Restrictive condition of the Chew and Low extrapolation procedure

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ABSTRACT

From the analysis of a weakly connected graph, it is shown that the extrapolation procedure proposed by Chew and Low for production processes is possible only if suitable configurations are chosen.
I. Introduction

The "polology" applied to the scattering experiments has provided a consistency test of the theory by showing, for example, that the pion-nucleon renormalized coupling constant calculated with the meson-nucleon dispersion relations in the energy, is the same as the one deduced from an extrapolation in the momentum transfer both of the nucleon-nucleon scattering cross-section and of the photoproduction cross-section \(^1\).

Although these results have been shown to hold in the domain of the perturbation theory, it is expected that they are more generally true.

Some authors \(^2\), by a generalization of these methods of extrapolation, have given the recipe which allows to get the cross-section involving \(n\) particles from suitable experimental data involving \(n+1\) particles. These recipes rest on the hypothesis that the

\[
\frac{\mathcal{C}^2 \mathcal{G}(W^2, \omega^2, \Delta^2)}{\mathcal{C}W^2 \mathcal{C} \omega^2}
\]

of the process:

\[p_1 + p_2 \rightarrow p_1' + p_2' + p_3' + \cdots \cdots,\]

where

\[W^2 = (p_1 + p_2)^2, \quad \Delta^2 = (p_1 - p_1')^2 \quad (*),\]

and

\[\omega^2 = (p_2' + p_3' + \cdots \cdots)^2\]

\(*)\) We choose the metric \((1, -1, -1, -1)\)
is a function of $\Delta^2$ which, for fixed values of $W^2$ and $\omega^2$, has the following properties:

a) it possesses a pole at $\Delta^2 = m^2$, $m$ being a suitable minimum mass;

b) $(\Delta^2 - m^2)^2 \cdot \frac{\partial^2 G}{\partial W^2 \partial \omega^2}$ is a regular function of $\Delta^2$ in a neighborhood of the real $\Delta^2$-axis

$$\Delta_{\text{max}}^2 < \Re \Delta^2 < \varepsilon m^2 \quad \text{with } \varepsilon > 1$$

where $\Delta_{\text{max}}^2$ is the maximum value of $\Delta^2$ allowed by the fixed physical value of $W^2$ and $\omega^2$.

a) and b) allow to calculate the residuum of the pole of the second order of $\frac{\partial^2 G}{\partial W^2 \partial \omega^2}$ and, from this, the cross-section involving the $n-1$ particles which is proportional to this residuum.

It has been shown with examples taken from the perturbation theory that $\frac{\partial^2 G}{\partial W^2 \partial \omega^2}$ possesses complex singularities and that singularities can occur on the real $\Delta^2$-axis with $\varepsilon$ very near to the unity.

We went to show in this paper that there are also configurations in which b) condition is violated, such that real singularities of $\Delta^2$ occur in between $\Delta_{\text{max}}^2$ and $m^2$. In the particular case we have investigated (scalar particles with equal masses) the b) condition is violated for a very large set of values of $W$ and $\omega$. 

II. Localization of the singularities

Let us consider the two graphs of figs. 1) and 2). The second differing from the first only due to the line $p_2'$.

![Graph 1]

![Graph 2]

All the particles are scalar and with equal masses. We will investigate the analytical properties in the variable $\Delta^2 = (p_1 - p_1')^2$ of the amplitude and cross-section of the graph in figs. 1) and 2) at fixed $W^2 = (p_1 + p_2)^2$ and $\omega^2 = (p_1' + p_2')^2$. The graph 1) has a pole at $\Delta^2 = \omega^2$ both in the amplitude and in the $\frac{\Delta^2 \omega^2}{3W^2 \omega^1}$. The conditions a) and b) hold not only for the graph 1) but for more general graphs as discussed in 4).

The singularities of the amplitude of the graph in fig. 2 are the zeros of

$$D = \Delta^2 + 2p_2' (p_1' - p_1') \quad (1)$$
First of all we note from (1) that the special configuration \( p_1' = p_2' \) gives

\[
D = 2 \cdot \Delta = 2 \langle p_1 - p_1' \rangle^2 \quad (*)
\]

(1')

If \( p_1'^2 \neq p_2'^2 \) it is sufficient to take

\[
P_1' = \sqrt{p_1'^2 - p_2'^2}
\]

in order to get a configuration such that the corresponding amplitude has at pole \( \Delta^2 = 0 \).

Let us investigate in detail how \( \Delta^2 \)-singularities move as a function of the invariants. Let us introduce a polar system of coordinates such that the polar axis is defined by \( \frac{\bar{q}}{|\bar{q}|} \) with \( \bar{q} = (p_1' - p_2') \) and the polar plane by the directions \( \bar{q} \) and \( \bar{p}_1 \). In this system let us call \( \chi' \) and \( \varphi \) the polar coordinates of the direction \( \frac{\bar{q}}{|\bar{q}|} = \frac{p_1' - p_2'}{|p_1' - p_2'|} \). The expression (1) as a function of \( W^2, \Delta^2, \omega^2, \chi' \) and \( \varphi \) is then

\[
D = \frac{1}{2} \left[ 3m^2 - W^2 - \omega^2 + \sqrt{W^2 - 4m^2} \cdot Q \cos \vartheta + W \left[ \frac{Q^2 - \chi^2 + 2Q \chi \cos \chi' + 4m^2}{4m^2} \right]^{1/2} \right. + \\
\left. \sqrt{W^2 - 4m^2} \chi' \left[ \sin \vartheta \sin \chi' \cos \varphi - \cos \vartheta \cos \chi' \right] \right]
\]

(2)

where

\[
Q^2 = \frac{(W^2 + m^2 - \omega^2)^2}{4W^2} - m^2
\]

\[
\chi^2 = \frac{(W - P)^2 (W^2 - 4m^2)}{(W - P)^2 - \frac{4m^2}{Q^2 - \chi^2}} \quad P = \sqrt{Q^2 + m^2}
\]

and \( \chi' \) is the angle between \( \bar{p}_1 \) and \( \bar{p}_1' \).

