hypercns
```

**0^{th} generation**  \hspace{1cm} **1^{st} generation** \ldots observed end products

where each fireball in any generation is the beginning of a new cascade
of this same structure, until we reach the resonance region, where the
thermodynamic description fails and must be replaced by an individual
analysis of each single decay. Each decay in each generation will,
of course, yield \(\ldots\) among the daughter-fireballs (or are they sons?), also
pions, kaons, etc., and the final numbers will be the sum of all these
contributions. It should be said that the calculation of these final
multiplicities has never been done, but is worth trying (Monte Carlo?);
for that one would need what follows.

We shall treat here only the first step of this problem: how many
fireballs, without regard to their nature (heavy, light, boson, fermion,
any charge), are present in a fireball and with what probability? In
other words: what is the probability that a fireball decays into
n daughter-fireballs in the first generation? Once more in other words: what is the probability \( W(n) \) to find the system (i.e. the fireballs) in any one of all those states that are characterized by \( \sum_{\alpha \gamma} v_{\alpha \gamma} = n \)? Thus, the set \( S \) contains all matrices \( \{v\} \) with the same sum \( n \) of all their elements:

\[
S_n = \left\{ \{v\} : \sum_{\alpha \gamma} v_{\alpha \gamma} = n \right\}
\]

(7.21)

and the rule (7.11) gives

\[
\bar{W}(n) = \frac{1}{Z} \sum_{\{v\}} (S_n) \prod_{\alpha \gamma} x_{\alpha \gamma}^{v_{\alpha \gamma}}.
\]

(7.22)

Hence all products entering into this sum are homogeneous of order \( n = \sum v_{\alpha \gamma} \) in the variables \( x_{\alpha \gamma} \), and all such \( n \)-products are indeed present. Thus, the problem of finding \( W(n) \) is to find the partial partition function of all \( n \)-particle states

\[
Z'(n) = \sum_{\{v\}} (S_n) \prod_{\alpha \gamma} x_{\alpha \gamma}^{v_{\alpha \gamma}}.
\]

(7.23)

Clearly, \( Z = \sum Z'(n) \) because \( \sum W(n) = 1 \).

We find this partial partition function by a well-known trick of statistical mechanics: we introduce the auxiliary variable \( z \), by which we multiply each \( x_{\alpha \gamma} \) and define a generalized partition function

\[
Z(Z, V, T, z) = \sum_{\{v\}} \prod_{\alpha \gamma} (z x_{\alpha \gamma})^{v_{\alpha \gamma}}.
\]

(7.24)
Obviously \( Z(V, T, z = 1) = Z(V, T) \), and equally obvious is it that a factor \( z^n \) can be factorized out of all products that are homogeneous of order \( n \) in \( x_{\alpha \gamma} \); thus

\[
Z^{(n)}(V, T, z) = z^n Z^{(n)}(V, T, z = 1)
\]

\[
Z(V, T, z) = \sum_{n=0}^{\infty} z^n \left[ \sum_{\nu} \left( \frac{S_{\nu}}{\nu} \right) \frac{\nu^{\alpha}}{\nu^{\beta}} \right]
\]

\[
= Z(V, T) \cdot \sum_{n=0}^{\infty} z^n W^{(n)}(\nu).
\]

(7.25)

Hence our wanted \( \sum^{(S_{\nu})} \) is the coefficient of \( z^n \) in the power expansion of \( Z(V, T, z) \) and could be calculated by

\[
\sum^{(S_{\nu})} = Z(V, T) W^{(n)} = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} Z(V, T, z).
\]

(7.26)

To see what the structure of \( Z(V, T, z) \) is, we go to eq. (5.13) which becomes with \( x_{pm} \) replaced by \( (zx_{pm}) \)

\[
\ln Z(V, T, z) = \frac{V}{2\lambda^2} \sum_{n} \frac{z^n}{n!} \int_{0}^{\infty} \rho(w; \nu) \exp \left\{ \frac{\nu}{\lambda^2} \int_{0}^{\infty} \rho(w; \nu) K_2 \left( \frac{\nu w}{T} \right) d\nu \right\}
\]

leading after momentum integration to the analogon of eq. (5.19), namely

\[
Z(V, T, z) = \exp \left\{ \frac{V T}{2\lambda^2} \sum_{n} \frac{z^n}{n!} \int_{0}^{\infty} \rho(w; \nu) \nu^2 K_2 \left( \frac{\nu w}{T} \right) d\nu \right\}.
\]

(7.27)

(7.28)

It is difficult to extract from this the coefficient of \( z^n \), because of the exponential function.
We obtain help from our knowledge about the numerical behaviour of the sum in the exponent: with the experimental mass spectrum for $m \leq 1$ GeV one sees that for $m \geq 1$ GeV, all terms with $n \geq 2$ of the sum can be neglected altogether and that the first term ($n = 1$) yields 98% of the total sum at $T = 120$ MeV. At lower $T$ the rest of the sum can be neglected with even better reason: the $K_2$-functions go like $[\exp(-m/T)]^n$ and for $T \to T_0$ the first term diverges -- at least if the "weak bootstrap" solution $\rho(m) \to C\ m^{-\frac{3}{2}}\ \exp(m/T_0)$ is adopted. For this solution the first term is the only numerically important one; for the "strong bootstrap" solution $\rho(m) \to C\ m^{-3}\ \exp(m/T_0)$ the first term does not diverge, but is still the most important one. Thus

$$Z(v_1T, z) \approx \exp \left\{ z \left( \frac{v_1}{2k^2} \right) \int_0^\infty f(u) u^2 K_2 \left( \frac{u}{T} \right) du \right\}. \quad (7.29)$$

To the same approximation the expression multiplying $z$ in the exponent is $\ln Z(V, T, z = 1) = \ln Z(V, T)$. Thus, neglecting all but the first term:

$$Z(v_1T, z) \approx \exp \left\{ z \ln Z(v_1T) \right\} = \sum_{n=0}^{\infty} \frac{z^n (\ln Z)^n}{n!}. \quad (7.30)$$

So that finally -- see (7.25) --

$$W(n) \approx \left[ \frac{\ln Z(v_1T)}{n!} \right]^n e^{-\ln Z(v_1T)}. \quad (7.31)$$

This is a Poisson distribution with mean value

$$N \equiv \bar{n} = \sum_{n} n W(n) \approx \ln Z(v_1T). \quad (7.32)$$
8. PHYSICAL PROPERTIES OF A FIREBALL

8.1 Mean particle number, pressure, mass distribution

decay scheme, multiplicities

Remember: our $N$, which was derived in the last section, is
the mean number of fireballs in a given fireball of volume $V$ and tempera-
ture $T$. As $\ln Z$ is proportional to $V$, the relation (3.25) implies

$$ P = T \frac{\partial}{\partial V} \ln Z = \frac{T}{V} \ln Z \approx \frac{T}{V} N $$

thus

$$ PV \approx NT. \tag{8.1} $$

This is formally the equation of state of an ideal gas of $N$ particles;
no surprise, as we have calculated the partition function of a gas of
formally non-interacting particles. Strong interactions are hidden in $N$,
which is itself a function of $T$ and $V$.

Since the result of (7.32) is important, we shall derive it in three
other illustrative ways.

1. $N$ is the mean number of particles, hence $N = \sum_{\alpha \gamma} v_{\alpha \gamma}$. With eq. (7.4),
   we obtain

$$ N = \sum_{\alpha \gamma} x_{\alpha \gamma} \frac{\partial}{\partial x_{\alpha \gamma}} \ln Z(v_i, T). \tag{8.2} $$

If now $\ln Z$ would be a homogeneous function of order one of
the $x_{\alpha \gamma}$, then Euler's theorem would tell us immediately $N = \ln Z$.
But as $\ln Z$ contains terms of higher order, this is only approxi-
mately true [see, e.g. (7.27); the higher terms are those of the sum
over $n$].
2. Eq. (7.25) says

\[ Z(v, T, z) = Z(v_T) \sum_{n=0}^{\infty} z^n W(n) \]  

(8.3)

so that

\[ N = \overline{n} = \sum_n n W(n) = \left. \frac{d}{dz} \ln Z(v_T, z) \right|_{z=1} \]  

(8.4)

This applied to eq. (7.28) gives

\[ N(v, T) = \frac{vT}{2\pi^2} \sum_{n} \frac{1}{n} \int_{0}^{\infty} \frac{1}{n} \int_{0}^{\infty} p w K_2 \left( \frac{w}{vT} \right) dw. \]  

(8.5)

This exact relation differs from \( \ln Z \), by having only a factor \( 1/n \) instead of \( 1/n^2 \) in each term. Thus, still exact:

\[ N(v, T) = \ln Z(v, T) + \]

\[ + \frac{vT}{2\pi^2} \sum_{n} \left( \frac{1}{n} - \frac{1}{n^2} \right) \int_{0}^{\infty} p w K_2 \left( \frac{w}{vT} \right) dw. \]  

(8.6)

The first term \( n = 1 \) of the correction vanishes; if the sum converges rapidly or if \( \ln Z \) diverges, the whole correction is negligible and \( N = \ln Z \).

3. If, in the particle spectra, we neglect the \( \pm 1 \), then

\[ f_w(p^2, T) \, dp \sim \frac{v_{2w}}{2\pi^2} \exp \left( -\frac{\sqrt{p^2 + w^2}}{T} \right) p^2 \, dp. \]  

(8.7)
and the total number of particles with discrete mass \( m \) becomes with the help of (5.16)

\[
\nu(T) = \int \rho_{m} (\rho, T) d^3 \rho = \frac{V_{2} m}{2 \pi^2} m^2 T K_2 \left( \frac{m}{T} \right)
\]

(8.8)

or, for \( m \gg T \)

\[
\nu(T) \approx V_m M_0 \left( \frac{M_0 T}{2 \pi} \right)^{3/2} e^{-\frac{m}{T}}
\]

(8.9)

which can be used as a rough rule to estimate production rates of particles of mass \( m \), whose creation is not limited by conservation laws. We see here:

- production rates decrease exponentially with the mass \( m \) to be created (Boltzmann factor corresponding to the jump over an energy difference \( m \) between non-existence and existence).
- If the process requires pair- or associated production, \( m = m_1 + m_2 \).

The last remark is clear from the fact that a pair with mass \( m_1 + m_2 \) might be considered as kinematical entity of mass \( m = m_1 + m_2 \), created without restriction to conservation laws and then dissociating. We shall, however, prove that this is so and derive the formula for pair creation, which essentially is the square of (8.9). Note that this formula \( \nu(m) \sim \exp (-m/T) \) has two important consequences for the application of the theory to real processes (collisions):

- if particles are created \( \sim \exp (-m/T) \), then even for the lightest ones, pions, this rate is very small for all \( T \ll T_0 \) and it raises rapidly when \( T \rightarrow T_0 \) or \( T \rightarrow \) maximum temperature < \( T_1 \) allowed in the given collision. This is the basis of our Postulate 2 in section 2.2, where we assumed adiabatic heating.
The mass spectrum rises like $\exp\left(\frac{m}{T_0}\right)$ and the rate of production of a mass $m$ decreases like $\exp\left(-\frac{m}{T}\right)$, its maximum production cross-section for $E \to \infty$ is bounded by $\exp\left(-\frac{m}{T_0}\right)$. Therefore, even at very high energies, there is little hope to push the experimental mass spectrum much beyond what Fig. 7 shows: resolution difficulties increase proportionally to $\exp\left(\frac{m}{T_0}\right)$, while counting rates for a given resonance go down like $\exp\left(-\frac{m}{T}\right)$.

