THE STABILIZING INFLUENCE OF NON-LINEAR SPACE-CHARGE ON TRANSVERSE COHERENT OSCILLATIONS

by

G. Merle* and D. Mühll

CM-P00063647

October 1969

* Visitor to the MPS Division from University of Karlsruhe, Germany
# TABLE OF CONTENTS

## SUMMARY

1. INTRODUCTION 1

2. PHYSICAL CONSIDERATIONS 2
   2.1. The model 2
   2.2. Origin of the Q-spread 4

3. DERIVATION OF THE DISPERSION RELATION 5

4. CALCULATION OF THE AMPLITUDE DEPENDENCE OF THE Q-SHIFTS 8

5. SOME REMARKS ON THE EFFECT OF SPACE-CHARGE NON-LINEARITY COMING FROM THE TRANSVERSE DIRECTION PERPENDICULAR TO THE COHERENT OSCILLATIONS 11

6. NUMERICAL EXAMPLES 13
   6.1. "Bell Shaped" amplitude distribution 14
       6.1.1. $u+v+iv = \text{const.}$ 16
       6.1.2. $u+v+iv = g(a) \cdot (\hat{u}+\hat{v}+i\hat{v})$ 19
   6.2. Parabolic density distribution in ordinary space 20

7. CONCLUSIONS 22

ACKNOWLEDGEMENTS 23

REFERENCES 23
SUMMARY

Non-linear space-charge forces tend to reduce the growth rate of transverse coherent instabilities of dipole type. Coherent dipole oscillations may become practically stabilized by this mechanism in cases where the standard theory would predict instability.

1. INTRODUCTION

This paper deals with transverse coherent instabilities of a beam in a circular accelerator. These instabilities may have their origin in different phenomena. Finite conductivity of the vacuum chamber leads to components of the electromagnetic self-field which are in phase with the velocity of the particles. Such field components introduce imaginary parts to the Laslett Q-shifts and can therefore lead to an exponential growth of coherent oscillations.

More generally coherent oscillations can be amplified through any interaction of the beam with surrounding media or structures such as ions of the rest gas, clearing electrodes, cavities, etc.

A spread in the betatron frequencies of the particles has a stabilizing effect (Landau damping). The case where this spread is due to a spread in longitudinal momentum or due to a non-linearity of the external focusing has been treated in literature. However, an additional frequency spread comes from the non-linearity of space-charge forces and the question arises whether this internal non-linearity produces Landau damping. Some preliminary answer on this question will be given in this report.
2. PHYSICAL CONSIDERATION

2.1. The model

We confine ourselves to dipole oscillations of the beam center $\overline{Z}$,

$$\overline{Z} = \zeta \cdot \exp \left[ i(Q_c \delta) \right]$$

$\delta$ is the azimuthal angle and represents the variable in longitudinal direction. $Q_c$ is in general a complex number, $Q_c = q_c + i\tau$, where $q_c$ is the oscillation frequency of the dipole mode and $\tau$ is the revolution frequency ($\tau > 0$).

We shall restrict our considerations to a beam with a horizontal diameter $2\delta$ large compared to its vertical dimension. Let $Z$ be the coordinate in the short (vertical) direction, where the coherent oscillations take place.

$$Z$$

Given an ensemble of $N$ particles we start looking at the $k^{th}$ particle with the equation of motion

$$Z_k'' + Q_0^2 Z_k - 2Q_o \Delta Q_c(k) \overline{Z} - 2Q_o \Delta Q_{ic}^{(k)}(Z_k - \overline{Z}) - W^{(k)} \times \overline{Z} = 0 \quad (2.1)$$

where the following forces are assumed to act on this particle:

1. External focusing $-Q_0^2 Z_k$, which for simplicity we assume to be linear. We also neglect the spread in angular momentum and assume that all particles have the same revolution frequency.

2. Wake fields $W^{(k)} \times \overline{Z}$, with $\overline{Z}$ equal to the displacement of the beam. The wake field term results from wall resistivity or any other of the mechanisms discussed above.
3. Coherent space-charge \( 2Q_o \cdot \Delta \zeta_c^{(k)} \cdot \overline{Z} \).

4. Incoherent space-charge \( 2Q_o \cdot \Delta Q_{ic}^{(k)} \cdot (z_k - \overline{Z}) \).

In the presence of non-linear space-charge forces \( \Delta Q_{ic} \), \( \Delta Q_c \) and \( \overline{W} \) are in general different for different particles and even for the same particles they vary in time as the particle oscillates. A general analysis of the influence of non-linear space-charge on coherent stability is difficult. However, the following idealized cases seem to allow an analytical treatment.

a. A flat coasting beam with homogeneous density in the large (horizontal) direction and inhomogeneous density in the short (vertical) direction.

As will be seen later the electric field component \( E_Z \) is non-linear in \( Z \) in this case and through an appropriate averaging process one obtains \( \Delta Q_{ic}^{(k)} = f(a_k) \), \( a_k \) being the amplitude of the free (incoherent) betatron motion of the \( k \)-th particle.

b. A flat beam again, but now with the density homogeneous in the short and inhomogeneous in the large direction, so that \( \rho = \rho(r) \) is a slowly varying function of the horizontal coordinate. The electric field component \( E_Z \) is now a function of \( r \) and \( \Delta Q_{ic}^{(k)} = f(b_k) \), with \( b_k \) equal to the amplitude of the horizontal betatron motion.

c. Let the density be homogeneous in both transverse directions of the beam, but consider a bunched beam with inhomogeneous line charge density in longitudinal direction. Then \( \Delta Q_{ic} = f(q_k) \), \( q_k \) being the longitudinal position of the particle in the bunch.