*) This result depends on a specific choice of the invariants which define the amplitude and it leaves out the variable \( \left( p_2 - p_1' \right)^2 \). We will see later that (1') is also a singularity of \( \frac{\partial^2}{\partial W \partial \omega} \).
Let us note that the choice of these variables is useful for studying the analytical properties of \( \frac{\partial^3 G}{\partial W^2 \partial \omega^2} \). In fact, with this choice the integration over the phase space of the particles \( p_2 \) and \( p_4 \) reduces to an integration over the full solid angle of the \( \Delta^2 \), \( W \), and \( \omega \). These variables, however, destroy the analyticity of the amplitude in the small Lehmann ellipse due to the appearance in (2) of a term in \( \sin \theta \). These extra singularities can be tackled accordingly to the procedure of ref. 5) and they are not present in \( \frac{\partial^3 G}{\partial W^2 \partial \omega^2} \).

From (2) the cross-section \( \frac{\partial^2 G}{\partial W^2 \partial \omega^2} \) corresponding to the production process of graph in fig. 2) is

\[
\frac{\partial^2 G}{\partial W^2 \partial \omega^2} = k \int_{-1}^{+1} dz f(W^2, \omega^2, z) \int_0^2 d\varphi \frac{d\varphi}{D^2}
\]

\( (3) \)

where \( f(W^2, \omega^2, z) \) do not depend either on \( \varphi \) or on \( \Delta^2 \) variables.

After the \( \varphi \)-integration one gets:

\[
\frac{\partial^2 G}{\partial W^2 \partial \omega^2} = k \int_{-1}^{+1} f(W^2, \omega^2, z) \frac{S}{\sqrt{S^2 - R^2}^{3/2}} dz
\]

\( (4) \)

where

\[
S^2 = \frac{1}{2} \left( \frac{3m^2 - W^2 - \omega^2}{\sqrt{W^2 - 4m^2}} \cos \theta + \right.
\]

\[
\left. + W \left[ \frac{q^2 + \gamma^2 + q \gamma (z + 4m^2)}{2} - \sqrt{W^2 - 4m^2} \gamma \cos \theta \cdot z \right]^2 \right)
\]

\( (5) \)

\[
R^2 = \frac{1}{4}(W^2 - 4m^2) \gamma^2 (1 - \cos^2 \theta - z^2 + \cos^2 \theta \cdot z^2)
\]

\( (6) \)
We look now at the end-point singularity of (4) occurring at \( z = 1 \).

These singularities are defined by \( S = 0 \), i.e.

\[
\Delta^2 = \frac{1}{2} \left[ 3m^2 - W^2 + \omega^2 + 2Q \frac{W^2 + \omega^2 - 3m^2 - W [(Q - \chi)^2 + 4m^2]}{Q + \chi} \right]^\frac{1}{2}
\]

(7)

One can check that the singularities (7) are always outside the small Lehmann ellipse \(^5\). It can be shown that the right hand side of (7) is an increasing function of \( W \) whatever is \( \omega \). Furthermore

\[
\lim_{W \to \omega + m, Q \to 0} \Delta^2 = \lim_{W \to \omega - m} \Delta^2 = m
\]

\[
\Delta^2 = \left( \frac{\omega^2 - m^2}{m} \right)
\]

(8)

\[
\Delta^2 \left( W^2 = 2m^2 + \omega^2 \right) = 0 \quad \lim_{W \to \infty (\omega = \text{const.})} \Delta^2 = \frac{1}{4m^2} \left[ (\omega^2 - 2m)^2 - 4m^2 \right]^{\frac{1}{2}}
\]

Therefore, while for \( W < \frac{\omega^2 - m^2}{m} \) the singularity (7) is always lower than \( m^2 \), for \( W > \frac{\omega^2 - m^2}{m} \) is larger than \( m^2 \), staying very close to this value except for a small region of values of \( \omega \) in the neighbourhood of the threshold. For example, for \( \omega > 3m \) and \( W > 8m^2 \) (7) is varying between \( m^2 \) and 1.168 \( m^2 \).

III. Conclusion

The case that we have investigated shows that the method of extrapolation suggested by Chew and Low does not hold in general.
The problem now arises to investigate what is the domain of $W$ and $\omega$-variables for which it holds. This region will depend obviously on the parity and on the masses of the particles involved.

In this connection, we hope to investigate the problem of the pion-pion interaction. Because of the odd parity of the $\pi$-meson it is likely that the domain of $W$ and $\omega$-values in which the extrapolation procedure of Chew and Low can be applied is larger than that found in the case treated in this paper.

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