BEAM ENVELOPE OSCILLATION WITH SPACE-CHARGE FORCES

A NUMERICAL STUDY FOR CIRCULAR MACHINES

by

P.M. Lapostolle
L. Thorndahl

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Summary

Space-charge induced non-linear envelope resonances are discussed, assuming a uniform density distribution.

In a first part properties of matched solutions are treated as well as mismatches and their stability. Above certain intensities more than one matched solution is found. In the second part, envelope resonances are crossed as the beam is accelerated. The relevance of a former proposal, see Ref. 3), to inject beams above space-charge induced resonances, is confirmed.
Introduction

With the intensity increases which have been achieved on present AG proton synchrotrons, intensity levels are certainly reached where limitations should occur.

In trying to obtain any further increase a better understanding of such phenomena is then extremely important; it is also essential for the design of new machines.

With the present terminology listing such intensity limitations, we may distinguish between instabilities and resonances.

In instabilities energy is exchanged between the oscillating beam and its surrounding medium. RF fields are created in this medium which can increase the beam oscillations either transversely or longitudinally.

Similar instabilities may occur also by excitation of unwanted modes in resonant cavities, in a way similar to beam breakup in linacs. In contradiction to the classical resistive wall instabilities, this type would take place at the resonant frequency of the cavities or close to it.

Nevertheless, this is not what is called resonance limitations where at certain intensities, due to space charge forces, the number of betatron oscillations per turn inside the machine is assumed to become integral or half-integral.

A distinction is usually made between betatron oscillations of the beam as a whole, called coherent betatron oscillations, and betatron oscillations of single particles around the centre of the beam, called incoherent betatron oscillations.

The purpose of the present report is to present some results of a numerical study on incoherent betatron oscillations and their space-charge induced resonances:

Previous studies on the behaviour of beams oscillating incoherently under space charge conditions, using a physical model, have been done by Kapchinskij and Vladimirskij\(^1\), Lloyd Smith\(^2\), Lapostolle\(^3\), and Hardt\(^4\).

This model assumes, in order to have linear equations for the single-particle movement, uniform space-charge density in any two-dimensional projection of the four-dimensional transverse density distribution. This
way, the effect of space-charge forces at a given moment and a given place in the machine can be simulated by a constant-gradient magnet where the gradients are defocusing in both transverse directions and a function of the two semi-axes of the elliptical beam cross-section. In four-dimensional phase space this corresponds to the particles being distributed homogeneously on the surface of a four-dimensional hyper-ellipsoid (for transverse motion only, supposed independent from the longitudinal or azimuthal movement). As with this density distribution, space-charge forces are linear in the displacement, only integral and half integral Q-resonances can be expected. The model does not exhibit any resonances due to non-linear fields.

In Ref. 1), the differential equations for the two principal semi-axes of the beam envelope have been derived and applied to a linac with an alternating gradient focusing system.

Ref. 2) treats the case of a constant-gradient machine with a small perturbation of this gradient. The Q-values, the emittances and the perturbations are equal in the horizontal and the vertical planes.

In Ref. 3), the influence of the amplitude of envelope oscillations on the oscillation wavelength has been studied, disregarding gradient perturbations. It is shown that for increasing envelope oscillations this wavelength tends towards its value without space charge. It is mentioned that for a given intensity one may have smaller maximum beam dimensions if injecting a rather small beam above space-charge induced non-linear resonances and crossing those resonances during the acceleration, than if injecting a large beam below the resonances.

Ref. 4) considers non-perturbed machines with different Q-values, different emittances and constant-gradient focusing. It is shown for which intensities resonances of the beam envelope are to be expected.

The investigations described in the present report are divided into two parts. A first part assumes a beam of constant energy and discusses mainly properties of matched beams. In the second part the energy of the beam increases with time, so that space-charge induced resonances are crossed.
PART I

Envelope Oscillations at Constant Energy and Q-Values

In this first part we shall start with the study of a so-called matched beam with the most trivial parameters: equal Q-values, equal emittances in the two transverse oscillation planes and no perturbations. After some understanding of this simple case, complicating features, such as different Q-values, different emittances in the two planes and gradient perturbations, are added one by one, bringing us nearer an actual case.

Our treatment is purely numerical and is based on a set of equations which already assume several approximations, as mentioned above. These approximations will not be discussed here. They may constitute the basis for further work.

Using the previously mentioned homogeneous space-charge density and a constant gradient, and also the previous studies, the following equations may be obtained for the single-particle motion:

\[
\begin{align*}
(1) & \quad \frac{d^2 y}{dt^2} + \sqrt{q_y^2} + \sum_{M=1}^{\infty} A_{yM} \cos(M\theta - \varphi_{yM}) - \frac{e^2 N e}{2\pi v^2 e_0 m y^2 B} \left( \frac{1}{y} \right) y = 0 \\
(2) & \quad \frac{d^2 z}{dt^2} + \sqrt{q_z^2} + \sum_{M=1}^{\infty} A_{zM} \cos(M\theta - \varphi_{zM}) - \frac{e^2 N e}{2\pi v^2 e_0 m y^2 B} \left( \frac{1}{z} \right) z = 0.
\end{align*}
\]

These equations take into account both the Coulomb forces and the forces due to the magnetic field created by the beam itself, which is assumed to be in free space.

- \( Q_y, Q_z \): zero-intensity Q-values
- \( N \): number of particles in the machine
- \( v \): particle velocity
- \( A_{yM}, A_{zM} \): amplitudes of \( M^{th} \) harmonic gradient perturbations
- \( \varphi_{yM}, \varphi_{zM} \): phases of \( M^{th} \) harmonic gradient perturbations (If not stated differently, \( \varphi_{yM} = \varphi_{zM} = 0 \))
- \( \theta \): azimuthal angle
- \( m \): particle mass
- \( B \): bunching factor \( \leq 1 \).

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The constant gradient assumption has been justified to a certain degree in Ref. 1, where correcting terms have been introduced into the computations. Some more work has been done by W. Hardt 4. It really seems legitimate to consider the constant gradient problem as a representation of alternating gradient focusing.

Following 1), we obtain the two differential equations for the two semi-axes of the beam envelope:

\[
\begin{align*}
(3) \quad & \frac{d^2 r_y}{d \varepsilon^2} + \frac{Q_y^2}{2} + \sum_{N=1}^{\infty} A_y \cos(\Theta - \phi_y) \gamma_y - \frac{E_{c y} R_y}{2} - \frac{2 \gamma_y \gammaBP}{2 \gamma^2 \varepsilon_c m \gamma B (r_y + r_z)} = 0 \\
(4) \quad & \frac{d^2 r_z}{d \varepsilon^2} + \frac{Q_z^2}{2} + \sum_{N=1}^{\infty} A_z \cos(\Theta - \phi_z) \gamma_z - \frac{E_{c z} R_z}{2} - \frac{2 \gamma_z \gammaBP}{2 \gamma^2 \varepsilon_c m \gamma B (r_y + r_z)} = 0
\end{align*}
\]

where

\[
\begin{align*}
E_{y} & \quad : \text{emittance in the } y, y' \text{-plane} \\
E_{z} & \quad : \text{emittance in the } z, z' \text{-plane}.
\end{align*}
\]

As it results from our computations and as was already shown in 3), the beam envelope oscillates with space charge at a frequency which is higher than the double of the individual-particle betatron-oscillation frequency.

But it seems that no resonance can occur on the individual-particle oscillation. This may be the subject of a separate paper. We shall concentrate here only on the problem of envelope oscillations.

Again following 1), we minimize the number of parameters by normalizing the beam dimensions in the two transverse oscillation planes. This normalization is such that the beam semi-axis in each plane is expressed in units of the square root of the product of the relative emittance by the radius of the machine.

\[
\begin{align*}
(5) \quad & \frac{\gamma^2}{2} \frac{d^2 Y}{d \varepsilon^2} + \frac{Q_y^2}{2} + \sum_{N=1}^{\infty} A_y \cos(\Theta - \phi_y) \gamma_y - \frac{E_{c y} Y}{2} - \frac{2 \gamma_y \gammaBP}{\sqrt{E_{c y} \gamma Y + Z}} = 0 \\
(6) \quad & \frac{\gamma^2}{2} \frac{d^2 Z}{d \varepsilon^2} + \frac{Q_z^2}{2} + \sum_{N=1}^{\infty} A_z \cos(\Theta - \phi_z) \gamma_z - \frac{E_{c z} Z}{2} - \frac{2 \gamma_z \gammaBP}{\sqrt{E_{c z} \gamma Y + Z}} = 0
\end{align*}
\]

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\[ Y = \frac{x_y}{\sqrt{RE_y/\pi}} \quad (7) \quad Z = \frac{x_z}{\sqrt{RE_z/\pi}} \quad (8) \]

\[ \delta = \frac{e^2}{4\pi\varepsilon_0 c^2 \beta^2 \gamma^2 B \sqrt{E_y E_z}} \quad (9) \]

\( \delta \) is the parameter for space-charge forces.

