1996–1997 ACADEMIC TRAINING PROGRAMME

LECTURE SERIES FOR POSTGRADUATE STUDENTS

SPEAKER : L. ALVAREZ-GAUME / CERN-TH
TITLE : Quantum Field Theory
TIME : 7, 8, 9, 10 & 11 October, from 11.00 to 12.00 hrs
PLACE : Auditorium

ABSTRACT

Quantum Field Theory provides the most fundamental language known to express the fundamental laws of Nature. It is the consequence of trying to describe physical phenomena within the conceptual framework of Quantum Mechanics and Special Relativity. The aim of these lectures will be to present a number of concepts and methods in the subject which many of us find difficult to understand. They may include (depending on time): the need to introduce quantum fields, the realization of symmetries, the renormalization group, non-perturbative phenomena, infrared divergences and jets, etc.

Some familiarity with the rudiments of Feynman diagrams and relativistic quantum mechanics will be appreciated.
A Small Course in
Quantum Field Theory

Luis Alvarez-Gaumé
INTRODUCTION

These notes are a text transcription of a set of lectures given in CERN’s academic training programme during the week of the 7th of November 1996. In five hours it is difficult to give a reasonable introduction to a subject as vast as Quantum Field Theory. Fortunately there are now good books with a modern presentation of the material. The only reasonable option for such a course is to select a number of topics that we are accustomed to but that we often have difficulties in understanding and to give a detailed presentation. The original plan was to include a few other topics than those covered in these notes, but I preferred to keep the material as close as possible to the original lectures. For further reading I would recommend looking at the two recent books:

- M.E. Peskin and D.V. Schroeder: An Introduction to Quantum Field Theory. Addison Wesley.

In these references the interested reader can find an authoritative treatment of the subjects covered in this minicourse as well as many other topics with extensive references to the literature.
LECTURE 1

The need for Quantum Fields

QFT is the most basic language known so far to express the fundamental laws of nature. When we try to put together Quantum Mechanics (QM) and Special Relativity we are led inevitably to the theory of quantum fields and to a many (\(\infty\)) body description of even the simplest systems. There are several ways to argue this basic point.

In QM we are used to the notion that the Hilbert space contains one, possibly few well-defined 'bodies'. The hydrogen atom \(H\) if ionized leads to an electron and a proton: \(e^- + p\). Hence it suffices \(\mathcal{H}_e \otimes \mathcal{H}_p\) to describe the system. We say \(H\) is a bound state of \(e^-, p\). In the non-relativistic approximation we say that the proton \(p\) is a bound state of three quarks \(uud\), but this does not mean that we can restrict ourselves to \(\mathcal{H}_u \otimes \mathcal{H}_u \otimes \mathcal{H}_d\). The proton is much more than that. If we look at energies \(E > 100\) MeV, we also see the pion cloud and other structures that cannot be simply described in terms of a one-body wave function. The relativistic dispersion relation allows for negative energy solutions, and this always leads to problems with fixing the number of particles (this can be seen in terms of the old Klein paradox described below). Let us briefly see what happens if we insist on maintaining the notion that relativistic wave equations should describe the quantum behaviour of particles in the relativistic domain. For a free scalar particle the relation between energy and momentum is,

\[
E^2 = p^2 + m^2 \quad E = \pm \sqrt{p^2 + m^2}.
\]

Using the correspondence principle (we use throughout natural units where \(\hbar = c = 1\)):

\[
\begin{align*}
E &\rightarrow i\partial_t & \vec{p} &\rightarrow -i\vec{\nabla} \\
-\partial_t^2 &\rightarrow -\nabla^2 + m^2
\end{align*}
\]

\(\left\{ (\partial_t^2 - \nabla^2 + m^2)\Psi = 0 \right\}

\]
whose plane-wave solutions are:

\[ \psi(x,t) = e^{-ik_xx} = e^{-i(\omega t - k_x x)} \ 
\omega^2 = k^2 + m^2 . \]

Note that both signs of \( \omega \) are necessary to describe a complete set of solutions to the wave equation. For complex solutions we can always construct a conserved current. However the time-like component is not positive definite and it cannot be used to define a probability density.

\[ j_\mu = \frac{1}{2} i (\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*) \]

\[ \partial_\mu j^\mu = 0 \quad j^0 \quad \text{not positive definite} \]

\[ f_p(x) = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega_p}} e^{-ipx} \]

\[ f_{-p}(x) = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega_p}} e^{ipx} \]

\[ \langle \psi_1|\psi_2 \rangle = i \int d^3x (\psi_1^* \partial_0 \psi_2 - \psi_2 \partial_0 \psi_1^*) \]

\[ \langle f_p | f_{p'} \rangle = \delta(p - p') \quad \langle f_{-p} | f_{-p'} \rangle = -\delta(p - p') \quad \langle f_p | f_{-p} \rangle = 0 \]

The \( f_p \) are the positive energy solutions and the \( f_{-p} \) the negative energy ones. Note that the scalar product is negative on the negative solutions. The spectrum is schematically drawn in the figure.

For a single particle of mass \( m \) in a box of size \( L^3 \) with \( L \gg 1/m \), we can say with confidence that there is a particle on the box. If we slowly decrease \( L \), \( \Delta x \sim L, \Delta p \sim 1/L, \Delta E \sim 1/L^2 \). For \( L \gg 1/m \) \( \Delta E \ll m \). As \( L \) is decreased, we eventually reach the Compton wavelength \( \Delta E \sim m \), particle pair production begins to be possible, and we can no longer say that there is a single particle inside the box. A way of saying the same thing in equations is the so-called Klein paradox briefly explained below.
Klein paradox

Consider a potential with the profile in the figure. To solve the Klein-Gordon equation we consider the wave function in region I containing the incoming and reflected wave, and in region II where we have the transmitted wave. The energy of the wave is taken to be $E$. 

$$
\Psi_I = e^{ip_1x} + Re^{-ip_2x}
$$

$$
\Psi_{II} = Te^{ip_2x}
$$

$$
E = (p_1^2 + m^2)^{1/2} \quad E = (p_2^2 + m^2)^{1/2} + V_0
$$

Matching at the barrier, we compute the reflection and transmission coefficients.

$$
\Psi_I(0) = \Psi_{II}(0) \quad \Psi_I'(0) = \Psi_{II}'(0) \quad T = \frac{2p_1}{p_1 + p_2}
$$

$$
1 + R = T \quad p_1(1 - R) = p_2T \quad R = \frac{p_1 - p_2}{p_1 + p_2}
$$

$$
p_1 = (E^2 - m^2)^{1/2}
$$

$$
p_2 = ((E - V_0)^2 - m^2)^{1/2}
$$

$E = m + \epsilon$ fixed. If $\epsilon > V_0$, the standard picture applies. We have reflected and transmitted waves.

$$
p_2 = [(m + (\epsilon - V_0))^2 - m^2]^{1/2} = (2m(\epsilon - V_0) + (\epsilon - V_0)^2)^{1/2}
$$

$$
= (x(2m + x))^{1/2} \quad x = \epsilon - V_0
$$
If $x < 0$, but $2m + x > 0$, we have total reflection.
If $x < 0$, but $2m + x < 0$, ($V_0 > 2m$), $p_2$ is real, and we have a transmitted wave. This is interpreted by saying that the barrier has created a pair. We cannot avoid a many body interpretation.

In QM any Hermitian operator is an observable. Not so in QFT. Measurements can be localized, but we have to respect microscopic causality. This is a restriction imposed by the causal structure of space-time in special relativity. If $A_{R_1 R_2}$ are two operators associated to measurements taken at space-like separated regions, they must commute:

$$[A_{R_1}, A_{R_2}] = 0$$

$$(R_1 - R_2)^2 < 0$$

Observables must know about space-time position. Since we can localize measurements, the basic operators on the Heisenberg representation will be of the form $A(x)$

$x = (t, \vec{x})$ is a label and not an operator as in QM. Thus we have an infinite number of degrees of freedom.