This is a similar situation as case (b), except that \( \Delta Q_{ic} \) is now in general slowly varying compared to the period of the betatron motion.
In the present note only some preliminary results for the cases (a) and (b) will be derived.

2.2. Origin of the Q-spread

We rewrite the eq. of motion for the k-th particle, regarding all terms that depend on \( \overline{Z} \) as "driving forces"

\[
Z''_k (a) + Q^2_k Z'_k = W_k \overline{Z}
\]  

(2.2)

with the abbreviations

\[
W_k = w^{(k)} + 2Q_o (\Delta Q_c^{(k)} - \Delta Q_i^{(k)})
\]

\[
Q^2_k = Q^2_o - 2Q_o \Delta Q_i^{(k)}
\]  

(2.3)

In as much as the left hand side of (2.2) is concerned, a spread in the incoherent Laslett detuning has the same effect as a spread in the external focusing \( Q_o \) [1].

However, any non-linearity of space-charge will automatically introduce a spread in the driving force term \( W_k \). In the following we will see that the non-linearity of the \( W \)-term is essential for Landau damping. In case (b) and (c) our model leads automatically to a dependence \( \overline{W}_k (b) \) or \( \overline{W}_k (\varphi) \) that introduces Landau damping whereas in case (a) more general forms of \( \overline{W}_k (a) \) are possible and not all of them tend to damp dipole oscillations. The spread due to a \( r \)- or \( \varphi \)- and under certain assumptions also due to a \( Z \)-dependence of the Laslett Q-shift, has therefore a similar effect on the coherent \( Z \)-motion as a spread in the external focusing [1].

No analysis of multipole modes has been made but it seems that any \( Z \)-, \( r \)- or \( \varphi \)-dependence of \( W_k \) will complicate the behaviour of these modes and may give rise to more involved "coupling modes". Some comparison of our nomenclature with the standard notation seems in order at this place. The "wake field"-term \( W_k \) in (2.2) is
commonly denoted by \( U/\omega + (i+1)V/\omega \) \([1]\), where \( \omega \) is the angular
revolution frequency. Therefore, from (2.3) we identify \( (\Delta Q_c - \Delta Q_{ic}) \)
with \( U/\omega \) and \( W^{(i)}/2Q_c \) with \((i+1) \cdot V/\omega \). Henceforth the quantities
\( U/\omega \) and \( V/\omega \) will be denoted by \( u \) and \( v \) respectively.

In our model \( u \) is simply determined by the difference of
the two Laslett Q-shifts and is therefore negative. However, in
general \( u \) may well be positive, when inductive forces are taken
into account (for instance in the presence of clearing electrodes).

The rest of this paper is devoted to a more mathematical
derivation of the qualitative arguments presented above. Our
conclusions may be found in Section 7.

3. DERIVATION OF THE DISPERSION RELATION

In this section we want to consider case (a), leaving
case (b) to Section 5.

As pointed out a beam is studied with a horizontal dia-
meter \( 2b \) large compared to its vertical dimension. The only
non-linear space-charge contribution comes from the vertical
direction.

Equation (2.2) is rewritten in the form

\[
Z_k'' + Q_o^2 Z_k - 2Q_o \cdot \Delta Q_{ic}(Z_k) Z_k = 2Q_o \left[ \Delta Q_c(Z_k) - \Delta Q_{ic}(Z_k) \right] + (i+1) \cdot v \cdot (Z_k) \cdot \frac{\bar{Z}}{Z}
\]

(3.1)

We content ourselves with the limiting case where the
driving force term (r.h.s. of (3.1)) is "small enough", so that
we can write

\[
Z_k \approx Z_{ko} + \delta_k
\]

(3.2)
where $Z_{k0}$ is the solution of (3.1) with the r.h.s. omitted and $\xi_k$ is a "correction" which describes the response to the driving force.

Expanding $\Delta Q_{ic}(Z_k) = \Delta Q_{ic}(Z_{k0}) + \frac{\partial \Delta Q_{ic}}{\partial Z_k} \xi_k$ equation (3.1)
can be split into two equations describing the coherent and incoherent motion separately. Let us write $\Delta Q_{ic}'$ for $\frac{\partial \Delta Q_{ic}}{\partial Z_k}(Z_{k0})$ for shortness. Then retaining only terms up to first order in $\xi_k$:

$$Z_{k0}'' + Q_o^2 Z_{k0} - 2Q_o \Delta Q_{ic}(Z_{k0}) Z_{k0} = 0$$ (3.3)

$$\xi_k'' + Q_o^2 \xi_k - 2Q_o \left[ \Delta Q_{ic}' Z_{k0} + \Delta Q_{ic}(Z_{k0}) \right] \xi_k =$$

$$2Q_o \left[ \Delta Q_c(Z_k) - \Delta Q_{ic}' Z_{k0} - \Delta Q_{ic}(Z_{k0}) + (1+i)\nu(Z_k) \right] \xi_k$$ (3.4)

The solution of (3.3) (neglecting higher harmonics in the solution of the non-linear equation) is written as

$$Z_{k0} = a_k \exp \left[ i (\bar{\omega}_k(a_k)^4 + \psi_k) \right]$$ (3.5)

