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THE BREMSSTRAHLUNG IN CRYSTALS
AND OTHER CONNECTED PHENOMENA

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First of all, why? Why should we discuss these topics? I can perhaps give three different reasons: first, the behaviour of high-energy ionizing particles and gamma-rays in crystals give rise to a quite interesting and amusing phenomenology. Second, this phenomenology might be technically utilized in ways which may be of interest for high-energy physicists. The third reason - and for me personally the most stimulating - is that the study of this subject may give an opportunity for thinking of high-energy physics not in terms of $S$ matrix only.

The first two reasons are quite obvious and are the conventional sources of interest for our subject. The third reason is most speculative in nature, less commonly recognized, less obvious, but perhaps it will not be useless to spend a part of this first lecture to try to give an idea of what I have in mind when speaking of "thinking not in terms of $S$ matrix only in high-energy physics", and particularly what the crystals have to do with that.

The question has to do with the following general problem: which are the observables in relativistic quantum mechanics? One may perhaps recall that it has been just when discussing a question of this kind that Heisenberg in 1943 had stressed the concept of $S$ matrix, and given the fundamental outline of the $S$ matrix theory. As a matter of fact, the philosophy that scattering amplitudes and a few structural parameters, like masses, angular momenta, magnetic moments, lifetime, etc., are the only things which are observable in high energy-physics, and therefore the only things which are interesting to talk about, are largely spread. True enough, in $S$ matrix theory, we talk directly of asymptotical behaviour of particles, i.e., of the behaviour of particles at a very large distance. This way they are no longer interacting, and therefore, at times infinitely remote, in past or future, with respect to the time in which the interaction is thought to take place.

On the other hand, the facts of real life do take place in finite space regions and during finite time. One could argue that knowledge of
the scattering amplitude is perfectly sufficient to allow for a space
time description of the real life phenomena by means of suitable wave
packets; and in particular, to describe the space time correlation
which characterizes the supposedly causal behaviour of the physical
systems.

To stress this point of view, it is perhaps suitable to recall
a simple argument, due to Eisenbud and Wigner. According to their

\[ \psi_{\text{inc.}} = \frac{1}{\pi} \left[ e^{-i(k_0 r - i(\nu - \delta \nu) t)} + e^{-i(k_0 + \delta k) r - i(\nu + \delta \nu) t)} \right] \] (1.1)

According to a well known elementary argument, one can find the "group
velocity" of this "wave packet" by imposing that the difference of phase
of the two waves is for instance equal to zero.

One gets this way, as "group velocity"

\[ \nu = - \frac{\delta \nu}{\delta k} \] (1.2)

(the minus sign is due to the wave packet going towards the scattering
centre, supposed at the origin).

Now, the expression for the scattered wave will have the form

\[ \psi_{\text{scat.}} = \frac{1}{\pi} \left[ e^{i(k_0 r - i(\nu - \delta \nu) t)} + e^{i(k_0 + \delta k) r - i(\nu + \delta \nu) t)} \right] \] (1.3)
Imposing again that the phase difference of the two monochromatic waves is equal to zero, we get

\[ \delta \nu \left( t - \frac{\delta \eta}{\delta \nu} \right) = \delta k \cdot r \]  

(1.4)

This may be interpreted by saying that the scattered wave has a delay \( \delta \eta \)/\( \delta \nu \), due to the interaction with the scattering centre.

If we suppose for instance that the radius of action of the scattering centre is \( a \), the causality principle would imply that the scattered wave could not anticipate, with respect to waves just scattered by the origin, by more — say — than \( 2a/(\delta \nu / \delta k) \). This implies a limitation on the derivative of \( \eta \) with respect to \( \nu \) or with respect to \( k \), which, in the case considered, would be

\[ \frac{\delta \eta}{\delta k} \geq - 2a \]  

(1.5)

So one could argue that there is no need, for instance, of a direct experiment implying space time correlations (as the measurement of the velocity of signals), for an experimental control of causality validity. It appears indeed sufficient to study the scattering amplitudes and their dependence on the energy of the scattered particles.

All this is very good. One should remark, however, that the argument relies on the supposed unrestrained possibility of forming wave packets arbitrarily, according to the superposition principle of quantum mechanics, and of applying to these wave packets the usual interpretation rules of the quantum mechanics itself.