If we insist still on the one-particle interpretation, we find a direct conflict with causality. Imagine we consider at time $t = 0$ a wavefunction concentrated near $\vec{x} = 0$, $\psi(\vec{x}, 0) \sim \delta^3(\vec{x})$, and we evolve it in time with the relativistic Hamiltonian $H = \sqrt{-\nabla^2 + m^2}$. The result is:

$$\psi(x, 0) = \delta^3(\vec{x})$$
$$\psi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x} - i\sqrt{k^2 + m^2} t}$$
$$H = \sqrt{-\nabla^2 + m^2}$$
\[ \psi(\vec{x}, t) = -\frac{1}{2\pi^2 r} \frac{\partial}{\partial r} \int_0^\infty dk \, \cos kr \, e^{-\mu(k^2 + m^2)^{1/2}} \]

For any \( t > 0 \) the wave function spreads outside the light-cone, i.e. \( \psi(r, t) \neq 0 \) for \( r > t > 0 \), as can be seen by evaluating the integral using the contour on the figure. This is again bad new for the one-body interpretation.

**Fock Space**

Let us take a different point of view. After all in one way or another when we obtain information about a physical system we do it through a scattering experiment. In a scattering process we assume that the initial and final states are well described by wave packets made of free particles. In the non-relativistic case a free moving particle of given momentum \( \vec{k} \) is described by a ket \( |\vec{k}\rangle \), with normalization:

\[ |\vec{k}\rangle \; ; \; \langle \vec{k}|\vec{k}'\rangle = \delta^3(\vec{k} - \vec{k}') \]

The rotation group acts unitarity:

\[ U(R)|\vec{k}\rangle = |R\vec{k}\rangle \; ; \; \vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle \]

Some basic relations are:

\[ 1 = \int d^3k |\vec{k}\rangle \langle \vec{k}| \]

\[ \vec{P} = \int d^3k |\vec{k}\rangle \langle \vec{k}| \]

\[ U^{-1}(R)\vec{p}U(R) = \int d^3k |R^{-1}\vec{k}\rangle \langle R^{-1}\vec{k}| = \int d^3k |\vec{k}\rangle R\vec{k}\langle k| = R\vec{P} \]

By analogy, since we observe weakly interacting particles on approximate momentum states we want to construct a multiparticle Hilbert
space with a unitary representation of the Poincaré group (ISO (3,1)) to be the basic object to describe physical processes. As with the rotation group above, to construct a unitary representation of the Lorentz group we need:

- An invariant measure with respect to Lorentz transformations to count states.
- An invariant normalization

The measure is:

\[ d^3k \rightarrow \frac{d^3k}{2\omega_k(2\pi)^3}, \quad \omega_k = \sqrt{k^2 + m^2} \]

We have states \(|p\rangle\) which are eigenstates of four-momentum

\[
\begin{cases}
P^0|p\rangle = \sqrt{p^2 + m^2}|p\rangle \\
\vec{p}|p\rangle = \vec{p}|p\rangle
\end{cases}
\]

Starting with the non-relativistic states \(|\vec{k}\rangle\) we construct:

\[
|p\rangle = (2\pi)^{3/2}(2\omega_p)^{1/2}|\vec{p}\rangle
\]

\[
\langle p|p'\rangle = (2\pi)^32\omega_p\delta^3(\vec{p} - \vec{p}')
\]

\[2\omega_p\delta^3(\vec{p} - \vec{p}')\] is a relativistic invariant. It is clearly invariant under rotations. To show invariance under boosts, using rotational invariance it suffices to prove invariance under boosts along the \(x\)-direction. Consider motion along the \(x\)-direction:

\[2\omega_p\delta(p_x' - p_x)\]
Since

\[
\begin{align*}
\omega_p &= m \cosh \phi_p \quad \Rightarrow \quad 2\omega_p \delta(p - p') = 2m \delta(\sinh \phi_p - \sinh \phi_{p'}) \\
p_z &= m \sinh \phi_p \quad \Rightarrow \quad 2m \frac{1}{\cosh \phi_p} \delta(\phi_p - \phi_{p'}) = 2\delta(\phi_p - \phi_{p'})
\end{align*}
\]

The Lorentz boost acts by \( \phi \rightarrow \phi + \text{const} \), and invariance follows.

The action of the Lorentz group is:

\[
\left\{ \begin{array}{l}
P^\mu |p\rangle = p^\mu |p\rangle \\
U(\Lambda)|p\rangle = |\Lambda p\rangle
\end{array} \right.
\]

On one particle states

\[
P^\mu = \frac{d^3p}{(2\pi)^3 2\omega_p} |p\rangle p^\mu \langle p|
\]

Unitarity: \( \langle p'^\mu |U^+(\Lambda) U(\Lambda) |p\rangle = \langle \Lambda p'| \Lambda p\rangle = \langle p|p'\rangle \) (inv. measure)

\[
U^{-1}(\Lambda) P^\mu U(\Lambda) = \Lambda_\mu^\nu P^\nu
\]

Multiparticle states follow in the same way, but there is a simple way using Fock operators.

In the non-relativistic case we introduce creation and annihilation operators \( a(k), a(k)^+ \) satisfying:

\[
[a(k), a(k')^+] = \delta^{(3)}(k - k')
\]

\[
|k\rangle = a^+(k)|0\rangle \quad \langle 0|0\rangle = 1
\]

Now introduce:

\[
\begin{align*}
\alpha(k) &\equiv (2\pi)^{3/2}(2\omega_k)^{1/2} a(k) \\
\alpha^+(k) &\equiv (2\pi)^{3/2}(2\omega_k)^{1/2} a^+(k)
\end{align*}
\]

\[
[a(k), a^+(k')] = (2\pi)^3 2\omega_k \delta^3(k - k')
\]
General 1-particle state:

\[ |f\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \alpha^+(k) |0\rangle f(k) \]

\[ \langle f | f' \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} f^*(k) f'(k) , \]

for multiparticle states:

\[ |f\rangle = \int \prod_i \frac{d^3k_i}{(2\pi)^3 2\omega_{k_i}} f(k_1, \ldots, k_n) \alpha^+(k_1) \cdots \alpha^+(k_n) |0\rangle . \]

This way we obtain the Fock space generated by acting with arbitrary linear combinations of polynomials in the creation operators on the vacuum:

\[ \alpha^+(k) |0\rangle = |k\rangle \]

\[ U(\Lambda) \alpha^+(k) |0\rangle = U(\alpha) \alpha^+(k) U^{-1}(\Lambda) U(\Lambda) |0\rangle = |\Lambda k\rangle = \alpha^+(\Lambda k) |0\rangle \]

Simple observable: Scalar Field

Since measurements can be localized, we can try to construct observables of the form \( \phi(x) \). We make some assumptions.

i) \( \phi^+(x) = \phi(x) \).

ii) \( [\phi(x), \phi(y)] = 0 \ (x - y)^2 < 0 \).
iii) Translational invariance \( e^{iP_0 \phi(x)}e^{-iP_\alpha} = \phi(x - a) \), is not really an assumption it means that \( P_0, P_\alpha \) generate \( t, x \) translations.

iv) Lorentz invariance:
\[
U^+(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)
\]

v) As a simplifying assumption take \( \phi \) linear on \( \alpha, \alpha^+ \). Since \( \alpha, \alpha^+ \) form a complete set of operators, any observable can be expressed in terms of them. These constraints lead to a unique answer
\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (f(k)\alpha(k) + g(k)\alpha^+(k))
\]
\[
U^+(\Lambda)\phi(0)U(\Lambda) = \phi(0) \Rightarrow f(k, 0), \ g(k, 0) \text{ independent of } k
\]
then:
\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (e^{ikx}\alpha(k) + e^{ikx}\alpha^+(k))
\]
This field satisfies:

a) \( (\Box + m^2)\phi(x) = 0 \)

b) \([\phi(\vec{x}, t), \phi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})\]

\[
[\phi(x), \phi(y)] = i\Delta(x - y)
\]

\[
\Delta(x - y) = \Delta_+(x - y) - \Delta_+(y - x) \quad \Delta_+(x - y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x - y)}
\]

\[
i\Delta(x - y) = \int \frac{d^4k}{(2\pi)^3} e^{-ikx} \epsilon(k^0) \delta(k^2 - m^2)
\]

