INTRODUCTION TO FIELD THEORY AND DISPERSION RELATIONS

by

R. Hagedorn

GENEVE
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Introduction to Field Theory and Dispersion Relations

FOREWORD

Sometimes it is useful, for didactical reasons, that a textbook or lectures notes be written by an outsider. This is the case if the intention is not to give a complete account on what has been achieved in a field of research but rather to provide an introduction for new-comers. Under these conditions the outsider is more likely to emphasize difficulties which are generally met by the beginner and which are frequently skipped by the experts, to whom they have become trivial in the course of time. The outsider, however, must fight them down and traces of this struggle will show up in his text.

This is one of the two excuses which I have for making these notes available.

They are based to a large extent on lectures which were given by H. Lehmann and W. Zimmermann at CERN in September 1959. The demand of many colleagues to have a written account of these lectures, and the fact that I had taken notes, is the second excuse for this publication. It would seem that this text consequently should have been called "lecture notes" and that it should bear the names of the lecturers. That this is not the case is due to the fact that the present "Introduction" grew to more than twice the size of the original lectures because of my additions (which are mainly collected in the appendices). Through these additions, however, the whole has been changed to such an extent (see above), that
it could not be called "lecture notes" any longer. I thank Dr. Lehmann (who, for the afore-mentioned reason, suggested that I should publish it under my name) and Dr. Zimmermann for their permission to publish it in the present form and for many discussions and suggestions during the course of preparation. I, of course, am entirely responsible for any faults or clumsy representation.

The aim of this Introduction is to acquaint those who know already a little about field theory with the more recent developments which lead to the dispersion relations. It would be an advantage for the reader to have read already some such textbooks as G. Wentzel's "Quantum Theory of Fields", Jauch and Rohrlich's "Theory of Photons and Electrons", or Schröder-Bethe-de Hoffmann's "Fields".

The most recent developments, connected with the "Mandelstam representation", have not been included. It should be easy, however, after having gone through these notes, to start reading the corresponding original paper. Since the reader will find there more than enough references to earlier work, no references are given in the present text, except at a few places where a lengthy calculation has been suppressed and reference is made to where details may be found.

In the present text only real scalar fields have been treated for simplicity. The metric of space time is sometimes taken as $x^2 = x^2 - x_0^2$, and sometimes as $x^2 = x_0^2 - x^2$.

I wish to express my appreciation to Mrs. Schibly, secretary of the Theoretical Study Division, for the careful typing of this report.

R.H.

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*) The metric is $x^2 = x_0^2 - x^2$ in all chapters down to D.

**) The metric is $x^2 = x_0^2 - \zeta^2$ from E. onwards.
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*) (The metric used - if relevant - is indicated at the heading of each appendix).
A) The Hamiltonian Formalism

The old fashioned field theory used the Hamiltonian formalism, which in principle should be able to describe the particle interaction but leads to certain difficulties. Take the case of a pseudoscalar meson interacting with a nucleon field:

\[
\begin{align*}
\left( \partial_{\mu} \gamma_{\mu} + M_0 \right) \psi_0 &= -i g_0 A_0 \gamma_5 \psi_0 , \\
\left( \Box - m_0^2 \right) A_0 &= \frac{i g_0}{2} \left[ \overline{\psi}_0, \gamma_5 \psi_0 \right] + \lambda_0 A_0^3 .
\end{align*}
\]

The term \( \lambda_0 A_0^3 \) is necessary for renormalisation.

To this add the commutation relations

\[
\{ A_0 (\vec{x}, t), A_0 (\vec{x}', t') \} = i \delta (\vec{x} - \vec{x}')
\]

\[
\left\{ \psi_0 (\vec{x}, t), \overline{\psi}_0 (\vec{x}', t') \right\} = \gamma_0 \vec{x}_\rho \delta (\vec{x} - \vec{x}')
\]

Here \( M_0, m_0, g_0, \lambda_0 \) are not yet the physical values of masses and coupling constants. How then can such a theory describe interaction of particles?