We have no serious reasons to doubt this possibility in non-relativistic quantum mechanics: the unrestricted validity of the superposition principle does correspond, in this case, to the assumption that any Hermitian operator defined in the relevant Hilbert space of the states represents an observable which can be measured.*

*) The possibility of forming in an operative way any wave packet \( \psi \rangle \) does correspond to the possibility of measuring the projection operator: \( P_\psi = |\psi\rangle\langle\psi| \).
In the relativistic quantum mechanics, however, the issue is not so clear. In particular, if we think — bringing the S matrix philosophy to its extreme consequences — that local quantities, like fields or currents, cannot really be measurable, not merely for technical difficulties, but for some fundamental reason, then the physical interpretation of the superposition principle has to be clarified. We risk, otherwise, to run into a circular argument.

One way of trying to clarify this point is to try to think whether other kinds of experiments in high energy physics are possible. Now, observations which imply space correlation on a scale which, in some aspects, may already be thought of as microscopic, in trivial real cases have already been performed.

Consider, for instance, the creation of electron-positron pairs by a gamma-ray in a fairly high-energy region, say, of the order of a few thousand GeV (as it is not extremely rare to find in cosmic rays); then the angular spread of the pair at the creation will be of the order of $10^{-7}$-$10^{-6}$ rad. We might think that the pair has originated in a region of linear dimensions of the order of the Compton wavelength of the electron. It will then have to travel at a distance of the order of one millimetre for getting the electron and the positron at distances greater than an atomic radius. But until the two particles are at a distance considerably greater than an atomic radius, their interactions with atoms (excitation, ionization,...) will not be the same as those produced by separate particles. In this case the probability of excitation or ionization of an atom cannot be expressed simply in terms of cross-sections anymore; it will depend essentially on the distance between the atoms, which for instance have to be ionized, and the target on which the conversion of the primitive gamma-rays in a pair is supposed to have taken place.

As a matter of fact, to be complete one should consider the interference effect of the ionization by Compton scattering of the primitive gamma-rays and by Coulomb interaction with the pair.
interference effect would be of course of relatively increasing importance, when the distance of the atom to be ionized from the converting target will be very small.

This effect has been actually observed in plates, at proximity of the origin of a very high energy pair, it appears to ionize much less than a particle at a minimum of ionization; along the path, the ionization is progressively increased, till it appears to be about twice the ionization of a single particle, and eventually the two particles do appear separately. It is in this last region that the usual ideal scheme underlying the S matrix description of high energy physics does apply directly, without need of any special consideration.

*) In the real case of a pair created and observed in a nuclear plate, the separation angle between electrons and positrons is changing and increasing rapidly due to scattering, but this fact does not change in a radical way the orders of magnitude involved in our consideration.

This very trivial example gives also a clue to the method which might be used in a quite general way for establishing space or space-time correlation, in a microscopic domain, by making observations on distances, say, of the order of a micron, which are actually in direct reach of the observer. This is obtained merely by using the Lorentz contraction device, which can transform a length of a few microns in the reference frame of the observer into a really microscopic length in the reference frame of the observed system.

Granted that there is a problem of clarifying which are the quantities directly observable in relativistic quantum mechanics, and particularly of establishing as far as one can push observations of a local character, one might ask what crystals have to do with this general problem.

I do not claim, obviously, that they are in any way essential for this purpose, but merely that they may be a useful device to start thinking in a concrete way about the problem itself.
The general reason is quite obvious, keeping in mind the consideration which we just made. Our problem is that of testing the superposition principle, and, possibly, in conditions which may vary arbitrarily at our command; but this means studying under such conditions interference phenomena, and for such a purpose the regular structure of the crystals appears to be quite useful. It is particularly important to notice the fact that the conditions of interference phenomena which we can see can be easily changed in a controlled way, for instance, by changing the angle of incidence of a beam of incoming particles, with the axis of the crystal.