The same results can be obtained more straightforwardly using canonical quantization of classical fields. It is nevertheless useful to see how far we can go if we insist on the existence of free asymptotic states, locality, and relativistic invariance.
LECTURE 2

In the following table we summarize the standard rules of canonical quantization in particle mechanics and its extension to the field theory context. We start with a Lagrangian \( L(q_i, \dot{q}_i) \) which in field theory becomes a Lagrangian density

<table>
<thead>
<tr>
<th>Lagrangian</th>
<th>( L(q_i, \dot{q}_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi^a(\vec{x}, t) )</td>
<td>( L(\phi^a(\vec{x}, t), \partial_\mu \phi^a(\vec{x}, t)) )</td>
</tr>
<tr>
<td>( q^i(t) \to \phi^a(\vec{x}, t) )</td>
<td>( i \to (\vec{x}, a) )</td>
</tr>
<tr>
<td>Action</td>
<td>( S = \int dt L(q, \dot{q}) )</td>
</tr>
<tr>
<td></td>
<td>( S = \int d^4x L(\phi, \partial_\mu \phi) )</td>
</tr>
<tr>
<td>Equations of motion</td>
<td>( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} = \frac{\partial L}{\partial q^j} )</td>
</tr>
<tr>
<td></td>
<td>( \partial_\mu \frac{\delta L}{\delta \phi^a} = \frac{\delta S}{\delta \phi^a} )</td>
</tr>
<tr>
<td>Canonical momenta</td>
<td>( p_i = \frac{\partial L}{\partial \dot{q}^i} )</td>
</tr>
<tr>
<td></td>
<td>( \Pi_\phi(\vec{x}, t) = \frac{\partial L}{\partial \partial_\mu \phi^a(\vec{x}, t)} )</td>
</tr>
<tr>
<td>Canonical commutation relation</td>
<td>([q_i, p_j] = i\delta_{ij} )</td>
</tr>
<tr>
<td></td>
<td>([\phi^a(\vec{x}, t), \pi_b(\vec{y}, t)] = i\delta_{ab} \delta(\vec{x} - \vec{y}) )</td>
</tr>
<tr>
<td>Hamiltonian</td>
<td>( H = \sum_i p_i \dot{q}_i - L )</td>
</tr>
<tr>
<td></td>
<td>( H = \int d^3x (\Pi_\phi(\vec{x}) \dot{\phi^a(\vec{x})} - L) )</td>
</tr>
<tr>
<td>Time evolution of any operator</td>
<td>( \frac{idA}{dt} = [A, H] )</td>
</tr>
<tr>
<td></td>
<td>( i\frac{dA}{dt} = [A, H] )</td>
</tr>
</tbody>
</table>

The simplest example is a free massive scalar fields:

\[
L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \left\{ \begin{array}{c}
(\Box + m^2)\phi = 0 \\
\Pi = \dot{\phi}
\end{array} \right.
\]
\[ [\phi(x,t), \Pi(y,t)] = i\delta^3(x - y) \]

\[
H = \int d^3x \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right)
\]

\[
\phi = \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} (a(k)e^{-ikx} + a^+(k)e^{ikx})
\]

\[
H = \int d^3k \frac{\omega_k}{2} (a^+(k)a(k) + a(k)a^+(k))
\]

\[
= \int d^3k \frac{\omega_k}{2} a^+(k)a(k) + \frac{1}{2} \omega_k \delta^3(0)
\]

\[ :H: = H - \langle 0 | H | 0 \rangle = \int d^3k \omega_k a^+(k)a(k) \]

In the last line we have defined the notion of a normal ordered operator. Normal ordering is the instruction to place all creation operators to the right and the annihilation operators to the left. This removes the (infinite) zero point energy

\[ \delta^3(0) \int d^3k \frac{1}{2} \omega_k , \]

which in finite volume \( V \) is

\[ V \sum_k \frac{1}{2} \omega_k . \]

Similar arguments can be carried out for more complicated fields.

As a final remark we should point out that the parameter \( m \) appearing in the Lagrangian for the free massless scalar can be identified with the physical mass of the associated particles. For interacting Lagrangians this identification between parameters in the Lagrangian and physical observables is not necessarily so straightforward, and the machinery of renormalization and the renormalization group provides a more systematic approach to set up these relationships.
Lorentz Group Representations

To understand better the type of fields and free wave equations involved in the theory of relativistic fields it is useful to review the representation theory of the Lorentz group. A Lorentz transformation can be represented as a composition of a rotation and a boost. If we denote by $\tilde{J}$ (resp. $\tilde{M}$) the generators of infinitesimal rotations (resp. boosts); finite rotations (resp. boosts) with axis the unit vector $\vec{e}$, and angle (resp. rapidity) $\phi$ can be written as:

$$R(\vec{e}, \phi) = e^{-i\phi \tilde{J}} \quad B(\vec{e}, \phi) = e^{-i\phi \tilde{M}}$$

(rotations) (boosts)

We also have translations for the inhomogeneous group. We can compute the commutation relations of $J$ and $M$ by looking at a particular representation:

$$[J_i, J_j] = i \epsilon_{ijk} J_k \quad \tilde{J}^{(\pm)} = (\tilde{J} \pm i\tilde{M})/2$$
$$[J_i, M_j] = i \epsilon_{ijk} M_k \quad [J^\pm_i, J^\pm_j] = i \epsilon_{ijk} J^\pm_k$$
$$[M_i, M_j] = -i \epsilon_{ijk} J_k \quad [J^+_i, J^-_j] = 0$$

On the left hand side we have the commutators in terms of $\tilde{J}, \tilde{M}$, while on the right hand side we have the commutators in terms of the new generators $\tilde{J}^{(\pm)}$. In terms of the latter the algebra becomes two copies of the angular momentum algebra. Since the representations of this group are labelled in terms of a spin $S = 0, 1/2, 1, \cdots$; with dimension $2S + 1$, the finite dimensional irreducible representations of the Lorentz group are labelled by two spins $S_+, S_- = 0, 1/2, 1, \cdots$ and have dimension $(2S_+ + 1)(2S_- + 1)$.

Under parity $J \rightarrow J$ and $M \rightarrow -M$, $J^{(+)} \rightarrow J^{(-)}$ while under complex conjugation $J^{(+)} \leftrightarrow J^{(-)}$,

- $\text{SO}(3) \ D^{(S)}(R)$ with dimension $2S+1$
- $\text{SO}(3,1) \ D^{(S_+, S_-)}(A)$ with dimension $(2S_+ + 1)(2S_- + 1)$
\[ \mathcal{J}^+ + \mathcal{J}^- \] are the rotation group generators. Hence the rotations properties follow from the Clebsch-Gordan decomposition of \( S_+ \otimes S_- \). Some examples:

\[ \mathcal{J} = \mathcal{J}^+ + \mathcal{J}^-, \quad \vec{M} = \frac{1}{i} (\mathcal{J}^+ - \mathcal{J}^-) \]

1) \( S_+ = S_- = 0 \) scalar field.

2) \( S_+ = 1/2, S_- = 0 \) \((1/2, 0)\) right-handed spinor.

3) \( S_+ = 0, S_- = 1/2 \) \((0, 1/2)\) left-handed spinor.

4) \( S_+ = S_- = 1/2 \) \((1/2, 1/2)\) ordinary vector.

5) \( S_+ = 1, S_- = 0 \) \((1, 0)\) self-dual antisymmetry tensor \( \vec{E} + i \vec{H} \), circularly polarized electromagnetic wave with positive polarization.