We continue to calculate $N(V, T)$ by defining the number

$$N(w, T)dw = N(w, T) \int f(w)dw$$

where now — by definition of $\rho(m)$ — the multiplicity factor $z_m$ is absorbed into $\rho(m)$. Thus, with (8.8)

$$N(V, T) = \int_0^\infty N(w, T)dw = \frac{VT}{2\xi^2} \int_0^\infty \rho(m) m^2 K_2\left(\frac{m}{T}\right)dw$$

which is, as expected, the first term ($n = 1$) of the sum (5.19), representing $\ln Z(V, T)$. Here we see that taking this first term only is just equivalent to neglecting the $\pm 1$ in the spectra. Hence, again

$$N(V, T) \approx \ln Z(V, T).$$

Whether we take the approximate $N \approx \ln Z$ or the exact expression (8.6), one important observation applies to both:
\[ N(V,T) \text{ diverges for } T \to T_0 \text{ when the "weak bootstrap" solution } \rho = c m^{-\frac{\gamma}{2}} \exp \left( \frac{m}{T_0} \right) \text{ is chosen; it converges (like ln } Z \text{) to a constant value if the "strong bootstrap condition" } [m^{3/2} \rho \approx \sigma], \rho \sim c m^{-3} \exp \left( \frac{m}{T_0} \right) \text{ is taken.} \]

Specializing \( V = V_0 \) and idealizing to a "central" collision, where all c.m. energy is transformed into heat and no collective motions remain, we obtain an upper bound for the \textit{first generation} (not final!) multiplicity [see (6.21); insert there (6.28)]

\[
\frac{\alpha_0 \ln \frac{T_0}{T_0 - 1}}{1} \approx \alpha_0 \ln \left( \frac{E}{E_{\text{GeV}}} \right) \approx 5 \ln \left( \frac{E}{1 \text{ GeV}} \right)
\]

for \( f(u) \to c m^{-3/2} \exp \left( \frac{u}{T_0} \right) \)

\[
\text{const} \quad \text{for } f(u) \to c u^{-3} \exp \left( \frac{u}{T_0} \right)
\]

(8.13)

For \( p\bar{p} \) annihilation at rest the first gives \( \approx 4 \) which is not bad (further decay of first generation resonances increases it to 5-6). But (8.13) cannot be used to eliminate the second possibility, because even if \( N \to \text{const}, \) the observed multiplicity at the end of the decay cascade may still grow with the energy. Frautschi (1970) points out that for his solution \( \rho(m) \sim c m^a \exp \left( \frac{m}{T_0} \right) \) with \( a < -5/2 \) low dimensional phase space is favoured; in fact \( N(V_0, T) \to \approx 2 \) and heavy resonances would decay preferentially like this:
(where $\pi$ might be also a light resonance or a $K$, etc.). Here, the increase with energy of total observed multiplicity would be due to the length of the decay chain; and if each emitted particle carries off a mean energy $E = m + 3/2 \ T$ (with $T$ of order $T_0$), then a fireball of mass $m_1$ decays like $m_1 \rightarrow (m_\pi + 3/2 \ T) + (m_{i+1} + 3/2 \ T) = m_\pi + 3T + m_{i+1}$. Summing up, one expects $m \rightarrow n(m_\pi + 3T)$. As $T = m_\pi$, one sees that the mean observed multiplicity $n$ coming from the decay of a fireball of mass $m$ is of order $m/4m_\pi$. In $p\bar{p}$ annihilation $m = 2m_p = 13.5 \ m_\pi$, so that the mean multiplicity of observed particles (mainly pions) would be about 3.5 -- the same as in the other case! [See eq. (8.13).]

On the other hand: for the solution $\rho(m) = c \ m^{-\frac{5}{2}} \exp (m/T_0)$ the number $N$ of fireballs in a fireball diverges (only logarithmically), which means that low-dimensional phase space is not favoured as in the other case. Thus, in the $m^{-\frac{5}{2}}$ case the picture is like the one drawn at the beginning of section 7.4. In this case, however, the logarithmic behaviour is really asymptotic and a decay scheme based on it is extremely approximate. In addition to that, a calculation of the whole decay chain would involve an integration (in each generation) over the mass distribution of the parent-fireballs, while in the case of the $m^{-3}$-solution, the mass of the parent-fireball is $m_i = m_{i-1} - 4m_\pi$.

Finally, it is worth pointing to another related problem, the limited (transverse) momenta. In the case of the $m^{-3}$-solution, each particle emitted from a fireball starts with a Planck spectrum (eq. 7.7) and leaves the parent fireball also with a Planck spectrum, but, as its mass is large, with negligible recoil velocity; the many decays will even tend to average out the randomly oriented recoils, so that nearly all emitted light particles (except the last two, three) have a Planck spectrum in the rest frame of the first fireball. Differently in the $m^{-\frac{5}{2}}$ solution, where the first fireball decays into $N$ radially diverging daughters, which decay the same way. Here each fireball of each generation decays in another -- his private -- rest frame and all these are in moderately relativistic motion with respect to each other. What the momentum spectrum of the last (observed) generation will be after this decay chain, is unknown.
But it is a well-defined, soluble problem [see Appendix IV of (II) for a first step]. Of course, all these considerations apply to the multiplicities of a decaying fireball—the multiplicities of a collision are still much more difficult to obtain, because the velocity weight functions $F(\lambda)$ and $F_0(\lambda)$ come in. Some simple guesses, based on the assumption that the "local" fireballs whose longitudinal motion is described by $F(\lambda)$, $F_0(\lambda)$ are never very heavy and thus decay in one, at most two, steps, have been presented in a paper "Remarks on the thermodynamical model" (Hagedorn 1970 b). We shall not go into the question of multiplicities further, except for one warning:

while no model of particle production should violently contradict the so far observed smooth increase of multiplicities with energy, the present experimental situation is not clear enough to allow a distinction between multiplicities which rise like a small power ($\sim E^{0.5}_{cm}$) or logarithmically with energy or even tend to a constant!

The reader should convince himself that $AE^{0.5} + B$ and $C \ln E/E_0$ can, by suitable choice of the constants, be made nearly equal over enormously large energy intervals! If anybody's theory "explains" the observed multiplicities, ask: so what?
8.2 Average mass and energy per particle

The \( N \) particles in a fireball cannot have kinetic energies much larger than \( 3/2 \; T \). If the average mass of them is \( \overline{m} \), then we expect that the total energy \( E \) of a fireball is just \( E \approx N(\overline{m} + 3/2 \; T) \). Since \( E \) diverges (for both bootstrap solutions) much more strongly than \( N \) (which in the \( m^{-3} \) solution tends even to a constant), it is clear that \( \overline{m} \) must roughly diverge like \( E \) itself. Thus, the fireballs inside the fireball must become rapidly heavier.

We show this now. The energy per particle is

\[
\mathcal{E}_1 = \frac{\int \rho^2 \mu^2 \left[ \frac{P}{\mu T} \right] \rho(\mu) \, d\mu \, d^3 \rho}{\int \rho(\mu) \, d\mu \, d^3 \rho} = \frac{E}{N}.
\tag{8.14}
\]

Using the Planck spectrum (7.7) with \( \pm \) neglected (which is, for this integration over \( m \) and \( \overline{p} \) an excellent approximation), one sees that (8.14) can be written as

\[
\mathcal{E}_1 = -\frac{d}{d \overline{\mu}} \ln \int \rho(\mu) \, d\mu \, d^3 \rho.
\tag{8.15}
\]

Now, in (8.8), we have defined the mean number of particles with mass \( m \):

\[
\nu(\mu, T) = \int \rho(\mu) \, d^3 \rho = \frac{V^2 m}{2\pi^2} \mu^2 T K_2(\mu/T)
\]

for \( \mu >> T \) : \( \approx \frac{V^2 m}{2\pi^2} \frac{(\mu T)^{3/2}}{\mu T} e^{-\mu/T} \),

\tag{8.16}

so that, absorbing \( z_m \) into \( \rho(m) \), we have

\[
\mathcal{E}_1 = -\frac{d}{d \overline{\mu}} \ln \frac{V T}{2\pi^2} \int \nu(\mu) \mu^2 K_2(\mu/T) \, d\mu
\]

\[= -\frac{d}{d \overline{\mu}} \ln \int \nu(\mu, T) \rho(\mu) \, d\mu.\]

\tag{8.17}
Under the logarithm we see again the first term \((n = 1)\) of \(\ln Z(V, T)\) [see (5.19)], thus, with (3.25) we obtain an apparently catastrophic result:

\[
\varepsilon_1 \approx -\frac{d}{T} \ln Z(V, T) = \frac{T^2}{\theta T} \ln Z(V, T) = E. \tag{8.18}
\]

Within this approximation the energy per particle equals the total energy! This is less of a disaster as it would seem, since the energy diverges fast and the number of particles only logarithmically \(m^{-\frac{3}{2}}\) solution or not at all \(m^{-3}\) solution. We can still do a little better by carrying out the differentiation in (8.17), using (8.11) and (8.16)

\[
\varepsilon_1 = \frac{1}{N} \left\{ -\frac{d}{d_\mu} \int \nu(\mu, T) \rho(\mu) d\mu \right\}. \tag{8.19}
\]

If \(\nu(m, T)\) is written in the approximate form (8.16) for \(m >> T\), then the \(T\)-differentiation gives

\[
-\frac{d}{d_\mu} \nu(\mu, T) = \left( \frac{3}{2} \frac{T}{T} + \mu \right) \nu(\mu, T) \tag{8.20}
\]

so that

\[
\varepsilon_1 = \frac{1}{N} \int \left( \frac{3}{2} \frac{T}{T} + \mu \right) \nu(\mu, T) \rho(\mu) d\mu = \frac{3}{2} \frac{T}{T} + \overline{\mu}(T), \tag{8.21}
\]

where \(\overline{m}(T)\) is defined by this equation. More exactly:

\[
\overline{\mu}(T) = \frac{\int \mu \nu(\mu, T) \rho(\mu) d\mu d^3p}{\int \nu(\mu, T) \rho(\mu) d\mu d^3p}. \tag{8.22}
\]
On the other hand, (8.21) shows the obvious result (not catastrophic, the factor 1/N makes it right):

\[ \mathcal{E}_1 = \frac{3}{2} T + \bar{m}(T) = \frac{E}{N} \]  \hspace{1cm} (8.23)

so that, to the approximation \( m \gg T \) \( [\pm 1 \text{ neglected and asymptotic form of } K_2(m/T) \text{ -- both good in integrals over all } m \text{ for } T \to T_0] \)

\[ \bar{m}(T) = \frac{E}{N} - \frac{3}{2} T. \]  \hspace{1cm} (8.24)

In (8.24) one sees how, in this thermodynamics, it is managed to match a constant temperature \( T \to T_0 \) and a constant or almost constant particle number \( N \) with diverging energy: the average mass \( \bar{m}(T) \) diverges in just the right way to achieve the balance.