Unfortunately, the practical use of crystals for this kind of purpose is strongly limited by the fact that crystals are not perfect, have finite dimensions, and, mainly, because of the thermal motion, or even the zero energy point motion, which the atoms composing the crystals have to undergo. These circumstances, and the fact that the energy of the particles which we have at hand in our laboratories now (and even of those of the next machine generation), seems to be by far insufficient, will probably limit in a rather drastic way the possibility of using crystals for the really fundamental purpose of checking directly micro-causality, etc.

However, it is always difficult to foresee which are the real possibilities of a method without quite accurate investigation. For this reason I hope that at the end of this course we shall discuss these topics.

The possibility of a coherence effect and of the interference between the radiative effects due to different nuclei in a solid, was first mentioned in a paper by Williams, where an estimate is also given


for the order of magnitude of the incoming electron energy that would produce effects of a considerable intensity. Williams realized correctly that in general these effects will decrease the radiation intensity,
but also that sometimes there may be an enhancement effect. For this purpose I quote his remarks "ad litteram":

"The interference effect will however not always be a "cutting-out" effect, because there will be cases when successive nuclei are on the same side of the path of the electron, thus enhancing the radiative effect."

Successively, without knowing Williams' remark, I independently realized that there could be interesting effects of interference in radiative interaction of high energy particles in solids, and thought it worth while to try and give a quantitative theory of this effect.

The results of my investigations appeared in a paper published in the Nuovo Cimento.


Then again, independently of the preceding authors, Landau and Pomeranchuk have given a theory of this phenomenon, whose results were comparable to mine. Later on, when comparison with experiment became possible, the theory has been again taken up by Dyson and Überall,


and particularly Überall, who performed extensive numerical calculations.

After these papers, quite a large number of experimental and theoretical papers has been devoted to this question, and we shall have the opportunity of discussing some of them in this course.

In my first research, I used the so-called Weizsäcker-Williams method for the calculation of bremsstrahlung. This method is generally considered only semi-quantitative, and for this reason subsequent authors did prefer a direct utilization of the Bethe-Heitler formula for the bremsstrahlung, supposing to be able to obtain thus more accurate results.
We shall see, however, that just in the case of the crystals, the Weizsäcker-Williams method gives as accurate results as any "exact" calculation using the first order Born approximation. This fact is due, as we shall see, to the thermal motion of the atoms belonging to the crystal lattice.

For this reason, and for the fact that the Weizsäcker-Williams method allows often a more "anschaulich" description of the phenomenon than a Bethe-Heitler calculation, we shall use in these lectures extensively the Weizsäcker-Williams method.

I shall therefore devote the last part of this lecture to a description of the Weizsäcker method for calculating the bremsstrahlung in the collision of a very fast electron against an isolated atom.

The second lecture will essentially be devoted to the description of the investigation contained in my first paper.

The third lecture will be devoted to quantitative discussion of the effect of the thermal motion on the radiative interaction.

The fourth lecture deals with the comparison of the numerical results of the theory with some experiments, with the polarization of the radiation and with pair creation.

The subsequent lectures will be mainly devoted to results which cannot be found in the literature and which are due to an investigation still in progress.

The first problem that we shall discuss in these lectures is the question of the validity of the Born approximation. We shall discuss approximate solutions of the problem of the motion of very high energy passing particles in crystals by means of the W.K.B. method in some cases, suitably improved.

The fifth and sixth lectures will be devoted to the foundations of this topic and the seventh lecture to the results obtained and to the discussion of possible technical applications for the collimation
of high-energy particles and the possibility of using crystals for distinguishing ionizing particles of different masses in the region of very high energy.

The remaining lectures, if any, will be devoted to the discussion of the possibility of using crystals for producing phenomena in which space time correlation in a fairly microscopical range might be detected.

Let us now try to remind you how the Weissflicker method works. First of all, instead of considering — as usual — a plane or spherical incoming wave, consider an incoming wave packet having rather small dimensions in transversal directions with respect to the average momentum. This allows to make use of semiclassical considerations in the treatment of the phenomenon, which helps a lot — in many cases — the physical insight. What will be the transversal dimensions of the wave packet? Quite small, but not arbitrarily small. This is because we wish that in the incoming state only one physical electron be contained. Now, the wave packet can be obtained by means of a superposition of plane waves which are solutions of the Dirac equation for the electron. As you know, this solution belongs both to positive and negative values of the energy. But the solutions that belong to positive values of the energy only, do not form a complete system of functions, and therefore we cannot describe an arbitrary wave packet by superposing positive energy solutions only. On the other hand, if we like to describe a state in which we have one physical electron, we have to use positive energy solutions only. Physically, we would otherwise get an electron plus one or more pairs. This condition imposes that the transversal dimensions of the wave packet cannot be smaller than the Compton wavelength of the electron.