6) \( S_+ = 0, S_- = 1 \) \((0, 1)\) anti-dual antisymmetry tensor \( \vec{E} - i \vec{H} \), idem with negative polarization.

**Spinors**

To illustrate the previous classification we study some important cases; in particular the spinors. We can consider either \((1/2, 0)\) or \((0, 1/2)\). Respectively

\[
\begin{align*}
\mathcal{J}^+(+) &= \frac{1}{2} \vec{\sigma} \\
\mathcal{J}^-(+) &= 0 \\
\mathcal{J}^+(+) &= 0 \\
\mathcal{J}^-(+)^- &= \frac{1}{2} \vec{\sigma} \\
\mathcal{J}^-(+)^- &= (1/2, 0) \\
\mathcal{J}^-(+)^- &= (0, 1/2) \\
\mathcal{J}^- &= \frac{1}{2} \vec{\sigma} \\
\vec{M} &= -\frac{1}{2} i \vec{\sigma} \\
\vec{M} &= +\frac{1}{2} i \vec{\sigma}
\end{align*}
\]

We can construct four-vectors:

\[
\begin{align*}
u_+^\mu \sigma^\mu u_+ \\
u_-^\mu \sigma^\mu u_-
\end{align*}
\]

\[ \sigma_+^\mu = (1, \vec{\sigma}) \],

\[ \sigma_-^\mu = (1, -\vec{\sigma}) \].
and the free Lagrangian follows if we require, that the Lagrangian be quadratic in \( u_\pm \), symmetric under \( u_\pm \rightarrow e^{i\alpha}u_\pm \) and the lowest possible number of derivatives

\[
\mathcal{L}^{(+)} = \pm (\pm i u_+^* (\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) u_+) \\
\mathcal{L}^{(-)} = \pm i u_-^* (\partial_0 - i \vec{\sigma} \cdot \vec{\nabla}) u_-
\]

With equations of motion:

\[
(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) u_+ = 0 \Rightarrow \Box u_+ = 0 \\
(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) u_- = 0 \Rightarrow \Box u_- = 0
\]

Hence the plane wave solutions are massless \( k^2 = 0 \). Further,

\[
u_+(x) = u_+(k)e^{-ik\cdot x}
\]

\[
(k^0 - \vec{k} \cdot \vec{\sigma})u_+(k) = 0
\]

Hence \( \vec{\sigma} \cdot \vec{k} / |\vec{k}| = +1 \), and we have positive helicity \( +1/2 \). Helicity is the projection of angular momentum along the direction of motion. For \( u_- \) we have negative helicity \( -1/2 \). \((0, 1/2)\) represents the neutrino field.

Notice that the requirement of symmetry under \( u_+ \rightarrow e^{i\alpha}u_+ \) excludes one possible extra term in the quadratic Lagrangian. Since \( u_+ \) is a spin \((1/2, 0)\) representation, recalling that \( 1/2 \otimes 1/2 = 1 \otimes 0 \), we can form a Lorentz scalar by an antisymmetric combination of two \( u_+ \) spinors. \( u_+ \) has two components \( u_{+A}, A = 1, 2 \); and the combination \( \epsilon^{AB}u_{+A}u_{+B}(\epsilon^{12} = -\epsilon^{12} = 1) \) is certainly an invariant. In classical field theory you would be tempted to believe that \( u_{+A}, u_{+B} \) are commuting objects: c-numbers. It is a fundamental consequence of quantum field theory that fields of integer spin should be quantized according to Bose-Einstein statistics (i.e. with commutators) while fields of half-integer spin should be quantized according to Fermi-Dirac statistics.
(i.e. with anti-commutators); this is the celebrated Spin-Statistics Theorem. This also opens now the possibility of writing a mass term for a single Weyl spinor:

$$\frac{1}{2} m (\epsilon^{AB} u_{+A} u_{+B} + \text{c.c.}) ,$$

$m$ is the mass parameter. This term would vanish in a classical theory. In fact since in the Fermi-Dirac statistics the occupation number of a given state cannot exceed one, there is no well-defined classical limit for fermionic fields. The mass term we have just written is called a Majorana mass. Finally, notice that since complex conjugation exchanges $J^{(+)}$ and $J^{(-)}$, it also exchanges left and right-handed fields. This means that instead of working with $u_-$, we can work with $i\sigma_2 u_-^*$ (ex. check that this spinor transforms like $u_+$ under Lorentz transformations). In other words, we can write a theory containing both $+$ and $-$ spinors in terms of a theory with only $+$ or only $-$ spinors. If we do not worry about symmetries of the form $u_+ \rightarrow e^{i\alpha} u_+$, the most general mass term we can write down for a collection of spinors $u_{+A}^i = 1, \ldots, N$ is a Majorana mass:

$$\frac{1}{2} M_{ij} u_{+A}^i u_{+B}^j \epsilon^{AB} + \text{c.c.}$$

where $M$ is a complex $N \times N$ symmetric matrix.

**LECTURE 3**

If we want a parity invariant theory the simplest thing to do is to include a pair of spinors, $(u_+, u_-)$. This is a Dirac spinor, with equations of motion

$$i\sigma^\mu_\nu \partial_\mu u_+ = m u_-$$
$$i\sigma^\mu_\nu \partial_\mu u_- = m u_+$$

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$
and Lagrangian:

$$\mathcal{L} = i\bar{\psi}^+ \begin{pmatrix} \sigma^\mu \partial_\mu & 0 \\ 0 & \sigma^\nu \partial_\nu \end{pmatrix} \psi - m \bar{\psi}^+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi$$

Defining the Dirac conjugate:

$$\gamma^0 \equiv \beta \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{\psi} \equiv \psi^* \gamma^0$$

The Lagrangian becomes

$$\mathcal{L} = i\bar{\psi}^+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \sigma^\nu \end{pmatrix} \psi - m \bar{\psi} \psi =$$

$$= i\bar{\psi}^+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \sigma^\nu \end{pmatrix} \psi - m \bar{\psi} \psi \quad \gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

And the Dirac matrices satisfy:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^\mu\nu, \quad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \quad P_\pm = \frac{1 \pm \gamma^5}{2}$$

Basic bilinears are:

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad \bar{\psi}\gamma^5\gamma^\mu\psi, \quad \bar{\psi} \frac{1}{2} [\gamma^\mu, \gamma^\nu] \psi, \quad \bar{\psi}\gamma_5\psi$$

The positive and negative energy solutions of \((i\eth - m)\psi = 0\) give 4 states:

$$u(k, s)e^{-ikx}, \quad \nu(k, s)e^{ikx}, \quad s = \pm 1/2$$

\((k - m)u = 0, \quad (k + m)v = 0\) with properties
The Hamiltonian is:

\[ \begin{align*}
\bar{u}u &= 2m \\
\bar{v}v &= -2m \\
\sum_s u\bar{u} &= \frac{k}{2} + m \\
\sum_s v\bar{v} &= \frac{k}{2} - m
\end{align*} \]

In quantizing we have

\[ \psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s (u(k,s)b(k,s)e^{ikx} + v(k,s)d^*(k,s)e^{ikx}) \]

Only if we use ACCR the Hamiltonian is bounded from below. This reminds us again of the spin statistic theorem:

- Integer spin quantized with commutators
- \((n + 1/2)\)-spin quantized with anticommutators

Furthermore, a consequence of the equations of motion is that \( j^\mu = \bar{\psi}\gamma^\mu\psi \) is a conserved current i.e. \( \partial_\mu j^\mu = 0 \). This current is associated with the electric charge current in Quantum Electrodynamics.

**Spin 1**

The best known example of a spin one particle is the photon, and its properties are described by the electromagnetic theory. The basic Lagrangian describing free photons is:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\vec{E}^2 - \vec{B}^2) \]

\[ \begin{align*}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
F_{0i} &= \partial_0 A_i - \partial_i A_0 = E_i \\
F_{ij} &= \partial_i A_j - \partial_j A_i = -\epsilon_{ijk} B^k
\end{align*} \]

where \( \vec{E} \) and \( \vec{B} \) are the standard electric and magnetic fields.