8.3 How could it happen that \( \bar{m}(T) \to \infty \)?

If, at given \( T \), one picks up at random one of the particles inside a fireball, then the probability that its mass is \( m \) is given by (8.9) to be \( \propto \exp(-m/T) \). When \( T \to T_0 \), this tends to a constant; to a terribly small constant when \( m \) is large. How, then, can it happen, that if at \( T \to T_0 \) I pick up particles, I will find heavier and heavier ones with average value \( \bar{m}(T) \to \infty \)? It is the exponential mass spectrum \( \rho(m) \), which produces the miracle: as long as we ask for a definite, well-defined particle with mass \( m \), the probability to see it is \( \propto \exp(-m/T) \); what we really do is, however, looking for a mass lying in an interval \( (m, dm) \); and since there is not just one, but \( \rho(m)dm \) of them available, the probability to find a particle in \( (m, dm) \) is \( \propto (m, T) \rho(m) dm \), and \( \bar{m} \) is the average mass with respect to this distribution which, for \( T \to T_0 \) and \( m \to \infty \), decreases very slowly and yields \( \bar{m}(T) \to \infty \) when \( T \to T_0 \). For the same reason, \( N(V, T) \) diverges (in the \( m^{-3/2} \) solution) for \( T \to T_0 \): although the production rate of a given mass does hardly increase, there are so
many mass states that the infinite number of infinitely small increments makes the integral \( \int \sqrt{v(m,T)} \rho(m) dm \) diverge.

Suppose all water molecules on earth were labelled with numbers 1, 2, 3 ... If I pour out a glass of water, then the chance \( P \) to find molecule No. 123336 in it is extremely small \((\approx 10^{-45})\). But equally small is the probability for any of the \( \approx 10^{23} \) other ones which I have in my glass and the probability for having just those in my glass which I actually do have, is \((10^{23})! \approx (10^{-22})^{10^{23}} \). One might not believe that one can get a glass of water. But one can. And for similar reasons \( N \) and \( \bar{m} \) do diverge; can fireball structures be observed in high energy collisions and will it yet be impossible to extend the region, where we know the hadronic mass spectrum particle by particle, to much higher masses.

9. **CONSERVATION LAWS**

All distributions, calculated from our present \( \ln Z(V,T) \), are always without regard to what else happens. Thus we can calculate the probability for antiproton production --- wrongly, because it does matter what else happens: there must be a nucleon (maybe excited) created with the antiproton. We never told our partition function to watch that this be so, and so it does not watch. We may repair this at the end by saying: I know what \( Z \) does not know; thus I require production of an antiproton together with a nucleon and expect the probability to observe an antiproton to be equal to the probability to create an antiproton *times* the probability to create a nucleon (both calculated from the ignorant \( Z \)). In other words: all probabilities calculated from our \( Z \) are unconditional probabilities \( W \), and we expect that if events \( A \) and \( B \) must occur together, one can calculate the conditional probability \( W_c \) as follows:

\[
W_c(A \& B) = W_c(A) = W_c(B) \approx W(A) \cdot W(B)
\]

(9.1)
while, with $\bar{A} \equiv \text{non-} A \ (\text{not anti-} A) \\

W_c (A \& B) = W_c (\bar{A} \& \bar{B}) = 0 . \quad \text{(9.2)}

This is what we expect from the discussion in section 7.3, following Eq. (7.16). We shall now show that this is true, provided $W(A), W(B) \ll 1$. 

The way to the proof is this: instead of leaving the partition function ignorant, we can tell it that $A$ must never occur unless also $B$. In other words: in the sum over states we include only those states which explicitly obey the conservation laws.

9.1 Pair- and associated production

We start with (5.7)

$$Z (v_{\bar{T}}) = \sum_{\{v\}} \prod_{\alpha \delta} X_{\alpha \delta}^{v_{\alpha \delta}} . \quad \text{(9.3)}$$

Consider a conservation law such that a particle of kind "a" can only appear together with a particle of kind "b" (e.g. $a = \text{nucleon}, b = \text{anti-nucleon}$). Then, by singling out $a$ and $b$, we write

$$Z (v_{\bar{T}}) = \sum_{\{v\}} \left[ \prod_{\alpha a} X_{\alpha a}^{v_{\alpha a}} \right] \cdot \left[ \prod_{\alpha b} X_{\alpha b}^{v_{\alpha b}} \right] \cdot \left[ \prod_{\alpha \delta} X_{\alpha \delta}^{v_{\alpha \delta}} \right] \quad \text{(9.4)}$$

rearranged

$$Z = \left[ \sum_{\{v\}} \prod_{\alpha a} X_{\alpha a}^{v_{\alpha a}} \right] \cdot \left[ \sum_{\{v\}} \prod_{\alpha b} X_{\alpha b}^{v_{\alpha b}} \right] \cdot Z' . \quad \text{(9.5)}$$
The prime indicates that particles a and b are not counted; \( Z' \) is what the partition function would become, if particles a and b did not exist in nature [the first two factors of (9.5) would then be equal to one]; clearly \( Z' = Z \), because removing two sorts of particles from an infinite spectrum does not matter -- except for all questions concerning just these two particles.

The sums over \( \{ \nu \}_{a} \) and \( \{ \nu \}_{b} \) in (9.5) go over all allowed \( 2 \times \infty \)
occupation matrices for particles a and b; for instance:

\[
\{ \nu \}_{a}^{(i)} = \begin{pmatrix} 1 & 0 & 7 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 0 & \ldots \end{pmatrix}
\]

\[
\{ \nu \}_{b}^{(i)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 3 & 1 & 1 & 0 & \ldots \end{pmatrix}
\]  

We must assume now, for the further argument, that \( m_{a} \) and \( m_{b} \) both \( \gg T_{0} \); otherwise our result will not be true. Fortunately, this condition is not badly fulfilled for \( K \bar{K} \), rather well for \( NN \) and well for heavier masses.

If \( m_{a} \) and \( m_{b} \) \( \gg T_{0} \), then \( x_{a}^{a} \ll 1 \) and \( x_{a}^{b} \ll 1 \) for all \( a \). Thus only those states contribute to (9.5), in which the occupation numbers \( \nu_{a}^{a} \) and \( \nu_{a}^{b} \) are the smallest possible. Indeed, the first term can be written

\[
\left[ \sum_{\{ \nu \}_{a}^{a}} \prod_{a} x_{a}^{a} \nu_{a}^{a} \right] = 1 + \sum_{a}^{(1)} + \sum_{a}^{(2)} + \ldots \]  

where the 1 corresponds to "no particle a present", \( \sum_{a}^{(1)} \) is the sum over all one-a-particle states, \( \sum_{a}^{(2)} \) counts all two-a-particles states, etc. The same for the sum concerning particle b. As the \( \nu_{a}^{(n)} \) are homogeneous of order \( n \) in \( x_{a}^{a} \), they very rapidly decrease with increasing order \( n \), and one need not go to a higher order than necessary to answer the question and to obey a given conservation law. If, for instance, somebody asks me: how probable is it to find five \(^{3}\) particles of sort a (no conservation law)?

\(^{3}\) or more
then I must consider $\sum^{(5)}$, but $\sum^{(6)}$ and all higher ones are only an insignificant correction. And if the question does not concern particles a or b at all -- e.g. if the spectrum of some particle c is to be calculated, or the total pressure or what not -- then we can be content with $\sum^{(0)} = 1$ and put $Z = Z'$, as if particles a and b did not exist in nature.

If, however, we ask: how probable is it to see one particle of kind a, then it matters enormously, whether or not particle b is tied up with a by conservation laws. We discuss both cases:

i) No conservation law

We need not worry about particle b, which may be included in $Z'$. The sum $\sum_{a}^{(1)}$ goes over all $1 \times \infty$ occupation matrices contributing to $\sum_{a}^{(1)}$; that is, containing one "1" and zeros elsewhere:

$$
\sum_{a}^{(1)} = \sum_{a} x_{a}.
$$

The probability to see exactly one particle a is, according to the general rule (7.11), -- where now S is the set of all states containing one particle a:

$$
W(a) = \frac{Z' \cdot \sum_{a} x_{a}}{Z' \left( 1 + \sum_{a} x_{a} + \cdots \right)} \approx \sum_{a} x_{a} \approx \int f_{m_{a}} \left( \hat{p}, \hat{T} \right) d^{3}p
$$

-- a familiar result.

If, in the numerator, we had written $Z' \left[ \sum_{a} x_{a} + \sum^{(2)} + \cdots \right]$, we had obtained the probability not to see exactly one but at least one particle a; however, since $\sum^{(1)} \gg \sum^{(2)} \gg \sum^{(3)} \cdots$, the result is numerically the same.
ii) Conservation law requiring \((a-b)\) pair creation

Here the partition function -- no longer ignorant of the conservation law -- contains only such states in which the number of particles \(a\) and \(b\) are equal: \(n_a = n_b\), but otherwise unrestricted.

Thus, if we decompose in \((9.5)\) the two factors corresponding to \(a\) and \(b\) again into homogeneous terms

\[
\sum_{\{\nu\}_a} \prod_{\nu_a} \chi^{\nu_a} = 1 + \sum^{(1)}_a + \sum^{(2)}_a + \ldots
\]

\[
\sum_{\{\nu\}_b} \prod_{\nu_b} \chi^{\nu_b} = 1 + \sum^{(1)}_b + \sum^{(2)}_b + \ldots
\]

then \((9.5)\) gives

\[
\mathcal{Z} = \left(1 + \sum^{(1)}_a + \sum^{(2)}_a + \ldots\right) \left(1 + \sum^{(1)}_b + \sum^{(2)}_b + \ldots\right) \cdot \mathcal{Z}'
\]

but now the conservation law requires that we eliminate all states in which \(n_a \neq n_b\); thus

\[
\mathcal{Z}_{\mathcal{C}}(\nu_{a,b}) = \left(1 + \sum^{(1)}_a \sum^{(1)}_b + \sum^{(2)}_a \sum^{(2)}_b + \ldots\right) \cdot \mathcal{Z}'
\]

since all cross-terms \(\sum^{(i)}_a \sum^{(k)}_b\) with \(i \neq k\) must be omitted.

Hence, the probability to see one particle \(a\) is now, according to the general rule \((7.11)\),

\[
\mathcal{W}_{\mathcal{C}}(a) = \frac{\mathcal{Z}' \cdot \sum^{(1)}_a \sum^{(1)}_b}{\mathcal{Z}' \cdot \left(1 + \sum^{(1)}_a \sum^{(1)}_b + \ldots\right)} \approx \sum^{(1)}_a \cdot \sum^{(1)}_b
\]
Since for the unrestricted production (9.9) we had \( \mathcal{W}(a) = \mathcal{Z}_a^{(1)} \), the result is, as anticipated in (9.1)

\[
\mathcal{W}_c(a) = \mathcal{W}_c(b) = \mathcal{W}(a).\mathcal{W}(b).
\]  

(9.14)

Also (9.2) is proved, since states with \( a \) and \( \bar{b} \) do not occur in \( Z_c(V,T) \).

In fact, (9.1) and (9.2) are somewhat more general than (9.13), because \( A \) can mean "\( a \) in momentum state \( \alpha \)" or "\( a \) or any of its excited states \( a^* \)" and any other such specialization or generalization. In particular, let us ask:

If the conservation law relates two families of particles, \( A = \{a, a^*, \ldots\} \) and \( B = \{b, b^* \ldots\} \), such that to the birth of a particle of family \( A \), family \( B \) has to delegate one -- any one! -- of its members to make it a double birth: what then is the probability to find one particle "\( a \)" at momentum \( \{\mathbf{p}, d^3p\} \)?

In this case, in (9.13) \( \mathcal{Z}_a^{(1)} = \sum_{\alpha} \mathcal{X}_\alpha a \) is to be restricted to the momenta \( \alpha \) in \( \{\mathbf{p}, d^3p\} \), hence the first factor of (9.13) is to be replaced by

\[
\frac{V_{Z,\mu}}{c^3} \int d^3p \approx \int \mu \alpha (\mathbf{p}, T) d^3p.
