

## Curiosities on Free Fock Spaces

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### Abstract

We consider some curious aspects of single-species free Fock spaces, such as novel bosonization and fermionization formulae and relations to various physical properties of bosonic particles. We comment on generalizations of these properties to physically more interesting many-species free Fock spaces.

1. Introduction and background: The old idea of master fields in large  $N$  gauge theories [1] has been recently revived by Douglas [2],[3] and Gopakumar and Gross [4]. It appears that the master field concept is naturally described in the framework of non-commutative probability theory [5]. One of the crucial ingredients of non-commutative probability theory is the idea of free Fock spaces [5], [6], [7]. The free Fock space is defined to be a Hilbert space with orthonormal basis vectors on which strings of creation ( $a_i^\dagger$ ) and annihilation operators ( $a_i$ ) act, satisfying no relations whatsoever. Hence the adjective free. The basic operations of  $a_i^\dagger$  and  $a_i$  are given by

$$\begin{aligned} a_p^\dagger |p_1, \dots, p_n \rangle &= |p, p_1, \dots, p_n \rangle \\ a_p |p_1, \dots, p_n \rangle &= \delta_{p, p_1} |p_2, \dots, p_n \rangle, \end{aligned} \tag{1}$$

where  $a|0 \rangle = 0$ . It follows then that  $a_j a_k^\dagger = \delta_{jk}$  (this relation defines the so called Cuntz algebra). The same expression also defines the so called infinite statistics [8], in which any representation of the symmetric group can exist. Free Fock spaces appear to have many unusual properties. For example, it was argued by Greenberg [8] that if one attempts to use infinite statistics in order to construct a second-quantized relativistic field theory of free fields, the resulting theory does not possess the property of locality and there is no analog of spin-statistics theorem, even though CPT theorem and cluster decomposition seem to hold. Note as well that from the Cuntz algebra structure there appears not to exist the usual classical limit, with the well defined classical phase-space structure.

In order to understand the mechanics of the free Fock space it is important to address some basic questions. In the previous note [9] we asked what the naive analog of the Gaussian coherent states is for the case of infinite statistics. This was motivated by the fact that the knowledge of the  $N = \infty$  master field is equivalent to the knowledge of the ground state of the  $N = \infty$  theory [10] and the fact that Gaussian coherent states describe rather well the ground states of large  $N$  vector models. Unfortunately, the naive analog

of Gaussian coherent states constructed in [9] turns out to possess rather complicated properties. It is natural to ask next if one can work directly with Lagrangians and path integrals, appropriately constructed for the case of free Fock spaces, by taking as a working hypothesis the notion that the latter describe reasonably well the non-perturbative Fock spaces of the large  $N$  matrix models. (Recently Migdal has addressed a similar question of construction of second quantized  $N = \infty$  effective planar theory [11], albeit within the framework of momentum loop equations.)

In this note we consider certain peculiar properties of single-species free Fock spaces. In particular we construct an analog of the free harmonic oscillator, and examine the resulting path integral. We resolve an apparent puzzle presented by the resulting path integral (namely that it describes a free bosonic particle, even though the underlying commutation relations are not bosonic), by pointing out the existence of novel bosonization as well as fermionization formulae and find out how ordinary Gaussian coherent states fit into this picture. We also examine, as an extra check, the quantum partition function for this system, and find that it corresponds to the well known result for the bosonic harmonic oscillator. We discuss generalizations of the above mentioned results to physically more relevant many-species free Fock spaces, having applications to realistic large  $N$  matrix models.

2. Single-species spaces: Consider the number operator  $N$ , defined by, as usual,  $[N, a^\dagger] = a^\dagger$ , for a slightly more general "deformed" commutation relation  $aa^\dagger - qa^\dagger a = 1$ . (Infinite statistics formally corresponds to the  $q \rightarrow 0$  limit.) Insert the following ansatz

$$N = \sum_{i=1}^{\infty} c_i (a^\dagger)^i a^i. \tag{2}$$

Then it is easy to show that

$$c_n = \frac{(1-q)^n}{1-q^n}. \quad (3)$$

Here  $(a^\dagger)^n |0\rangle = \sqrt{1(1+q)\dots(1+q+\dots+q^{n-1})} |n\rangle$ .