The Hamiltonian is:

\[ H = \int d^3x \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \]
and the oscillator quantization of $A_\mu$ is given by

$$A_\mu = \int \frac{d^3k}{(2\pi)^3 2|k|} \sum_\chi \left( \epsilon^{(\lambda)}_\mu(k) a^{(\lambda)} (k) e^{-ikx} + \epsilon^{(\lambda)}_\mu(k)^* a^{(\lambda)+} (k) e^{ikx} \right)$$

In the Landau gauge $\partial_\mu A_\mu = 0$, the photon’s equation of motion is $\square A_\mu = 0$ showing the masslessness of photons.

In classical physics we do not need to use the vector potential $A_\mu$. The equation of motion involve only $\vec{E}$ and $\vec{B}$. In the quantum theory however, in order to write down the Schrödinger equation $A$ is absolutely necessary. A dramatic confirmation of the need of using $A_\mu$ is provided by the Aharonov-Bohm effect. As soon as we work with $A_\mu$ we notice that the same gauge field strength $F_{\mu\nu}$ can obtained with $A_\mu$ and $A_\mu + \partial_\mu \chi$. This is gauge invariance. Physics should not depend on whether we use one vector potential or the other. In QM the wave function (or the fields) of particles of charge $Q$ also change under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \chi$$

$$= g \left( A_\mu + \frac{1}{ie} \partial_\mu \right) g^{-1}$$

$$\psi \to g^2 \psi \quad ; \quad g = e^{-ie\chi(x)\hat{\imath}}$$

$$= e^{-ieQ\chi} \psi$$

It should be noticed that a gauge symmetry is more a redundancy in the description of a physical system. A physical photon has two degrees of freedom while the vector potential has four. Quantum Electrodynamics (QED) is obtained by coupling photons to Dirac fermions. The Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m) \psi$$
The general Lagrangian coupling gauge fields to scalars and fermions

\[ D_\mu \equiv \gamma^\mu D_\mu, \quad D_\mu = \partial_\mu + ieA_\mu Q \]

The equations of motion for the photon field are:

\[ \partial_\mu F^{\mu\nu} = eQ \bar{\psi} \gamma^\mu \psi \]

which is consistent only if the electromagnetic current is conserved (just contract the equation with \( \partial_\nu \) and use the antisymmetry of \( F^{\mu\nu} \)).

An important generalization of QED is provided by non-abelian gauge theories. The standard model of strong, weak and electromagnetic interactions is indeed based on these theories. To set up a gauge theory we begin with a continuous group \( G \) and choose a set of fields transforming as some irreducible representations of \( G \). If \( T^a, a = 1, \cdots, \text{dim} G \) are the generators of the group \( G \), to obtain a covariant derivative of the field \( \Phi \), we introduce a photon like vector field \( A^a_\mu \) for each generator of \( G \) and define:

\[ D_\mu \Phi = \partial_\mu \Phi + ig \frac{A^a_\mu T^a}{\text{dim}_G} \Phi \]

Gauge covariance is achieved by the following transformations:

\[ \phi \rightarrow U \psi, \quad A_\mu \rightarrow U (A_\mu + \frac{1}{ig} \partial_\mu) U^{-1}; \quad U = e^{iT^a x^a} \]

The generalization of the gauge field strength is:

\[ F^{\mu\nu}_a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \]

\[ F_{\mu\nu} \equiv F^{a}_\mu T^a \rightarrow UF_{\mu\nu} U^{-1} \]

The general Lagrangian coupling gauge fields to scalars and fermions is:
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} D\psi + \bar{\psi}_\mu \phi^\mu \phi - V(\phi) \]

\[ -\bar{\psi} (M_1(\phi) + i \gamma_5 M_2(\phi)) \psi \]

For a renormalizable Lagrangian \( V \) should be at most quartic and \( M_1, M_2 \) at most linear in the scalar fields.

**LECTURE 4**

**Symmetries. Noether Theorem**

In field theory there is a general consequence of the existence of continuous symmetries. To each continuous symmetry we can associate a conservation law and a conserved current (Noether’s Theorem).

Consider the active action of symmetries, for instance for translations:

\[
\phi'(x') = \phi(x) \quad \phi'(x) - \phi(x) = -a^\mu \partial_\mu \phi \\
x' = x - a
\]

The statement of Noether’s theorem is:

If without using the equations of motion one can show that the Lagrangian density changes by a total divergence under an infinitesimal transformation:

\[ \delta \mathcal{L} = \partial_\mu K^\mu \Rightarrow \delta S = 0, \]

then there is a conserved current:

\[ \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\mu \delta \phi + \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi \Rightarrow \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \phi \right) = \partial_\mu K^\mu \Rightarrow EOM \]
Two important currents associated to phase rotations are:

- Dirac fermion

\[
\bar{\psi} (i\gamma^\mu - m) \psi \quad \psi \rightarrow e^{i\alpha} \psi
\]

\[
j^\mu = \bar{\psi} \partial^\mu \psi
\]

If \( \partial_\mu j^\mu = 0 \), there are some important consequences

- \( Q = \int d^3x \; j^0(\vec{x}, t) \) is conserved \( \frac{dQ}{dt} = 0 \), and it is a Lorentz scalar.
- After canonical quantization \( i[Q, \phi] = -\delta \phi \), hence \( Q \) generates the symmetry acting on the fields.

Two important currents associated to phase rotations are:

- Dirac fermion
The notion of running coupling constants comes about consider the process including some comments. To understand how renormalization and the fact that the centre of mass of the system moves with constant velocity.

Flavour symmetries

\[ \begin{align*}
\mathcal{L} &= \bar{\psi} i\gamma\cdot \sigma (i\gamma - m) \psi \quad \delta \psi = i\epsilon \sigma T^a \psi \\
\nonumber
j^a_\mu &= \bar{\psi} T^a \gamma_\mu \psi
\end{align*} \]

etc.

Lorentz invariance implies conservation of angular momentum and the fact that the centre of mass of the system moves with constant velocity.

**Renormalization, Renormalization Group**

This section was not covered in the material presented in the lectures for lack of time; however, the subject is important enough to justify including some comments. To understand how renormalization and the notion of running coupling constants comes about consider the process

\[ e^+ e^- \rightarrow \mu^+ \mu^- \]
If we compute the scattering amplitude in perturbation theory up to order $\alpha^2$ we have the following graphs.

We look at the last diagram, and analyze the modified photon propagator. In the gauge $\alpha = 1$
We have arbitrary $e^+e^-$ intermediate states with any energy. Take $q^2 \gg m_e^2$, (the photon is time-like (off-shell) in our process) hence if we keep $q^2$ large we can work in the limit $m_e = 0$, as we will not have an infrared divergence to worry about.

\[
\Pi_{\alpha\beta}(q) = -e^2 \int d^4k (2\pi)^4 \frac{Tr \gamma_\alpha(k + q) \gamma_\beta}{(k^2 + i\epsilon)((k + q)^2 + i\epsilon)}
\]

\[
= (-q_\alpha q_\beta + \eta_{\alpha\beta} q^2) \Pi(q^2)
\]
\[ \Pi(q^2) \approx \frac{\alpha}{3\pi} \ln \frac{q^2}{\Lambda^2}, \quad \alpha = \frac{e^2}{4\pi} \]

We seem to be in trouble. Do we have to abandon? The full process we are computing is:

\[ \mathcal{M}(q, q') = (\bar{u}\gamma^\mu u)(\bar{v}\gamma_\mu v). \]

\[ \cdot \frac{e^2}{q^4} \left( 1 + \frac{\alpha}{3\pi} \ln \frac{q^2}{\Lambda^2} \right) \]

We have correction to Coulomb’s law (1/q^2 term), however, to understand what is going on we must ask what is the coupling constant \( e \). \( e \) was a parameter in the Lagrangian but now we have to know how to measure it. We can define the renormalized coupling to be the strength of Coulomb’s law at some scale \( \mu \):

\[ e^2(\mu) = e^2(\Lambda) \left( 1 + \frac{\alpha\Lambda}{3\pi} \ln \frac{\mu^2}{\Lambda^2} \right) \]