Assuming that the initial phases $\psi_k$ are randomly distributed, one has $Z_{k0} = 0$ and $Z = \frac{1}{N} \sum_{k=0}^{N} \xi_k = \xi$ (by virtue of (3.2)). Thus we can rewrite (3.4)

$$\xi_k'' + Q_o^2 \xi_k - 2Q_o \left[ \Delta Q_{ic}'(Z_{k0}) Z_{k0} + \Delta Q_{ic}(Z_{k0}) \right] \xi_k =$$

$$2Q_o \left[ \Delta Q_c(Z_k) - \Delta Q_{ic}'(Z_{k0}) Z_{k0} - \Delta Q_{ic}(Z_{k0}) + (1+i)\nu(Z_k) \right] \xi_k$$ (3.6)

Let us now examine, under which conditions $\xi_k = \xi$ (i.e. the forced motion is the same for all particles). It is easily verified that the assumption $\xi_k = \xi$ is in fact consistent for the solution of (3.6) provided that there is no spread in $\Delta Q_c$ and $\nu$, i.e. if $\Delta Q_c$ and $\nu$ are independent of $Z_{k0}$. 
Under this condition the incoherent space-charge terms
\[ [\Delta Q_{ic} Z_{ko} + \Delta Q_{ic}] \xi_k = [\Delta Q_{ic} Z_{ko} + \Delta Q_{ic}] \xi \] "drop out" from \( \xi \) (3.6)
and is no Q-spread left. We can therefore conclude that the dependence of the coherent Q-shifts \( \Delta Q_c \) and/or \( v \) on the incoherent motion \( Z_{ko} \) is essential for the Landau damping mechanism we are considering. This is in agreement with the qualitative argument that non-linearities concerning only the relative motion of the particles with respect to the beam center should have no effect on the motion of the beam center [2].

Henceforth, we will assume that a spread in the coherent Q-shifts is present. Considering dipole oscillations we assume that
\[ \xi = \zeta \exp [iQ_c \phi] \] (3.8)
\[ \xi_k = \zeta \exp [iQ_c \phi] \] (3.9)

Using an appropriate averaging method, the \( Z_{ko} \) dependence in (3.6) is converted into a dependence, which we denote by \( \overline{\Delta Q_c(a_k)} \), \( \overline{V(a_k)} \), etc. Upon substitution of (3.8)(3.9) into (3.6) (using \( Q_o + Q_c \approx 2Q_o \))
\[ \zeta_k = \zeta \frac{\overline{\Delta Q_c(a_k)} + (1+i) \overline{V(a_k)} - \overline{\Delta Q_{ic}(a_k)}}{Q_o - Q_c - \overline{\Delta Q_{ic}(a_k)}} \] (3.10)

The solution (3.10) fulfills the exact initial conditions if we assume that the perturbation causing the coherent oscillations has been applied at \( \phi \rightarrow \infty \) and the system (3.1) and (3.6) has at least a small damping term.

Let us deduce a dispersion relation for \( Q_c \) from (3.10) using the formalism discussed in [1]. We use the fact that \( \overline{\zeta_k} = \zeta \), and replace the average \( \overline{\zeta_k} = \frac{1}{N} \sum_{k=1}^{N} \zeta_k(a_k) \) by an integral \( \zeta_k = \int f(a) \zeta_k(a) \, da \), where \( f(a) \) is the amplitude distribution function properly normalized. We then obtain the dispersion relation for \( Q_c \)
\[ 1 = \int_{0}^{\infty} \frac{\overline{\Delta Q_c(a)} - \overline{\Delta Q_{ic}(a)} + (1+i) \overline{V(a)}}{Q_o - Q_c - \overline{\Delta Q_{ic}(a)}} f(a) \, da \] (3.11)
or alternatively

\[ l = \int_0^\infty \frac{u(a) + (1+1) v(a)}{Q_0 - Q_c - \Delta Q_{1c}(a)} f(a) \, da \]

In general the integral in (3.11) has now a pole which will lead to Landau damping. Let us e.g. assume that the \( a \)-dependence of \( u \) and \( v \) is of the same functional form, \( u = \hat{u} \cdot g(a) \), \( v = \hat{v} \cdot g(a) \). We may then write

\[ l = (\hat{u} + \hat{v} + iv) \int_0^\infty \frac{g(a)f(a) \, da}{Q_0 - Q_c - \Delta Q_{1c}(a)} \]  

(3.12)

This dispersion relation is of exactly the same type as those discussed in literature (e.g. [1]).

However, some reservation about (3.11) has to be added at this stage. The calculations of this section followed along the same line as used in ref. [1] for the case of external non-linearities. The criticism expressed to this approach in a recent paper [3] therefore also applies to our calculations. It seems that the more rigorous approach of ref. [3] which takes into account "second order" terms of the non-linear equation (3.1) would lead to a somewhat modified dispersion relation. In general this more exact relation exhibits the same principle properties as (3.11), also numerical results are somewhat different.

4. CALCULATION OF THE AMPLITUDE DEPENDENCE OF THE Q-SHIFTS

Given a certain amplitude distribution function \( f(a) \) (considering case a), Section 2.1) we want to calculate the electric field component \( E_Z(Z,\overline{Z}) \) at the point \( Z \) assuming that the beam center is displaced from \( Z = 0 \) by \( \overline{Z} \). Some general conclusions can be drawn by expanding \( E_Z(Z,\overline{Z}) \) into a power series in \((Z-\overline{Z})\) and \( \overline{Z} \).
Retaining only terms up to first order in \( \overline{Z} \) we write

\[
E_Z(Z-\overline{Z}, \overline{Z}) = \sum_{n=1}^{\infty} A_n (Z-\overline{Z})^n + \sum_{n=1}^{\infty} B_n (Z-\overline{Z})^{n-1} \overline{Z},
\]

(4.1)

where \( A_n \) and \( B_n \) are constants and higher powers of \( Z \) are neglected as we are interested in small amplitude coherent oscillations. In coherent space-charge forces are a function of the deviation \( (Z-\overline{Z}) \) from the beam center. Coherent space-charge is a function of the displacement \( \overline{Z} \) of the beam center and of \( (Z-\overline{Z}) \).

