PREDICTION OF LONG-TERM STABILITY IN LARGE HADRON COLLIDERS*

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Short-term tracking data are used to compute early indicators of long-term stability in large hadron colliders. A phenomenological picture of the mechanisms that lead to long-term particle loss is outlined, using numerical simulations. The Lyapunov exponent and the variation of the nonlinear tunes with time are computed with an automated procedure and compared to model-independent thresholds: the motion of particles is considered stable when the early indicators are below the threshold. In addition, a law is proposed to interpolate the dynamic aperture as a function of the number of turns, and its extrapolation allows one to predict the dynamic aperture at large number of turns. The law has some analogies with the estimate given by the Nekhoroshev theorem. Comparisons of our predictions with the results of standard element-by-element tracking are carried out for a realistic model of the CERN-LHC.

1 INTRODUCTION

In hadron accelerators, the long-term stability of particle motion is a crucial issue. In a large circular machine such as the planned Large Hadron Collider,\textsuperscript{1} charged particles circulate for $10^7$ turns before energy ramping. Numerical simulations can hardly evaluate the stability for a few $10^5$ turns with modern computers. Even though one could argue that supercomputers able to carry out simulations for $10^7$ turns will be available in the near future, it must be pointed out that the optimization of the lattice parameters requires the analysis of a large number of configurations, and therefore methods alternative to brute-force element-by-element tracking should be worked out.

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Over the past years, three main approaches have been proposed in order to predict the long-term behaviour. The first one is based on the evaluation of early indicators that distinguish regular from chaotic motion. Under the assumption that all the chaotic particles are lost after a sufficiently high number of turns, one obtains a criterion for long-term stability. Two early indicators, borrowed from celestial mechanics, have been used: the Lyapunov exponent and the variation of the instantaneous nonlinear frequencies. A check of the Lyapunov against long-term tracking has been carried out for realistic accelerator models in Refs. The second approach is based on the spirit of the Nekhoroshev theorem and its generalizations to symplectic mappings. The basic idea is to define precise nonlinear invariants, and to bound the particle loss by computing the drift in the space of invariants for a limited number of turns. Clever methods to evaluate such invariants have been proposed, based for instance on an interpolation of the actions, or on frequency analysis. The intrinsic limit of these methods lies in the error associated to the determination of the nonlinear invariants. The third approach is based on tracking: the number of turns where particle loss occurs is plotted versus the initial amplitude, and an extrapolation to the required number of turns is worked out (survival plots, see Refs. ). The main difficulty in this approach is due to a rather irregular behaviour of the dynamic aperture as a function of the number of turns, that does not allow a precise extrapolation. We will show that this difficulty can be overcome by using a more precise definition of dynamic aperture that involves an average over the phase space.

We believe that in order to understand the mechanism of long-term losses and to define the limits of the early indicators predictivity, a check against long-term tracking is crucial. For this reason we carried out extensive simulations for the 4D Hénon map to understand the dynamics of the problem; we then switched to a realistic LHC model, finding out similar results. We first propose a way to visualize resonances in phase space (which has been used also in Ref. ) and we determine the relation between the network of resonances and the long-term losses. It turns out that very wide resonances are stable when they are not too close to the dynamic aperture, and that long-term phenomena are mainly due to macroscopic bands of chaoticity. This is in agreement with previous results.

The use of early indicators in general implies to evaluate the trend of some dynamical quantity with only a finite set of simulation data. This evaluation can be somewhat arbitrary and it can hardly be implemented
in computer codes. In a previous paper\textsuperscript{21} we have proposed a procedure based on thresholds to automatically determine whether the early indicator predicts stability or instability. We review this procedure, and we show that for the analysed models the thresholds turn out to be the same. This feature is very important, since it allows to use the early indicators for the systematic evaluation of the dynamic aperture of the lattice.

Using the definition of dynamic aperture given in Ref.\textsuperscript{16}, we show that a simple law interpolates very well the dynamic aperture evaluated through tracking as a function of the number of turns. The dynamic aperture turns out to decay with the inverse of the logarithm of the number of turns. This \textit{law of the inverse logarithm} can be derived by assuming that the phase space is divided in a inner region stable for infinite times, and an outer region of chaoticity where the escape rate agrees with the Nekhoroshev bound. This interpolation proves to be extremely powerful. A complementary approach is based on the measurement of the escape rates rather than the size of the stability domain; numerical evaluations of the escape rates in discrete hamiltonian systems are in progress\textsuperscript{22} and early results show a good agreement with our scenario.

The plan of the paper is the following: in Section 2 we carry out the phenomenological analysis of the mechanism of long-term particle loss, developing numerical tools based on frequency analysis. In Section 3 we discuss two early indicators (Lyapunov and tune variation), defining thresholds to provide automated criteria for selecting chaotic from stable motion. In Section 4 we discuss the definition of dynamic aperture. In Section 5 we use this definition to produce a refined version of survival plots that allow a precise interpolation of the dynamic aperture versus the number of turns. These early indicators are used in Section 6 to predict particle loss for the Hénon map and for a realistic model of the LHC; a check with long-term tracking is presented.