The numerical integration method that has been used finds the value \( F_{n+1} \) of the function \( F \) at the argument \( \theta_{n+1} = \theta_n + \text{steplength} \) by fitting a second order polynomial at \( \theta_n \).

**Known:** \( F_n, \ F'_n, \ F''_n \) at \( \theta_n \)

**Then:**

\[ \theta_{n+1} = \theta_n + \text{steplength} \]
\[ F_{n+1} = F_n + F'_n \cdot \text{steplength} + F''_n \cdot (\text{steplength})^2/2 \]
\[ F''_n = f(F_{n+1}) \text{ (using differential equation)} \]
\[ F'_{n+1} = F'_n + F''_n \cdot \text{steplength} \]

With the present results, it was convenient to express maximum beam dimensions in units of the **constant matched-beam dimensions** in an unperturbed machine as given by the constant solution \( Y, Z \) of (5) (6), where \( \Delta y^M = \Delta z^M = 0 \).

To simplify the calculation furthermore, we assumed first that this unperturbed machine had the same \( Q \)-value in both planes and equal emittance.

The constant solution, as can easily be shown, is then:

\[ Z = Y = \text{const} = \sqrt{\frac{\delta}{2Q^2} + \frac{1}{Q^2}} \quad (10) \]

Maximum beam dimensions normalized in the way described above become:

\[ Y_{\text{MAX REL}} = \frac{Y_{\text{MAX}}}{\sqrt{\frac{\delta}{2Q_y^2} + \left(\frac{\delta}{2Q_y^2}\right)^2 + \frac{1}{Q_y^2}}} \quad (11) \]
\[ Z_{\text{MAX REL}} = \frac{Z_{\text{MAX}}}{\sqrt{\frac{\delta}{2Q_z^2} + \left(\frac{\delta}{2Q_z^2}\right)^2 + \frac{1}{Q_z^2}}} \quad (12) \]

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We shall deal with values of \( Y_{\text{max rel}} \) and \( Z_{\text{max rel}} \) not very much above unity, which covers most practical cases of interest.

In this first part we shall assume that all the parameters of the envelope equations remain constant with time, in particular the space-charge parameter \( \delta \). This would be the case in practice for a constant energy (and constant bunching factor).

We shall in this report use the notion of **matched beams**.

For a beam without space-charge effects, the matched conditions are interesting namely for two reasons:

(i) They are the conditions for minimum beam dimensions

(ii) At a geometrical point inside the machine the beam dimensions are time-independent.

In a more general way, here, we shall call a **matched beam** a beam whose envelope oscillates only with the periodicity of the machine, but is time independent.

In the unperturbed case this periodicity is \( \frac{2\pi}{k} \) in \( \Theta \), \( k \) being any integer. In the perturbed case it is the periodicity of the perturbation.

1.1. **Unperturbed beams** \[ A_{\nu M} - A_{\nu} = 0 \]

Although the gradient perturbation is an essential factor for increased envelope oscillations around resonances, we start like Ref. 3) with unperturbed machines, in order to understand better relations between the space charge parameter \( \delta \), the Q-value of the envelope oscillations and the amplitude of the envelope oscillations.

1.1.1 **Matched beams**

A computer program was written to find the matched solution for a given set of beam and machine parameters. The program works with an iterative method. From approximate initial conditions for the matched beam one computes the dimensions of the beam after one turn; from a comparison with the dimensions at the beginning of the turn one tries to improve the initial conditions. When the beam is matched the dimensions at the
beginning and after one turn are equal.

1.1.1.2 Same Q-values and emittances in the 2 planes

\[ Q_y = Q_z \quad E_y = E_z \]

It is the simplest case we can think of. By setting \( \frac{d^2 y}{d \theta^2} \) and \( \frac{d^2 z}{d \theta^2} \) equal to zero in equations (5) and (6), we can easily obtain an algebraical expression for a matched solution, constant in \( \theta \):

\[ Y = Z = \sqrt{\frac{\delta^2}{2Q^2} + \left(\frac{\delta_0}{2Q^2}\right)^2 + \frac{1}{Q^2}} \]  

This expression increases with \( \delta \), i.e. with the intensity. But we can also find another so-called matched solution.

a) Symmetrical envelope oscillations \( Y = Z \)

Let us suppose that the \( \delta \)-value is slightly above an integer, for example 6.4. If \( \delta \approx 0 \), the number of envelope oscillations of the matched beam is \( Q_e = 12.8 \) (2\( \delta \)).

With an increasing \( \delta \), the number of envelope oscillations per turn, \( Q_e \), decreases. For a certain space charge \( \delta_c \), \( Q_e \) crosses the first integral value, in our example 12. This point is of special interest. When \( Q_e \) becomes an integer, the envelope oscillates with the periodicity of the machine and the oscillating beam is again matched. To represent this second matched solution it is convenient to use a plane where the vertical axis gives the maximum amplitude of the matched envelope normalized with respect to the constant solution (see Fig. 1).

In this plane the first matched solution, constant in \( \theta \), is given by the horizontal line

\[ Y_{\text{MAX REL}} = 1 \]

The line representing the oscillating matched solution leaves the above line vertically and turns to the right for increasing amplitudes of oscillation.

To understand qualitatively most of the present work, and in particular the above-mentioned phenomena, it is essential to notice the following:
For small amplitudes of envelope oscillations, around the matched constant solution in \( \theta \), the \( Q \)-value of the envelope remains constant. (Reason for the vertical intersection between lines in Fig. 1.)

But for larger amplitudes \( Q_M \) tends to go back towards its value for \( \delta = 0 \). Thus, in order to keep \( Q_M \) constant, so that the oscillating solution remains matched, it is necessary to increase \( \delta \), when the envelope oscillation increases. This is precisely the reason why the upper branch in Fig. 1, showing the normalized maximum envelope for symmetrical oscillations, is bent to be right.

b) Antisymmetrical envelope oscillations

We have so far assumed that the envelope oscillations are the same in both planes. Abandoning this restriction we discover, as in Refs. 2) and 4), that in the absence of perturbations only envelope oscillations in phase and antiphase can satisfy the requirements for matched beams. One way of explaining this phenomenon is the following:

The matched beam requires that the extrema of its envelope in each plane coincides with extrema of the gradient in each plane.

In the absence of perturbations the terms of the gradients in the two planes that are variable in \( \theta \) are

\[ \frac{2 \delta}{Y \left( \frac{E_Y}{E_Z} Y + Z \right)} \quad \text{and} \quad \frac{2 \delta}{Z \left( Y + \sqrt{\frac{E_Y}{E_Z} Z} \right)} \]

from Eqs. (5) and (6).

The extrema of the first term is then where:

\[ \frac{d}{d\theta} \left[ \frac{\delta}{Y \left( \frac{E_Y}{E_Z} Y + Z \right)} \right] = 0 \]

and at this extrema \( \frac{dY}{d\theta} \) is also zero. It is easy to show that in order to fulfill the above equation \( \frac{dZ}{d\theta} \) must also be zero.

Thus, extrema in \( Y \) and \( Z \) must coincide, which means that only symmetrical and antisymmetrical envelope oscillations are possible for the matched beams.

When the envelope oscillations in the two planes are in antiphase, the critical value for \( \delta \), where \( Q_M \) crosses an integral value, for infinitely
small oscillations, is lower by approximately 1/3 compared with the symmetrical case. This can easily be shown by linearization of Eqs. (5) and (6), as has been done in Refs. 2) and 4).

Another branch must then be added in Fig. 1. It shows the normalized maximum envelope for matched antisymmetrical oscillations.

c) Other resonances for higher values of $\delta$

When $\delta$ increases sufficiently $Q_{E}$ crosses lower integral values such as 11 and 10. It can be shown by linearization of (5) and (6) that when $\delta$ goes to infinity, the Q-value of the envelope does not go to zero but to $\sqrt{2Q}$ for small symmetrical envelope oscillations and to $Q$ for small antisymmetrical envelope oscillations. For the example we have just considered, the lowest possible space-charge induced symmetrical envelope resonance is the 9th and the lowest possible antisymmetrical is the 7th.

It is evident that for every resonance crossing one more matched solution appears. This is schematically shown in Fig. 2.

An interesting feature is that the branches for oscillating solutions become more and more bent to the right, the larger the Q-shift of the envelope is. The explanation is that with an increasing $\delta$, $Q_{E}$ becomes more and more dependent on the amplitude of oscillation of the envelope; a small increase in amplitude changes $Q_{E}$ very much and a large increase in $\delta$ is necessary to bring $Q_{E}$ back to the integral value.