Since we do not know in principle what \( \Lambda \) is, we need to measure \( e \) at some scale, and everything can be expressed with respect to \( e^2(\mu) \) at that scale. This coupling constant is called the running coupling.
constant. To see how it runs:

\[
\mu \frac{d}{d\mu} \alpha(\mu) = \frac{2\alpha^2(\mu)}{3\pi} \quad \begin{aligned}
\Rightarrow \\
\alpha(\mu) &= \frac{\alpha(\mu_0)}{1 - (2\alpha(\mu_0)/3\pi) \ln(\mu/\mu_0)} \\
\end{aligned}
\]

\[
\frac{\epsilon^2(\mu_1)}{\epsilon^2(\mu_2)} = 1 + \frac{2\alpha}{3\pi} \ln \frac{\mu_1}{\mu_2}
\]

Normally we define \(\alpha(m_e)\), the fine structure constant at the scale of the electron mass, (Thompson scattering), \(\alpha(m_e) \approx 1/137\). However, if we want to know the value at the Z-mass, include contributions from all charged particles:

\[
\alpha(M_Z) = \alpha(m_e) \left(1 + \frac{2\alpha(m_e)}{3\pi} \sum Q_i^2 \ln \frac{M_Z}{m_e}\right) \\
\approx 1/128 \quad \text{it grows!}
\]

The logarithmic change of \(\alpha\) with the scale \(\mu\) is known as the \(\beta\)-function

\[
\mu \frac{d\alpha}{d\mu} = \beta(\alpha) \quad (\beta - \text{function})
\]

or using \(\ln \mu = t\):

\[
\frac{d\alpha}{dt} = \beta(\alpha)
\]

At weak coupling the possible qualitative behaviour of \(\beta\) is shown in the figures below. The left hand side corresponds to asymptotic freedom (A.F.), while the right hand side corresponds to infrared freedom.
The origin in the two pictures corresponds to a fixed point; i.e. if we tune \( \alpha = \alpha_0 \), then \( \alpha(\mu) = \alpha_0 \) for all \( \mu \).

If \( \beta' < 0 \) at \( \beta(\alpha_0) = 0 \) \( \Rightarrow \) as \( E \to \infty \), the coupling becomes weak (AF).

If \( \beta' > 0 \) at \( \beta(\alpha_0) = 0 \) \( \Rightarrow \) as \( E \to 0 \), the coupling becomes weak (IF).

In principle low energy states influence low-energy physics by renormalizing the coefficients of leading operators (also anomalous dimensions).

Low energy physics is dominated by infrared fixed points.

High energy physics is dominated by ultraviolet fixed points.

(if they exist).

In general it is too naïve to imagine that a theory can exist to arbitrary (large) energy, although AF theories may have this property (in contrast the infrared may be very complicated).

In QFT it seems that the only singularities of the \( \beta \)-function equations that appear are fixed points in contrast to the case of dynamical systems where limit cycles or strange attractors are also possible. In
two-dimensions, using locality, unitarity and Lorentz invariance and positivity of the energy. A. Zamolodchikov showed that fixed points are the only singularities. In four dimensions we are still lacking a similar proof (which may not exist).

\[
\{ \begin{align*}
&\text{Important principle (Wilson K.),} \\
&\text{All QFT look renormalizable at energy scales small} \\
&\text{compared to their intrinsic scale.}
\end{align*} \}
\]

Think of thermodynamics and the microscopic description of different systems. There is an universality in thermodynamic behaviour which is to a large extent independent of the specific microscopic interactions. In modern RG parlance we can divide operators (interactions) with respect to a given fixed point into:

i) Irrelevant.

ii) Relevant.

iii) Marginal.

Given some Lagrangian we can count dimensions.

\[
\partial_\mu \phi \partial^\mu \phi \left\{ \begin{align*}
[\mathcal{L}] &= 4 \quad \text{(energy units)} \\
[\partial_\mu] &= 1 \\
[\phi] &= 1
\end{align*} \right.
\]

For:

- scalars \( [\phi] = 1 \)
- fermions \( [\psi] = 3/2 \)
- gauge fields \( [A_\mu] = 1 \)

All coupling can be given dimensions.
tive. Although the machinery of the RG is complicated, the physical

Operators of dimension > 4 are supposed to be non-renormalizable and
from our point of view they are called irrelevant operators. Operators
of dimension 4 are called marginal, and operators of dimension < 4 are
called relevant operators.

Irrelevant operators are weak at low energies

\[ \Delta \mathcal{L} = \frac{1}{M^2} (\bar{\psi} \psi)^2 \Rightarrow \sigma \sim \frac{E^2}{M^4}. \]

For instance in Fermi theory \( e^- + \nu_e \rightarrow e^- + \nu_e \quad \sigma \sim G_F^2 E^2 \). In
GUT theories \( p \rightarrow \pi^0 e^+ , \hat{\sigma} \sim uud e^- \), it is suppressed by \( M_{\text{GUT}}^{-2} \) \( \Gamma \sim m_p/M_{\text{GUT}}^2 \).

Higher order corrections by light fields also introduce corrections in
the coefficients of irrelevant operators, but they are normally small if
we make all coefficients on the Lagrangian at low energies be running
coupling constants

\[
\begin{align*}
\mathcal{L} &= m\bar{\psi}\psi \\
\lambda \phi^4 &= [\phi^4] = 4 \Rightarrow [\lambda] = 0 \\
g\phi^6 &= [\phi^6] = 6 \Rightarrow [g] = -2 \quad g = \tilde{g}/M^2 \\
g\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4 &= [\psi^4] = 6 \Rightarrow [g] = -2 \quad \text{etc.}
\end{align*}
\]

The corrections can be systematically taken into account by including
the appropriate running. Below the characteristic scale \( E \ll M \) the
irrelevant (non-renormalizable) couplings are indeed irrelevant. Predictions
at low \( E \) are really independent of high energies, or very insensitive. Although the machinery of the RG is complicated, the physical
ideas are simple and intuitive. You should no longer be afraid of working with effective Lagrangian if you understand the physical picture just described. This ends our brief interlude into renormalization theory.

Goldstone’s Theorem: General remarks

There are two ways of realizing symmetric on a QFT. Even if we have that the charge $Q$ we obtain from Noether’s Theorem commutes with the Hamiltonian, $[Q, H] = 0$, we are not guaranteed that $Q|0 >= 0$. There are two modes of realizing the symmetry:

1) Wigner-Weyl: $[Q, H] = 0$, $Q|0 >= 0$ and the spectrum falls in explicit multiplets of the symmetry group.

2) Nambu-Goldstone: $[Q, H] = 0$, but $Q|0 >\neq 0$. In this case the symmetry is not manifest on the spectrum, and for every broken generator there is a massless scalar particle associated.

There is a third “away” of understanding symmetries in QFT. These are the anomalous symmetries. This is the case when the classical theory respects a symmetry that is violated by quantum fluctuations. Two classic examples are scale invariance and axial symmetries. If a classical theory does not contain mass parameters or dimension-full coupling constants, the classical theory is scale invariant, physics will look the same at every scale. In the quantum theory generically this is not the case. We have seen in our brief analysis of renormalization that the coupling constants depend on the scale. Thus the quantum theory violates scale invariance. The other example we will deal with later has to do with global chiral rotations of fermions.