\section{PHENOMENOLOGICAL ANALYSIS OF MECHANISMS OF LONG-TERM LOSS}

\subsection{The Models}

In this section we use the results of numerical simulations to illustrate a phenomenological picture of long-term losses and their relation with the network of resonances. We will first restrict ourselves to the analysis of
four-dimensional (4D) betatron motion. We consider a generalization of the 4D Hénon map\textsuperscript{17} that includes also an octupolar term:

\begin{equation}
\begin{pmatrix}
x' \\
p'_{x} \\
y' \\
p'_{y}
\end{pmatrix} = \begin{pmatrix}
R(\omega_{x}) & 0 \\
0 & R(\omega_{y})
\end{pmatrix} \begin{pmatrix}
x \\
p_{x} + x^2 - y^2 + \alpha_3(x^3 - 3xy^2) \\
y \\
p_{y} - 2xy - \alpha_3(3x^2y - y^3)
\end{pmatrix}
\end{equation}

where \( R(\omega) \) is the 2D rotation matrix of an angle \( \omega \) in each of the transverse phase planes. In the above expression \((x, y)\) denote the coordinates transversal to the longitudinal motion, and \((p_{x}, p_{y})\) the conjugated momenta. We use Courant-Snyder coordinates where the linear part of the map is the direct product of rotations of angles \( \omega_{x} \) and \( \omega_{y} \). This map can be obtained, after rescaling the phase space coordinates, from the map of a linear lattice with a sextupolar and an octupolar kick placed where the horizontal and vertical beta functions have the same value. For this reason, the phase space coordinates are expressed in arbitrary units. The parameter \( \alpha_3 \) is related to the ratio of the sextupole and octupole strengths.

For more realistic investigations we also consider a full LHC lattice model with the expected field-shape errors in the superconducting magnets. The chromaticity is corrected in both planes down to 3 units of \( Q' \) using two families of sextupoles. Cases with and without RF-cavities are investigated. In the former case, the test particles have a relative momentum spread of \( 0.75 \cdot 10^{-3} \), that is about three quarters of the momentum spread of the bucket. In the latter case only 4D motion is investigated.

### 2.2 Tracking and Frequency Analysis Tools

The numerical exploration of the dynamics of the above models is based on the iteration of the mapping and on the computation of the frequencies. The initial conditions are chosen along the plane \((x, y)\), using either rectangular or polar grid, and fixing the momenta \((p_{x}, p_{y})\) to zero. The orbit is computed for \( N \) turns; then the nonlinear frequencies in each phase plane \((x, p_{x})\) and \((y, p_{y})\) are evaluated through the method based on the interpolation of the FFT with Hanning filter\textsuperscript{23} which provides very precise estimates. The final data are presented in the following ways.

- **Long-term diagram.** The initial conditions are iterated for a very long number of turns \((10^{5} - 10^{7})\) and the stable ones are plotted in the space
(x, y); this diagram provides the shape and the dimension of the stability domain. It is also interesting to plot initial conditions that are unstable using different markers according to the number of turns where the particle is lost. This procedure allows one to visualize the rate of escape of unstable particles. Unfortunately, these diagrams are very CPU time consuming and therefore a fine scan in the initial conditions cannot be carried out.

Footprint diagram. A dense set of initial conditions (typically at least $10^4$) are iterated and the frequencies of the stable initial conditions are plotted in the frequency space. Since one can use very precise algorithms\textsuperscript{5,23,24} to compute the tune, a low number of turns (1024 or 2048) can be used. This plot, as proposed in Ref.\textsuperscript{24}, provides the set of resonances that are involved in the nonlinear motion. Moreover, a qualitative measure of the strength of the resonances is given by the width of the depletion area around the resonance lines; indeed, a rigorous relation between these quantities has not yet been worked out.

Resonance network diagram. We propose to visualize the network of resonances in the space (x, y). In this case one has to consider a very dense set (typically $10^5$–$10^6$) of initial conditions, iterated for a low number of turns (1024 or 2048). Then, the frequencies are computed and only the initial conditions whose nonlinear frequencies ($v_x$, $v_y$) satisfy the resonant condition

\[ n_1 v_x + n_2 v_y = l + \epsilon \]  

with $\epsilon \ll 1$ are plotted. The main advantage of this plot is that the channels of resonances and their widths are directly visualized, and one can analyse their relation with the dynamic aperture. Unfortunately, one needs a very dense scan in the plane of initial conditions to obtain significant plots. Moreover, contrary to the footprint, this plot is invariant under the choice of the initial phases only close to the origin. This difficulty can be avoided by plotting the resonance network in the space of the nonlinear invariants, as it has been suggested in Ref.\textsuperscript{25}.

2.3 Analysis of 4D Hénon Map and 4D LHC

We apply the above-defined numerical tools to the map (1) with $\alpha_3 = 1$. We set the linear tunes to the LHC values $\omega_x/2\pi = 0.28$ and $\omega_y/2\pi = 0.31$. In
Figure 1 the long-term diagram is shown. Empty circles are initial conditions that are stable for more than $10^7$ turns. Black circles are initial conditions that are lost between $10^7$ and $10^3$ turns; the size of the circles is roughly proportional to the number of turns where particle loss occurs (see the figure caption). The related footprint is shown in Figure 2, where $10^4$ initial conditions were iterated for 2048 turns: resonances $(1, -1)$ and $(1, -4)$ strongly influence the motion. The graph of the network of resonances with an indication of the resonance number above the channels is given in Figure 3; in this case $3 \cdot 10^5$ initial conditions were iterated for 2048 turns, and only initial conditions whose nonlinear frequencies satisfy the condition (2) with $\epsilon = 10^{-4}$ and $|n_1| + |n_2| \leq 15$ were plotted. The diagram shows that resonances $(1, -1)$ and $(1, -4)$ produce very large resonant channels. The channels that pass through the origin are due to resonances $8\nu_x - 4\nu_y = 1$ and $\nu_x + 12\nu_y = 4$ that are exactly satisfied by the linear frequencies. For this model long-term losses are not very relevant (see Figure 1); moreover, notwithstanding the strong network of resonances shown in Figure 3, there is a full domain of initial conditions that are stable for a very long number of turns. Particles that are locked on islands are stable even when they are rather close to dynamic aperture. Long-term losses occur where the resonant channels meet the dynamic aperture, and on both sides of the channels (where the hyperbolic structures are located) close to the dynamic aperture. However, the most relevant long-term losses occur in a narrow band between resonances $(1, -1)$ and $(1, -4)$ that appears in Figure 3 as a region with scattered dots, i.e., where some initial conditions are locked on resonances and some not. This band seems to be a region of chaoticity where the motion does not take place on 2D tori. All the resonant initial conditions in this region are locked on resonance $(1, -1)$.