\]  

(9.15)

On the other hand, we are not interested in the details concerning the particle of family \( B \), neither in its momentum, nor in its excitation.

Thus, we not only sum over all momentum states \( \alpha \) of \( b \) [this is already done in \( \mathcal{Z}_b^{(1)} = \sum_{\alpha} \mathcal{X}_{\alpha b} \)], but also over all its excited states \( b, b^*, b^{**} \ldots \).

If we denote this sum by \( \mathcal{Z}_B^{(1)} \), then the second factor in (9.13) is, for the present question, replaced by

\[
N_B(T) \equiv \sum_{\mathcal{B}} \left( \sum_{\alpha} \mathcal{C}_{\alpha \mathcal{B}} \right) = \sum_{\mathcal{B}} \int \mu \mathcal{C} (\mathbf{p}, T) d^3p.
\]  

(9.16)
and the spectrum of particle $a$, tied up by a conservation law with family $B$, becomes

$$f_{\mu_a}^{(A,B)}(\vec{p},T) = N_B(T) \cdot f_{\mu_a}(\vec{p},T).$$

(9.17)

It is obvious [and easy to show by an inverse argument; details see (II), App. I] that if "$a$" obeys a conservation law tying it up with its anti-particle ($B = \text{anti } A$) and one particle "$a$" has been there from the beginning ("brought-in" particle; e.g. $ip$ in a collision: p is brought in) then [the superscript $(1,A)$ indicates: one brought-in and conserved particle of family $A$]

$$f^{(1,A)}_{\mu_a}(\vec{p},T) = \frac{f_{\mu_a}(\vec{p},T)}{N_A(T)}$$

(9.18)

so that automatically [see (9.16)]

$$\sum_{A}^{\text{states}} \int f^{(1,A)}_{\mu_a}(\vec{p},T) \, d^3p = 1$$

(9.19)

as it should be: the one particle a brought in must still be there at the end -- in one excited state or another, but summed over all its possible states, it will be there with probability 1, as (9.19) indeed says.

If we introduce a graphical symbol for the spectrum of a particle, then there are three cases:
i) Free creation (no conservation law)

\[ \circ \to \nu \quad \text{contains} \quad \mathcal{f}_\nu (\mathbf{p}, T) \quad (9.20) \]

ii) Associated creation (conservation law A\( \rightarrow \)B)

\[ \bullet \to \nu_a \quad \text{contains} \quad N_B(T) \cdot \mathcal{f}_\nu_a (\mathbf{p}, T) \quad (9.21) \]

iii) Through-going particles (brought in and conserved)

\[ \circ \to \nu_a \quad \text{contains} \quad \frac{\mathcal{f}_\nu_a (\mathbf{p}, T)}{N_A(T)} \quad (9.22) \]

We shall specify this graphical notation in detail later on. Clearly, if the factor \( N_B(T) \) is called "creation factor" for a particle of family B, then \( 1/N_A(T) \) can be called "absorption factor" for a particle of family A and the last process can be read: the incoming particle is absorbed (in any state; factor \( 1/N_A \)) and then recreated without restriction [factor \( \mathcal{f}_\nu_a (\mathbf{p}, T) \)].

The total pair-creation rate of pairs \( a \rightarrow b \) -- which equals the rate of observed \( a \)-particles (and \( b \)-particles) -- is given directly by (9.13):
\[ W_c(a) = W_c(b) = W_c(a \& b) \equiv \sum_{a}^{(1)} \cdot \sum_{b}^{(1)} \equiv \left[ \int f_{a}(\vec{p}_1 T) d^3 p \right] \cdot \left[ \int f_{b}(\vec{p}_2 T) d^3 p \right] \]
\[ \equiv \frac{V V_0}{(2\pi)^3} T^3 \left( \frac{m_a m_b}{3} \right)^{3/2} \gtrsim 2 \exp \left( -\frac{m_a + m_b}{T} \right). \] (9.23)

We write \( VV_0 \) and not \( V^2 \), because while one of the two may be born anywhere in \( V \), the second one must be created in its immediate neighbourhood: conservation laws of this type act locally (see section 10.1.2). For particle physics one has \( V_0^2 \), but in astrophysics the more general form with \( VV_0 \) applies.

Note that one obtains the wrong result \( W \sim \exp (-m_a / T) \) if one tries to fulfill this type of conservation law by introducing a chemical potential [Landau & Lifschitz (1968)]; the reason is that we treat here rare events --- we supposed \( m_{a, b} \gg T \), hence all \( x_{aa}, x_{ab} \ll 1 \) --- where the conservation law must strictly be obeyed in each single case, while with chemical potential one can only enforce its fulfilment in the average.

When pair creation becomes very frequent --- that is: when our supposition \( 1 \gg \gamma^{(1)} \gg \gamma^{(2)} \gg \gamma^{(3)} \ldots \) is no longer true, then our proof of (9.13) breaks down and then the use of the chemical potential becomes admissible; indeed: when both, \( a \) and \( b \) are produced abundantly (in \( V_0 \)), then I need not bother about matching each single \( a \) with some \( b \); things will arrange themselves and the production rate of particle \( a \) can be calculated as if there were no conservation law and the result \( \sim \exp (-m_a / T) \) is correct.

This, however, seems to be true only for pions with respect to charge (which strictly speaking should be conserved); it fails already for kaons. Our result (9.23) --- with some obvious generalizations for the case that \( a \) and/or \( b \) have an excitation spectrum --- is well confirmed by experiments (Fig. 8) from K to anti-He\(^3\) over a range of 12 orders of magnitude (a formula à la Landau & Lifschitz would fail by 7 orders of magnitude for anti-He\(^3\)). Details are presented in [Hagedorn, (1968 b), quoted as (II1)]; the anti-He\(^3\) production was reported in [Antipov et al., (1970)].
Note the drastic consequences of our formula (9.23): roughly speaking it says (for \( m_a = m_b \)) that the production rate drops by about \( 10^4 \) to \( 10^5 \) for each increment \( \Delta m \cdot m_p \). Example: anti-He\(^3\) was produced at a rate of about 1 per week; to produce one anti-He\(^4\) the Serpukhov machine would have to run day and night for about 200 years! Quarks may exist without ever been seen; a mass \( \gtrsim 4 \, m_p \) would be sufficient. It should be clear that for high enough energy (\( T \to T_0 \)), these production rates become independent of the collision energy; it will not help to build a larger machine.

![Graph](image)

**Fig. 8** Production rates for heavy pairs. It drops by roughly \( 10^5 \) for every increment by \( m_0 \) of the produced particle. With some phase space correction applicable to anti-nuclei production, the recently produced \( \overline{\text{He}^3} \) fits in very well. Details in (III) from which the figure is taken.
9.2 Energy-momentum conservation

As we have chosen the canonical description with given temperature, the energy is not conserved. Thus, if we start with given energy $E$, then we define a temperature $T(E)$, such that $\frac{\Delta E}{E} = T^2 \frac{\partial T}{\partial T} \ln Z$ equals $E$, but then that system may have any other energy than $E$. Indeed, for $T \to T_g$ the fluctuations [see (I) and Hagedorn (1970 a)] become infinite. This is not serious, as all quantities which interest us depend on $T$, not directly on $E$. In (III) the question: how serious is all this? was discussed in some more detail; the reader is referred to that article.

Also momentum is conserved only on the average. What we have learned in the previous section is, however, that neglected conservation laws may be repaired at the end with (in most cases) the same result, as if they had been built into the partition function from the start and carried along all the way — which is very inconvenient and becomes unfeasible when not only two, but all particles, are involved as in energy-momentum conservation; in fact: if one tries, one gets back to phase space.

Thus, energy-momentum conservation will be enforced by a correction of the final result: if a spectrum $f_m(p, T)$ has been calculated, we cut it off at the kinematical limit $p_{\text{max}}$ which is given by the mass of the decaying fireball, by the mass of the particle in question and by perhaps other masses, if a conservation law forbids two-body decay [see (II)]. A refinement consists in not just cutting off $f_m(p, T)$ at $p_{\text{max}}$, but replacing it already before $p_{\text{max}}$ by low-dimensional phase space and fitting both in some way together to obtain a smooth transition from $f_m(p, T)$ to $\phi_{\text{ph.sp.}}(p, T)$; the latter will not only vanish at $p_{\text{max}}$, but also approach zero in the kinematically correct way. The details are tedious (not published).
10. APPLICATIONS

We have two main applications: collision processes and astrophysics (big bang, hadron stars).

10.1 Collision processes

Our whole thermodynamic description applies to "ideal fireballs", that is: to something that we hope to be identical to what the co-moving observer of section 2 sees in his immediate neighbourhood. In other words, we believe that our thermodynamics is able to fully describe the local situation; essentially: to calculate $T$, when the energy density $\varepsilon$ is given, and from $T$ all local spectra, production rates, etc., correctly observing all conservation laws. We shall construct the expressions for such a description now.

10.1.1 Combination with kinematics

Going back to section 2, we see that the simplest spectrum is given by eq. (2.9), where we specialize $R$ to be the c.m. frame [superscripts omitted]:

$$W(p) d^3p = \int F(\lambda) L(\lambda, \gamma_0) \left\{ f_m (p', T) d^3p' \right\} d\lambda.$$  

(10.1)

From the thermodynamical part of the theory $T = T(\varepsilon) = T_0(\varepsilon(\lambda))$ is a calculable function of $\lambda$ and $\gamma_0$ [see (II)], which we suppose to be known (numerical table, approximate analytical expression -- anything of that sort, which can be incorporated in a computer program and therefore does no longer interest us).

In the simplest case, eq. (10.1) describes the unrestricted production of a particle of mass $m$, where $f_m (p', T)$ is given by the Planck-spectrum (7.7). Actually, there is not a single sort of particle -- not even pions -- which follow such a simple rule. But eq. (10.1) is the basic expression, from which others will be derived. We introduce the graphical symbol (it has nothing to do with Feynman graphs!).

$$O \rightarrow m \equiv F(\lambda) L(\lambda, \gamma_0) \left\{ f_m (p', T(\lambda)) \right\}$$  

(10.2)
for the integrand of (10.1). The explicit form of the Lorentz-transformed spectrum has been given in (II), App. V.

This Lorentz-transformation can be simulated in a very approximate way by taking the untransformed spectrum, but assigning to the longitudinal momentum a "longitudinal temperature" $T_\gamma$ which is about $\gamma$ times larger than our $T$. Such a "longitudinal temperature" is a purely computational quantity without physical significance. The so-called "two-temperature-model" [Wayland and Bowen (1967); Wayland (1968); La Pointe and Wayland (1969)] exploits this approximation to simplify the appearance of formulae. There the volume $V_0$ and the temperatures $T_\gamma$ and $T$ are treated as free parameters which vary from energy to energy and from particle to particle. This is the price they have to pay for using essentially only one formula corresponding to our eq. (10.1).