1.1.1.2 Different Q-values in the two transverse planes but same emittances

In most machines the Q-values in the two planes are kept different to avoid bad coupling effects. If it is so, the differential equations (5) and (6) are different and it becomes difficult to calculate algebraically the non-oscillating matched solutions. The starting points, however, for $\delta = 0$, are given by:

$$Y(\delta = 0) = \sqrt{\frac{1}{Q_{y}}}$$
$$Z(\delta = 0) = \sqrt{\frac{1}{Q_{z}}}$$

When the coupling between the two movements increases, with an increasing $\delta$, the two non-oscillating solutions are likely to approach each other.
In Fig. 3 the solutions are normalized with respect to the constant solution for a round beam. For each plane the constant solution used for normalization corresponds to what it should be if the other plane had the same Q-value.

$Q_z$ has been set equal to 6.3 and the $Q_y$ equal to 6.4. The Y-reference amplitude corresponds to both $Q_y$ as well as $Q_z$, being equal to 6.4, and for the Z-reference both Q-values equal to 6.3.

The constant matched solution in the z-plane is smaller than in the other. For increasing $\delta$ it tends to increase with respect to the solution for the round beam. This increase is, however, so small that it cannot be seen in Fig. 3. Similarly, the Y-solution decreases when $\delta$ increases. For the non-oscillating solution, one can say that the round beam represents a very good approximation. Similarly to the previous case, symmetrically and antisymmetrically oscillating matched solutions exist for sufficiently high $\delta$.

The $\delta$ with which they appear are lower due to the fact that $Q_z$ in the z-plane is only to be diminished by 0.6 instead of 0.8 as in the previous case.

One notices also that the z-plane responds very much to the antisymmetrical resonance whereas the y-plane mostly to the symmetrical. The antisymmetrical envelope oscillations are larger in the plane with the lower Q-value than in the plane with the higher Q-value.

1.1.1.3. Different emittances in the two planes but same Q-values

It is interesting to obtain some information on the influence of unequal emittances in the two planes; as in most machines the acceptances are different. Furthermore, when multturn injection is used, the finally injected beam ends up with larger dimensions in the plane where multturn injection is done, than in the other.

With increasing $\delta$ the Q-constant solutions for Y and Z tend to differ from what they would be if both planes had the same emittance. The normalized dimension in the plane with the small emittance is increased, the normalized dimension in the other plane is diminished. See Fig 4.

The starting points are, of course, given by the Q-values only.

\[ Y(\delta = 0) = \frac{1}{Q_y} \]  
\[ Z(\delta = 0) = \frac{1}{Q_z} \]

Again, the oscillating matched solutions are different in the 2 planes.

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1.1.1.4 The most general case without perturbation

The $Q$-values and the emittances are different in the two transverse planes. From the previous results it should be possible to draw the most important conclusions as to the general behaviour:

(i) The constant solution for the plane with the smaller emittance tends to increase with increasing coupling, i.e. with increasing $\delta$, compared to a round beam (beam where both planes have the small emittance).

The constant solution for the plane with the larger emittance decreases, with increasing $\delta$, compared to a round beam where both transverse planes have the large emittance.

(ii) For the oscillating matched solution the oscillations in one plane are larger than in the other. In the plane with large oscillations we may speak of a proper or natural non-linear resonance and in the other plane of a forced non-linear resonance (forced by the coupling).

(iii) Diminishing the emittance or the $Q$-value to approach an integral or half integral value in one plane, favours the amplitude of antisymmetrically oscillating matched solutions of this plane. The same amplitude for the other plane is diminished. On the contrary, the amplitudes of the symmetrically oscillating matched solutions are favoured in the second plane and diminished in the first.

Increasing the emittance or the $Q$-values (putting it further away from a resonance), in one plane, has the opposite effect.

The values of $\delta$, where oscillating solutions are possible, can conveniently be deduced from Ref. 4).

1.1.2 Mismatched beams

It is complicated to study influences of mismatches in the most general case; due to the coupling between the two transverse planes. It would be necessary to use a four-dimensional volume $Y, Y', Z$ and $Z'$, to follow the evolution of a mismatch over many turns inside the machine.

In the simplest case, however, where the movements in the two planes are identical, such a study can be done by considering one plane only,
either $Y$, $Y'$ or $Z$, $Z'$.

Fig. 5 gives the evolution of mismatches at a fixed azimuth along the circumference of a machine with $Q_y = Q_z = 6.4$ and $\delta = 2.02$.

The mismatches described in Fig. 5 are, of course, purely symmetrical. For this $\delta$-value there are two symmetrical matched solutions: 1 constant and 1 oscillating solution. Point $A$ represents the constant solution. Around $A$ one finds an infinity of ellipse-like curves. All the curves are closed. Curve $C$ represents the oscillating matched solutions. All beams whose envelopes are given by points on $C$ are matched. Inside $C$, $Q_E$ is smaller than $12$; outside $C$, the influence of space-charge is so much reduced that $Q_E$ is larger than $12$.

In the outside region points turn in the negative direction; inside, in the positive direction. Points on $C$ are stationary, in accordance with the definition of matched beams.

When $\delta$ has diminished and there is no symmetrically oscillating matched solution, curve $C$ degenerates into Point $A$.

For increasing values of $\delta$, curve $C$ goes more outwards. Every time $Q_E$ of the non-oscillating beam crosses an integral value, a new curve, with the same properties as $C$ is created from point $A$. Point $A$, representing the non-oscillating solution, is the point where $Q_E$ is lowest. Every departure from $A$ increases $Q_E$. Infinitely far away from $Q_E = Q_E(\delta = 0) = 26$.

It is furthermore of interest that the oscillating matched beam is unstable. An infinitely small departure from curve $C$ will give rise to a finite change of beam dimensions at a fixed azimuth over many turns. On the contrary, the constant matched solution is stable. An infinitely small departure from $A$ gives rise to infinitely small changes in beam dimensions.

The explanation is straightforward. The $Q_E$-value of the oscillating matched solution is an integer. We are in the presence of a resonance. On the contrary, the constant solution has a non-integral value for $Q_E$; there is no resonance.

One may say that the instability on curve $C$ is not an instability of amplitude but only an instability of phase. An infinitely small departure from $C$ does not lead to a finite increase in envelope oscillations but only makes the phase between oscillation and machine variable with time.
1.2. Gradient perturbed beams

Gradient perturbations are present in any machine, due for instance to magnet imperfections, beam-handling or targetting devices near the closed orbit. These perturbations may be responsible for rather large incoherent betatron oscillations, if they occur at frequencies close to the betatron frequency (or more precisely the envelope frequency).

It is convenient to express the gradient perturbations as a Fourier expansion.

Supposing all the amplitudes of the various harmonics to be of the same order of magnitude, one may expect one harmonic to be more perturbing than the others around a given resonance of envelope oscillations. The present results confirm this expectation; the harmonic with the harmonic number nearest to the number of envelope oscillations per turn is the most perturbing.

1.2.1 Matched beams

The cases we have studied under 1.1 are limiting cases where the perturbation amplitude goes to zero. For small perturbations these cases represent, of course, a good approximation, as we shall see below.

1.2.1.1 Same Q-values and emittances in the two planes

a) Symmetrical perturbations

We shall consider first a perturbation, which for the sake of simplicity, consists of one single harmonic. Symmetrical perturbation means that the perturbation is the same in both planes.

Figure 6 gives the maximum amplitude of the envelope normalized with respect to the constant solution without perturbations. The harmonic of the perturbation is 12. From the numerical results represented in Fig. 6 and in Fig. 6a, we note the following:

(I) All matched solutions oscillate with the periodicity of the perturbation.

(II) The solution which was constant in $\theta$ in the non-perturbed machine, is slightly modulated by the perturbation.
(III) Each branch, representing oscillating solutions in the non-
perturbed case, is divided into two branches when a perturbation
is introduced. One branch is above the branch for the non-
perturbed case, the other branch is below.

(IV) On the higher branch the phase between envelope oscillation and
perturbation is such that the perturbation has a decreasing
influence on $q_E$. (The beam has large dimensions, where the
perturbation is defocusing.) The decrease in $q_E$ is cancelled
by additional envelope oscillations; that is the reason why
this branch lies above the branch for no perturbations.

(V) On the lower branch the phase is inversed. The perturbation
increases $q_E$; therefore, the envelope must oscillate less than
in the non-perturbed case.

(VI) The perturbation being purely symmetrical, the beam has at
least one matched solution, $Y = Z$ (for any value of $\delta$).
This, however, does not exclude the matched antisymmetrical mode.
This antisymmetrical mode appears for lower values of $\delta$.
To accentuate the influence of the symmetrical perturbation on
this mode, a rather large perturbation amplitude was introduced:
$A_{y12} = A_{z12} = 1.0$ (see Figs. 6 and 6a).

(VII) Curves c and d represent the antisymmetrical solution for
$\delta > \delta_A$ (see Fig. 6a). According to the definition, the $y$- and
$z$-envelope oscillations are then dephased by half the oscilla-
tion period. When the beam oscillates antisymmetrically, the
maximum beam dimensions on the two axes are not the same, the
phases between perturbations and envelope oscillations being
different.