Consider a scalar theory with Lagrangian:
\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi), \]

invariant under some symmetry group

\[ \delta \phi^i = \epsilon^a (T^a)_{ij} \phi_j. \]

Invariance means that \( \delta V(\phi) = 0. \) The ground state configurations for this classical system are obtained as follows. The Hamiltonian for \( \mathcal{L} \) is:

\[ H = \int d^3x \left( \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi^i)^2 + V(\phi) \right) \]

Hence the potential energy is:

\[ V(\phi) = \int d^3x \left[ \frac{1}{2} (\nabla \phi^i)^2 + V(\phi) \right] \]

The minima are given by those configurations such that \( \nabla \phi^i = 0 \) \( V(\phi) = 0. \) Call these minima vacuum expectation values (VEV's). The Wigner-Weyl mode corresponds to the case \( \delta \langle \phi^i \rangle = 0. \) The Nambu-Goldstone to \( \delta \langle \phi^i \rangle \neq 0. \) We can divide the generators \( \{T^a\} \) into two subsets \( \{H^i, K^A\}. \) The \( H^i \)'s correspond to unbroken symmetries; i.e. \( H^i \langle \phi \rangle = 0, \) the \( H^i \)'s generate a subgroup of the symmetry group known as the unbroken subgroup. The broken generators \( K^A \) do not leave the vacuum invariant \( K^A \langle \phi \rangle \neq 0. \) We now show that for each \( K^A \) there is massless particle in the spectrum. The mass matrix is given by:

\[ \frac{\partial^2}{\partial \phi_i \partial \phi_j} V(\phi)|_{\phi = \langle \phi \rangle} = M^2(\langle \phi \rangle)_{ij} \]

Invariance of \( V(\phi) \) under the symmetry group means:

\[ \delta V = \frac{\partial V}{\partial \phi^i} (T^a)_{ij} \phi_j = 0 \]

Take one more derivative with respect to \( \phi^i \) and then set \( \phi^i = \langle \phi^i \rangle \)
should be careful when talking about degenerate vacua. It simply says that we can only determine the classical potential by the effective potential. In other words, we cannot interpolate between any pair of different vacua. This is an incorrect statement. In fact if we have multiple vacua, this proves the classical form of Goldstone theory. The QFT proof follows the same lines except that action is replaced by the effective action and the classical potential by the effective potential.

**Remark**

In the quantum theory we have to be careful about the statement that we have multiple vacua. This is an incorrect statement. In fact if we have a collection of states satisfying

\[
P^\mu |0,a\rangle = 0 \\
\langle 0,a|0,b\rangle = \delta_{ab} \quad a,b = 1, \ldots, N
\]

Then for any collection of local observables \( A_1, \ldots, A_n \) it is not difficult to show that we can always choose a basic such that

\[
\langle 0,a|A_1 \cdots A_n|0,b\rangle = 0
\]

In other words we cannot interpolate between any pair of different vacua by means of any local process. This is however not in contradiction with the existence of massless states. It simply says that we should be careful when talking about degenerate vacua.
Some examples of Goldstone bosons can be taken from condensed matter physics. Perhaps the most familiar are the phonons in a crystal in the acoustic branch of the dispersion relation. While the material is in the liquid phase we have translational invariance; however, when the temperature drops the ground state becomes a crystal and the unbroken group is the group of lattice translations. The associated Goldstone bosons are precisely the phonons. In High Energy Physics one of the better known example is provided by the pions which are Goldstone bosons in the limit that the up and down quark masses vanish.

LECTURE 5

In the last lecture we would like to study some properties of anomalous symmetries. To exhibit them in the simplest possible example but keeping all the right ingredients, consider an ant world: \( S^1 \times \mathbb{R} \). Space is taken to be one-dimensional with the topology of a circle of length \( L \), and \( \mathbb{R} \) is time. The light-cone wraps around the space-time cylinder as shown in the figure:

\[
x^\pm \approx \frac{1}{\sqrt{2}} (t \pm x)
\]
We wish to study massless fermions in two-dimensions in the presence of external electric fields. The two-dimensional Dirac algebra is:

\[ \{\gamma^\mu, \gamma^\nu\} = 2\eta^\mu\nu; \quad \mu, \nu = 0, 1 \quad \eta^\mu\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \gamma^0 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^1 = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

The analogue of \( \gamma_5 \) is

\[ \gamma^5 = -\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \{\gamma^5, \gamma^\mu\} = 0 \]

A two-dimensional Dirac spinor has two components

\[ \psi(t, x) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \]

and the free Dirac equation for massless particles becomes:

\[ (\partial_0 - \partial_1)\psi_+ = 0 \quad \psi_+ = \psi_+(x^0 + x^1) \]

\[ (\partial_0 + \partial_1)\psi_- = 0 \quad \psi_- = \psi_-(x^0 - x^1) \]

Hence \( \psi_+ \) (resp. \( \psi_- \)) represents left (resp. right) moving wave-packets.

Looking for plane wave solutions:

\[ \psi_\pm(x, p) = e^{-ip_\mu x^\mu} \quad p^0 \pm p^1 = 0 \]

For \( \psi_+ \) we have:

- Positive energy solutions \( p^0 = E > 0, p^1 < 0 \)
- Negative energy solutions $p^0 = -E < 0$, $p^1 > 0$

(and the reverse for $\psi_-$). Choosing periodic boundary conditions in the ant world, $\psi_\pm(x + L) = \psi_\pm(x)$ the momentum $p^1$ is quantized: $p^1 = 2\pi n/L$, $n \in \mathbb{Z}$. The spectrum is easily portrayed in a picture:

![Wave equation](image)

When we quantize we associate to positive energy solutions annihilation operators, and creation operators to negative energy solutions of the wave equation:

$$\psi(x) = \sum_{E>0} a_E u_E + \sum_{E<0} b_E^* \tilde{u}_E$$

$$\{a_E, a_E^*\} = \delta_{EE'}, \quad \{b_E, b_E^*\} = \delta_{EE'}, \quad \{a_E, b_E\} = 0$$

The free Lagrangian for $\psi_\pm$ is

$$\mathcal{L} = i\bar{\psi}_\pm^\dagger \left( \frac{\partial_0}{\gamma_0 + \gamma_0} \right) \gamma_\pm + i\partial_\pm \left( \frac{\partial_0}{\gamma_0 - \gamma_0} \right) \psi_\pm$$

and we have two conserved currents associated to independent phase rotations of $\psi_\pm$. In terms of a Dirac fermion they are:

$$V^\mu = \bar{\psi} \gamma^\mu \psi = (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- , -\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-)$$

$$A^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi = (\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- , -\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-)$$
or

\[ \bar{\psi} \gamma^\mu P_+ \psi = (\psi_+^\dagger \psi_+ , \ -\psi_+^\dagger \psi_+) \]
\[ \bar{\psi} \gamma^\mu P_- \psi = (\psi_-^\dagger \psi_- , \ -\psi_-^\dagger \psi_-) \]

\[ P_\pm = \frac{1}{2} (1 \pm \gamma_5) \]

The charge associated to the vector current \( V^\mu \) (resp. \( A^\mu \)) count the number of \( \psi_+ \) fermions plus (resp. minus) the number of \( \psi_- \) fermions. In QFT it is not enough to present the creation and annihilation operators we also need to construct the vacuum state. In our case we want to construct \( |0_\pm \rangle \). One convenient way to describe \( |0_\pm \rangle \) is in terms of the Dirac sea. Imagine that we fill out all the negative energy states of \( \psi_+, \psi_- \); or in many body language, we construct the Hartree-Fock wave function associated to all the negative energy states. For many purposes this gives a good semiclassical description of the fermionic ground states. Representing filled states with filled dots, the left and right Dirac seas can be pictured as:

If necessary some infinite constants are subtracted so the \( P^\mu |0_\pm \rangle = 0 \). Next we turn a constant electric field \( E \). If \( \psi_\pm \) have unit charge with respect to it (i.e. the electromagnetic field couples to the vector current \( V^\mu \) as in four dimensions) the electric field induces currents in the vacuum:
In the $\psi_-$ vacuum $E$ creates holes which are seen as antiparticles, while on the $\psi_+$ vacuum it produces particles. In one case $E$ decreases the Fermi surface and in the other it augments it. This means that $E$ creates $e^+ e^-$ pairs. If we count the number of $\psi_+$ plus $\psi_-$ particles, it is clear from the picture that it is conserved, hence $\partial_\mu V^\mu = 0$ in the presence of gauge fields. On the other hand if we count the number of $\psi_+$ minus the number of $\psi_-$ particles we see current conservation violation:

$$\partial_\mu A^\mu \sim E$$

This is the two-dimensional analogue of the Adler-Bell-Jackiw anomaly. A current that is conserved classically (no Dirac sea) is violated by quantum effects.