There exists an energy momentum operator \( P_{\mu} \) (which is a functional of the field operators) such that for instance

\[
\frac{\partial A_0}{\partial x_{\mu}} = -i \left[ P_{\mu}, A_0 \right]
\]

If this \( P_{\mu} \) has a meaning at all, one should have among the eigenvalues of

\[
P^2 = P_{\mu} P^\mu \]

a lowest one, which might be put \( = 0 \) (vacuum) and the next higher one should be \( m^2 \), namely the physical mass of the meson. But this makes sense only for free particles, since otherwise \( k_0^2 - k^2 = m^2 \) is not valid. We know, on the other hand, the equations of motion for free particles (denoted by \( \psi_{\text{in}} \) and \( A_{\text{in}} \)):

\[
\begin{align*}
\left( \partial_{\mu} \gamma_{\mu} + M \right) \psi_{\text{in}} &= 0 , \\
\left( \Box - m^2 \right) A_{\text{in}} &= 0
\end{align*}
\]

(plus the commutation relations).
Therefore, if the field equations with interaction should make sense, then there must be a connection between equations (I) and (II). This is established through renormalisation:

For \( t \to \pm \infty \) there is a relation between \( A_0(x) \) and \( A_{\text{out}}(x) \) and between \( \Psi_0(x) \) and \( \Psi_{\text{out}}(x) \).

To find this, one puts
\[
m^2 = m_0^2 + \delta m^2 \quad ; \quad M = M_0 + \delta M
\]
and considers \( m^2 \) and \( M \) as given, the others \( m_0, M_0, \delta m^2 \) and \( \delta M \) as undetermined (and in fact infinite) parameters.

Furthermore, one puts
\[
\Psi = Z_2^{-\frac{\lambda}{2}} \Psi_0 \quad ; \quad A = Z_3^{-\frac{\lambda}{2}} A_0
\]
\[
\gamma = Z_1^{-1} Z_2 Z_3^{-2} \gamma_0 \quad ; \quad \lambda = Z_4^{-1} Z_3^2 \lambda_0
\]
and writes (I) in these new variables. One obtains
\[
\begin{align*}
\{ (\partial_\mu \gamma_\mu + M) \Psi &= -i q Z_1 Z_2^{-1} \gamma_5 A \Psi + \delta M \Psi \\
(\Box - m^2) A &= \frac{i}{2} q Z_1^{-1} Z_3^{-1} [\bar{\Psi} \gamma_5 \Psi] - \delta m^2 A + \lambda Z_4 Z_3^{-1} A^3.
\end{align*}
\]

To this, one may now apply perturbation theory with the result that the operators \( A \) and \( \Psi \) have finite matrix elements in all powers of \( \gamma \) and \( \lambda \) and that for \( t \to \pm \infty \) they approach to the matrix elements of \( A_{\text{out}} \) and \( \Psi_{\text{out}} \) respectively.

This shows that – as far as perturbation theory is accepted as a possible tool – the equations (I) or (I') may be able to describe meson-nucleon interaction.

However, the convergence of perturbation theory has not been established, and is in fact very unlikely. Even the existence of solutions has not been proved (which is quite another thing, since the solutions need not necessarily be analytic functions of \( \gamma \) and \( \lambda \)). To make the disaster complete,
perturbation theory has in the case of strong interactions never been able to produce any agreement with experimental facts (in contradistinction to quantum electrodynamics, where it is a practical tool of great precision in spite of the fact that there also nothing is known concerning convergence and existence of solutions).

One should therefore try to find formulations which are nearer to measurable quantities and operate with mathematical objects which have a better chance to exist than the operators of interacting fields. It will turn out, however, that the latter concept cannot be completely excluded and that one is forced to consider again quantities of which one does not yet know whether they exist. The point is that one does not start from them, but rather from generally agreed principles. If then the derived or constructed objects should turn out to be non-existent, the starting principles must be wrong.

B) The New Formulation of Field Theory

i) The S-matrix theory.