(VIII) In the $\delta$-interval between $\delta_A$ and $\delta_P$, envelope oscillations
in the two planes are in phase, but the oscillation amplitudes
are not equal, with the exception of junction point $F$.

We have so far only considered solutions oscillating symmetrically or
antisymmetrically. For a symmetric perturbation there exists, however, a
third solution\textsuperscript{4)}, which we call the solution with variable phase:

\textsuperscript{4)} Solution first found by F. Sacherer, Berkeley, private
communication.

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This solution, represented by curve $c$ in Fig. 6c, has equal oscillation amplitudes in the two transverse planes:

$X_{\text{MAX REL}} = Z_{\text{MAX REL}}$.

The oscillation phases in the two planes vary with $\delta$, with respect to the perturbation. If the oscillation phase in one plane is $\psi$, it is $-\psi$ in the other. At junction point $E$ with the symmetry branch, $\psi$ vanishes. When $\delta$ increases from $\delta_{E}$, $\psi$ goes asymptotically towards $\pi/2$; thus, the solution is antisymmetric for infinitely large oscillations.

The $\psi$-convergence towards $\pi/2$ becomes faster, when the perturbation amplitude decreases. For zero perturbation, curves $c$, $d$, and $e$ coincide and become the curve for antisymmetric solutions of Fig. 1.

In the case of non-perturbed beams, it was relevant to draw a distinction between non-oscillating and oscillating matched beams. In the presence of perturbations, where all solutions oscillate at least in one plane, one may distinguish between weakly and strongly oscillating solutions.

The change from one type of solution into the other is gradual around the resonance for symmetrical envelope oscillations. Around the antisymmetrical resonance, the change is abrupt, more like the case with no perturbations. As we shall see later, this is because in the present case, the perturbation is purely symmetric.

When the Q-value in a gradient-perturbed machine without space charge crosses integral or half-integral values, no periodic solutions exist for the envelope equations. These Q-intervals are called stopbands. They depend on the strength of the perturbation.

Each value of $A_{12}$ corresponds to a certain stopband. If, instead of a 12th harmonic perturbation, a 6th harmonic perturbation is used, with an amplitude $A_{6}$ such that the stopband width is the same, a comparison of numerical results shows that the curves obtained in Fig. 6 are very much the same.

It seems that to a first approximation the departures from the curves for the unperturbed case are a function of the stopband width. A similar conclusion has also been reached in Ref. 2).

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(XIII) If all the perturbation harmonics have the same amplitude, only the harmonic with the harmonic number equal to the closest integral value of \( Q \) gives an appreciable stopband. The stopband width due to the other harmonics is zero or insignificant, as can be shown from the theory of Mathieu functions.

Thus, if all perturbation harmonics have amplitudes of the same order of magnitude, it is only necessary to introduce in this study perturbation harmonics around each possible resonance of the envelope.

(XIV) Sufficiently far away from point D in Fig. 6, the departures from the curves for the unperturbed case seem linear with the width of the stopband. Towards D the departure becomes proportional to the cubic root of the width of the stopband. This cubic-root dependence has also been previously indicated in 2).

Additional remarks on stopbands (see Fig. 6b)

We have also studied numerically the matched solution as a function of the Q-value, without and with space charge (\( \delta \) is a parameter), in the vicinity of half the perturbation frequency chosen here equal to 12, i.e. near \( Q = 6.0 \) (the behaviour near other resonances should be very much the same). One notices the following when space charge is introduced (see Fig. 6b):

(i) Provided \( \delta \) is sufficiently large, the stopband around \( Q = 6.0 \) disappears.

(ii) In a certain Q-interval, roughly equal to the Q-shift due to space charge, there exist three matched solutions instead of only one. This is in accordance with Fig. 6.

(iii) The two upper matched solutions go to infinity when \( Q \) goes to 6.0, whereas the lower solution remains finite and is approximately equal to the value one would have if there were no stopband (no perturbations) and no space charge.
b) **Antisymmetrical perturbation** \( A_{y12} = -A_{z12} \)

The matched beam in the case of a purely antisymmetrical perturbation is similar to the matched beam with purely symmetrical perturbation. The main differences are summed up below:

(I) The weakly oscillating solution oscillates antisymmetically.

(II) There is a gradual change from the weakly oscillating anti-symmetrical matched solution into the strongly oscillating anti-symmetrical solution \( \delta \).

(III) The symmetrical solutions show different amplitudes of oscillations in the two planes, due to the difference in phase between perturbations and envelope oscillations.

(IV) The solution with variable phase starts at the symmetrical resonance \( \delta \approx 1.8 \). Supposing the perturbation in the \( y \)-plane to be \( A \cos (12\theta - \varphi_y) \), and in the \( z \)-plane \( A \cos (12\theta - (\varphi_z + \pi)) \), the maximum beam dimensions in the two planes are at \( \varphi_y + \psi \) and at \( \varphi_z + \pi - \psi \), where \( \psi \) goes asymptotically towards \( \pi/2 \), when \( \delta \) increases. For \( \delta = \infty \), we have infinitely large symmetric envelope oscillations. When the antisymmetric perturbation vanishes, the curves for \( Y_{\text{MAX REL}} \) and \( Z_{\text{MAX REL}} \) of the symmetric oscillation mentioned under (III) coincide with the curve \( Y_{\text{MAX REL}} = Z_{\text{MAX REL}} \) of the solution with variable phase. The resulting curve is the same as the curve for symmetric envelope oscillations shown in Fig. 1.

c) **General perturbation phases**

We have so far discussed two extreme perturbation types: the symmetric and the antisymmetric. To understand the general case, it seems relevant to study the continuous evolution of the various solutions when the perturbation goes from one type to the other.

This continuity is studied in Annex 1.

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\( \delta \) Similar results for the antisymmetrical perturbation have been found analytically in a recent work by P. Sachser (3). Instead of expressing \( Y_{\text{MAX REL}} \) as a function of \( \delta \), it is expressed as a function of the space-charge induced Q-shift divided by the zero intensity Q-distance to the next lower stopband. This way, more general results than ours (in the sense of Q-values) have been obtained at the cost of some imprecision for large Q-shifts, which for the understanding of the space-charge phenomenon and for many applications is, however, of little importance.
It is shown that, in the general case with arbitrary perturbation phases, every time \( t \) increases, such that an envelope resonance is crossed, 4 more matched solutions appear. Two of those four are however of little interest as they generally only exist for very large envelope oscillations. The 2 other solutions exhibit the expected resonances as well as reasonable off-resonance beam dimensions.

In the further study we have therefore dropped the 2 first solutions, which may be a particularity of the ellipsoidal density distribution and therefore of mainly academic interest, and concentrated on the 2 last, which apparently correspond to a more realistic beam.

Fig. 7 shows these 2 matched solutions in a case where the sum of a symmetric and an antisymmetric perturbation gives zero perturbation in one plane.

1.2.1.2. Different \( Q \)-values in the 2 planes but same emittances

a) Symmetrical perturbation

When the \( Q \)-values of the 2 planes are not the same the movements in \( y \) and \( z \) become different and we have to consider what happens in both planes. See Fig. 8.

We notice the following:

(i) although the perturbation is purely symmetrical there is an interval on the \( \delta \)-axis, where the matched solution is antisymmetrical only.

This can be explained as follows: The \( \Theta \)-variable gradient being dependant upon both the perturbation and the beam shape, it can become antisymmetrical.

(ii) When there is a change from the weakly oscillating antisymmetrical matched beam to the symmetrical or vice versa, the envelope stops oscillating in one plane.
(iii) When the matched envelope oscillates strongly we may say that in one plane there is a natural non-linear resonance whereas in the other plane the non-linear resonance is forced. In this latter plane the amplitudes of oscillation are smaller than in the first (at least if we consider the normalized variables $Y$ and $Z$). There is also an intersection of the two curves for the 2 matched solutions in the first plane. The plane with the lower $Q$-value has its proper non-linear resonance in the antisymmetrical mode and the plane with the higher $Q$-value its proper non-linear resonance in the symmetrical mode.

b) Other perturbations

Introducing antisymmetrical or mixed perturbations does not change the general behaviour of the curves in Fig. 8.

Fig. 9 gives the results for an antisymmetrical perturbation of the same amplitude as in Fig. 8.

Fig. 10 gives the results for a mixed perturbation; half the antisymmetrical perturbation of Fig. 9 has been added to half the symmetrical perturbation of Fig. 8.

1.2.1.3 Unequal emittances in the 2 planes but same $Q$-values

a) Symmetrical perturbations

The movements in the 2 planes are different, due to $\frac{E_Y}{E_Z} \neq 1$.

From the case represented in Fig. 11 we draw the following conclusions:

(i) Again, although the perturbation is purely symmetrical, there is an interval, on the $\delta$-axis, where matched solutions can only be antisymmetrical. The explanation is the same as in the previous case with different $Q$-values.

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(ii) Again the change from the weakly oscillating, antisymmetrical matched beam to the symmetrical or vice versa is characterized by the envelope oscillation vanishing in one plane.