By comparison with (3.1) it is easily seen that one can identify the first sum in (4.1) with the incoherent space-charge term and the second sum with the coherent space-charge forces. More precisely

\[
\frac{e(1-p^2)}{m Q_0 w^2} \sum_{n=1}^{\infty} A_n (Z-\overline{Z})^{n-1} = \Delta Q_{ic} (Z-\overline{Z})
\]

(4.2)

\[
\frac{e(1-p^2)}{m Q_0 w^2} \sum_{n=1}^{\infty} B_n (Z-\overline{Z})^{n-1} = \Delta Q_c (Z) + (1+i) \nu(Z)
\]

We now calculate \( \Delta Q_{ic} (Z-\overline{Z}) \) neglecting contributions from image charges and currents. We assume that the phases \( \psi_k \) of \( z_0 \) in (3.3) are randomly distributed between 0 and \( 2\pi \).

Given the amplitude distribution \( f(a) \) the density \( \rho(\tilde{Z}) \) in ordinary space is then in the one dimensional case under considerations given by [4]

\[
\rho(\tilde{Z}) = \frac{A \cdot e}{\pi b} \int \frac{f(a) \, da}{\sqrt{a^2 - \tilde{Z}^2}}
\]

(4.3)
Notation:

\( \lambda \) : number of particles per unit length of the beam

\( \hat{b} \) : transverse beam radius in the large (horizontal) direction

\( \hat{Z} \) is for shortness written for \( (Z - \bar{Z}) \).

The electric field \( E_Z(\hat{Z}) \) is found from div. \( E = 4\pi \rho \)
to be

\[
E_Z(\hat{Z}) = 4\pi \int \rho(\hat{Z}) \, d\hat{Z} \quad (4.4)
\]

This force is inserted into (3.6) and averaged along the assumed orbits (3.3).

We will use the averaging relation [5]

\[
\overline{\Delta Q_{ic}(a)} = \frac{e(1 - \beta^2)}{2\pi m Q_0 \alpha^2 a} \int_0^{2\pi} \sin\phi \, E_Z(a \sin\phi) \, d\phi \quad (4.5)
\]

The results obtained from (4.5) are in agreement with the standard theory [1], which uses a similar averaging relation. Nevertheless, we are aware of the fact that it would be more correct to replace

\[
\sin\phi \cdot E_Z(a \sin\phi)
\]

in (4.5) by \( \frac{\partial E_Z}{\partial Z}(a \sin\phi) \), as long as we study case a). This is obvious from (3.6) where the term to be averaged is

\[
\frac{\partial E_Z}{\partial Z} = \left[ \Delta Q_{ic}(Z_{ko}) + \Delta Q_{ic}(Z_{ko}) \right] \quad \text{rather then} \quad E_Z = \Delta Q_{ic}(Z_{ko}) Z_{ko} \quad \text{which appears in (3.3)}.
\]

With the aid of (4.5) we are able to convert any \( E_Z(Z) \) into

\( \overline{\Delta Q_{ic}(a)} \) dependence.

Finally we have to evaluate and average the coherent space-charge forces. This is a more complicated process since these forces are a strong function of the beam- and chamber geometry and of the mechanism responsible for the \( v \)-term.
We content ourselves with a few remarks here. If the beam dimension is much smaller than the chamber radius, but larger than the amplitude of the coherent displacement, the $B_1$-term in (4.1) is expected to predominate. The coherent Q-shifts are then constant and Landau damping does not occur.

Therefore, we will assume that the beam tightly fills the chamber, although this is not fully consistent with some assumptions we make in the numerical examples (Section 6). Terms higher than $B_1$ may then be of importance and these terms can again be averaged over the $(Z-Z)$-motion assumed, thus converting $\Delta Q_c(Z-Z)$ into another function $\overline{\Delta Q_c}(a)$.

In the examples in Section 6, we will discuss rather arbitrarily two special cases only. We will either assume that the numerator of the dispersion relation (3.11) is independent of a or that the functional form of $\Delta Q_c(a)$ and $v(a)$ is the same as that of $\Delta Q_{ic}(a)$.

5. SOME REMARKS ON THE EFFECT OF SPACE-CHARGE NON-LINEARITY COMING FROM THE TRANSVERSE DIRECTION PERPENDICULAR TO THE COHERENT OSCILLATIONS

We now consider the case b) defined in Section 2. Here the quantities $\Delta Q_{ic}$, $\Delta Q_c$ and $v$ are functions of the horizontal position $r_k$ of the test particle. Equation (3.1) is rewritten in the form

\[
\frac{d}{dr} \left[ Z_k' + Q_0^2 Z_k - 2Q_0 \Delta Q_{ic}(r_k) \right] \cdot Z_k = 2Q_0 \left[ \Delta Q_c(r_k) + (1+i)v(r_k) - \Delta Q_{ic}(r_k) \right] \cdot Z_k
\]  

(5.1)

Let $r_k$ be

\[
r_k = h_k \exp \left[ i (Q r_k + \omega_k) \right]
\]  

(5.2)

i.e. we only consider the incoherent r-motion.
Again by averaging the r-dependence is converted into an amplitude dependence and the dispersion relation

$$\frac{1}{Q_0 - Q_c - \bar{\Delta} Q_{ic}} \int_0^\infty \frac{\bar{\Delta} Q_c(b) + (1+i)\bar{v}(b) - \bar{\Delta} Q_{ic}(b)}{\bar{f}(b)} \, db$$  \hspace{1cm} (5.3)$$

is obtained by exactly the same formalism as used above in case a). The function $f(b)$ in (5.3) is the distribution function of the horizontal amplitudes of the particles.