With some effort we can generalize the previous argument to four-dimensions. One of the most relevant examples is a theory analogous to QCD with $N_c$ colors and $N_f$ massless flavors of quarks $Q^f_a$, $f = 1, \cdots, N_f$; $a = 1, \cdots, N_c$. The purely fermionic part of the Lagrangian is:

$$\mathcal{L} = i\bar{Q}^f \slashed{D} Q^f + i\bar{Q}^f \slashed{D} P^- Q^f + \cdots$$
where color is ignored, $P_\pm = (1 \pm \gamma_5)/2$, and $P_\pm Q$ (resp. $P_\mp Q$) is the same as $u_+$ (resp. $u_-$) from the spinor point of view. In fact the Dirac field $Q'_{a\pm}$ can be written as:

$$Q'_{a\pm} = \begin{pmatrix} Q'_{a+} \\ Q'_{a-} \end{pmatrix}$$

The symmetry of $\mathcal{L}$ corresponds to arbitrary unitary rotations of left- and right-quark flavors. Hence the global symmetry group is:

$$U(N_f)_+ \times U(N_f)_- = SU(N_f)_+ \times SU(N_f)_- \times U(1)_B \times U(1)_A$$

One of global $U(1)$'s is identified with baryon number and it corresponds to the same phase rotation for $Q_+, Q_-$. If we make opposite phase rotations however:

$$Q \rightarrow e^{i\alpha_\gamma} Q = \begin{pmatrix} e^{i\alpha} & Q_+ \\ e^{-i\alpha} & Q_- \end{pmatrix}$$

we are in the same situation as before, and it can be shown that the corresponding current $A^\mu$ has an anomaly:

$$A^\mu = \sum_f \bar{Q}'^\dagger \gamma^\mu \gamma^5 Q'$$

$$\partial_\mu A^\mu \sim F^a_{\mu\nu} F^a_{\alpha\beta} \epsilon^{\mu\nu\alpha\beta},$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the four-dimensional Levi-Civita density. This result is obtained technically by computing a triangle diagram coupling the axial current to two vector currents:
imposing current conservation and Bose symmetry on the gluon lines fixes the ambiguities in the graph and leads to the anomaly in $A_\mu$. One useful application of the triangle graph is in the computation of $\pi^0$ decay:

\[ N_c T \tau T_3 Q^2 \]

where $Q$ is the electric charge of the quarks around the loop, and $T_3$ is the third component of isospin. Further $N_c$ is the number of colors of quarks running on the loop. The results provides an accurate prediction of the $\pi^0$-lifetime, and it is an efficient color counter. It is one of the simplest ways of measuring that the number of colors is three.

The triangle graph can be understood in two ways: if we compute it in terms of quark propagators the diagram is linearly divergent, hence it could be evaluated by using the high energy degrees of freedom of the theory (quarks and gluons); and apart from some kinematical factors the answer is proportional to the trace.
of the three generators of the global symmetry appearing in the vertices; hence the anomaly looks like:

$$\partial_\mu A^\mu \sim Tr T_1\{T_2, T_3\}$$ (kinematical factor)

This anomaly explains not only $\pi^0$-decay but it was also used by 'tHooft to solve the $U(1)$-problem. By looking carefully at the analytic structure of the graph one finds that the origin of the anomaly can also be identified with a singularity at zero value of $q$ of the triangle diagram. This means (as discovered by 'tHooft) that we can relate low- and high-energy information about the same theory. In other words the coefficient of the kinematical factor should be the same whether we compute it in terms of the low or high energy degrees of freedom. If the theory is described by different degrees of freedom in the ultraviolet and the infrared regimes (as in QCD) we obtain a powerful constraint on the theory known as the 'tHooft anomaly matching conditions. In fact for $N_c = 3$ and $N_f > 2$ this matching can be used to show that in QCD we should have the breaking of the chiral symmetry $SU(N_f)_+ \times SU(N_f)_- \rightarrow SU(N_f)_{\text{vector}}$. In this case the field acquiring a vacuum expectation value (VEV) is a fermion composite $\langle Q\bar{Q} \rangle \sim \Lambda_{\chi S_B}^3$, analogous in some respect to the formation of Cooper pairs in superconductors.

We should mention one final application of the anomaly which is related to the family structure of the Standard Model. We saw in QED that
currents coupled to the electromagnetic field should be conserved if the theory is to be consistent. We have seen now that anomalies may appear in the quantum theory and therefore we should verify whether a given theory will or will not be anomaly free. For a given gauge group $G$ fermions will transform under some collection of representations $T_{R_+}$ and $T_{R_-}$. Then the anomaly will be proportional to:

$$\sum_{R_+} \text{tr} T_{R_+} \{ T_{R_+}, T_{R_+} \} - \sum_{R_-} \text{tr} T_{R_-} \{ T_{R_-}, T_{R_-} \}$$

Hence anomaly freedom is guaranteed for example in vector-like theories: theories where the $+$ and $-$ representations coincide. In this case the corresponding currents can be coupled to gauge fields consistently, and a theorem due to Adler and Bardeen guarantees that if the anomaly is removed to the first non-trivial order (the triangle graph), it will not reappear to any order of perturbation theory. If we now look at the Standard Model of weak strong and electromagnetic interactions, the gauge group is:

$$G = SU(3)_c \times SU(2)_w \times U(1)_Y$$

and for a single family of quarks and leptons the assignments of left- and right-handed fields is quite different. For instance for the first family $(e, \nu_e, u, d)$ the representations of $G$ involved are:

$$\begin{pmatrix}
  u^{(\alpha)}_+ \\
  d^{(+1/6)}_+ \\
  e^{(-1/2)}_+
\end{pmatrix}, \quad \begin{pmatrix}
  \nu_e^{(\alpha)} \\
  u^{(-2/3)}_- \\
  d^{(-1/3)}_-
\end{pmatrix}, \quad e_-, -1
$$

$$\alpha = 1, 2, 3$$

or equivalently:

$$(3, 2, 1/6) \quad (1, 2, -1/2) \quad (1, 1, +2/3), \quad (1, 1 - 1/3), \quad (1, 1, -1)$$
where the first entry represents the color representation, the second the weak $SU(2)$ representation and the third the weak hypercharge. The electric charge is given by $Q = T_3 + Y$ ($T_3$ is the 3rd component of weak isospin, and $Y$ the weak hypercharge). Verifying the cancellation of anomalies within a family is a very instructive exercise. For example if $T_1,T_2,T_3$ are all $Y$ we find:

$$Tr Y^3_+ - Tr Y^3_- =$$

$$= 3 \cdot 2 \left(\frac{1}{6}\right)^3 + 2 \left(-\frac{1}{2}\right)^3 - 3 \left(-\frac{2}{3}\right)^3 - 3 \left(-\frac{1}{3}\right)^3 - (-1)^3 =$$

$$= \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right) = 0$$

So that the anomalies of quarks and leptons cancel each other. The same cancellation can be checked for other choices of $T_1,T_2,T_3$. Why the quarks and leptons in a family are so chirally asymmetric, and why this structure repeats at least three times are some of the biggest mysteries of High Energy Physics.

This concludes these lectures. I hope they were useful to refresh some basic concepts of QFT, and to motivate the students to pursue some of these (and other) issue in the literature.

Acknowledgements

I would like to thank A. De Rujula and E. Verlinde for inviting me to participate in the Academic Training Programme and for giving me the opportunity to present these lectures. I would also like to thank I. Canon and A. Coudert from CERN’s DTP service for their diligence and excellent work in making these notes readily available.