(iii) As previously when the envelope oscillates strongly, we can consider one plane as performing a natural non-linear resonance whereas the non-linear resonance is forced in the other plane. For the Q-values and emittance, the plane with the small emittance participates most in the antisymmetrical resonance and the plane with the large emittance most in the symmetrical resonance.

1.2.2 Gradient perturbed mismatched beams

a) Symmetrical mismatches

Only the simplest case where the movements are identical in both planes can be studied in a simple way. We shall consider a machine with some emittances, some Q-values and some perturbations in both transverse planes. Furthermore, we shall also assume the mismatches to be the same.

Fig. 12 gives a transverse phase plane at the beginning of a perturbation period \( \left( \theta = \frac{2\pi}{12} \right) \).

The curves show how a mismatch, equal in both planes, can change during many revolutions. The 3 matched solutions are given by F, G and I corresponding to F, G and I in Fig. 6.

In G and J the matched beam oscillates strongly. In F we have the slightly oscillating matched solution. The central point F represents a beam with a Q-value of free oscillation of the envelope well below the integral number. The small oscillation in F is only the forced oscillation induced by the perturbation. There is no risk of instability. This point corresponds to point A in graph 5.

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Considering the strongly oscillating matched solutions from the point of view of stability, one notices that the phase between envelope oscillation and perturbation is of great significance.

In the case of point J this phase is such that the perturbation prevents the growth of an infinitely small mismatch. Point J is stable. At point G on the contrary, the phase has the opposite effect. This point is unstable.

We remember from Fig. 5 that in the machine without gradient perturbations strongly oscillating matched solutions were unstable. Introducing a gradient perturbation limits the number of strongly matched solutions for a given 5 to two, but has the beneficial influence of making one of these solutions stable.

b) Non-symmetrical mismatches

With the computer we have also tried to mismatch slightly in one plane only, keeping the beam matched in the other. The curves for the 2 planes giving the evolution of the mismatch over many turns as in Fig. 12 are of course different (see Fig. 13). They do not seem periodical. The envelopes of the 2 curves for the 2 planes, however, appear to be identical and are very close to the closed curve in Fig. 12 for the same initial mismatch in both planes. This is valid for small mismatches around the 2 matched solutions that are stable. (F and J).
PART 2

Envelope oscillations at variable energy or under variable conditions.

The main purpose of this second part is to investigate about the increase in amplitude of envelope oscillations when a resonance is crossed. The crossing will normally be done by increasing the energy of the particles. When the energy increases $\delta$ decreases. In practice, resonance crossings could also result from other variations like changes in $Q$ or $B$ values. It should be possible to extrapolate some of our results to these cases but we have limited most of our discussions, for sake of simplicity, to the acceleration effect.

The differential equations for the envelope being non-linear, the number of oscillations per turn depends on the amplitude of envelope oscillations. One may therefore expect that for a given value of $\delta$ this amplitude cannot grow indefinitely. The growth in amplitude could furthermore be reduced by the fact that the resonance is crossed with a certain speed.

In the numerical computations, however, we cross resonances sufficiently slowly to make the amplitude increase practically independent on the speed of the crossing.

This way, mainly the non-linearity of the problem is the physical phenomenon that stops the growth of envelope oscillations.

Beam with variable parameters

In order to study such a problem the differential equations (5) and (6) have to be slightly modified. As we have said, we shall restrict ourselves to the case of an energy increase.

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Then, when the energy is varied the emittance in each of the 2 transverse planes change with $1/\gamma^2$. The bunching factor $B$ will be assumed to change with $\gamma^{3/4}$. Thus, considering the definition of $\delta$:

\[
\delta = \frac{e N}{4 \pi e_0^2 n_0 c^2 \gamma^2 B \sqrt{2 \gamma^2}}
\]

It results that the following relation for $\delta$ is valid:

\[
\delta = \delta_0 \frac{e_0}{\epsilon} \left( \frac{\gamma_0}{\gamma} \right)^{5/4}
\]

Although the transverse beam dimensions become smaller with increasing energy and the particle density higher, the importance of space charge forces diminishes compared with the importance of the external focusing forces.

The beam dimensions change with the square root of the emittances. But our variables $Y$ and $Z$ are transverse beam dimensions normalized with respect to the square root of the emittances. Thus, small variations in beam dimensions, due to energy increase, may be cancelled by the variations in the emittances.

After these general remarks we shall now look into more specific cases.

2.1 Round beam without gradient-perturbations

As a first case let us consider the acceleration of a round and unperturbed beam, which at the beginning is matched at 50 MeV and $\delta = 2.1$. This beam remains well matched during the acceleration, although one resonance is crossed. There are no perturbations and the small envelope oscillations during the acceleration as we shall see later with more general cases, are due to the imprecision of the numerical method and to the crossing speed not being infinitely small.
2.2 Round beam with gradient-perturbations

2.2.1 One-resonance crossing

A first case was studied where the beam was initially matched at 50 MeV and $\delta_0 = 2.1$. What happened during the acceleration is illustrated in Fig. 14, Fig 15 and Fig. 16.

Fig. 14 shows the maximum value of $Y$ as a function of the number of revolutions or time, normalized with respect to the matched solution without perturbation. This graph corresponds to Fig. 6.

Fig. 15 gives a curve proportional to the maximum physical dimension of every revolution in the machine; the maximum value of $Y$ has simply been divided by the square root of $\gamma$.

Fig. 16 gives the same plane as Fig. 13. For a certain value of $\Theta$ in our case $\Theta = 0$, after each revolution inside the machine the value of $Y$ is normalized with respect to the matched value without perturbations. This matched value is a function of $\delta$ and therefore of time. The logarithm of this normalized beam dimension is plotted in the horizontal direction. The vertical axis indicates $Y'$.

One can make the following remarks:

(i) As long as the beam is in the $\delta$-interval with 3 matched solutions it remains well matched. When it enters the interval with only one solution rather large envelope oscillations occur.

(ii) Before entering this interval the beam has been oscillating (with practically no amplitude) around a slowly changing matched condition which suddenly becomes unstable and then disappears. Afterwards, the beam being rather far away from the only remaining matched condition, a rather large mismatch is present.
(iii) The growth of the maximum beam semi-axes during the resonance crossing is about twice what it would be if the beam was matched also after the crossing; i.e. the crossing gives rise to a mismatch.

(iv) After having crossed the resonance the normalized maximum beam semi-axes remain about constant, apart from fast oscillations. In real dimensions, however, the beam size is strongly shrunk when energy increases, as about $(\beta \gamma)^{-1/2}$.

This can be expressed in a different way: the beam enters a stopband and the envelope oscillations increase. This increase changes $Q_B$ such that the beam leaves the stopband again on the other side with an amplitude twice as large as it would be if the beam was also matched after the crossing. We are in presence of two phenomena that limit the growth of envelope oscillations:

The property of the initially matched beam to remain matched when matched conditions change sufficiently slowly (adiabatic damping with energy increase) and the non-linearity of the space charge phenomena, described above and also in the introduction to Part 2. To observe this first property and check that it also applied to the so called matched solutions two further computer runs were done.

1. A beam initially matched at $\delta = 2.02$ to a strongly oscillating solution (corresponding to the stable point J in graphs 12 and 6), was accelerated. The matched condition was found to change adiabatically and the beam remained well matched.

2. A beam matched below the resonance was decelerated; i.e. $\delta$ was increased. The beam remained matched and followed the upper curve in graph 6.

From these three computer runs we can draw the conclusion (which we shall find later to be valid for general beams): a matched beam remains matched as long as the matched conditions, which must be stable, change sufficiently slowly, which is the case at the standard acceleration rates.

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2.2.2 Two-resonance crossings (Fig. 17, 18, 19)

A case where a beam crosses the 11th and then the 12th resonances of the envelope was also considered. With an initially matched beam at $\delta = 5.7 \ (q_2 < 11);$ one notices the following:

(i) For the crossing of the 11th resonance mainly a 11th harmonic perturbation matters. It was given the same amplitude as the 12th harmonic. $(A_{y11} = A_{z11} = A_{y12} = A_{z12} = 0.1)$

(ii) The resonance that was crossed first, namely the 11th, was crossed in a way very similar to the 12th in the previous case. At the beginning the beam remained well matched. When the stable condition, that had been followed, became unstable and disappeared a mismatch occurred.

(iii) This time the mismatch was less important. The increase in amplitude due to the crossing of the 11th resonance was about 23%. In the previous crossing of the 12th harmonic it was about 40%. That checks with the property mentioned in (iii) of paragraph 2.2.1. A qualitative explanation of this difference can easily be found: the non-linearity of the differential equation increases with $\delta$. With an increasing non-linearity less amplitude increase is necessary to bring the envelope out of resonance.

(iv) After having crossed the 11th resonance the beam oscillated with a mismatch of somewhat constant amplitude (see iv of 2.2.1).