In order to find out a toy-model that exhibits a strong long-term loss, we considered the Hénon map [see Equation (1), with $a_3 = 0$] with a linear tune close to resonances $(6, 0)$ and $(0, 5)$: we set $\omega_x / 2\pi = 0.168$ and $\omega_y / 2\pi = 0.201$. The long-term diagram (see Figure 4) shows a rather large region of particle loss between $10^3$ and $10^7$ turns. The tune footprint, shown in Figure 5, is very wide and short, contrary to the previous case. The resonance network diagram (see Figure 6) shows very wide resonances that do not intersect. In fact, due to the peculiar shape of the footprint, almost all the resonant channels are parallel. Notwithstanding the absence of a network of crossing resonances, the long-term phenomena are very relevant; they seem to be due to the large band of stochasticity analogous to the band shown in Figure 3.
FIGURE 1 Long-term diagram for the Hénon map with octupolar term at $\omega_x/2\pi = 0.28$, $\omega_y/2\pi = 0.31$; particles stable for at least $10^7$ turns (empty circles), lost between $10^7$ and $10^5$ turns (large black circles), lost between $10^5$ and $10^3$ turns (medium black circles), lost between $10^4$ and $10^3$ turns (small black circles). Horizontal and vertical Courant-Snyder coordinates are in the coordinate axes.

FIGURE 2 Tune footprint for the Hénon map with octupolar term at $\omega_x/2\pi = 0.28$, $\omega_y/2\pi = 0.31$; initial conditions are iterated for $10^4$ turns; nonlinear frequencies are in the coordinate axes, and resonances up to order 9 are shown.
FIGURE 3  Network of resonances in the Hénon map with octupolar term at $\omega_x/2\pi = 0.28$, $\omega_y/2\pi = 0.31$; only initial conditions whose frequencies lie on a resonance of order smaller than 15 are plotted. Some resonance numbers are indicated above the channels. Horizontal and vertical Courant-Snyder coordinates are in the coordinate axes.

FIGURE 4  Long-term diagram for the Hénon map at $\omega_x/2\pi = 0.168$, $\omega_y/2\pi = 0.201$; particles stable for at least $10^7$ turns (empty circles), lost between $10^7$ and $10^5$ turns (large black circles), lost between $10^5$ and $10^4$ turns (medium black circles), lost between $10^4$ and $10^3$ turns (small black circles). Horizontal and vertical Courant-Snyder coordinates are in the coordinate axes.
FIGURE 5 Tune footprint for the Hénon map at $\omega_x/2\pi = 0.168$, $\omega_y/2\pi = 0.201$; initial conditions are iterated for 1024 turns; nonlinear frequencies are in the coordinate axes, and resonances up to order 9 are shown.

FIGURE 6 Network of resonances in the Hénon map with octupolar term at $\omega_x/2\pi = 0.168$, $\omega_y/2\pi = 0.201$; only initial conditions whose frequencies lie on a resonance of order smaller than 15 are plotted. Some resonances number are indicated above the channels. Horizontal and vertical Courant-Snyder coordinates are in the coordinate axes.
Also in this case the resonant initial conditions in the band are locked on resonance $(1, -1)$, and long-term losses occur either in the band, or at the end of the resonant channels, or on their border when they are close to the dynamic aperture.

Finally, we carry out the same analysis for a 4D model of LHC, version 4.2, with all the nonlinear field errors included as indicated in Ref. 1. The diagrams are shown in Figures 7, 8 and 9; the initial conditions are given in mm at the center of the focusing quadrupole. Long-term losses are rather relevant, and also in this case there is a stochastic band on the dynamic aperture, whose resonant conditions are locked on resonances $(1, -1)$ and $(4, 0)$. In fact, it seems that the phenomenology described for the toy-models is qualitatively very similar to that one of a realistic 4D LHC model. We summarize the results of our analysis as follows.

- The domain of stability is practically full, i.e. it is very rare to find unstable initial conditions that are surrounded by stable ones. Even though very strong resonances are present in phase space (see Figures 3, 6 and 9), the stability domain basically features no holes. The mechanism of diffusion along the network of resonances (etheroclinic intersections, i.e. intersections between different resonances), which is sometimes improperly called Arnold diffusion, seems to be negligible.

- We have observed long-term losses in three distinct cases:
  - At the end of a resonant channel, i.e. where the resonance meets the dynamic aperture.
  - On both sides of a resonant channel (where hyperbolic structures are present), but close to the dynamic aperture.
  - In macroscopic bands of chaoticity, characterized by the presence of a scattered set of initial conditions that are locked on low order resonances. These bands produce the most relevant part of long-term phenomena.

- The resonant channels are in general stable inside the dynamic aperture, even if they are very wide.

3 AUTOMATED EARLY INDICATORS

The Lyapunov exponent$^{2-4}$ and the tune variation$^{5}$ are used as early indicators to predict long-term particle loss with a limited number of turns.
FIGURE 7 Long-term diagram for the 4D LHC model; particles stable for at least $10^5$ turns (empty circles), lost between $10^5$ and $10^4$ turns (large black circles), lost between $10^4$ and $10^3$ turns (medium black circles), lost between $10^3$ and $10^2$ turns (small black circles). Horizontal and vertical physical coordinates are used. They are expressed in mm at $\beta_H = \beta_V = 181 \text{ m}$.