The forces acting on the electron due to the atom against which it is going to collide, will be due, mainly, to the Coulomb electric field, owing to the atomic charges. As usually we may consider the Coulomb field of the nucleus which, at relatively considerable distance, will be progressively screened out by electric charges. If the electron is ultra-relativistic, i.e., if \( \gamma = \frac{E}{mc^2} \gg 1 \) (where \( E \) is the energy and \( m \) the rest mass of the electron), the electromagnetic field
due to the atom, as seen by the electron - i.e., in a reference frame in which the average momentum of the incoming electron is zero - will look, however, very similar to a superposition of light waves. This may be immediately seen, remembering how an electromagnetic field transforms under a Lorentz transformation

\[
\begin{align*}
\vec{E}_{\parallel}' &= \vec{E}_{\parallel}, \\
\vec{E}_{\perp}' &= \left( \frac{\vec{E} + \frac{1}{c} [\vec{\nabla} \vec{B}]_{\perp}}{\sqrt{1-\beta^2}} \right), \\
\vec{B}_{\parallel}' &= \vec{B}_{\parallel}, \\
\vec{B}_{\perp}' &= \left( \frac{\vec{B} - \frac{1}{c} [\vec{\nabla} \vec{E}]_{\perp}}{\sqrt{1-\beta^2}} \right)
\end{align*}
\]

(The symbols $\parallel$ and $\perp$ indicate respectively the parallel or normal components with respect to the direction of motion.) If the electron is ultrarelativistic, the component of the field parallel to the direction of motion becomes negligible with respect to the normal component, and so, with an error of the order of $1/\gamma$, we can say that the field seen by the incoming electron is a transversal field. Furthermore, considering that in the reference frame of the atom the magnetic field is zero, the electron will see in its frame an electric and magnetic field, which will be orthogonal to each other, and of equal intensity, (with an error of the order of $1-\beta \approx 1/2 \gamma^2$). On the other hand, obviously, this electromagnetic field will travel towards the electron with a velocity parallel to the average momentum of the wave packet, and equal in modulus to $\beta c$, which is the velocity of light, again with a relative error of $1-\beta \approx 1/2 \gamma^2$.

It is for this reason that, allowing for errors which are at maximum of the order of $1/\gamma$, we can say that the electron will see the atomic field like a superposition of light waves.

Obviously, the intensity and the shape of the light pulse seen by the electron will depend on the distance of the path of the electron from the nucleus. If this distance is, however, considerably larger than the Compton wavelength of the electron, the intensity and the shape of the field seen by the electron on different points of any cross-section of the wave packet orthogonal to the direction of the motion, will not be appreciably different.
Consequently, even taking into account the finite transverse dimensions of the electron wave packet, we will be in a position to compute the effect of the incoming field on the electron, as one of a superposition of plane light waves.

The radiation emitted by the electron under the effect of this light wave will be about the same as that emitted by the effect of a superposition of real quanta states, taken with suitable amplitudes needed to describe the incoming field. It may therefore be calculated by applying the Klein-Nishina formula for the scattering of quanta.

So the calculation will be performed in the following way: first, we have to evaluate the spectrum intensity of the incoming field for any collision distance from the electron, second, we calculate, for any distance \( r \), the spectral intensity of the radiation in function of the emission angle, just by applying the Klein-Nishina formula. Then we transform the radiation obtained to the reference frame of the atom, and eventually we integrate with respect to all collision distances. It is in performing this integration that the chief approximations and uncertainties of this semi-classical method came out. This is due to the fact that all the preceding considerations are valid only for distances which are large with respect to the Compton wavelength of the electron. Thus we have to cut off the integration with respect to the collision distance \( r \) to a distance which cannot be smaller than the Compton wavelength of the electron.