(v) Due to this mismatch the 12th resonance is not situated at the same $\delta$-interval as in the case where the beam is matched before the resonance. The mismatch has a tendency to increase $q_2$. With a mismatch therefore, the resonance is at a higher $\delta$-interval.

(vi) As far as maximum beam dimensions are concerned, for the perturbations used in the present case, illustrated by Fig. 17, Fig. 18 and Fig. 19, the resonance the beam crosses first is the most dangerous one.

Between the first and the second crossing the adiabatic shrinking of the beam dimensions, characterized by $(\gamma_0)^{-1/2}$, saves the beam from having larger dimensions after the second crossing than after the first.
Only with a higher perturbation amplitude for the 12th... harmonic the dimensions after the second crossing can exceed those after the first.

(vii) One might think that the phase with which the mismatched beam enters the second resonance would be of importance. If the speed with which $\delta$ varies is sufficiently slow this phase is, however, irrelevant as we have seen from different computer runs. This is probably because, when entering the resonance, the movement in the plane of Fig. 19 is slowed down and instead of turning in the positive direction around the matched value, it starts turning in the negative direction, and the exact phase of reversal does not matter appreciably.

2.3 Different movements in the 2 transverse planes with mixed gradient perturbation

The results we have previously found for the round beam explain very well what happens in a more general case; let us consider the case where the 2 planes have different $Q$-values, different emittances and where only one plane is perturbed (Fig. 20, 21, 22, 23).

We notice the following:

(i) The initially matched beam remains matched up to the first resonance as in the example of the round beam we have just considered.

(ii) The amount of increase in maximum beam-dimensions at each resonance can again be estimated as previously by considering in the two planes the curves for matched solutions.

(iii) The beam envelope still resonates twice (and only twice) with the frequency 12. Once symmetrically and another time antisymmetrically. Each time for different values of $\delta$.

(iv) The value of $\delta$ for which the antisymmetrical resonance is crossed (second crossing) is dependent on the mismatch from the crossing of the symmetrical resonance. The same is valid for the envelope increase (or mismatch) at the crossing of the antisymmetrical resonance. The explanation is the same as in the previous case: due to the non-linearity of the equations the $\delta$-value of the resonance depends on the amplitude of oscillation.
(v) The fact that symmetrical and antisymmetrical resonances take place at different values of $\delta$ is beneficial as far as maximum beam dimensions are concerned. Each resonance causes an increase in beam dimensions, but a part of the first increase is cancelled before the second resonance by the adiabatic shrinking of the emittance.

(vi) In most cases the resonance crossed first is the resonance that gives rise to the largest beam dimensions. In some cases it may be better to avoid the first resonance by injecting below it. To make it interesting to cross a resonance it is necessary to start sufficiently above it so that the initial dimensions are not exceeded during the crossing.

To check that the introduction of general perturbation phases in the two transverse planes do not change appreciably, the resonance blow-ups from what we have found so far, resonance crossings where $\delta_{y12} = \delta_{z12} = \pi/2$, were computed; see Figs. 24, 25, 26, and 27.

At both the symmetrical as well as at the antisymmetrical resonance, the beam dimension blow-ups can be understood from the matched solutions drawn in Figs. 24 and 26.

We remember from paragraph 1.2.1.1.c, that when $|\delta_{y12} - \delta_{z12}| \neq \pi$, there appear per resonance two more solutions. At the crossings these do not seem to interfere, at least in the case considered. Probably they most frequently only exist for very strongly oscillating beams and high $\delta$-values.
DISCUSSION AND CONCLUSIONS

DISCUSSION

The previous work can be used to examine what would be the best injection conditions in a synchrotron for very large intensities only.

In particular, what would be the optimum Q-values to use, the best emittances (we might also say the best bunching factor) and what gradient errors could be permitted?

The answer which can be given to these questions will, of course, only be valid if the effects which have been considered here are the effective intensity limitations and if all instabilities or coherent space charge effects do not appear. It also supposes that there is no additional intensity dependent effect of a still different nature which could modify the behaviour of the beam.

Under these very general conditions, we have assumed, in the present computations the Q's fixed during acceleration and a certain dependence of $\delta$ with the energy of the particles, $\delta$ being the space-charge parameter in 4-dimensional transverse phase space. For other $\delta$-variations the results we have obtained in the planes

$$Y_{\text{MAX REL}} \text{ versus } \delta \text{ and } Z_{\text{MAX REL}} \text{ versus } \delta$$

remain valid, provided $\delta$ still decreases with time and changes sufficiently slowly, and this requires in particular slow enough variations of the bunching factor $B$. $Y_{\text{MAX REL}}$ and $Z_{\text{MAX REL}}$ are maximum beam semi-axes normalized with respect to the matched solution in an unperturbed machine, (see (11) and (12) page 6). The maximum beam dimensions as a function of $\delta$, however, depend on the $\delta$ variation with energy. Some of our results could also be extended to cases where the Q-values change with time. The computations are based on a series of simplifying assumptions; the following 2 being probably the most serious:

a) uniform density in the transverse phase planes

b) no coupling between longitudinal and transverse movements.

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The non-uniform distribution should not change basically the property of the non-linear resonance we have discussed. It is to be expected that, at the crossing of a resonance, first the outer part of the beam resonates and that the resonance goes inwards to the centre as the particle-energy increases; the envelope oscillations are probably not very much affected.

The influence of the longitudinal movement on the transverse is much harder to estimate. There is probably an important dependence upon the ratio synchrotron oscillation frequency to envelope oscillation frequency. This may be the subject of a further study.

In addition, large changes in $\delta$ especially during the trapping process might produce fast $\delta$ variations, falling out of the cases referred to.

Having pointed out these assumptions and limitations, let us come now to the questions we have raised above and concentrate for instance on the problem of the best emittances.

Suppose as an example it would be possible to choose and adjust the two transverse emittances, keeping a fixed ratio between them (This case would occur if the bunching factor was the variable). Would it in this case be better to inject a beam with small or with large emittances or is there an optimum? In other words, making use of the space charge parameter, is there an optimum for $\delta$?

To answer those questions, we must find out how to express the chamber-wall limitations in the planes:

\[ Y_{\text{MAX REL}} \text{ versus } \delta \]

and

\[ Z_{\text{MAX REL}} \text{ versus } \delta \]

for variable emittances $E_Y$ and $E_Z/E_Y = \text{constant}$.

Chamber wall limitations

Let $a$ and $b$ be the chamber half axies. From (7) we obtain the condition:

\[ Y_{\text{MAX}} \leq \frac{a}{\sqrt{\text{RE}_Y/\pi}} \]
on the other hand

\[ \delta = \frac{a^2}{4 \pi e_0 c^2 \beta^2 \gamma^3 B E_y \sqrt{E_z / E_y}} \tag{9} \]

If for given values of \( N, \sqrt{E_z / E_y}, \gamma \) and \( B \) the emittance \( E_y \) increases, \( \delta \) decreases and so does also the permissible \( Y_{\text{MAX}} \).

Combining (14) and (9) we can eliminate \( E_y \) and we obtain:

\[ Y_{\text{MAX}} \ll \frac{a}{Q_y} \sqrt{\delta} \tag{15} \]

and similarly:

\[ Z_{\text{MAX}} \ll \frac{b}{Q_z} \sqrt{\delta} \tag{16} \]

where

\[ Q_{y,z} = \frac{c \sqrt{E_{y,z}}}{4 \pi e_0 c^2 \beta^2 \gamma^3 B} \tag{17} \]

If \( N, E_y/E_z \) and \( B \) are kept constant and \( E_y \) varies, the constant chamber wall limitations \( a \) and \( b \) appear as parabolas in the planes \( Y_{\text{MAX}} \) versus \( \delta \) and \( Z_{\text{MAX}} \) versus \( \delta \).

Dividing both sides of (15) with the matched unperturbed solution we obtain the condition:

\[ Y_{\text{MAX REL}} \ll \frac{a}{Q_y} \frac{1}{\sqrt{\frac{\delta}{2Q_y^2} + \sqrt{\left(\frac{\delta}{2Q_y^2}\right)^2 + \frac{1}{Q_y^2}}}} \tag{18} \]

similarly in the other plane:

\[ Z_{\text{MAX REL}} \ll \frac{b}{Q_z} \frac{1}{\sqrt{\frac{\delta}{2Q_z^2} + \sqrt{\left(\frac{\delta}{2Q_z^2}\right)^2 + \frac{1}{Q_z^2}}}} \tag{19} \]

For small values of \( \delta \) (18) and (19) become:

\[ Y_{\text{MAX REL}} \ll \frac{a}{Q_y} \sqrt{\delta Q_y} \tag{20} \]
\[ (21) \quad Z_{\text{MAX REL}} \leq \frac{b}{q_z} \sqrt{b q_z} \]

For very large values of \( \delta \) we can write:

\[ (22) \quad Y_{\text{MAX REL}} \leq \frac{\alpha}{q_y} q_y \]

\[ (23) \quad Z_{\text{MAX REL}} \leq \frac{b}{q_z} q_z \]

For small values of \( \delta \) the limiting curves in the planes \( Y_{\text{MAX REL}} \) versus \( \delta \) and \( Z_{\text{MAX REL}} \) versus \( \delta \) are parabolas, for large values of \( \delta \) they tend asymptotically towards \( \frac{\alpha}{q_y} q_y \) and \( \frac{b}{q_z} q_z \) respectively. See Fig. 28.