**10.1.2 Associated and/or pair creation; locality**

In order to calculate the spectrum of particle "a", when some "b" $\in B$ must be created along with it, we use what we have learned in section 9.1: its spectrum is

$$\hat{f}_{\lambda\alpha}(p', T) = N_B(T) \hat{f}_{\lambda\alpha}(p', T). \quad (10.3)$$

It would be wrong to simply insert this into the curly brackets of (10.1), because $T = T(\lambda)$ belongs to some given $\lambda$, and it might well be that when "a" is created at one place and time, then "b" is created at another place and another time during this same collision; thus "a" and "b" might belong to different $\lambda$ and to different $T(\lambda)$. Since $F(\lambda)$ is the probability to find $\lambda$, we must write the differential contribution to $\hat{W}(p)$ coming from $\lambda$ and $\lambda'$:

$$d^2 W_\lambda(p) = K(\lambda, \lambda') \left[ N_B(\alpha') F(\alpha') d\alpha' \right] \times$$

$$\times F(\lambda) d\lambda \ L(\lambda, \lambda') \left\{ \hat{f}_{\lambda\alpha}(p', T(\lambda)) \right\}. \quad (10.4)$$
where $K(\lambda, \lambda')$ is a correlation function which we will discuss in a moment. The second factor $[\ldots]$ is for the (free) creation of any particle $b \in B$ coming from $\{\lambda', C\lambda'\}$ -- no Lorentz-transformation, since the number $N_B(\lambda')F(\lambda')$ is invariant -- and the rest is as before.

In (II) it has been shown that for all reasonably heavy particles $\{K, N, Y \ldots\}$ the conservation laws act so locally that:

$$K(\lambda, \lambda') \sim \delta(\lambda - \lambda').$$  \hspace{1cm} (10.5)

This is an experience from our attempts (II) to fit spectra: $K(\lambda, \lambda')$ was first left free, but narrowed to $\delta(\lambda - \lambda')$ within our computing accuracy.

Secondly we checked it by calculating the invariant mass distribution $f(M^2, T, \gamma)$ for $\bar{K}K$ pairs: if $K$ and $\bar{K}$ are created at the same velocity, $f(M^2, T, \gamma = 1)$ has the narrowest possible form; if they can come from different $\lambda'$s, that is: if the birth places have a relative velocity measured by $\gamma$, then $f(M^2, T, \gamma)$ widens. Figure 9 shows the result of a calculation of $f(M^2, T, \gamma)$, supposing thermodynamical spectra for $K$ and $\bar{K}$ but varying $\gamma$. The experiment favours locality. Something similar might be true even for I-spin, because pions seem to be created locally pairwise [Wang (1969 a, b, c); Horn and Silver (1970)]. In Regge-language: quantum numbers do not travel across many vertices of multi-Regge graphs.

Introducing (10.5) into (10.4), we can integrate over $\lambda'$ and $\lambda$ with the result

$$W_a(p^a) = \int \left[ F(\lambda') \right]^{2} N_B(T(\lambda)) L(\lambda, \lambda') \left\{ f_{\mu_\lambda}(p^a, T(\lambda)) \right\} d\lambda. \hspace{1cm} (10.6)$$
Fig. 9 Invariant mass distribution $f(m^2,T,\gamma)$ for fixed $T$ and two values of $\gamma$ corresponding to local and non-local strangeness conservation, compared to experimental distribution in 1.2 GeV/c pp annihilation. Figure taken from (II).
Hence, graphically:

\[
\begin{align*}
  a & \equiv [F(\lambda)]^2 N_B(\lambda) \cdot L(\lambda, \chi_0) \cdot \left\{ \tilde{f}_{w_d}(\vec{p}', T(\lambda)) \right\} \\
\end{align*}
\]  

(10.7)

Note that not only a factor \( N_B \) appeared, but also that \( F(\lambda) \) is squared!

The generalization is obvious:

\[
\begin{align*}
  \overrightarrow{d} & \equiv [F(\lambda)]^3 [N_A(\lambda)]^2 \cdot L(\lambda, \chi_0) \cdot \left\{ \tilde{f}_{w_d}(\vec{p}', T(\eta)) \right\} \\
\end{align*}
\]  

(10.8)

describes anti-deuteron production.

10.1.3 Through-going particles

If a particle need not be newly created, because it is one of the colliding two, then the spectrum, as we have seen in section (9.1), is divided by the factor \( N_A(T) \). We might thus expect to find for a through-going particle in analogy to above

\[
\begin{align*}
  \frac{F(\lambda) \cdot L(\lambda, \chi_0)}{[F(\lambda) \cdot N_A(\lambda)]} \left\{ \tilde{f}_{w_d}(\vec{p}', T(\lambda)) \right\}
\end{align*}
\]

where \([N_A(T(\lambda)) \cdot F(\lambda)]^{-1}\) stands for absorption at \( \lambda \). Thus, \( F(\lambda) \) would drop out. But it turns out that this is not sufficient, and that indeed an extra function \( F_0(\lambda) \) for through-going particles must be introduced which is not flat \([\text{as } F(\lambda)/F(\lambda) = 1]\), but even peaked towards \( \lambda \rightarrow 1 \), thus emphasizing the mostly peripheral way of the through-going particle. We thus write

\[
\begin{align*}
  \overrightarrow{\Theta} & \equiv \frac{F_0(\lambda)}{N_A(T(\lambda))} \cdot L(\lambda, \chi_0) \cdot \left\{ \tilde{f}_{w_d}(\vec{p}', T(\lambda)) \right\} \\
\end{align*}
\]  

(10.9)

for the through-going particle.
It may happen that an incoming proton is transformed into with simultaneous creation of a K; obviously then

\[ \sum_{\lambda} \equiv \frac{F_{0}(\lambda) F(\lambda)}{N_{\lambda}(\lambda)} \cdot N_{K}(\lambda, \gamma_{0}) \{ f_{\mu_{\lambda}}(p, T(\lambda)) \} \]  \hspace{1cm} (10.10)

where the extra factors \( F(\lambda)N_{K}(\lambda) \) take care of the K-production.

### 10.1.4 Isobars and resonances

Consider the \( \pi \)-spectrum. It obtains contributions from all generations (see section 7.4). Let then \( m^{*} \) be one of the intermediate fireballs of a decay chain, and let \( f_{\pi, m^{*}}^{*}(p, T) \) be the \( \pi \)-spectrum component, due to the pions emerging directly from \( m^{*} \). The total \( \pi \)-spectrum will be the sum of all these \( \sum_{m^{*}} \) goes over all intermediate fireballs of all generations)

\[ f_{\pi} = f_{\pi, \pi} + \sum_{m^{*}} f_{\pi, m^{*}}^{*} \]  \hspace{1cm} (10.11)

where \( f_{\pi, \pi} \) is our thermodynamical spectrum, describing the pions coming from the first generation. Now, for large \( m^{*} \) the pion spectrum coming from \( m^{*} \) is again a thermodynamic spectrum. Even a three-body decay yields a spectrum with a shape similar to a Planck-spectrum, only the two-body decay is very different: the spectrum is a \( \delta \)-function. Thus, except for the two-body contributions, we can say that \( \sum_{m^{*}} f_{\pi, m^{*}}^{*} \) yields...
again a superposition of thermodynamic spectra very similar to \( f_{m_\pi} \) (with slightly different \( T \)-values) which simply may be added to \( f_{m_\pi} \). Thus, taking two-body decay apart

\[
\hat{f}_\pi (\vec{p}_1, T) \approx Q(E) \hat{f}_{m_\pi} (\vec{p}_1, T) + \sum_{\nu \nu^*}^{(2)} \rho_{\nu, \nu^*} \hat{f}_{\nu, \nu^*} (\vec{p}_1, T). \tag{10.12}
\]

Here \( Q(E) = 1 + \text{total contribution of the rest (except from two-body decay)}. \) Lacking a theory of multiplicities (see Section 8.1), these factors \( Q(E) \) can be calculated as follows: suppose all spectra of all particles are known (section 10.1.5; in particular Fig. 11), then at a given collision energy \( E \), the whole set can be computed and the mean energies per kind of particle can be determined. Then, roughly speaking, multiplying these by the factors \( Q(E) \) and adding them up, the total energy must result. This -- in reality a little more complicated procedure -- yields factors \( Q(E) \) at various energies. We found that \( Q(E) \sim \sqrt{E_{\text{c.m.}}} \) is compatible with such calculations. For details see (II).

Now, once we know the spectrum of \( m^* \) -- which we assume to be the usual Planck-spectrum -- the two-body decay spectrum \( f_{1,m^*}^{\pi} \) of particle 1 in the decay \( m^* \rightarrow m_k + m_1 \) can be calculated analytically:

\[
f_{1,m^*}^{\pi} (\vec{p}, T) \equiv \text{momentum spectrum of particle 1 in the decay } m^* \rightarrow m_k + m_1, \text{ is a known function; similar to, but not a Planck spectrum [explicit formula given in (II), eq. (A-IV-10)]}.
\]

Again, it turns out that the second term of (10.12) is most important, if \( m^* \) is an excited through-going particle, which has already by itself a very peaked momentum distribution. But, of course, one can write down \( \rho \)-decay, etc., and include it.
The production probability of \( m^* \) is already contained in \( \ell^*_1 m^* \), but it remains to specify the range of the sum. Let \( m^* \) belong to a particular family (e.g. baryons), then we might replace the sum by

\[
\sum_{m^*} (2) \ell^*_1 m^* \Rightarrow \int \rho_B^{(2)}(m^*) \ell^*_1 m^* \, dm^*
\]  

(10.13)

where \( \rho_B^{(2)} \) counts only the baryon resonances which decay predominantly into two particles. This integral is therefore not extending to very high masses, and we might write again a sum:

\[
\int \rho_B^{(2)}(m^*) \ell^*_1 m^* \, dm^* \Rightarrow \sum_{m^*} \rho_B^{(2)}(m^*) \Delta m^* \ell^*_1 m^*
\]  

(10.14)

where we now do not sum over all, but only over a few representative resonances with weight factors

\[
A(m^*) \approx \rho_B^{(2)}(m^*) \Delta m^* .
\]  

(10.15)

As \( \rho(m^*) \) is expected to grow exponentially, the same is true for the \( A(m^*) \). In (II) these \( A(m^*) \) were left free as fitting-parameters: they pleased us by taking values as expected from the above argument (Fig. 10).

Finally, the whole \( \sum \) may have to be multiplied by a weight factor related to isospin conservation: from isobars in pp collisions more \( \pi^+ \) than \( \pi^- \) will be emitted [for the principle, see (II); there is an error
Fig. 10  The weight factors with which the isobars, \( J^{a*} \) contribute to the spectra of \( p, \pi^+ \) and \( K^+ \). The proton weight is fixed, the others left free and obtained from a fit to the experimental spectra. They all lie scattered about the straight line \( \text{exp}\left[\frac{\pi^*}{132 \, \text{MeV}}\right] \).

\[ \bullet, A_{J^{a*}} \text{ for } K^+; \quad \circ, A_{J^{a*}} \text{ for } \pi^+ \text{ and } p. \]

in the numerical values; corrected in "Atlas", p. 120 Grote, Hagedorn, Ranft, 1970\). We write for the contribution from such decay processes:

\[
\begin{align*}
\text{(10.16)} \\
\sum A(w^*) F(\lambda) L(\lambda, \chi_0) \left\{^{\rho^+ \rightarrow \pi^+} \setminus T \right\}
\end{align*}
\]

\[
\sum A(w^*) \frac{F_0(\lambda)}{N_N(\lambda)} L(\lambda, \delta_0) \left\{^{\rho^+ \rightarrow \pi^+} \setminus T \right\}
\]

The first applying e.g. to \( \rho \rightarrow 2\pi \) and the second to \( N^* \rightarrow N + \pi \).