FIGURE 8 Tune footprint for the 4D LHC model; initial conditions are iterated for 1024 turns; nonlinear frequencies are in the coordinate axes, and resonances up to order 9 are shown.
FIGURE 9 Network of resonances for the 4D LHC model: initial conditions whose frequencies lie on a resonance of order smaller than 12 are plotted. Some resonances number are indicated above the channels. Horizontal and vertical physical coordinates are used. They are expressed in mm at $\beta_H = \beta_V = 181 \text{ m}$.

Following Ref. 21, we predict long-term behaviour using short-term tracking data by evaluating the indicators over $N$ turns, and comparing them with their thresholds, that depend as well on $N$: if the indicators are greater than the threshold, then we predict instability.

3.1 Lyapunov Exponent

The maximal Lyapunov exponent is related to the ratio of divergence of two orbits whose initial conditions are close in phase space. The estimate of the maximal Lyapunov exponent after $N$ turns is

$$\lambda(N) = \frac{1}{N} \log \frac{|x^{(N)} - \hat{x}^{(N)}|}{\delta} \quad \delta \ll 1$$

(3)

where $x^{(N)}$ and $\hat{x}^{(N)}$ are the iterates of the initial conditions $x^{(0)}$ and $\hat{x}^{(0)}$ respectively, $\delta = |x^{(0)} - \hat{x}^{(0)}|$, and log is the natural logarithm. If the motion is regular, the distance between the orbits grows linearly with $N$, and $\lambda(N)$ tends to zero. If the motion is chaotic, the distance grows exponentially
with $N$ and the Lyapunov exponent tends to a positive value. Particles with regular orbits are assumed to be stable for ever. Particles with chaotic orbits are assumed to be lost sooner or later.

Let us now compute the Lyapunov exponent for the Hénon map using the initial conditions of Figure 4, and the four time windows given by $N = 64, 256, 1024, 4096$ turns. The results are shown in the histograms of Figure 10. Particles stable for $10^7$ turns (in white) are rather well separated from unstable ones (in black), that are lost before $10^7$ turns. The sharp fall of the rather narrow peak, that contains most of the stable particles, is the natural choice of the threshold $\sigma_\lambda(N)$ for long-term predictions. The four values of the thresholds extracted from the histograms of Figure 10 are very well interpolated by the equation

$$\sigma_\lambda(N) = \frac{1}{N} \log(N A_\lambda)$$

with $A_\lambda = 0.5$, that gives

$$\begin{align*}
\sigma_\lambda(64) &= 0.054 \\
\sigma_\lambda(256) &= 0.0190 \\
\sigma_\lambda(1024) &= 0.0061 \\
\sigma_\lambda(4096) &= 0.00186
\end{align*}$$

These thresholds are indicated in the histograms as dotted lines. This kind of dependence on the number of turns is quite natural, since it corresponds to the rate of decay of the Lyapunov exponent for stable particles. Indeed, one can prove that $A_\lambda$ is related to the maximum detuning of regular particles (see Appendix A).

In Figure 11 we show the same histograms for the case of a 4D model of the LHC. The similarity with the results of Figure 10 is impressive; the thresholds are interpolated by Equation (2), with the same constant $A_\lambda$. The well-defined upper bound that appears as a secondary peak in the distribution of Figure 11 is well understood. This is due to the algorithm used in the simulations for LHC, that does not renormalize the Lyapunov when the distance between the two particles becomes too large. This leads to a systematic underestimation of the Lyapunov exponent of strongly chaotic particles: fortunately, one can still distinguish the secondary peak from the main peak of stable particles. In the case of the Hénon map instead it was possible to compute the Lyapunov exponent by using the renormalization procedure that leads to a correct estimate (see Refs. 4,21 for more details).
FIGURE 10 Distribution of the Lyapunov evaluated at four different number of turns for the Hénon map at $\omega_x/2\pi = 0.168$, $\omega_y/2\pi = 0.201$; particles lost before $10^7$ turns are marked in black, and the dashed lines show the thresholds.

FIGURE 11 Distribution of the Lyapunov evaluated at four different number of turns for the 4D LHC model; particles lost before $10^5$ turns are marked in black, and the dashed lines show the thresholds.
3.2 Tune Variation

Another indicator of long-term stability is based on frequency analysis.\(^5,24\) The motion of a regular initial condition takes place on a 2D torus with fixed frequencies; on the other hand, if the motion is chaotic, the frequencies are not well-defined. Let \(v_x(1:N)\) and \(v_y(1:N)\) be the frequencies computed over the first \(N\) turns in the phase planes \((x, p_x)\) and \((y, p_y)\) respectively. We define the tune difference as

\[
\tau(N) = \sqrt{\frac{1}{2} \sum_{i=x,y} [v_i(1:N/2) - v_i(N/2 + 1 : N)]^2}.
\]

(6)

If the orbit is regular \(\tau(N)\) tends to zero for \(N \to +\infty\); otherwise \(\tau(N)\) is bounded away from zero.

In Figure 12 we plot the distribution of the tune variation evaluated for the Hénon map close to low-order resonances (see Figure 4). One can see that for sufficiently high number of turns (above \(10^3\)) there is a large fraction of the particles whose tune difference is practically zero: these particles are always
stable. Indeed, one can see that there is not a specific feature of the distribution that allows one to define the threshold as in the case of the Lyapunov. For this reason, we have empirically fixed the threshold optimizing it with the results of the long-term simulations ($10^7$ turns) according to

$$\sigma_T(N) = \frac{A_T}{N}. \quad (7)$$

with $A_T = 0.2$. This can be justified by heuristic arguments: in fact, the dependence on the inverse of the number of turns is an upper bound to the precision associated to the tune estimate with $N$ turns for generic signals. In Figure 13 we show the same plots of Figure 11 for the 4D LHC model: the distributions are very similar, and the threshold optimized for the Hénon map turns out to be valid also in this case. In Section 6 we will use the thresholds established here to give quantitative estimates of long-term dynamic aperture.