One can argue with Weizsäcker that collision at shorter distances will not give any very appreciable contribution to the total bremsstrahlung cross-section. The Weizsäcker argument runs like follows:

If the electron wave packet just passes through the nucleus, the different parts of the packet that sit on the opposite parts of the nucleus, will experience opposite accelerations, and therefore taking into account (we shall see the reason in a moment) that in the electron reference frame a radiation will be emitted with a wavelength not appreciably shorter than the Compton wavelength of the electron, a destructive interference between the waves emitted from the different parts of the wave packets will occur, and the total emission will be considerably reduced in this way.
Independently of this kind of considerations, the best reason probably for not worrying too much about the necessity of the cut-off is, first of all, that the cut-off distance enters only logarithmically in the final result, and in the second place, that the results obtained by just taking the cut-off equal to the Compton wavelength of the electron do agree well with those of Bethe and Heitler.

This "a posteriori" argument for a theory is indeed not very satisfying. Luckily enough, in the case of crystals, we shall not have to apply arguments of this kind because we shall need semiclassical considerations only in a region where they are fully justified.

Let us try now to see how our programme can be carried out. The intensity of the radiation falling on the electron at a distance \( r \) is given by the Poynting vector

\[
\mathbf{P} = \frac{\mathbf{c}}{4\pi} \mathbf{E} \times \mathbf{B}
\]

and as \( |\mathbf{B}| \sim |\mathbf{E}| \), \( |\mathbf{P}| = (\mathbf{c}/4\pi)|\mathbf{E}|^2 \). Now if we put

\[
E_{\perp} = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} f(\omega) e^{-\omega t} \, d\omega \tag{1.7}
\]

One gets, supposing for a moment to consider a Coulomb field due to the nuclear charge \( Z e \),

\[
f(\omega) = \frac{Z e^2 \gamma}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{r e^{-\omega t} \, dt}{(r^2 + \epsilon^2 + \zeta^2 t^2)^{3/2}} \tag{1.8}
\]

The intensity of radiation falling on the electron at distance \( r \), between the frequencies \( \omega \) and \( \omega + d\omega \), is given by

\[
F(\omega) = \frac{a}{4\pi} \int f^2(\omega) \tag{1.9}
\]

\( f(\omega) \) may be exactly evaluated by means of Hankel functions. In the case of the Weiss-Schäfer calculation, he approximated the Hankel function by means of a step function. This is not too bad. The critical frequency for the discontinuity in the step function is the inverse of the
duration of the collision. Measuring the frequency in natural units

\[ \omega_0 = \frac{mc^2}{h} \]  

one can then write

\[ F(\omega) d\omega = \left\{ \begin{array}{ll} \frac{Z^2 - \frac{e^2}{\pi^2 R^2 \lambda}}{\pi^2} d\omega, & (\omega < \frac{\lambda \gamma}{R}) \\ 0, & (\omega > \frac{\lambda \gamma}{R}) \end{array} \right. \]  

(1.91)

where \( \lambda \) is, as usual, the Compton wavelength of the electron.

Now, following Klein and Nishina, the radiation having frequency between \( \omega \) and \( \omega + d\omega \) will give rise to a diffused radiation, in the solid angle \( d\Omega \), making an angle \( \theta \) with the incoming direction, is given by

\[ I(\omega) \omega d\omega d\Omega = \frac{F(\omega) \frac{Z^2 - \frac{e^2}{\pi^2 R^2 \lambda}}{\pi^2}}{1 + \omega^2} \left\{ \begin{array}{ll} \frac{1 + \omega^2 (1 - \cos \theta)^2}{(1 + \omega (1 - \cos \theta)) [1 + \omega (1 - \cos \theta)]} d\omega d\Omega & (\omega < \frac{\lambda \gamma}{R}) \\ 0 & (\omega > \frac{\lambda \gamma}{R}) \end{array} \right. \]  

(1.11)

\( r_0 \) being the classical radius of the electron. The diffused frequency \( \omega' \) will be connected to the primary frequency by means of the usual relation

\[ \omega = \frac{\omega'}{1 - \omega' (1 - \cos \theta)} \]  

(1.12)

It remains then merely to perform the Lorentz transformation to the rest system of the atom.