10.1.5 Complete spectra, velocity functions

In Fig. 11 we present a complete self-explanatory set of spectra formulae in graphical notation. In this form the model has been used to fit our formulae to measured spectra and to determine thereby the empirical functions \( F(\lambda) \) and \( F_0(\lambda) \) as they are shown in Fig. 3.
This is the complete set of spectra constructed according to the rules discussed above. Although this set is only approximately complete [e.g. the contribution of pp pair production to the proton spectrum is neglected], one obtains already good fits to experimental spectra with two functions \( F(\lambda) \) and \( F_0(\lambda) \) which are the same for all particles and all energies [see Figs. 12 and 13 and the "Atlas"].
It was found that these two functions $F(\lambda)$, $F_0(\lambda)$ are, in some sense, universal: they can describe all particles at all energies with sufficient-to-excellent accuracy as far as measurements are available. In the "Atlas of Particle Spectra" [Grote, Hagedorn, Ranft (1970)] spectra are compared with experiments and others are predicted (up to 800 GeV/c primary momentum). The Atlas has been calculated using exactly the graphs of Fig. 11. The reader is referred to that report.

For the sake of completeness, we reproduce here from the "Atlas" two typical spectra, one for (mainly newly-created) $\pi^-$ (Fig. 12, Atlas p. 134) and one for (through-going) protons (Fig. 13, Atlas p. 159).

One remark to terminate this sub-section: one can, and one should extend the table of graphs given in Fig. 11. For instance, to the pions one should add $\gamma \rightarrow \pi$, and to the protons $\gamma \rightarrow \bar{p}$, etc. Since we have neglected the contribution of nucleon-antinucleon pair production to the proton spectrum, the function $F_0(\lambda)$ has tried to simulate it: this explains the little peak at $\lambda = 0$, which probably will disappear when $\gamma \rightarrow \bar{p}$ is included. For more details see (II) and the Atlas.

10.1.6 Transverse momentum distribution

Looking at Fig. 11, we see that the spectrum of a given particle comes from various processes -- except for $K^-$, $\bar{p}$, $\Xi$ and $\bar{d}$ which, in our approximation, consist each of one contribution only. This contribution has always a Planck spectrum in the integral, Lorentz-transformed and multiplied by $\left[F(\lambda)\right]^k \cdot \left\{N[T(\lambda)]\right\}^l$, where $k$ and $l$ are some powers;
Figs. 12 and 13 show the computer output for $\pi^-$ and $p$ spectra. The steps at high momenta come from the crude numerical $\lambda$-integration and kinematical cut-off [Section 9.2 and (II)]. Note the difference in shape: $p$ are through-going and peripheral, $\pi^-$ newly created and mostly from the central part of the collision.
in any case these factors result in some function \( G_i(\lambda) \), where \( i \) denotes the particle in question. Thus, for \( K^-, \overline{p}, \Xi \) and \( \overline{d} \)

\[
W_i(\vec{p}) = \int_{-1}^{1} G_i(\lambda) L(\lambda, \delta_0) \left\{ f_{w_i}(\vec{p}', T) \right\} d\lambda ;
\]  

(10.18)

\( G_i(\lambda) \) is composed of \( F(\lambda) \) and \( N[T(\lambda)] \) according to the graph assigned to particle \( i \). In these cases the transverse momentum distribution is most simple. In all other cases \((\pi^\pm, K^+, p, \Lambda, \Xi)\), it is more complicated by the contribution of decay spectra \( f^*_{1,m^*} \) which are not of the Planck type. We shall not discuss them here, we limit ourselves to \( K^-, \overline{p}, \Xi \) and \( \overline{d} \) transverse momentum distributions.

With

\[
\vec{p} = (p_\parallel, p_\perp)
\]

\[
d^3p = 2\pi p_\perp dp_\perp dp_\parallel
\]

(10.19)

we have, because of the invariance of \( p_\perp \) under \( L(\lambda, \gamma_0) \),

\[
W_i(p_\perp) dp_\perp = 2\pi p_\perp dp_\perp \int_{-1}^{1} d\lambda \int_{-\infty}^{\infty} dp_\parallel G_i(\lambda) L(\lambda, \delta_0) \left\{ f_{w_i}(p_\perp, p_\parallel, T) \right\}.
\]

(10.20)

Instead of integrating over \( dp_\parallel \), in the c.m. frame, we can as well integrate over \( dp_\parallel \), in the \( \lambda \)-frame, and omit the Lorentz-transformation, because the result of this integration is a number, invariant under \( L(\lambda, \gamma_0) \). Hence

\[
W_i(p_\perp) = 2\pi p_\perp \int_{-1}^{1} d\lambda G_i(\lambda) \int_{-\infty}^{\infty} f_{w_i}(p_\perp, p_\parallel, T) dp_\parallel.
\]

(10.21)
We use for \( f_n(p, T) \) the Planck spectrum (7.6) and (7.7)

\[
\tilde{f}_m(p, T) \, dp = \frac{V \, Z_m}{(2\pi)^3} \, \frac{d^3p}{\exp \left( \frac{\sqrt{p^2 + m^2}}{T} \right) + 1} = \frac{V \, Z_m}{(2\pi)^3} \, dp \, \frac{x_{\rho m}}{1 \pm x_{\rho m}} \tag{10.22}
\]

with

\[
\sqrt{p^2 + m^2} = \sqrt{p_{\perp}^2 + \mu^2}
\]
\[
\mu^2 = p_{\perp}^2 + m^2 \tag{10.23}
\]

we obtain with (5.16)

\[
\int_{-\infty}^{\infty} \tilde{f}_m(p, T) \, dp = \frac{V \, Z_m}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{x_{\rho m}}{1 \pm x_{\rho m}} \, dp =
\]

\[
= \frac{2 \, z \, v \, V}{(2\pi)^3} \sum_{n=0}^{\infty} \left( + \right)^n \int_{-\infty}^{\infty} x_{\rho m} \, dp =
\]

\[
= \frac{2 \, z \, v \, V}{(2\pi)^3} \sum_{n=0}^{\infty} \left( + \right)^n \int_{-\infty}^{\infty} \exp \left( - \frac{n+1}{T} \sqrt{p_{\perp}^2 + \mu^2} \right) =
\]

\[
= \frac{2 \, z \, v \, V}{(2\pi)^3} 2 \sqrt{p_{\perp}^2 + \mu^2} \sum_{n=0}^{\infty} \left( + \right)^n K_1 \left[ \frac{n+1}{T} \sqrt{p_{\perp}^2 + \mu^2} \right]. \tag{10.24}
\]

Inserting this into (10.21) yields

\[
\tilde{W}_i(p_{\perp}) \, dp_{\perp} = \frac{2 \, z \, v \, V}{2 \pi \mu^2} \, p_{\perp} \, \sqrt{p_{\perp}^2 + \mu^2} \times
\]

\[
\times \sum_{n=0}^{\infty} \left( + \right)^n \int G_i(\lambda) K_1 \left[ \frac{n+1}{T(\lambda)} \sqrt{p_{\perp}^2 + \mu^2} \right] \, d\lambda \tag{10.25}
\]

(+ bosons, - fermions).
With
\[ G_{K^{-}}(\lambda) = \left[ F(\lambda) \right]^2 N_{K}(T(\lambda)) \]
\[ G_{\bar{p}}(\lambda) = \left[ F(\lambda) \right]^2 N_{\bar{p}}(T(\lambda)) \]
\[ G_{\Xi}(\lambda) = \left[ F(\lambda) \right]^2 F_{0}(\lambda) \frac{\left[N_{K}(T(\lambda))\right]^2}{N_{\bar{p}}(T(\lambda))} \]
\[ G_{\Xi}(\lambda) = \left[ F(\lambda) \right]^2 \left[ N_{\bar{p}}(T(\lambda)) \right] \]

(10.26)

(10.25) is an exact expression, which can be evaluated numerically
\{the \( N_{\Xi}(T(\lambda)) \) are given in (II), eq. (A-I-38)\}. A similar expression --
without \( G_{i}(\lambda) \) and \( \lambda \)-integration -- was first derived by Imaeda (1967).

**Approximations**

We can do four approximations:

i) Neglect all terms, except \( n = 0 \), in the sum. This corresponds
exactly to neglecting the \( \pm \) in the Planck spectrum. In view of \( m_{K} \)
being the smallest mass, where (10.25) applies, and because of the
integration over \( p_{\Xi} \), this is a good approximation; this is also
seen by inspecting eq. (10.25):
\[ K_{i} \sim \left[ \exp \left( - \frac{\sqrt{p_{\Xi}^2 + u_{i}^2}}{T} \right) \right]^{n+1} \]

The result is then
\[ W_{i}(p_{\Xi}) \approx \frac{2i V_{0}}{2\alpha} \rho_{\Xi} \rho_{\Xi}^{2} \int_{-}^{1} G_{i}(\lambda) K_{i} \left[ \frac{\sqrt{p_{\Xi}^2 + u_{i}^2}}{T(\lambda)} \right] d\lambda. \quad (10.27) \]

*) Although \( \Xi \) is a through-going particle, the two \( K \) produced with it
and the locality of conservation laws require the \( \Xi \) to come from the
central part of the collision (unlike \( p \)).
For not too small $p_\perp$ this is an excellent approximation. It may be improved for $K^-$ (and $\pi^-$ which has only a small isobar contribution) by adding the next term $(n = 1)$.

ii) The factors $G_i(\lambda)$ decrease rapidly when $\lambda$ grows away from zero: partly because of their content of $F(\lambda)$, partly because the factors $N_i[T(\lambda)]$ decrease exponentially. Furthermore, the $K_1$-function also decreases like $\exp[-m/T(\lambda)]$. Thus, in all cases where no through-going particles or significant contributions from through-going isobars are involved \{through-going particles pass often near $\lambda = 1$; the function $F_0(\lambda)$ increases towards $\lambda \to 1$ and the exponential drop of $K_1$ is compensated by the absorption factors $1/N_i[T(\lambda)]$ which increase like $\exp[m/T(\lambda)]$\} -- when no significant influence from through-going particles are expected, I said, then the above integration over $\lambda$ may be replaced by a constant times the integrand at $\lambda = 0$; this amounts to neglecting all contributions from particles, which do not come from the hot central part of the collision:

\[
W_i^{(2)}(p_\perp) \simeq \text{const} \times p_\perp \sqrt{p_\perp^2 + m_i^2} T \left[ -\frac{\sqrt{p_\perp^2 + m_i^2}}{T} \right] \tag{10.28}
\]

where $T = T(\lambda = 0) = T(E_{\text{c.m.}}/2V_0)$ [for a table of $T(E)$ see (II), Appendix II].