Let us examine the Hamiltonian  $H = N$ . We take this to define an analog of the harmonic oscillator Hamiltonian. Given the definition of the Gaussian coherent states

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle, \quad (4)$$

with the usual properties of the resolution of unity

$$\frac{1}{\pi} \int d^2z |z\rangle \langle z| = 1$$

and exponential overlap

$$\langle z'|z\rangle = \exp\left(-\frac{|z'|^2}{2} - \frac{|z|^2}{2} + \bar{z}'z\right),$$

it follows that  $\langle z'|N|z\rangle = \bar{z}'z \langle z'|z\rangle$ . Note that we cannot write  $\langle z'|N|z\rangle = N(\bar{z}', z) \langle z'|z\rangle$ , as the Gaussian coherent states are not eigenstates of  $a$ . Let us consider the propagator

$$\langle z_f | \exp(-iNt) | z_i \rangle.$$

Using, as usual, the resolution of unity and the exponential overlap of the Gaussian coherent states, we obtain

$$\langle z_f | \exp(-iNt) | z_i \rangle = \int Dz D\bar{z} \exp\left(i \int_0^t dt \left[ \frac{1}{2i} \left( \frac{d\bar{z}}{dt} z - \bar{z} \frac{dz}{dt} \right) - \bar{z}z \right]\right), \quad (5)$$

which corresponds to the well-known holomorphic path integral representation of the bosonic harmonic oscillator. We are now faced with a slight puzzle. Namely, it appears that there exists an infinite ambiguity in quantization: for any  $q$ , say, between zero and

one, the path integral of the analog of the free harmonic oscillator looks the same. If we insist that there should be no relations between creation and annihilation operators of different index (see the comment after eq. (12)), then  $q = 0$  case is singled out. Thus we concentrate on infinite statistics in what follows.

The resolution of the puzzle is presented by a peculiar looking bosonization formula.

Let

$$b = \sum_{i=1}^{\infty} \alpha_i (a^\dagger)^{i-1} a^i. \quad (6)$$

Then it can be shown that  $[b, b^\dagger] = 1$  if

$$\alpha_n = \pm\sqrt{n} \mp \sqrt{n-1}. \quad (7)$$

Moreover, given these values for  $\alpha_n$  it follows that the number operators corresponding to bosonic and infinite statistics are the same, namely  $N_b = N_i$  where,  $N_b = b^\dagger b$  and  $N_i = \sum_{i=1}^{\infty} (a^\dagger)^i a^i$ . Note also that the Gaussian coherent states which are eigenstates of  $b$  ( $b|z\rangle = z|z\rangle$ ) are eigenstates of a very complicated infinite combination of  $a$  and  $a^\dagger$ , defined by (6).

It is true as well, that the quantum partition function for a system of noninteracting harmonic oscillators  $Z = \text{Tr} \exp(-\beta H)$ , with the hamiltonian  $H = N$ , is given by the well known partition function of the ideal Bose-Einstein gas. The resulting distribution function is therefore Planckian.

It is possible to construct fermionic operators in a similar manner. Let

$$f = \sum_{i=1}^{\infty} \beta_i (a^\dagger)^{i-1} a^i. \quad (8)$$

Then it can be likewise shown that  $\{f, f^\dagger\} = 1$  provided

$$\beta_{2n-1} = \pm 1, \quad \beta_{2n} = -\beta_{2n-1} \quad (9)$$

for  $n = 1, 2, \dots$ . One can also check that  $f^2 = f^{\dagger 2} = 0$ ; hence these are genuine fermion variables. Note that the above formulae differ from the well-known Jordan-Wigner type of bosonization formulae.