We shall first develop a pure S-matrix formalism. The underlying idea is that for $t = + \infty$ and $t = - \infty$ all interacting particles are so far apart from each other that they can be treated as free, although physical, particles. They carry the observed masses, charges, etc. and obey the free field equations, wherein only physical quantities occur. The states at $t = \pm \infty$ are those physical states which can be experimentally prepared. They correspond, e.g. in a scattering experiment, to the situation long before and long after the collision. Since these states are by definition physical states, there is no doubt that there exists a unitary transformation $S$ between $t = -\infty$ and $t = + \infty$, whose matrix elements describe all the physical information which may be extracted from actual experiments.

We now construct such a theory from the following principles:
perturbation theory has in the case of strong interactions never been able to produce any agreement with experimental facts (in contradistinction to quantum electrodynamics, where it is a practical tool of great precision in spite of the fact that there also nothing is known concerning convergence and existence of solutions).

One should therefore try to find formulations which are nearer to measurable quantities and operate with mathematical objects which have a better chance to exist than the operators of interacting fields. It will turn out, however, that the latter concept cannot be completely excluded and that one is forced to consider again quantities of which one does not yet know whether they exist. The point is that one does not start from them, but rather from generally agreed principles. If then the derived or constructed objects should turn out to be non-existent, the starting principles must be wrong.

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We now construct such a theory from the following principles:
\[ \begin{align*}
\{ & \text{a) the S-matrix exists} \\
& \text{b) quantum theory is valid} \\
& \text{c) the theory is Lorentz invariant (inhom. L.-group)} \\
& \text{d) initial and final states are described by the free field equations}
\end{align*} \]

To this we add three technical points:

\[ \begin{align*}
& \text{a) we use the Heisenberg representation,} \\
& \text{b) we assume no bound states to exist,} \\
& \text{c) for simplicity, we assume one single kind of self-interacting, scalar particles.}
\end{align*} \]

From (A) follows that there are two sets of operators \( A_{in} \) and \( A_{out} \) such that (we write everything for \( A_{in} \) only, similar formulae hold for \( A_{out} \)):

\[ \begin{align*}
& \left( \Box - m^2 \right) A_{in} (x) = 0 \\
& \left[ A_{in} (x', t), A_{in} (x'', t) \right] = i \delta (x' - x'') \\
\end{align*} \]  \( (1) \)

Expanding \( A_{in} (x) \) in momentum space and imposing \( \left( \Box - m^2 \right) A_{in} = 0 \) yields (the \( \delta \)-function inside the integral allows now \( \tilde{A}_{in} (k) \) to be any reasonable function of \( k \) without restrictions coming from the Klein-Gordon equation):

\[ \begin{align*}
A_{in} (x) = & \left( \frac{4}{2 \pi} \right)^{3/2} \int d^4 k e^{ik \cdot x} \delta (k^2 + m^2) \tilde{A}_{in} (k) \\
\end{align*} \]  \( (2) \)

where \( \tilde{A}_{in} (k) \) can be split into positive and negative frequency parts (absorption and creation operators respectively)

\[ \tilde{A}_{in} (k) = \Theta (k_0) A_{in} (k) + \Theta (-k_0) A_{in}^* (-k) \]  \( (3) \)

\[ \text{See (A2, 23')}. \]
The existence of these operators is postulated as they describe observable things, free particles. Note, however, that these particles are not bare particles in the sense of perturbation theory. The mass in (1) is the physical mass.

The $S$-matrix is defined now by means of in and out-states:

$$
\phi_{in}^{k_1 \ldots k_n} = a_{in}^{* k_1} \ldots a_{in}^{* k_n} \phi_{0}^{k_1 \ldots k_n}
$$

$$
S_{k_1 \ldots k_n | p_{1} \ldots p_{m}} = (\phi_{out}^{k_1 \ldots k_n} | \phi_{in}^{p_1 \ldots p_m})
$$

It follows

$$
\phi_{in} = S \phi_{out} ; \quad A_{out} = S^{+} A_{in} S
$$

In the old $S$-matrix theory used together with perturbation treatment, one defines the matrix elements by

$$
S_{fi} = (\text{final} | S | \text{initial})
$$

and this seems to define an $S$-matrix different from the above one. That is not the case. One has to notice that the definition in the old theory refers to the interaction representation, whereas we here deal with Heisenberg representation only. The $S$-operator is of course the same in both cases. See p. 20 and following pages.