iii) For not too small $p_\perp$ and for larger masses the asymptotic formula $K(x) \sim \sqrt{\pi/2x} e^{-x}$ may be used to write

\[
W_i^{(3)}(p_\perp) \approx \text{const} \cdot T \cdot p_\perp \cdot (p_\perp^2 + m_i^2)^{1/4} \exp \left( -\frac{\sqrt{p_\perp^2 + m_i^2}}{T} \right) \tag{10.29}
\]

where $T = T(\lambda = 0) = T(E_{\text{c.m.}}/2V_0)$ [for a table of $T(\epsilon)$ see (II), Appendix II].
iv) Finally, for \( p_\perp \gg m \) we expand the square roots:

\[
W_i^{(\gamma)}(p_\perp) \sim \omega \mu_\gamma \cdot T \cdot p_\perp^{3/2} \left( 1 + \frac{m_i^2}{4p_\perp^2} \right) e^{-\frac{p_\perp}{T}} \left( 1 + \frac{m_i^2}{2p_\perp^2} \right)
\]

(10.30)

\( T = T(\lambda = 0) \) as before,

valid only at the high \( p_\perp \) tail. If here \( m^2/2p_\perp^2 \) is neglected, we obtain the Boltzmann spectrum which has so often been tried but which is not the very best approximation one can imagine:

* * *

To conclude we stress again: All above formulae are true (within their approximations) only for \( K^- \), \( \bar{p} \), \( \Xi \) and \( \bar{d} \), while \( \pi \), \( p \), \( K^+ \), \( \Lambda \), \( \Sigma \) obtain contributions from through-going isobar decay (see Fig. 11) whose spectrum \( f_{1,m}^* \), though similar to it, is not a Planck spectrum [see (II), (A-IV-10)]. Thus, there are contributions of a slightly different \( p_\perp \) dependence. Furthermore, in particular our approximation (ii) is then not valid, because here \( F_0(\lambda) \) and \( 1/N[T(\lambda)] \) come in, which both increase with \( \lambda \) (peripherality). The only exception is \( \pi^- \) which follows still rather well (in pp collisions) our above formulae, because the isobar decay contribution is very small (the factor \( C_\pi^- = 0.15 \)).
10.1.7 Mean transverse momenta

We return to (10.25) and calculate the mean transverse momentum by

\[
\langle p_{\perp}(u) \rangle = \frac{\int p_{\perp} W(p_{\perp}) \, dp_{\perp}}{\int W(p_{\perp}) \, dp_{\perp}}
\]

\[
= \frac{\int \, d\lambda \, G(\lambda) \sum_{n=0}^{\infty} (\frac{a}{\pi})^{n} \int \rho_{\perp}^{2} \sqrt{\rho_{\perp}^{2} + u^{2}} \, K_{n} \left[ \frac{u+1}{T(\lambda)} \sqrt{\rho_{\perp}^{2} + u^{2}} \right] \, dp_{\perp}}{\int \, d\lambda \, G(\lambda) \sum_{n=0}^{\infty} (\frac{a}{\pi})^{n} \int \rho_{\perp} \sqrt{\rho_{\perp}^{2} + u^{2}} \, K_{n} \left[ \frac{u+1}{T(\lambda)} \sqrt{\rho_{\perp}^{2} + u^{2}} \right] \, dp_{\perp}}
\]

\[(10.31)\]

We use the formula [valid for Re (k) > -1]

\[
\int_{0}^{\infty} \frac{x^{k+1}}{\sqrt{x^{2} + u^{2}}} K_{k} \left( \frac{\sqrt{x^{2} + u^{2}}}{x} \right) \, dx = 2^{k} \Gamma(k+1) \mu^{k+2} T^{k+1} K_{k+2} \left( \frac{\mu}{T} \right)
\]

\[(10.32)\]

to obtain the exact result [first derived -- without G(\lambda) and \lambda-integration--
by Imaeda (1967)]

\[
\langle p_{\perp}(u) \rangle = \sqrt{\frac{\pi \mu}{2}} \frac{\int_{-1}^{1} [T(\lambda)]^{3/2} G(\lambda) \sum_{n=0}^{\infty} (\frac{a}{\pi})^{n} K_{5/2} \left( \frac{u+1}{T(\lambda)} \mu \right) \, d\lambda}{\int_{-1}^{1} T(\lambda) G(\lambda) \sum_{n=0}^{\infty} (\frac{a}{\pi})^{n} K_{2} \left( \frac{u+1}{T(\lambda)} \mu \right) \, d\lambda}
\]

\[(10.33)\]

Now we go through the same approximations as with \( W(p_{\perp}) \), but with better justification because of the \( p_{\perp} \)-integration:

i) Neglect all terms except \( n = 0 \) (good approximation)

\[
\langle p_{\perp}(u) \rangle_{i}^{(1)} \approx \sqrt{\frac{\pi \mu}{2}} \frac{\int_{-1}^{1} [T(\lambda)]^{3/2} G_{i}(\lambda) K_{5/2} \left( \frac{u_{i}}{T(\lambda)} \mu \right) \, d\lambda}{\int_{-1}^{1} T(\lambda) G_{i}(\lambda) K_{2} \left( \frac{u_{i}}{T(\lambda)} \mu \right) \, d\lambda}
\]

\[(10.34)\]

For \( G_{i}(\lambda) \) see (10.26).
ii) Here we can trust the previous formula even for \( m = m_{\pi} \); again we argue that the integrand has contributions only from small \( \lambda \) -- the hot, central part of the collision -- and the \( \lambda \)-integration may be suppressed, taking the integrand at \( \lambda = 0 \). Note that here again this is not allowed for through-going particles, and those which obtain significant contributions from through-going isobar decay. Such particles -- in particular protons in pp-collisions -- will go through peripherally and for them the integral would come mainly from \( \lambda \) near to \( \pm 1 \) (the factors \( 1/N[T(\lambda)] \) will compensate the exponential decrease of the \( K[m/T(\lambda)] \)). Thus, for \( \tau^-, K^-, p, \bar{p}, \bar{\Xi}, \bar{d} \) the temperature determining \( \langle p_\perp \rangle \) will be high (hot centre), while for \( \pi^+, p, K^+, \Lambda, \Sigma \), it will be low (peripheral parts). In any case we have approximately

\[
\langle p_\perp(u_1, T) \rangle_i^{(2)} \propto \sqrt{\frac{\pi m}{2T}} \frac{K_{5/2}\left(\frac{um}{T}\right)}{K_2\left(\frac{um}{T}\right)}
\]

where \( T(\pi^-, K^-, p, \Xi, \bar{d}) \) is near to the central, maximal temperature for the given collision energy \([T = T(E_{\text{c.m.}}/V_0)]\), while \( T(\pi^+, p, K^+, \Lambda, \Sigma) \) will be lower. Figure 14 shows (10.35) as function of \( m \) with \( T \) as parameter. As an illustration: in (II), Appendix II, we have calculated a table of values \( T(\varepsilon) \) from which we read off that to accelerator energies 12-30 GeV corresponds a "central" temperature of \( \approx 125-130 \) MeV. Pions from collisions with the highest cosmic ray energies -- \( \approx 10^5 \) GeV -- lie on the curve \( T = 160 \) MeV.

iii) For very large mass the two Hankel functions may be replaced by their asymptotic form, which does not depend on the index. Thus

\[
\langle p_\perp(u, T) \rangle_{\text{large } u}^{(3)} \Rightarrow \sqrt{\frac{\pi m}{2T}} .
\]

(10.36)
Fig. 14  Mean transverse momentum as a function of the mass, with the temperature labelling the curves. By a tricky cancellation of the zero of the square root with a divergence of the ratio of the Hankel functions, the resulting curve looks like a linear relation from pion to nucleon.

Note that antiprotons feel a higher average temperature than protons because the former come from the hot central, the latter from the cold peripheral regions. Figure taken from (11), references "our spectra" and 27 listed there.
This result is instructive: such masses move non-relativistically with momentum \( \equiv \langle p_L \rangle \). Thus, we square and find
\[
\frac{\langle p_L^2 \rangle}{2m} = \frac{\pi}{4} T \approx 2 \cdot \frac{1}{2} T ,
\]
which means that the two transverse degrees of freedom obtain the classical equipartition share of kinetic energy. That we found \( \pi/4 \) instead of 1 comes from calculating \( \langle p_L^2 \rangle^2 /2m \) instead of \( \langle p_L^2 \rangle /2m \). The difference is just due to the well-known fact that the square of the mean value is in general not equal to the mean value of the square. The reader may check this; eq. (10.32) allows to calculate \( \langle p_L^n \rangle \) for any \( n \). He will find that \( \langle p_L^2 \rangle /2m = T \) (exact result for \( m \to \infty \)).

**10.1.8 Collision of non-identical particles**

When the target (\( m_1 \)) and projectile (\( m_2 \)) particles are different, then two things change:

i) The kinematics is different from the one discussed in these lectures if \( m_1 \neq m_2 \). Section 2 has to be re-considered.

ii) The functions \( F(\lambda) \) and \( F_0(\lambda) \) have to be replaced by more detailed ones: a target function \( F^t(\lambda) \), which is \( \neq 0 \) mainly for \( \lambda < 0 \), but small or zero for \( \lambda > 0 \), and \( F^p(\lambda) \) for the projectile, where the opposite is the case. Making the **hypothesis** that these functions are universal, i.e. do not depend on the nature of the target or projectile (both hadrons), we see that for \( -1 < \lambda < 1 \)

\[
F^p(\lambda) = F^t(-\lambda) \\
F_0^p(\lambda) = F_0^t(-\lambda)
\]
In the symmetric case (pp collision) discussed in these lectures our functions are simply the sum: \( F(\lambda) = F^p(\lambda) + F^t(\lambda) \).

If it is furthermore true that \( F^p \) is very small or zero for \( \lambda < 0 \) (and \( F^t \) for \( \lambda > 0 \)), then

\[
\begin{align*}
F^p(\lambda) &= F^t(-\lambda) \approx F(\lambda) \quad \text{for} \quad \lambda > 0 \\
F^p(\lambda) &= F^t(-\lambda) \quad \text{very small} \quad \text{for} \quad \lambda < 0
\end{align*}
\]

(10.39)

(the same for \( F_0 \)).

These relations are only guess-work. So far the thermodynamical model has not been generalized to non-identical collision partners and not yet been applied to the analysis of such reactions.

10.2 Astrophysical applications

10.2.1 Big Bang

If the world has originated some \( 10^{10} \) years ago from a "big bang" with total charge, baryon number, lepton number, strangeness, all equal to zero, then, in its first moment, it was a hadronic black body radiation, as described in these lectures: the universe started as a fireball, which was never hotter than \( \approx 160 \) MeV = \( 1.8 \times 10^{12} \) K, and whose pressure was \( \sim \ln \epsilon \) or even constant, while the mass-energy fluctuations from one elementary volume \( V_0 = (4\pi/3) m^{-3} \) to the next were infinite. All this has been discussed in [Hagedorn, (1970 a)]. If the baryon number is \( \neq 0 \), the situation becomes more difficult and qualitatively different, although a maximum temperature is still there [Huang and Weinberg (1970)]. What happens later in such a universe is described in several papers [see e.g. Harrison (1970); Kundt (1970); Möllenhoff (1970)].

10.2.2 Hadron stars

The concept of a neutron star [Landau (1932); Oppenheimer and Volkoff (1939)] is old: the enormous gravitational pressure squeezes all electrons into the nuclei which then become neutral; they are so closely
packed that the whole star is a giant nucleus of some 10 km radius, consisting of neutrons only. Its density equals nuclear density \( \sim 10^{14} \) g cm\(^{-3}\). What happens if I press further? Here the exponential mass spectrum comes in: reactions like \( n + n \rightarrow N^* + N^{**} \) become possible, and the newly-born baryon resonances can fill up their own level scheme, since they do not obey a Pauli exclusion principle with respect to the neutrons, but only among themselves. If the star remains cold, it will become a star composed of baryon resonances; each sort filling a level scheme up to some common Fermi-energy \( E_F = \sqrt{p^2 + m^2} \); no resonance can decay, because all lower levels are occupied; no free bosons; the pressure will be lower than for a neutron star (a pure neutron star does not exist, since we know there are baryon resonances!), but the details depend on the baryon mass spectrum.