Case b) is in some respects simpler than case a). Especially the reservations added in connection with (3.11) do not hold for (5.3).

The next step consists in calculating the quantities $\Delta Q_{ic}(b)$, $\bar{\Delta} Q_c(b)$ and $\bar{v}(b)$, which are correlated to $f(b)$, since the space-charge forces depend on the distribution function $f(b)$.

One of the main differences between the cases a) and b) is that the relation between $\bar{\Delta} Q_{ic}(b)$, $\bar{\Delta} Q_c(b)$, $\bar{v}(b)$ and $f(b)$ is different from the relation linking $f(a)$ with $\Delta Q_{ic}(a)$, $\Delta Q_c(a)$ and $\bar{v}(a)$.

In fact, we have assumed that the density varies slowly in the r-direction. Thus solving Poisson's equation, one finds that the field $E_z(r)$ is to first order proportional to the density $\rho(r)$.

In this approximation we can perform the usual calculation of the Q-shifts $\tilde{\Delta} Q_{ic} = \Delta Q_{ic}(r=0)$, $\tilde{\Delta} Q_c = \Delta Q_c(r=0)$, $\tilde{v} = v(r=0)$ for the beam center and include the r-variation by a factor

$$g(r) = \frac{\rho(r)}{\rho(0)}$$

The density $\rho(r)$ is again obtained from $f(b)$ by means of an expression similar to (4.3)

$$\rho(r) = \frac{\lambda \cdot g}{\pi \hat{a}} \int_0^\infty \frac{f(b) db}{\sqrt{b^2 - r^2}}$$  \hspace{1cm} (5.4)$$
Finally, averaging over the motions (5.2), the r-dependence is converted into a b-dependence of $\Delta Q_{ic}$, $\Delta Q_c$ and $v$.

Comparing the cases a) and b) discussed hitherto it is seen that the field calculation in case b) is simpler. In addition, all the Q-shifts, $\Delta Q_{ic}$, $\Delta Q_c$ and $v$ in case b) have the same amplitude dependence, so the dispersion relation (5.3) can be written as

$$1 = \left( \hat{\alpha} + (1+i)\hat{\eta} \right) \int_0^\infty \frac{g(b) f(b) \, db}{Q_0 - Q_c - \Delta Q_{ic} \cdot g(b)} \quad (5.5)$$

This relation again is very similar to the usual dispersion relations and we can in general expect Landau damping as the integral has a pole.

Some of the assumptions made in Section 3 need a critical examination, e.g. the averaging processes (4.5) and the neglect of "second order terms" in going from (3.1) to (3.6). In the following section we give some numerical examples using these relations. We emphasize that case b) discussed in this sub-section is simpler and more transparent and our reservations do not hold for the dispersion relation (5.5) derived above.

6. NUMERICAL EXAMPLES

With the aid of the formalism described above we are able to calculate growth rates and to find stability limits. We are going to use two special distribution functions: A bell-shaped amplitude distribution, a model where the amount of non-linearity in the space-charge forces is rather large; and a parabolic density distribution for which the non-linearity is weak.
6.1. "Bell-shaped" amplitude distribution

We choose

\[ f(a) = \frac{2}{a_1^2} \frac{a}{(1 + (a/a_1)^2)^2} \]  \hspace{1cm} (6.1)

already normalized so that \[ \int_0^\infty f(a) da = 1. \]

Then by virtue of (4.3)

\[ \rho(\tilde{Z}) = \frac{2\lambda e}{\pi b a_1^2} \int_0^\infty \frac{a \ d a}{(1 + (\frac{a}{a_1})^2)^2 \sqrt{a^2 - \tilde{Z}^2}} \]  \hspace{1cm} (6.2)

Performing the integration we find

\[ \rho(\tilde{Z}) = \frac{\lambda e}{2a_1 b} \frac{1}{(1 + (\frac{\tilde{Z}}{a_1})^2)^{3/2}} \]  \hspace{1cm} (6.3)

The electric field is then

\[ E_z(\tilde{Z}) = 4\pi \int \rho(\tilde{Z}) \ d\tilde{Z} = \frac{4\pi e}{2b a_1} \frac{\tilde{Z}}{\sqrt{1 - (\frac{\tilde{Z}}{a_1})^2}} \]  \hspace{1cm} (6.4)

and the Q-spread (4.5) is

\[ \Delta Q_{ic}(a) = \Delta Q_{ic} \times \int_0^{2\pi} \frac{\sin^2 \psi \ d\psi}{\sqrt{1 + (\frac{a}{a_1})^2 \sin^2 \psi}} = \Delta Q_{ic} \times J \]  \hspace{1cm} (6.5)

with \( \Delta Q_{ic} \), the maximum space-charge Q-shift and J used as an abbreviation for the integral in (6.5).
\[ \hat{\Delta} Q_{iC} = \frac{4r_p R^2 \lambda}{Q_0 \beta \gamma^2 b a_1} , \quad r_p = 1.53 \ldots \times 10^{-16} \text{m for protons} \ (6.6) \]

Note that (6.6) is only valid at low enough energies, since we have neglected image forces.