### 3.3 Failures of the Indicators: Intermittency and Stable Chaos

In our predictions based on the Lyapunov exponent (see Figures 10 and 11) there are two types of failures.

- **Intermittency.** There are initial conditions with low Lyapunov exponent that will be lost: these particles have not yet developed a chaotic behaviour and therefore are erroneously considered as stable. They are shown in the histograms as black parts below the threshold: they are dominant for low number of turns (100–1000 turns). This kind of behaviour is well-known as intermittency: for such initial conditions it is very hard to predict the long-term behaviour with any method.

- **Stable chaos.** There are initial conditions with Lyapunov exponent above the threshold that are stable for a very high number of turns. These particles are visible in the histogram as white parts of the distribution above the threshold: they are dominant for high number of turns (more than 1000). In any case, our simulations have been carried out only for $10^7$ turns (Hénon map) or $10^5$ turns (LHC): it is not clear whether all the chaotic particles will be lost for a sufficiently high number of turns. A partial answer to this question is given in the next section. On the other hand, when one considers a high but finite number of turns, there are several indications$^{19,20}$ that the Lyapunov underestimates the dynamic aperture.
When the method of the tune variation is used, another difficulty is met. In fact, even though a large fraction of the stable initial conditions has a tune variation that collapses to zero very rapidly for low number of turns, there are some initial conditions (see Figures 12 and 13) whose tune difference is converging to zero more slowly, even if they are stable. There is a strong numerical evidence that this may happen when the particle is locked on a resonance: in this case the rate of convergence of the tune variation is very slow, and the particle is erroneously considered as unstable. This can happen even for initial conditions rather close to the origin: indeed, we have seen that islands can be very wide also at low amplitudes (see the resonance networks in Figures 3, 6 and 9). On the other hand, for these initial conditions the Lyapunov exponent stays always well below the threshold, and the particle is assumed as stable. In the next section we will propose a definition of dynamic aperture for the tune variation method that helps to avoid this type of error.
4 DYNAMIC APERTURE DEFINITION

In a previous paper\textsuperscript{16} we have proposed a definition of the dynamic aperture, providing an estimate of the error associated to the step of the initial conditions in phase space and a criterion for the optimization of the steps. The dynamic aperture is defined as the average over the whole phase space of the minimum amplitude where particle loss occurs. The average over two variables can be automatically taken into account using the dynamics, and therefore one must make a scan over two variables only. By defining a polar grid in the space \((x, y)\), one has

\[ x = r \cos \alpha \quad y = r \sin \alpha \]

with \(r > 0\) and \(\alpha \in [0, \pi/2]\). One performs a scan over \(\alpha\), and for each \(\alpha\) an initial condition is started along \(r\) (see also Figures 1, 4 and 7). Let \(r(\alpha)\) be the last stable initial condition along \(\alpha\) before the first loss at a turn number lower than \(N\) occurs, and let \([\bar{r}(\alpha)]^2\) be the sum of the squares of the nonlinear invariants of the orbit with initial condition \((r \cos \alpha, 0, r \sin \alpha, 0)\). Then, the dynamic aperture is defined as

\[ D(N) = \left( \int_0^{\pi/2} [\bar{r}(\alpha)]^4 \sin(2\alpha) d\alpha \right)^{1/4}. \]  

In this paper we used an approximate definition where the nonlinear invariant \(\bar{r}\) has been replaced with the linear one \(r\). One can show that the factor \(\sin(2\alpha)\) is due to the integration over the phase space variables (see Ref.\textsuperscript{16} for more details). The same kind of definition is given for the estimate through the Lyapunov exponent: in this case \(r(\alpha)\) stands for the amplitude of the particle immediately before the first particle along \(\alpha\) whose Lyapunov is greater than the threshold.

We have already pointed out in the previous section, that the method based on tune variation can lead to underestimation of the dynamic aperture due to phase locking on a resonance. For this reason we propose to define \(r(\alpha)\) as the total number of particles along \(\alpha\) whose tune variation is lower than the threshold, times the step along \(r\). In this way one can avoid severe underestimates due to resonances; unfortunately, overestimates can be produced by stable islands that lie outside the dynamic aperture. Even though this dynamic aperture definition is not consistent with the definitions
given for tracking and Lyapunov exponent, we will show in the next sections that it produces rather good results.

We would like to stress the importance, in the definition of dynamic aperture (9), of the average along the angle \( \alpha \). In fact, we have seen that both the Lyapunov and the tune variation can fail the long-term prediction for some values of \( \alpha \). Therefore, an average over \( \alpha \) greatly reduces the effect of these errors and moreover, it is crucial for obtaining a significant dynamic aperture for models that feature a wide phase space deformation.

5 INVERSE LOGARITHMIC EXTRAPOLATION OF SURVIVAL PLOTS

5.1 Interpolation of the Dynamic Aperture vs. Number of Turns

Survival plots\(^{13-15}\) have been used to predict long-term behaviour since many years. A typical survival plot is obtained by tracking initial conditions along the line \( A = x = y \), fixing the momenta to zero, and plotting the number of turns where the particle is lost versus the amplitude \( A \). In Figure 14 we show the survival plot for the Hénon map at \( \omega_x / 2\pi = 0.168, \omega_y / 2\pi = 0.201 \). The pattern is rather irregular, and an interpolation of these data seems hard to be worked out. Indeed, if we compute the dynamic aperture through tracking using the averaging procedure given in (9), we obtain a curve whose regular behaviour is rather striking (see Figure 15, large dots). What is even more striking is that this behaviour is very well interpolated (see Figure 15, solid line) by the simple law

\[
D(N) = D_\infty \left( 1 + \frac{b}{\log_{10} N} \right). \tag{10}
\]