A photon, of frequency \( \omega' \), with a scattering angle \( \theta \) in the rest system of the electron, will be changed into a photon of frequency \( \nu \) in the rest system of the atom, where \( \nu \) is given by

\[ \nu = \omega'(1 - \beta \cos \theta) \gamma \approx \omega'(1 - \cos \theta) \gamma \]  

(1.13)
Now, if we put the ratio between the energy of the photon in the rest system of the atom, to the energy of the electron, equal to $\varepsilon$, we have

$$\varepsilon = \frac{v}{\gamma} \tag{1.14}$$

(remember that we are taking $\omega_0$ as frequency unit). We get then

$$\omega' = \frac{\varepsilon}{1 - \beta \cos \theta} \approx \frac{\varepsilon}{1 - \cos \theta} \tag{1.15}$$

and

$$\omega = \frac{\varepsilon}{1 - \beta \cos \theta} - \varepsilon (1 - \cos \theta) \approx \frac{\varepsilon (1 - \varepsilon) (1 - \cos \theta)}{(1 - \varepsilon)^2 (1 - \cos \theta)} \tag{1.15'}$$

$$d\omega \approx \frac{d\varepsilon}{(1 - \varepsilon)^2 (1 - \cos \theta)}$$

The energy $I(\omega) \ d\omega \ d\Omega$ scattered in the electron rest system becomes then, in the rest system of the atom,

$$I_{\alpha}(\omega) \ d\omega \ d\Omega = \gamma F(\omega) \frac{r^2}{2} \left[ (1 + \frac{r^2}{4} \varepsilon^2) (1 - \varepsilon) + \varepsilon \right] \ d\varepsilon \ d\Omega \tag{1.16}$$

$F(\omega)$ has a constant value for $\omega < (\lambda \gamma / r)$ which, for the scattered quantum, gives

$$\cos \theta \leq \frac{1 - \varepsilon (1 + \frac{r}{\lambda \gamma})}{1 - \varepsilon} \tag{1.17}$$

and, putting $\cos \theta = -1$, we obtain

$$\frac{r}{\lambda \gamma} \leq \frac{2(1 - \varepsilon)}{\varepsilon} \tag{1.17'}$$

The limitation is due to the fact that in the incoming field the components with a frequency higher than a definite one, depending on the distance, have a practically vanishing intensity. To obtain...
a cross-section for the bremsstrahlung it is sufficient now to integrate with respect to the solid angle $d\Omega$ and the collision distance $r$, taking into account all the discussed limitations. For the case of a pure Coulomb field (neglecting screening) we get then for the cross-section relative to a photon of fractional energy $E$, the following expression

$$d\sigma(E) = 2\pi \int \frac{2\lambda y^{1-\frac{E}{\lambda}}}{\varepsilon} d\sigma(E) r dr =$$

$$\frac{4Z^2\varepsilon_0^2}{137} \left[ \frac{4}{3} (1-\varepsilon) + \varepsilon^2 \right] \log \frac{2\lambda y^{1-\frac{E}{\lambda}}}{\varepsilon} \frac{d\varepsilon}{\varepsilon}$$

(1.18)

In the case in which one takes into account the screening due to external electrons, what is changed is merely the logarithmic factor in the preceding expression, which becomes

$$\log \frac{13\lambda}{2\sqrt{3}}$$

if

$$\frac{2\lambda y^{1-\frac{E}{\lambda}}}{\varepsilon} \gg \frac{13\lambda}{2\sqrt{3}}$$

(1.19)

If we compare this result with the "exact" Bethe-Heitler formula, we find that the error is of the order of 10-20%.

Let us conclude by a remark: the essential reason for the success of the semiclassical method of Weizsäcker is due to the combination of two facts. The first is the long range character of the electromagnetic forces, and the second is that the Klein-Nishina cross-section is decreasing rapidly when the frequency of the incoming radiation becomes higher than $\omega_0$. Due to the first fact, we can think of the electron like a localized object, without committing too great errors. The second fact renders, in all cases, relatively unimportant the high frequency component of the atomic field, and consequently we get very small collision parameters. For this reason the dependence on the cut-off is merely logarithmic. One may understand, however, that any attempt to apply a method similar to that of Weizsäcker to radically different cases (for instance, short range forces) is very likely to be bound to insuccess.