The integral in (6.5) can be expressed by means of complete elliptic integrals

\[ J = \frac{1}{\sqrt{1+(a/a_1)^2}} \cdot \frac{(1+(a/a_1)^2) E(\pi/2, [a^2/(a^2+a_1^2)]^{1/2}) - K(\pi/2, [a^2/(a^2+a_1^2)]^{1/2})}{(a/a_1)^2} \]

When the amplitude \( a \) goes from 0 to \( \infty \), the term in the bracket changes between \( \pi/4 \) and 1, and for simplicity we shall drop this term, keeping in mind that this introduces an error up to 20 o/o.

So we write

\[ \bar{\Delta} Q_{iC}(a) = \hat{\Delta} Q_{iC} \frac{1}{\sqrt{1 + (a/a_1)^2}} \]

and the dispersion relation (3.11) is

\[ l = 2 \int_0^\infty \frac{\bar{u} + \bar{v} + i\bar{\nu}}{\hat{\Delta} Q} \frac{\alpha \cdot d \alpha}{(1+\alpha^2)^2 \left( \frac{Q_0 - Q_0}{\hat{\Delta} Q} - \frac{1}{\sqrt{1+\alpha^2}} \right)} \]

with \( \alpha = a/a_1 \). Henceforth in this section we will for simplicity suppress the bars which we used so far to indicate that \( \bar{u}(a) \) was obtained from \( u(z) \) by an appropriate averaging process. All \( u \)-s and \( v \)-s appearing below stand for \( \bar{u}(a) \), etc. Substituting \( x = (1+\alpha^2)^{-1/2} \), \( x_0 = (Q_0 - Q_0)/\hat{\Delta} Q_{iC} \), we have

\[ l = -2 \int_0^1 \frac{(u+v+iv)}{\Delta Q} \cdot \frac{x \cdot d \cdot x}{x + x_0} \]
We now introduce a cut-off for $f(a)$ at $a = a_{\text{wall}}$, and using the abbreviation $\epsilon = \left(1 + (a_{w}/a_1)^2\right)^{-1/2}$, we write

$$1 = -2 \int_{\epsilon}^{1} \frac{(u+v+i\nu)}{\Delta Q} \cdot \frac{x \, d \, x}{x + x_0} \quad (6.7)$$

Here $a_w$ is the half aperture of the vacuum chamber in the vertical direction. The distribution function $f(a)$ is normalised $\int_{a_{w}}^{a_{\text{wall}}} f(a) \, da = 1$ if we replace $\Delta Q$ by $\Delta Q/(1-\epsilon^2)$. The correction $\epsilon$ is usually small, since we are mainly interested in cases where $\epsilon \ll 0.1$.

We are now going to investigate two limiting cases. We either assume that $u$ and $v$ are the same for particles, or we treat $u$ and $v$ as a function of betatron amplitude $a$ taking the same functional dependence $g(a)$ for $u, v$ and $\Delta Q_{ic}$.

6.1.1. $u+v+i\nu$ = const.

The dispersion relation is

$$1 = -2 \frac{u+v+i\nu}{\Delta Q} \int_{\epsilon}^{1} \frac{x \, d \, x}{x + x_0} \quad (6.8)$$

In general, $x_0 = \frac{Q_c - Q_0}{\Delta Q}$ is complex,

$$x_0 = \xi + i \eta$$

and we want to compute $\eta$, since $-(\nu \cdot \Delta Q \cdot \eta)$ is the growth rate of the instability, and the coherent oscillations are unstable, when $\eta$ is negative.

Let us split the integrand into its real and imaginary part:

$$\frac{x}{x_0 + x} = \frac{x}{(\xi + x)^2 + \eta^2} - 1 \frac{\eta \, x}{(\xi + x)^2 + \eta^2} = \text{Re} + i \text{Im} \quad (6.9)$$
The dispersion relation (6.8) can be rewritten in the form:

\[
\frac{1}{2} \left[ \frac{\hat{\Delta} Q_{ic}}{-(u+v)} \frac{1}{1+\left( \frac{v}{u+v} \right)^2} + i \frac{\hat{\Delta} Q}{(u+v)} \frac{v}{u+v} \frac{1}{1+\left( \frac{v}{u+v} \right)^2} \right] = \frac{1}{\epsilon} \int (\text{Re}+i\text{Im}) dx \tag{6.8a}
\]

As mentioned in Section (2.2) the quantity \( u \) is negative in our model, as we have neglected inductive terms. Performing the integration and splitting into real and imaginary parts, we finally can write

\[
\frac{\hat{\Delta} Q_{ic}}{-2(u+v)} \frac{1}{1+\left( \frac{v}{u+v} \right)^2} = 1 - \epsilon + \eta \Phi - \frac{\eta}{2} \ln \left( \frac{(l+\xi)^2 + \eta^2}{(l+\xi)^2 + \eta^2} \right) \tag{6.10}
\]

and

\[
\frac{1}{2} \frac{\hat{\Delta} Q}{(u+v)} \frac{v}{(u+v)} \frac{1}{1+(v/u+v)^2} = -\frac{\xi}{2} \Phi - \frac{\eta}{2} \ln \left( \frac{(l+\xi)^2 + \eta^2}{(l+\xi)^2 + \eta^2} \right) \tag{6.11}
\]

with the abbreviation

\[
\Phi = \arctg \left( \frac{-\xi + \epsilon}{\eta} \right) - \arctg \left( \frac{-\xi + 1}{\eta} \right)
\]

Equations (6.10) and (6.11) have been solved by means of a computer programme. Some typical results for the case \( u+v<0 \) (\( v<|u| \)) are shown in Fig. 1.