Completeness of the description is achieved by assuming that the two sets

$$
\left\{ \phi_{in}^{k_1 \ldots k_n} \right\} \quad \text{and} \quad \left\{ \phi_{out}^{k_1 \ldots k_n} \right\}
$$

each form a complete basis of the Hilbert space. The completeness is a postulate which follows from the empirical fact that all states of non-interacting particles can obviously be described by such Hilbert vectors. This implies of course that one introduces a separate field for each kind of particle considered. There is no distinction between elementary and composite particles. Bound states must therefore be described by new field operators (which however can be constructed from the operators of the bounded particles).
The interaction of fields shall not be introduced as in the old theory by assuming field equations to be known. We rather express everything in the quantities we really know (free fields and S-matrix).

Now we need the concept of a complete system of operators:

It can be shown that the following three conditions are equivalent:

(I) a set \( \{ A(x) \} \) of operators is complete, if and only if every operator \( B \) can be approximated by polynomials of operators of the set \( \{ A(x) \} \); symbolically:

\[
B = \sum_{n=0}^{\infty} \int b_n(x_1, \ldots, x_n) A(x_1) \ldots A(x_n) \, d^nx_1 \ldots d^nx_n
\]

(II) a set \( \{ A(x) \} \) is complete if there exists no operator \( C \) which commutes with all \( A(x) \) of the set, except a multiple of the unit operator,

(III) a set \( \{ A(x) \} \) is complete if it is possible to transform any given state \( \phi \) into a given state \( \phi' \) by means of applying

\[
\sum_{n=0}^{\infty} \int b_n(x_1, \ldots, x_n) A(x_1) \ldots A(x_n) \, d^nx_1 \ldots d^nx_n \quad \text{on} \quad \phi.
\]

The proof of (I)-(III) is not trivial and must be omitted here. It seems, however, plausible if a set \( \{ A(x) \} \) of operators suffices to "build up" the whole Hilbert space then also all operators can be built from this set - that is the content of (III).

Now (III) is evidently fulfilled if \( \{ \phi_{in}^{k_1 \ldots k_n} \} \) is complete since then any given state \( \phi \) can be represented by a superposition of such states. It can be transformed into any other given state \( \phi' \) by applying first a certain combination of destruction operators \( a_{kin}^{k_1 \ldots k_n} \) such that \( \phi \to \phi_0 \) (vacuum) and then by applying another combination of creation operators \( a_{kin}^+ \) which makes \( \phi_0 \to \phi' \).
If the set of the \( \{ a_{kin} \} \) is complete, then so also is the set \( \{ A_{in}(x) \} \).
Hence every operator - in particular the \( S \)-operator - can be written as an expansion (power series) in the operators \( A_{\vec{n}}(x) \):

\[
S = 1 + \sum_{n=1}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int d^4x_1 \ldots d^4x_n \ \tau_n'(x_1 \ldots x_n) : A_{in}(x_1) \ldots A_{in}(x_n) : \quad (5)
\]

Supposing now the \( S \)-operator to be given, can we determine the functions \( \tau_n' \)?