As \( Y_{\text{MAX REL}} \) and \( Z_{\text{MAX REL}} \) cannot be smaller than unity, there exist minimum conditions for \( \delta \); these can be taken from (20) and (21):

\[ (24) \quad \delta > \frac{c_y^2}{2 q_y^2} \]

\[ (25) \quad \delta > \frac{c_z^2}{b^2 q_z^2} \]

In the cases we have studied when the beam energy increases, the limiting curves move upwards with the factor \( \beta^2 \gamma^2 \) and the situation becomes less and less stringent, the real limitation taking place near injection.

CONCLUSIONS

From the present work and under the assumption made, it is then possible to draw the following conclusions about the optimum \( \delta \) for injection.

From Fig. 28 and expressions (24) and (25) it is clear that for given values of \( N, \frac{E_y}{E_z}, \gamma, B, a \) and \( b \), \( \delta \) cannot be made infinitely small; that simply means that there is a maximum value for the emittances and that is obvious.

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But then, apart from the resonances, there is no other limitation for \( \delta \); and even in order to leave more space between beam and vacuum chamber, the larger \( \delta \) (the smaller the emittances), the better it is.

On the other hand, if the beam is not allowed to cross any resonance, choosing \( \delta_{\text{inj}} \) too high means that one must inject a strongly oscillating matched beam, having rather large maximum dimensions. Therefore \( \delta_{\text{inj}} \) has in this case an upper limit. For sufficiently small perturbations this limit is in first order given by the \( \delta \)-value of the first resonance in the machine without perturbations. It corresponds to the so-called "incoherent space charge limit".

In order to discuss the possibility of crossing resonances one may be a little more specific and make a few more remarks. It is good to inject a matched beam in a \( \delta \)-interval between 2 non-linear resonances, provided the following conditions are fulfilled:

a) The shrinkage of the beam dimensions, (adiabatically with \((\gamma \beta)^{-1/2}\)), between injection and the beginning of the first resonance encountered, should be larger or comparable with the increase in beam dimensions during the crossing of this resonance. If not, one would probably lose so many particles that it would be better to inject at a lower value of \( \delta \). An estimate of the increase in beam dimensions, in the two transverse planes, at the crossing of a resonance can be obtained from the graphs showing \( y_{\text{MAX REL}} \) and \( z_{\text{MAX REL}} \) of the matched beam as a function of \( \delta \). The increase is twice what it would be if the beam was matched also after the resonance. Condition a) fixes the inferior boundary of any usable \( \delta_{\text{inj}} \)-interval.

b) The upper boundary of any \( \delta_{\text{inj}} \) interval is given by the next resonance which cannot be approached too much without having to increase the beam dimensions in order to keep it matched on the strongly oscillating solution.
c) If due to a possible longitudinal bunching (not treated in this report), $\delta$ increases after injection, the initially weakly oscillating matched beam may become a strongly oscillating matched beam. To avoid an excessive growth in transverse beam dimensions the previously defined upper limit for $\delta_{\text{inj}}$ should be reduced by the amount that $\delta$ increases during the bunching process.

d) Generally speaking, the width of the use $\delta_{\text{inj}}$ interval between 2 resonances increases when the perturbations decrease. Therefore, whether or not there is any sense in injecting at intensities above one or several resonances depends mainly on the strength of the perturbations.

We have not been able to obtain very accurate information on the amplitudes of the gradient-perturbation harmonics in the CERN-PS. It seems, however, that they are of the same order of magnitude as the amplitudes we have used in the computations.

One may conclude that the computations confirm, at least in the case of the PS, the relevance of the proposal made in (3) to inject beams at intensities above space-charge induced incoherent resonances, provided that no other important beam-dimension increases take place than the one caused by the incoherent resonance and also provided enough care can be taken as to the exact injection conditions.
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References:


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ANNEX I

Starting from the purely symmetrical perturbation, there are 3 different ways to reach a purely antisymmetrical perturbation:

a) by introducing an additional antisymmetrical perturbation, with $\gamma_{12} = \gamma$ and $\gamma_{21} = \gamma + \pi^+$ and by letting the symmetrical perturbation vanish. This ought to create resonances around curves c and d in Fig. 6.

b) by introducing an additional antisymmetrical perturbation with $\gamma_{12} = \gamma - \frac{\pi}{2}$ and $\gamma_{21} = \gamma + \frac{\pi}{2}$ and letting the symmetrical perturbation vanish. With this perturbation the resonance around curve c (solution with variable phases) ought to appear. Changing progressively the perturbation phases in the two planes towards antisymmetry, and keeping the amplitudes constant is a special case of b/.

c) by introducing additionally both kinds of antisymmetrical perturbation and by letting the symmetrical perturbation vanish. In this way, the very general behaviour should be seen. Resonances around curves c, d and e in Fig. 6b should appear.

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+ We call this an antisymmetrical perturbation of the first kind.
++ We call this an antisymmetrical perturbation of the second kind.
Fig. 29 shows the influence of a small antisymmetrical perturbation of the first kind, as defined under a), added to the large symmetrical one.

Branches a, b, c and d represent solutions oscillating either in phase (symmetrically) or in antiphase (antisymmetrically).

Branches f and g represent solutions with variable phases. For this type of perturbation we notice that the differential equations for Y and Z are symmetric with respect to $\varphi = \varphi$. Due to these equations being of second order, $\varphi$ can be replaced by $-\varphi$ and it is therefore obvious that if there is a matched solution commencing with:

$$Y_1 (\varphi), Y_1' (\varphi), Z_1 (\varphi) \text{ and } Z_1' (\varphi)$$

there must be a second solution starting with:

$$Y_2 (\varphi) = Y_1 (\varphi) \quad Y_2' (\varphi) = -Y_1 (\varphi)$$

$$Z_2 (\varphi) = Z_1 (\varphi) \quad Z_2' (\varphi) = -Z_1 (\varphi)$$

where:

$$Y_{2\text{MAX REL}} (\delta) = Y_{1\text{MAX REL}} (\delta)$$

$$Z_{2\text{MAX REL}} (\delta) = Z_{1\text{MAX REL}} (\delta)$$

Branches of f and g are thus double branches, in the sense that they represent two different matched solutions.

Points F and G are junction points between the two types of solutions.

+ Here we generalize the notion of symmetrical oscillations to oscillations with different amplitudes in y- and z- but with the same phases. Antisymmetrical oscillations are oscillations in antiphase. Symmetrical or antisymmetrical perturbations, however, are still perturbations of same amplitude or in phase resp. in antiphase.
When the antisymmetrical perturbation increases and the symmetrical perturbation vanishes we should end up with curves for matched solutions like those discussed under 1.2.2.b.

Fig. 29a is a sketch of how branches a, b, c and d are supposed to change during that process.

If a small antisymmetrical perturbation of the second kind, as defined under b) is introduced, curves like those of fig. 30 are valid for the matched solutions.

As perturbations in this case can be interchanged in the two transverse planes by replacing $\phi$ by $-\phi$, it can be shown that if there is a solution, starting at $\phi$ with:

\[ Y_1(\phi), Y_1'(\phi), Z_1(\phi), Z_1'(\phi) \]

there must be a second solution, starting with:

\[ Y_2(\phi) = Z_1(\phi), \quad Z_2(\phi) = Y_1(\phi) \]
\[ Y_2'(\phi) = -Z_1'(\phi), \quad Z_2'(\phi) = -Y_1'(\phi) \]

where:

\[ Y_{2\text{MAX REL}}(\delta) = Z_{1\text{MAX REL}}(\delta) \]
\[ Z_{2\text{MAX REL}}(\delta) = Y_{1\text{MAX REL}}(\delta) \]

The most continuous solutions c and d are formed around the solution with variable phase as shown in Fig. 6a, i.e., the small antisymmetrical perturbation of the second kind drives the variable phase resonance. Along branches c and d of this resonance one notes that:

\[ Y_{\text{MAX REL}} = Z_{\text{MAX REL}} \]

and the initial conditions are:

\[ Y(\phi) = Z(\phi) \]
\[ Y'(\phi) = -Z'(\phi) \]
Considering this solution as a first solution and looking for a second as described above, one obtains the relations:

\[ Y_2(\theta) = Y_1(\theta) \]

\[ Z_2(\theta) = Z_1(\theta) \]

The two solutions are thus identical.

With this perturbation case it is much easier to anticipate the continuous transformation of the various branches when the perturbation becomes purely antisymmetrical.

Branches c and d have already taken the shape they will have at the end of the transformation process and branch c moves upwards along branch c, as already stated.