We want to know how much the growth rate with non-linear space-charge differs from the case where all particles have the same betatron frequency, and therefore an interesting quantity is the ratio of growth rate with space-charge to growth rate without. In Fig. 1 this ratio is plotted against the quantity \( \hat{\Delta} Q_{ic}/|u+v| \). As \((-u) = \Delta Q_{ic} - \Delta Q_c < \Delta Q_{ic} \) in our model, we find that

\[
\frac{\hat{\Delta} Q_{ic}}{|u+v|} \approx \frac{\hat{\Delta} Q_{ic}}{|u|} \frac{1}{1+|v/u|} > \frac{1}{1+|v/u|} \tag{6.12}
\]
From Fig. 1 we deduce that for a given value of \(|u+v|\) the growth rate becomes smaller with increasing \(\hat{\Delta}Q_{ic}\). For \(\Delta Q_{ic}/|u+v|\) greater than a certain threshold the growth rate is zero or negative and therefore the oscillations are stable. These threshold values are plotted in Fig. 2. A strange effect is observed in cases where \(f(a)\) has a cut-off \((c>0)\). Let us take e.g. the case \(v/|u+v| = 0.01\) and \(c = 0.1\) in Fig. 1. In this example, the growth rate does not vanish exactly until \(\Delta Q_{ic}/|u+v| = 64\), although it is very small for \(\Delta Q_{ic}/|u+v| \geq 3\). In the first instance this result seems to be discouraging. But we will nevertheless speak of a practical stability for \(\Delta Q_{ic}/|u+v| \geq 3\), as the growth rate is low in this range. Also it seems that for \(\Delta Q_{ic}/|u+v| \geq 3\) we would have stability in the mathematical sense, provided that the function \(f(a)\) went smoothly to zero at \(a=a_w\); rather than to have a sharp cut-off.

From Fig. 2a it is noted that the threshold for stability depends quite sensitively on \(v/|u+v|\) and on the cut-off value \(c\). With a cut-off \((c > 0)\) we observe again the somewhat unphysical fact that for small \(v/|u+v|\) (i.e. for small \(v/|u|\)) the stability limit tends to large values of \(\Delta Q_{ic}\). For reasons explained above we extrapolate the threshold curves as sketched in Figs. 2 and expect "practical stability" if the "working point" is on the right of this extrapolated curve.

From Fig. 2a we can further deduce that for small \(v/|u+v|\), or equivalently \(v/|u| \ll 1\) the threshold lies between \(\Delta Q_{ic} = 2|u|\) and \(\Delta Q_{ic} = 3|u|\) for small \(c\). In other words dipole oscillations are practically stable in low and medium energy machines \((U >> V)\) if \(\Delta Q_{ic} \gg 3u = 3(\hat{\Delta}Q_{ic} - \hat{\Delta}Q_{0})\). The curves drawn in Fig. 2a are valid only for \(u < 0\), \(v < |u|\). For the opposite extreme, \(v >> |u|\), \((u < 0)\) we find the threshold for stability, to be \((\Delta Q_{ic})_{thr} = 12.5\ V\).
The assumption \( u+v+iv = \text{const.} \) used above is evidently unrealistic to some extent, as non-linear space-charge will make \( u \) and \( v \) to be functions of the betatron amplitude \( a \) (see Section 3). In the following we therefore shall investigate the threshold, for a special case where \( u \) and \( v \) are not the same for all particles.

6.1.2. \( u+v+iv = g(a) \cdot (\hat{u}+\hat{v}+i\hat{v}) \)

In general, \( u \) and \( v \) will be different functions of \( a \), but in this subsection we will restrict ourselves to the case where \( u \) and \( v \) vary with the amplitude exactly like \( \Delta Q_{ic} \) does. We write

\[
u + v + iv = (\hat{u} + \hat{v} + i\hat{v}) \frac{1}{\sqrt{1 + (a/a_1)^2}}\]

The dispersion relation is now

\[1 = -2 \frac{\hat{u} + \hat{v} + i\hat{v}}{\Delta Q_{ic}} \cdot \int \frac{x^2 dx}{x + x_0} \]

Putting \( x_0 = \xi + i\eta \), integrating and separating into real and imaginary parts, we have

\[
\frac{1}{2} \frac{\Delta Q_{ic}}{|\hat{u} + \hat{v}|} \frac{1}{1 + (\hat{v}/\hat{u} + \hat{v})^2} = \frac{1}{2}(1-\epsilon^2) - \xi(1-\epsilon) + (\xi^2 - \eta^2) \ln \left( \frac{(\xi+1)^2 + \eta^2}{(\xi+\epsilon)^2 + \eta^2} \right) - 2 \xi \eta \phi \quad (6.14)
\]

\[
\frac{1}{2} \frac{\Delta Q_{ic}}{|\hat{u} + \hat{v}|} \frac{\hat{v}}{|\hat{u} + \hat{v}|} \frac{1}{1 + (\hat{v}/\hat{u} + \hat{v})^2} = -\eta(1-\epsilon) + \xi \eta \ln \left( \frac{(\xi+1)^2 + \eta^2}{(\xi+\epsilon)^2 + \eta^2} \right) + (\xi^2 - \eta^2) \phi \quad (6.15)
\]
with \( \theta = \arctg \left( \frac{x+c}{\eta} \right) - \arctg \left( \frac{x+1}{\eta} \right) \)

We want to discuss here only thresholds of the instability, which are plotted in Fig. 2b, again for \( u+v < 0 \). We remark that for very small \( \hat{v}/|\hat{u}+\hat{v}| \) and small \( \varepsilon \) the threshold condition is \( \Delta \hat{Q}_{ic} > |\hat{u}+\hat{v}| \). For \( \hat{v}/|\hat{u}| \ll 1 \) this relation generally holds in our model (see (6.12)).