There are only two constants to fit: the estimate of the dynamic aperture for infinite times \( D_\infty \) and the constant \( b \) that measures the relevance of the long-term losses. We used decimal logarithm to give a straightforward meaning to \( b \): for instance, for the analysed model one has \( b = 1.5 \), that means a 50% overestimate if 1000 turns tracking is used. Equivalently, with \( b = 1.5 \) one has that the tracking results with \( 10^7 \) turns are still 20% above \( D_\infty \). We have considered other toy-models based on the Hénon map, finding out a wide range of constants \( b \) (from 1.5 to 0.3). It must be pointed out that the interpolation (10) is safer and more reliable when the dynamic aperture is evaluated by the average over several different ratios \( x/y \).
FIGURE 14 Survival plot for the Hénon map at $\frac{\omega_x}{2\pi} = 0.168$, $\frac{\omega_y}{2\pi} = 0.201$.

FIGURE 15 Dynamic aperture as a function of the number of turns for the Hénon map at $\frac{\omega_x}{2\pi} = 0.168$, $\frac{\omega_y}{2\pi} = 0.201$: data extracted from tracking (dots) and interpolation through the law (11) (solid line). The dynamic aperture is evaluated as an average over 20 angles.
In Figure 16 we show the same plot of Figure 15 for the 4D LHC model. The fit of the tracking data is even better. Here the long-term simulation is carried out only up to $10^5$ turns. The constant $b$ is slightly smaller than in the previous case, but the long-term phenomena are still relevant. The estimate at $10^5$ turns is 20% above the dynamic aperture at infinite times, and 5% above the dynamic aperture at $10^7$ turns, that is the number of revolutions of the beam in LHC before the energy ramping.

5.2 Analogies with the Nekhoroshev Theorem

It has been pointed out$^{26}$ that the law of the inverse logarithm (10) has the same form of the Nekhoroshev estimate generalized to symplectic mappings of arbitrary dimension (see Refs. $^8,^10$). This could provide a link between numerical simulations of realistic models and the rather abstract theory of quasi-integrable symplectic mappings. According to the theorem, a particle of amplitude $r$ will stay inside a ball of radius $2r$ for a number of turns given by

$$N = N_0 \exp \left( \frac{R}{r} \right)^\eta$$

(11)

where $N_0$, $\eta$ and $R$ are suitable positive constants.
The Nekhoroshev theorem gives a very pessimistic upper bound to the diffusion in phase space. In fact, this bound does not ensure stability for infinite times, whilst, when \( r \to 0 \) almost all the phase space is foliated into invariant tori, where no diffusion is possible (KAM theorem);\(^{27}\) the complement of the invariant tori is a set whose dimension is exponentially small with the amplitude \( r \) and therefore it can be neglected for all practical purposes. The inversion of equation (11) provides

\[
    r(N) = \frac{R}{\log^{1/\eta}(N/N_0)}.
\]

If one adds the information provided by KAM theorem (i.e., the existence of a full domain of initial conditions that are stable for infinite times except a set of negligible measure), and assumes that in the region where there are no KAM tori the particles are emptying the phase space according to the Nekhoroshev law (12), one obtains

\[
    r(N) = r_\infty + \frac{R}{\log^{1/\eta}(N/N_0)}.
\]

The interpolation law can be rewritten as the above expression with \( \eta = 1 \) and \( N_0 = 1 \). Even though the interpolation with \( \eta = 1 \) and \( N_0 = 1 \) provides good results, it is interesting to optimize the constants \( \eta \) and \( N_0 \) and to check their agreement with the Nekhoroshev result: this work goes beyond the aims of the present paper, and it is still in progress.\(^{28}\)

### 5.3 Long-term Prediction

The law (10) provides a very powerful indicator since it can be extrapolated to predict not only the particle loss for infinite times, but also for a fixed number of turns. With this method, for instance, it is easy to evaluate the DA of the LHC for \( 10^7 \) turns, i.e. the duration of the injection plateau. However there are still many initial conditions that can be lost for infinite times. In fact the law of the inverse logarithm implies that the limiting value of the dynamic aperture \( D_\infty \) can be reached very slowly. A crucial issue is whether these particles will be lost due to some effect neglected in the accelerator model. If this is true the DA will practically coincide with \( D_\infty \). More investigations are necessary to clarify this point.
6 DYNAMIC APERTURE PREDICTION

6.1 Hénon Map

We consider the 4D Hénon with linear frequencies $\omega_x/2\pi = 0.168$ and $\omega_y/2\pi = 0.201$ that has been analyzed in Section 2. In Table I we give the relative errors of the dynamic aperture estimates with respect to tracking at $10^7$ turns. The first column indicates the number of turns that are used to carry out tracking or computing early indicators. In the second column the plain tracking results are given; in the third and in the fourth columns we give the estimates based on the Lyapunov exponents and on the tune variation respectively. In the last two columns the results of the extrapolation obtained using Equation (10) are given: in the fifth column extrapolation is carried out at $10^7$ turns; in the sixth column the extrapolation for infinite times is given, in order to check the relevance of the particle loss between $10^7$ and $\infty$.