Both kinds of antisymmetrical perturbations of small amplitude, compared with the also present symmetrical, give rise to curves for the matched solutions like those in Fig. 31.

All resonances are excited.

Fig. 32 is a similar case, but with larger antisymmetrical perturbations. The two branches drawn with thin dotted lines, are branches coming from and returning to \( \delta = 0^\circ \) and \( Y_{\text{MAX REL}} \to \infty \) respectively.

\( Z_{\text{MAX REL}} \to \infty \)

How far down these curves reach is of course strongly dependent on perturbations. By comparing Figs. 31 and 32 one can anticipate that for a large constant symmetrical perturbation they move upwards, when the small antisymmetrical perturbation increases.

We have not searched for then in our further investigations, assuming then in a normal perturbation case where symmetrical and antisymmetrical perturbations are of the same order of magnitude, only to exist for extremely large envelope oscillations. The justification of this assumption is confirmed later by a computational resonance crossing in a general case, where only the resonances one can expect from the other branches turned up. (See 2.3).
\[ Q = 6.4 \text{ in both planes} \]

\[ 2\pi Q = v \]

Symmetrically oscillating solution

\[ 2\pi Q = v \]

Antisymmetrically oscillating solution

\[ Y_{\text{MAX REL}} = \frac{\sqrt{\frac{6}{2Q^2}} + \frac{1}{2Q^2}}{2Q^2} \]
Fig 2

\[ Q_y = Q_z = 6.4 \]
\begin{align*}
Y_{\text{MAX REL.}} \\
Z_{\text{MAX REL.}} \\
\text{antisymmetrical oscillation} \quad Q_E = 12 \\
\text{sym. oscillation} \\
\text{different Q values} \\
Q_z = 6.3 \quad Q_y = 6.4 \\
\text{constant solution in } \Theta \quad \text{(the curves for the 2 planes are practically the same)}
\end{align*}
DIFFERENT EMITTANCES

\[ Q_y = Q_z = 6.4 \]

\[ \frac{E_z}{E_y} = 0.5 \]

Fig 4

constant solution in \( \Theta \)
SYM METRICAL MISMATCHES

\[ Z' = Y' \]

\[ Y = Z \]

\[ Q_e > 12 \]

\[ Q_e < 12 \]

\[ Q_e = 12 \]

\[ Q = 6.4 \]

\[ \delta = 2.02 \]

no perturbation

Fig 5

POINT A:
MATCHED SOLUTION CONSTANT IN \( \Theta \)

\[ Q_e < 12 \]

\[ \logn \left( \frac{Y}{\text{CONSTANT SOLUTION IN } \Theta} \right) \]
When the maked beam oscillates antisymmetrically the phases between envelope oscillations in the two planes and the symmetrical perturbation are different. Therefore the maximum dimension in one plane will not be the same as in the other. When $Y_{\text{MAX REL}}$ is on curve $c$, $Z_{\text{MAX REL}}$ must be on curve $d$ or vice versa.

**Remark:**

$A_{y12} = A_{z12} = A, \varphi_{y12} - \varphi_{z12} = 0$

Only a symmetrical 12th harmonic perturbation is present.

$Q_y = Q_z = 6.4$

**Fig. 6**
$Y_{\text{MAX REL}} = \frac{Q_2}{Q_1} = 6.4$
$A_{y12} = A_{z12} = 1$
$E_y/E_z = 1$

Large symmetric perturbation

Solution with variable phase

Symmetric solution $Y(\theta) = Z(\theta)$

Antisym. sol.
At $Y_{\text{MAX}} = \infty$ the two curves a and b are centered around $Q_y = 6.0$. The situation there is the same as if there was no space charge.

**Symmetrical Perturbations**

$A_y^{12} = A_z^{12} = 0.1$

**Fig. 6b**

**Space-Charge Modified Stopband**

Only symmetrical oscillations are considered.

The space charge forces cancel the stopband for small envelope oscillations. (provided $\delta$ is sufficiently large)
MIXED PERTURBATIONS
both symmetrical and antisymmetrical

Q_y = Q_z = 6.4
A_y12 = 0.1
A_z12 = 0.0

Fig 7
DIFFERENT Q VALUES
$Q_y = 6.4$
$Q_z = 6.3$

SYMMETRICAL PERTURBATION
$A_{yze} = A_{zye} = 0.1$

Fig 8
DIFFERENT Q VALUES

$Q_y = 6.4 \quad Q_x = 6.3$

ANTISYMMETRICAL PERTURBATIONS

$A_y \approx 0.1 \quad A_x \approx -0.1$

Fig 9
DIFFERENT Q VALUES

Q_y = 6.4  
Q_z = 6.3

MIXED PERTURBATIONS

A_yz = 0.1  
A_zyz = 0.0
DIFFERENT EMITTANCES
\[ \frac{E_z}{E_y} = 0.5 \]

SYMOMETRICAL PERTURBATION
\[ A_{yt} = A_{zt} = 0.1 \]

SAME Q VALUES
\[ Q_x = Q_z = 6.4 \]

Fig 11
The graph is symmetric with respect to this axis.

\[ \frac{dz}{d\theta} = \frac{dy}{d\theta} \]

**Fig 12**

\[ Q_y = Q_z = 6.4 \]
\[ A_{y12} = A_{z12} = 0.1 \]
\[ \delta = 2.02 \]

**EQUAL MISMATCHES IN BOTH PLANES**

\[ \log\left(\frac{y}{\sqrt{\frac{\delta}{2Q^2} + \frac{1}{\delta}}}ight) \quad y = z \]
Beam is initially only mismatched in the x-plane. The mismatch in the y-plane is due to the coupling. The mismatches are around point F of fig. 6 and fig. 12.

\[ a_y = a_z = 0.1 \]

\[ F_x/F_y = 1 \]

Various initial mismatches in the x-plane were followed in the y-plane over many turns. For each mismatch an envelope appears in each of the transverse phase planes. They seem to be identical and very close to the closed curve of fig. 12 for the same initial mismatch in both planes.
The beam remains well matched during the beginning of the acceleration. When the resonance is entered a mismatch occurs.

At the resonance crossing the increase in maximum beam dimension is about twice what it would be if the beam was matched also after the resonance.

acceleration:
To keV per turn?
The graph takes into account the adiabatic shrinking of the emittance with $(ya)^{-1}$.

The vertical axis gives $Y_{em} - Z_{em}$ divided by $(ya)$.

The curve is thus proportional to the physical beam dimensions during the acceleration.
$\gamma_{\text{max}}$, $z_{\text{max}}$

ACCELERATION:

350 kV per turn

$\gamma_{\text{max}}/\gamma_{\text{p}}$ versus $\delta$

The curve is proportional to the physical dimensions of the beam during the acceleration.
This curve is proportional to the maximum beam-dimension in the Z-plane during the acceleration.
Fig. 24

\[ \Delta y_{12} = \Delta z_{12} = 0.1 \]

\[ \frac{E_y}{E_z} = 2 \]

\[ \phi_{3/12} - \phi_{y_{12}} = 0.5 \pi \]

\[ Q_y = 6.4 \quad Q_z = 6.3 \]

MAX REL
$Y_{\text{MAX}}/ (\chi \beta)^{1/2}$

**Fig. 25**

This curve is proportional to the maximum physical dimensions in the $y$-plane during acceleration.
This curve is proportional to the maximum physical dimensions in the $z$-plane during acceleration.
Fig 28

$y_{\text{max rel}}$

chamber wall limitation in $y$ plane

$\frac{a}{\delta} = \text{constant}$

matched solution
PERTURBATIONS:

$y$: Plane: $0.975 \cos 12^\circ$
$z$: Plane: $1.025 \cos 12^\circ$

$same$ $sym.$ $perturbation$ $with$
$small$ $antisym.$ $of$ $the$ $first$ $kind$

$Q_y/Q_z = 6.4$
$E_y/E_z = 1$

---

**Fig. 29**

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- sym. oscillations
- antisym. "
- variable phase

---

const. phase

---

$F$, $a$, $b$, $c$, $d$, $g$
Perturbations: (if not stated differently)
y: Plane $\cos 12\theta - 0.025 \sin 12\theta$
z: Plane $\cos 12\theta + 0.025 \cos 12\theta$

Large sym. perturbation with small antisym. of the second kind

Perturbations:
y-plane: $0.975 \cos 12 - 0.0625 \sin 12\psi$
z-plane: $0.975 \cos 12 + 0.0625 \sin 12\psi$
Small antisym. perturbations of the first and second kind added to the large symmetric perturbation.
If not stated differently, perturbations are:

$A_{y_12} = 1.0$
$A_{y_2} = 1.05$
$\frac{\delta_y}{\delta} = 3.85$ 
$\pi - \frac{\pi}{12}$

Fig. 32

Sym. perturbation

$A_{y_12} = A_{y_2} = 1$
$\phi_{y_2} = \phi_{z_2}$