If the star is hot, its temperature cannot exceed \( T_0 \). If \( T \rightarrow T_0 \), the above arguments (filled levels) do no longer apply and boson creation sets in. The full mass spectrum \( \rho(m) \) becomes available! Now the total energy-mass density \( \varepsilon \) consists of two terms: \( \varepsilon_0 = \text{mass density at } T = 0, \) due to the brought-in material; hence \( \varepsilon_0 = M/V, \) where \( M = \text{total mass of the star at } T = 0, \) \( V \) its volume. Now let it be heated with \( V \) kept fixed, then, for \( T \rightarrow T_0 \), the matter produced in the black-body radiation has a density \( \varepsilon_1(T) \). Thus

\[
\varepsilon(T) = \varepsilon_0 + \varepsilon_1(T) \quad [V \text{ const}].
\] (10.40)

For \( T \rightarrow T_0 \), \( \varepsilon_1(T) \rightarrow \infty \), hence

\[
\lim_{T \rightarrow T_0} \frac{\varepsilon(T)}{\varepsilon_1(T)} = 1
\] (10.41)
where $\epsilon_1(T)$ is given by our theory; it either diverges like $(T_0 - T)^{-1}$ or like $(T_0 - T)^{-k}$, according to whether we prefer the bootstrap solution $m^{-5/2}$ or $m^{-3}$ respectively. The pressure becomes, for $T \approx T_0$

$$P \rightarrow \ln \epsilon_1(T) + \text{const} \quad (m^{-5/2} \text{ solution})$$

$$P \rightarrow \text{const} \quad (m^{-3} \text{ solution}) \quad (10.42)$$

Matter becomes much, much softer than in old theories (where $P \sim \epsilon$). The velocity of sound is given by $\sqrt{\partial P/\partial \epsilon}$, and in our theory tends to zero in all cases (hot or cold). There were theories in which it became greater than the velocity of light, thus violating causality. The stability against gravitational collapse of such a hadron star is, however, not very different from that of the old neutron star, because our new thermodynamics weakens the pressure — from $P \sim \epsilon$ to $P \sim \ln (\epsilon/\epsilon_0)$ in a hot, from $P \sim \epsilon$ to $P \sim \epsilon/\ln (\epsilon/\epsilon_0)$ in a cold star — without changing much else. As the pressure in older theories ($P \sim \epsilon$) did not suffice to prevent gravitational collapse, our pressure suffices even less; collapse just happens a little more easily.

It is only recently that this thermodynamics of strong interactions has been applied to hadron stars, and the field is in rapid development. The reader is referred to the literature [Koebeke, Hilf and Ebert (1970); Rhoades and Ruffini (1971); Wheeler (1971); Lee, Leung and Wang (1970); Leung and Wang (1970)]. The last paper is rather remarkable as an illustration and check of our philosophy. The authors calculate an equation of state at low densities, using potential theory of nuclear forces with all modern refinements; then they calculate another equation of state for high densities, using their version of our thermodynamics of formally free particles with exponential mass spectrum. When they extrapolate both equations to some common medium density, the pressure curves match rather well (slope and value), without any free adjusting parameter. This shows that our thermodynamics does indeed contain strong interactions.
10.3 A table of units converted from CGS to \( \hbar = c = k = 1 \)

\[
\begin{align*}
1 \text{ g} &= 5.612 \times 10^{26} \text{ MeV} \\
1 \text{ cm} &= 5.068 \times 10^{10} \text{ MeV}^{-1} \\
1 \text{ s} &= 1.519 \times 10^{21} \text{ MeV}^{-1} \\
1 \text{ MeV} &= 1.782 \times 10^{-27} \text{ g} \\
&= 5.068 \times 10^{10} \text{ cm}^{-1} \\
&= 1.519 \times 10^{21} \text{ s}^{-1}
\end{align*}
\]

Check:

\[
\frac{1 \text{ MeV}}{1 \text{ MeV}} = 1 = \frac{1.591}{5.068} \times 10^{11} \frac{\text{cm}}{\text{s}}
\]

\[
= 2.998 \times 10^{10} \frac{\text{cm}}{\text{s}} = c
\]

Derived quantities:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
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<td>(7.682 \times 10^{-33} \text{ MeV}^{3})</td>
</tr>
<tr>
<td></td>
<td>(1 \text{ g cm}^{-3})</td>
<td>(4.311 \times 10^{-6} \text{ MeV}^{4})</td>
</tr>
<tr>
<td>Velocity</td>
<td>(1 \text{ cm s}^{-1})</td>
<td>(3.336 \times 10^{-11})</td>
</tr>
<tr>
<td>Acceleration</td>
<td>(1 \text{ cm s}^{-2})</td>
<td>(2.196 \times 10^{-32} \text{ MeV})</td>
</tr>
<tr>
<td>Force</td>
<td>(1 \text{ dyn})</td>
<td>(1.232 \times 10^{-5} \text{ MeV}^{2})</td>
</tr>
<tr>
<td>Energy</td>
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<td>(6.242 \times 10^{5} \text{ MeV} = 624 \text{ GeV})</td>
</tr>
<tr>
<td>Temperature</td>
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<td>(8.616 \times 10^{-11} \text{ MeV})</td>
</tr>
<tr>
<td></td>
<td>(1 \text{ MeV})</td>
<td>(1.1606 \times 10^{10} \text{ K})</td>
</tr>
<tr>
<td></td>
<td>(T_{\odot})</td>
<td>(160 \text{ MeV} = 1.857 \times 10^{12} \text{ K})</td>
</tr>
</tbody>
</table>

Gravitational constant

\[
G = 6.668 \times 10^{-8} \text{ dyn cm}^{2} \text{ g}^{-2} = 6.70 \times 10^{-45} \text{ MeV}^{-2}
\]
11. CONCLUSION

Many things could not be treated here. The reader must consult the papers quoted in the text, in particular (I), (II), (III).

Comparison with experiments is mainly done in (II), but also the Atlas [Grote, Ranft, Hagedorn (1970)] contains experimental data drawn into the plots. Further checks are made in [Ranft (1970a, b)]. The last paper shows that the log $\tan \theta$-distribution comes out as observed: with increasing primary energy a two-bump shape appears — in both c.m. and lab. frame.

One question might have intrigued the reader all the time: what happens if the mass spectrum does not grow exponentially, at least not beyond some mass $M$? Well, then bootstrap is out, because there are of course fireballs beyond $M$ [even if $\rho(m)$ is cut off, $\sigma(m, V_0)$ continues to grow at least like $\exp (am^{3/4})$ — see section 5.2], but, for some reason or other, they are not admitted in $\rho(m)$. This "some reason or other" must, however, be an established physical principle for whose existence we have presently no indication. If such a principle reveals itself some day, then bootstrap will be limited. Nevertheless, the spectrum $\rho(m)$ has to grow exponentially up to some mass $M \gg 1$ GeV, because otherwise, at cosmic ray energies, we would be back to the Stefan-Boltzmann law $T \sim E^{4/3}$ (see section 5.2). A crude quantitative analysis [Hagedorn (1968 a)] shows that at $10^6$ GeV/c primary momentum, one would expect much larger mean transverse momenta (i.e. $T$ significantly greater than 160 MeV) than one actually observes, if $\rho(m)$ were cut off much below $M = 15$ GeV.

However, as far as curve fitting and prediction of spectra are the only aim, the present model with $\rho(m)$ cut off anywhere above, say, 20 GeV, would be perfectly good and not distinguishable from its form with $\rho(m)$ not cut off. It is a question not of practical applicability, but of insight in nature. Of course, somewhere all theories break down (the mass $M$ might be given by general relativity); however, as far as only strong interactions are in the game, we should not, without compelling reasons, break off a self-consistent scheme which has opened a new view
and has made unexpected predictions which came true -- we should not break it off only because "we cannot imagine that the spectrum grows and grows and grows".

The question of whether and how the model can be applied to exclusive experiments, has here been evaded; if it can be applied at all, then not in its present form which is too crude; firstly: energy-momentum conservation has to be handled more carefully; secondly: we deal with probabilities, not with amplitudes; in exclusive experiments phases and interferences are important -- they are neglected here; thirdly: our separation of collective motions from thermodynamics cannot be applied in exclusive reactions for the above first two reasons -- thus we had to change the whole model. Nevertheless, if one applies the present model with much handwaving to exclusive processes, then the miracle happens that some qualitative features come out, which are not sheer nonsense (this is the best one can say).

For details, and also for an extensive discussion of the misunderstanding that "transverse momentum limitation is only another word for the uncertainty principle", see the paper "Remarks on the Thermodynamical Model" [Hagedorn (1970 b)].

That paper ends with a section "philosophy" of which here I only wish to repeat that the obvious success of the thermodynamics discussed in these lectures presents a strong case for a general and complete hadronic bootstrap, where each hadron contains a world.
REFERENCES


Nahm W., private communication, March 1971.


The following pages should be considered as Section 12 of these lectures.

Equation (4.10), on p. 39, should read:

$$\sum \left[ \ldots \right] \Rightarrow \sum_{N=0}^{\infty} \int \left[ \ldots \right] \prod_{i=1}^{N} p(w_i) dw_i$$

Equation (6.2), on p. 61, should read:

$$\phi_{(\nu)} = \sum_{n=2}^{\infty} \left[ \frac{\nu_0}{(2\pi)^3} \right]^{n-1} \frac{1}{n!} \int d^3 \left( \sum_{i=1}^{n} p_i \right) \prod_{i=1}^{n} \delta (w_i - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}) \prod_{i=1}^{n} \rho (w_i) dw_i d^3 p_i.$$
The strong bootstrap version (with $m^{-3}$ in front of the exponential) has been introduced into the thermodynamical model for collisions (Hagedorn and Ranft 1972, Letessier and Tounsi 1972 a, b, c).

Scaling and limiting fragmentation have been reconsidered (Ranft 1971, Hagedorn and Ranft 1972).

Arguments for abandoning the view described here in Sections 10.1.2 and 10.1.3 have been given with the result that $F(\lambda)$ and $F_0(\lambda)$ are combined in one single function which never appears squared; also the necessity of factors $N(T)$ appearing in pair production might be doubled (Letessier and Tounsi 1972 a, b, Ranft 1972).

Gamma-spectra have been computed (Htun Than et al. 1972).

Good fits to data from 20 GeV to ISR energies are now made with one set of parameters (Ranft 1972, Letessier and Tounsi 1972 b, Htun Than and Ranft 1972).

Neutral spectra can be calculated (Letessier and Tounsi 1972 c).

Introducing one additional free parameter (not needed in single-particle spectra) describing the energy dependence of the average fireball mass, the model can be used to compute two (and more) particle spectra (correlations) as a function of this free parameter. The model can reproduce the results of multiperipheral as well as diffractive models, and continuously changes from the one to the other extreme. Comparison with experiments leads to a determination of the new parameter and to its interpretation in terms of an intermediate (more multiperipheral than diffractive) situation (Ranft and Ranft 1972 a, b).

On the whole it can be said that while the general spirit of the statistical bootstrap approach and its most prominent result, the exponential mass spectrum implying a limiting temperature, have not been changed, many details could be considerably improved. Parts of these lecture notes must now be considered as outdated, in particular all those in which the once-favoured solution with $m^{-3/2} \exp(m/T_0)$ is taken as a basis of discussion.
REFERENCES TO THE ADDENDUM


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Letessier J. and A. Tounsi, Inclusive thermodynamical spectra in the range 19 to 1500 GeV/c, Paris Univ. Preprint IPNO/TH 72-6 (1972 b).


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