As an aside we quote the most general expressions, that interpolate between formulae (6) and (8) and (7) and (9). Let

$$c = \sum_{i=1}^{\infty} \gamma_i (a^\dagger)^{i-1} a^i. \quad (10)$$

Then  $cc^\dagger - qc^\dagger c = 1$  if

$$\gamma_i = \mp \sqrt{1 + q + \dots + q^{i-2}} \pm \sqrt{1 + q + \dots + q^{i-1}}. \quad (11)$$

Here  $i = 2, 3, \dots$  and  $\gamma_1 = \pm 1$ .

Formulae (6)-(9) do not look that strange if we remember that any representation of the symmetric group can exist for infinite statistics. It is not surprising therefore that there exist realizations of symmetric and antisymmetric representations in terms of operators satisfying infinite statistics.

3. Many-species spaces: Can we generalize above formulae to the case of many-species spaces?

First of all we can immediately write down the general expression for the number operator  $N_i$  (see [8])

$$N_i = \sum_{j=1}^{\infty} \sum_{k_1, \dots, k_{j-1}} a_{k_1}^\dagger \dots a_{k_{j-1}}^\dagger a_i^\dagger a_i a_{k_{j-1}} \dots a_{k_1}. \quad (12)$$

Note that in the case of infinite statistics, this expression follows without any assumptions about relations between creation and annihilation operators of different index. This is not the case if  $a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}$  and  $q \neq 0$ .

What about the bosonization and fermionization formulae? If we demand

$$b_i = \sum_{j=1}^{\infty} \sum_{k_1 \dots k_{j-1}} \alpha_j a_{k_1}^\dagger \dots a_{k_{j-1}}^\dagger a_{k_{j-1}} \dots a_{k_1} a_i \quad (13)$$

then the following relation holds

$$b_i b_j^\dagger = \delta_{ij} a_j b_j^\dagger b_i a_i^\dagger, \quad (14)$$

which indeed reduces to the previously quoted expressions for  $i = 1$ . So the bosonization and fermionization formulae, defined through (13), are true only for  $i = 1$ .

Likewise, one can examine the quantum partition function for this generalized situation. This can be easily accomplished if we remember that there exist no relations between  $a_i$  and  $a_i^\dagger$ ; therefore the relevant combinatorial factor is Boltzmannian. The resulting quantum partition function is then (this result was established in conversations with V.P.Nair)

$$Z = (1 - \sum_k \exp(-\beta E_k))^{-1}. \quad (15)$$

Here  $E_k$  denotes the appropriate energy eigenvalue. In the  $k = 1$  limit we recover the previous result. Note that in this general case, the quantum partition function is not always well defined. In particular the large volume limit is singular and the expectation value for the number operator is not always positive definite. Note also that the purely classical partition function is of the Boltzmann type except for the omission of the Gibbs  $\frac{1}{N!}$  factor, and is therefore well defined (see [8]). In view of this observation and of Wigner's original work on random matrices, it seems that instead of the quantum partition function the appropriate physical quantity one should consider is the classical Gibbsian partition function, though appropriately "projected" [12].

Thus we clearly see that results valid for the single-species case are very special.

4. Conclusions: We have examined some basic properties of single-species free Fock spaces, in particular the path integral (and the quantum partition function) of an analog of the free harmonic oscillator and found that it corresponds to the usual path integral (and the quantum partition function) of a free bosonic particle. This slightly puzzling result was understood by utilizing a bit peculiar looking bosonization formula. (Similar fermionization formula was discussed as well.) These results do not generalize to many-species spaces indicating that properties of many-species free Fock spaces appear not to be straightforward extensions of the single-species case (as we are usually accustomed to). It is of utmost importance to understand these, before we can even attempt to address the physics behind realistic large N theories.

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