We thus arrive at the surprising result, that coherent oscillations are always stable (in our model) if \( \Delta Q_{ic}(a) \), \( \Delta Q_c(a) \) and \( v(a) \) have the same functional form, \( v/|u| \) is small and \( f(a) \) is bell shaped.

6.2. Parabolic density distribution in ordinary space

Our second example deals with a parabolic density distribution. This is a fairly realistic distribution for low intensity beams \([6]\) and it provides only a very weak "non-linear" part of the space-charge forces. Thus we expect much higher thresholds than in the previous example.

We shall only give a brief outline, as the technique of evaluation has already been treated in extenso. Starting from

\[
f(a) = \frac{3}{2} \cdot a \sqrt{1 - (a/a_1)^2}
\]

the density is found to be

\[
\rho(\tilde{z}) = \frac{3}{2b a_1} \left( 1 - \frac{\tilde{z}^2}{a_1^2} \right)
\]

and the electric field

\[
E_z(\tilde{z}) = \frac{6 \pi \varepsilon_0 a_1}{b} \left( 1 - \frac{1}{3} \frac{\tilde{z}^2}{a_1^2} \right)
\]
finally the incoherent Q-spread is

\[ \Delta Q_{ic}(a) = \Delta Q_{ic} \left( 1 - \frac{1}{4} \left( \frac{a}{\alpha_1} \right)^2 \right) \quad \text{with} \quad \Delta Q_{ic} = \frac{3\pi \lambda r R^2}{ba_1 Q_o \beta^2 \gamma^2} \]

The dispersion relation, here only discussed for \( u \) and \( v = \text{const.} \), is

\[ \frac{\Delta Q_{ic}}{12(u+v+iv)} = -\int_0^1 \frac{x^2 dx}{x^2 + z_0} \]

with \( z_0 = 3 + 4 x_o \), \( x_o = \frac{Q_o - Q}{\Delta Q_{ic}} \)

We shall only discuss the thresholds as a function of \( v/|u+v| \), derived from the equation

\[ \sqrt{|z_0|} = \frac{\Delta Q_{ic} \cdot v}{(u^2 + v)^2 (1 + (v/u)^2) \cdot 6\pi} \]

and

\[ \frac{\Delta Q_{ic}}{12|u+v| (1+(v/u)^2)} = 1 - \varepsilon + \frac{\sqrt{|z_0|}}{2} \ln \left| \frac{(1-\sqrt{|z_0|})(\varepsilon+\sqrt{|z_0|})}{(1+\sqrt{|z_0|})(\varepsilon-\sqrt{|z_0|})} \right| \]

The results - plotted in Fig. 3 for \( v < |u| \) - show that for \( v/|u| \ll 1 \) and \( \varepsilon \ll 1 \) the threshold for stability is \( (\Delta Q_{ic})_{\text{thr}} = 12|u| \), thus the thresholds are much higher than in the example of the bell-shaped amplitude distribution discussed above.
7. CONCLUSIONS

Space-charge - although being ultimately responsible for the existence of transverse coherent instabilities - may have a beneficial effect, since non-linear space-charge forces show a tendency to reduce the growth rate of coherent instabilities of the dipole type, and possibly of types of a higher polarity, too. Even complete stabilization may be obtained for beams with a sufficiently large amount on non-uniformity in the transverse density distribution. The non-linearity of both coherent- (images forces) and incoherent space-charge forces is essential for this stabilizing mechanism.

In cases where the internal non-linearity is strong, external non-linearities to produce a stabilizing Q-spread (e.g. octupole lenses) should be of such a polarity that they increase rather than reduce the space-charge Q-spread.

Our results are based on a simplified model and many of our assumptions need a critical examination.

Some conclusions, especially the numerical results of Section 6 are therefore preliminary and more work is needed to confine them.
ACKNOWLEDGEMENTS

We are grateful to H.G. Hereward for interesting and clarifying discussions. He has also read our manuscript and suggested several improvements. Further we have to thank C. Pellegrini and A.M. Sessler for discussion during their visit to CERN in early 1969, and K. Hübner, V.G. Vaccaro and B. Zotter for their comments on our calculations.

REFERENCES


Growth rates for bell shaped amplitude distribution
$u + v + iv = \text{const.}, v < |u|, u < 0$

### Dipole mode

$$
\nu / |u + v| = 0.01
$$

- $\varepsilon = 0$
- $\varepsilon = 0.1$

**Fig. 1**

### Extrapolated growth rate

- $\Delta \tilde{G}_{ic} / |u + v|$

### Growth rate with s.c.
- Growth rate without s.c.

$$
\nu / |u + v| = 2, \varepsilon = 0.1
$$

- $\varepsilon = 0$
- $\varepsilon = 0.1$

**Fig. 1b**
Thresholds for coherent stability as a function of $\nu/|\mu + \nu|$

Dipole mode, $\mu < 0$, $\nu < |\mu|$

$f(a)$: bell shaped
$\mu, \nu = \text{const.}$

Fig. 2a

$f(a)$: bell shaped
$\mu, \nu \propto (1 + (a/a_i)^2)^{-\nu_2}$

Fig. 2b

$f(a) \propto a(1-(a/a_i)^2)^{1/2}$
$\mu, \nu = \text{const.}$

Fig. 3