The main result is that extrapolation, Lyapunov and tune variation give a good guess of the dynamic aperture (only a few percent of error) already at 2048 turns. The Lyapunov at high number of turns becomes pessimistic with respect to the direct tracking at $10^7$ turns, and it approaches the value of the dynamic aperture at infinite number of turns. For instance, the Lyapunov threshold at $10^5$ turns is 12% below the dynamic aperture at $10^7$ turns. The tune variation is not predictive in this case, since the threshold has been optimized using long-term tracking at $10^7$ turns. Extrapolation results are rather stable from 2048 turns onward: $D_\infty$ is about 20% below D.A. at $10^7$ turns. Simulations carried out over 128 or 512 turns are not sufficiently precise for all the methods: this seems to be a lower limit to predictivity. Nevertheless, both the Lyapunov and the tune variation at 128 turns already recover more than two third of the dynamic aperture overestimate given by plain tracking.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Particle loss</th>
<th>Lyapunov</th>
<th>Tune variation</th>
<th>Extrapolation at $10^7$</th>
<th>Extrapolation at $\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>40%</td>
<td>13%</td>
<td>12%</td>
<td>13%</td>
<td>-1%</td>
</tr>
<tr>
<td>512</td>
<td>31%</td>
<td>8%</td>
<td>4%</td>
<td>1%</td>
<td>-16%</td>
</tr>
<tr>
<td>2048</td>
<td>18%</td>
<td>1%</td>
<td>0%</td>
<td>1%</td>
<td>-23%</td>
</tr>
<tr>
<td>8192</td>
<td>10%</td>
<td>-4%</td>
<td>-2%</td>
<td>-3%</td>
<td>-23%</td>
</tr>
<tr>
<td>$10^5$</td>
<td>5%</td>
<td>-12%</td>
<td>-4%</td>
<td>-4%</td>
<td>-24%</td>
</tr>
<tr>
<td>$10^7$</td>
<td>0%</td>
<td></td>
<td></td>
<td></td>
<td>-21%</td>
</tr>
</tbody>
</table>

TABLE I Relative errors of the dynamic aperture estimates with respect to tracking at $10^7$ turns for the Hénon map at $\omega_x/2\pi = 0.168$ and $\omega_y/2\pi = 0.201$. 
The results relative to the realistic 4D model of the LHC are given in Table II. All the dynamic aperture estimates are given as relative errors with respect to plain tracking at $10^5$ turns. Also in this case the extrapolation is reliable from 2048 turns onward. From $10^5$ to $10^7$ turns the estimated D.A. decreases by 5%, and another 15% is lost from $10^7$ to $\infty$. The Lyapunov is very pessimistic with respect to plain tracking at $10^5$ turns, but its estimate is consistent with extrapolation of $D_\infty$. The predictions with the tune variation are rather pessimistic with respect to tracking at $10^5$ turns. This is due to the fact that the threshold is optimized with Hénon map simulations for $10^7$ turns. Indeed, the prediction with the tune variation agrees very well with the inverse logarithm extrapolation for $10^7$ turns. Also in this case predictions with only 128 turns already give a clear indication about the relevance of long-term phenomena.

6.3 6D LHC

We have also considered a 6D model of the LHC; the momentum deviation is chosen to bring the particle close to the bucket separatrix. In Figure 16 we plot the long-term diagram equivalent to Figure 7: one can see that there is a reduction in the stability domain, and that the deformations in phase space are more relevant. The Lyapunov exponent was computed using the definition (3) generalized to the 6D phase space. The variation of the tune was computed only in the transversal plane, since the contribution due to the variation of synchrotron tune is negligible. In Figures 17 and 18 the distributions of the
LONG-TERM STABILITY IN LHCs

FIGURE 17 Long-term diagram for the 6D LHC model; particles stable for at least $10^5$ turns (empty circles), lost between $10^5$ and $10^4$ turns (large black circles), lost between $10^4$ and $10^3$ turns (medium black circles), lost between $10^3$ and $10^2$ turns (small black circles). Horizontal and vertical physical coordinates are used. They are expressed in mm at $\beta_H = \beta_V = 181 \text{ m}$.

FIGURE 18 Distribution of the Lyapunov exponent evaluated at four different number of turns for the 6D LHC model; particles lost before $10^5$ turns are marked in black, and the dashed lines show the thresholds.
FIGURE 19 Distribution of the tune variation evaluated at four different number of turns for the 6D LHC model; particles lost before $10^5$ turns are marked in black, and the dashed lines show the thresholds.

FIGURE 20 Dynamic aperture as a function of the number of turns for the 6D LHC model: data extracted from tracking (dots) and interpolation through the law (11) (solid line). The dynamic aperture is evaluated as an average over 9 angles.
early indicators are plotted. The Lyapunov exponent features a very similar pattern to Figures 10 and 11; the threshold is the same. On the other hand, the tune variation does not work as early indicator for low number of turns; in fact the coupling with synchrotron motion can be seen as a slow modulation of the frequencies, that cannot be detected before a complete synchrotron period. For the LHC case the synchrotron tune is $Q_s = 0.003$, and therefore only the tune variations measured over 1024+1024 and 4096+4096 turns are significant. In this case, the threshold seems to be the same as in the 4D case and the long-term estimates are consistent with extrapolation. The behaviour of the dynamic aperture as a function of the number of turns is shown in Figure 19: the interpolation is slightly worse, but still holds. One can see that there is a 30% reduction in the asymptotic value with respect to the 4D case. Fortunately, this reduction is only 15% at $10^7$ or $10^5$ turns.

The dynamic aperture estimates based on the early indicators and extrapolation are given in Table III. The Lyapunov still features a monotonic dependence of its estimates on the number of turns, and provides rather precise estimates for 2048 and 8192 turns. The only estimate based on tune variation is rather pessimistic. The extrapolation is not very stable at high number of turns, but is consistent with the indications of the Lyapunov. As in the 4D case, the loss from $10^5$ to $10^7$ turns is about 5%. A summary of the absolute values of the dynamic aperture for the 4D and 6D case is given in Table IV.

Successive independent checks of the law of the inverse logarithm have been carried out at CERN for the 6D LHC model; these studies confirm the validity of the law and the reliability of the extrapolation of the dynamic aperture to predict long-term particle loss.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Particle loss</th>
<th>Lyapunov</th>
<th>Tune variation</th>
<th>Extrapolation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>at $10^7$</td>
<td>at $\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>39%</td>
<td>12%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>512</td>
<td>29%</td>
<td>9%</td>
<td>−1%</td>
<td>−20%</td>
</tr>
<tr>
<td>2048</td>
<td>21%</td>
<td>0%</td>
<td>−4%</td>
<td>−2%</td>
</tr>
<tr>
<td>8192</td>
<td>13%</td>
<td>−6%</td>
<td>−7%</td>
<td>−2%</td>
</tr>
<tr>
<td>$10^5$</td>
<td>0%</td>
<td></td>
<td>−7%</td>
<td>−31%</td>
</tr>
</tbody>
</table>
TABLE IV Dynamic aperture (in mm) for the LHC models (4D and 6D) at different number of turns through plain tracking and extrapolation.

<table>
<thead>
<tr>
<th>n</th>
<th>Method</th>
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<th>6D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>Tracking</td>
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<td>11.3</td>
</tr>
<tr>
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<td>Tracking</td>
<td>10.8</td>
<td>9.1</td>
</tr>
<tr>
<td>$10^7$</td>
<td>Extrapolation</td>
<td>10.3</td>
<td>8.4</td>
</tr>
<tr>
<td>$\infty$</td>
<td>Extrapolation</td>
<td>8.8</td>
<td>6.3</td>
</tr>
</tbody>
</table>

7 CONCLUDING REMARKS

In this paper we have given a phenomenological analysis of the mechanisms of long-term particle loss using tracking and frequency analysis, and proposing a numerical method to evaluate the resonance widths. We show that relevant long-term losses are mainly due to the presence of stochastic bands where there are isolated particles whose nonlinear frequencies lie on a low-order resonance. The existence of these bands can be detected by short-term tracking data over a very dense set of initial conditions. Moreover, we have found very wide resonant channels where particles are stable for at least $10^7$ turns.

We have proposed automated methods based on thresholds to give long-term estimates using two early indicators: the Lyapunov exponent and the variation of the tune. The thresholds hold both for Hénon-like models and for the LHC. Both indicators are very predictive, allowing to give rather precise estimates with a low number of turns ($10^3-10^4$). We have also shown that the dynamic aperture as a function of the number of turns is very well interpolated by the law of the inverse logarithm. The interpolation is fully satisfactory for the analysed models, and the extrapolation of this law provides another very powerful early indicator. Comparison with long-term tracking shows that the Lyapunov exponent evaluated at large number of turns ($10^4 - 10^5$) always underestimates the dynamic aperture at $10^7$ turns, and that it seems to tend towards the extrapolation of the dynamic aperture at infinite number of turns. On the other hand the tune variation, whose threshold has been optimized through tracking at $10^7$ turns for the Hénon map, is in good agreement with tracking extrapolated at $10^7$ turns. One can conclude that
the three indicators (Lyapunov exponent, tune variation and extrapolation of tracking data) provide complementary information and allow one to carry out significant cross-checks.

**Acknowledgements**

We wish to acknowledge Prof. Turchetti for pointing out the relation between the empirical law of the decaying of the dynamic aperture and the Nekhoroshev and KAM theorem. We also want to thank R. Bartolini and A. Faus-Golfe for helping us to carry out the simulations for the LHC lattice, and M. Böge, A. Morbidelli and M. Vergassola for useful and stimulating discussions.

**References**


A. RELATIONS BETWEEN LYAPUNOV AND TUNESHIFT

In this appendix we derive the relation between the constant $A_\lambda$ of the decay of the Lyapunov exponent for regular particles and the tuneshift. We first consider a 2D map with phase space $(x, p_x)$. If the motion is regular, we approximate it with an amplitude-dependent rotation. Using the complex coordinates $z = x - ip_x$, one has the following expressions for the iterates of the initial condition $z_0$ and for its companion $\hat{z}_0 = z_0(1 + \delta)$:

$$
\begin{align*}
    z^{(n)} &= z_0 e^{i vn} \\
    \hat{z}^{(n)} &= \hat{z}_0 e^{i \nu n}.
\end{align*}
$$

(14)

It is not restrictive to consider $z_0$ and $\hat{z}_0$ real (i.e. the initial momenta are zero). Then the distance of the neighbour particles is

$$
|z_0^{(n)} - \hat{z}_0^{(n)}| = z_0|1 - (1 + \delta) e^{in\Delta \nu}|.
$$

(15)
where $\Delta v = \nu - \hat{\nu}$ is the tune difference between the two orbits. Assuming that $n \Delta v \ll 1$ for fixed $n$, we can develop the exponent and neglect the higher orders:

$$|z_0^{(n)} - \hat{z}_0^{(n)}| = z_0|1 - (1 + \delta)(1 + in\Delta v)| = z_0|\delta + in\Delta v|.$$  \hspace{1cm} (16)

It is reasonable to assume that the separation in phase space it is mainly due to the difference in the tunes and not to the initial distance $\delta$: therefore $\delta \ll n\Delta v$ and one has

$$|z_0^{(n)} - \hat{z}_0^{(n)}| = nz_0|\Delta v|.$$  \hspace{1cm} (17)

Using the Lyapunov definition one obtains

$$\lambda(n) = \frac{1}{n} \log \frac{|z_0^{(n)} - \hat{z}_0^{(n)}|}{|z_0 - \hat{z}_0|} = \frac{1}{n} \log n \frac{\Delta v}{\delta} \rightarrow \frac{1}{n} \log nz_0 \frac{\partial v}{\partial z}. \hspace{1cm} (18)$$

Therefore $A_\lambda$ is given by

$$A_\lambda = \max z_0 \frac{\partial v}{\partial z}. \hspace{1cm} (19)$$

where the max is taken over the set of regular initial conditions. The generalization to the 4D case is straightforward.