That is not the case as one sees if one goes to the momentum space:

put

\[
A_{in}(x) = \left( \frac{\lambda}{2\pi} \right)^{n/2} \int d^4k \ \delta(k^2 + m^2) \ \widetilde{A}_{in}(k) e^{ikx} \quad (2)
\]

\[
\tau_n'(x_1 \ldots x_n) = \left( \frac{n}{2\pi} \right)^{5n/2} \int d^4k_1 \ldots d^4k_n \ \delta(\Sigma k_i) \ \widetilde{\tau}_n'(k_1 \ldots k_n) e^{i \sum_{i=1}^{n} k_i x_i} \quad (6)
\]

insert in the integral in (5) and use

\[
\left\{ d^4x e^{i(k-k')x} = (2\pi)^4 \delta(k-k') \right. 
\]

with the result

\[
S = 1 + \sum_{n=1}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int d^4k_1 \ldots d^4k_n \ \widetilde{\tau}_n'(k_1 \ldots k_n) \delta(\Sigma k_i) \prod_{i=1}^{n} \delta(k_i^2 + m^2) \times \\
\times : \widetilde{A}_{in}(k_1) \ldots \widetilde{A}_{in}(k_n) : \quad (7)
\]
The expression (6) for \( \mathcal{T}' \) has been used because of the translation invariance: if \( S \) is translation invariant, i.e. if \( \mathcal{S} = \mathcal{S} \circ e^{-iP_\mu a_\mu} \), then from (5) follows

\[
\mathcal{S} = \mathcal{S} \circ e^{-iP_\mu a_\mu} = 1 + \sum_{\pi_1, \ldots, \pi_n} \int dx_1 \ldots dx_n \mathcal{T}'(x_1, \ldots, x_n) : e^{-iP_\mu A_\mu(x_1) \ldots A_\mu(x_n)} e^{iP_\mu} : \]

but

\[
\frac{\partial A_\mu}{\partial x_\mu} = -i [P_\mu, A_\mu] \]

implies that

\[
e^{-iP_\mu A_\mu(x)} e^{iP_\mu} = A_\mu(x+a) ,
\]

hence

\[
\mathcal{S} = 1 + \sum_{\pi_1, \ldots, \pi_n} \int dx_1 \ldots dx_n \mathcal{T}'(x_1, \ldots, x_n) : A_\mu(x_1+a) \ldots A_\mu(x_n+a) : \]

With \( x_1+\alpha \Rightarrow x_1' \) the old form (5) reappears with

\[
\mathcal{T}'(x_1-a, x_2-a, \ldots, x_n-a) \quad \text{replacing} \quad \mathcal{T}'(x_1, \ldots, x_n) .
\]

Hence \( \mathcal{T}' \) must be itself translation invariant, i.e. it can depend only on the differences of the four vectors \( x_1 \ldots x_n \). Replacing \( x_1 \) by \( x_1+\alpha \) in (6) changes the exponential into

\[
e^{i \sum k_\alpha x_\alpha + i \alpha \sum k_\alpha} .
\]

The \( \delta(\sum k_\alpha) \) makes \( \sum k_\alpha = 0 \) and thus guarantees the translation invariance.

Now one sees that \( \mathcal{T}'(x_1 \ldots x_n) \) cannot be uniquely defined simply by \( \mathcal{S} \) alone, since from (7) follows that the functions \( \mathcal{T}'(k_1 \ldots k_n) \) are defined only on the energy shell and there only for such values \( k_1 \ldots k_n \) that \( \sum k_\alpha = 0 \). Outside this region one may change them in any arbitrary way without effect on \( \mathcal{S} \). That implies that there is an infinite manifold of functions

\[
\mathcal{T}'(x_1 \ldots x_n) = \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} \int \delta(\sum k) \mathcal{T}'(k_1 \ldots k_n) e^{i \sum k_\alpha x_\alpha} d^4 k_1 \ldots d^4 k_n
\]

belonging to the infinite manifold of extrapolations of \( \mathcal{T}' \) off the energy shell - all of them leading to the same \( \mathcal{S} \)-operator.
We have expressed the S-operator by the $A_{\text{in}}$ fields (we could have done so also with the out-fields); these operators ($S$ and the set $\{A_{\text{in}}(x)\}$) are supposed to be sensible quantities. The infinitely many functions $\tau_n(x_1 \ldots x_n)$ describe the interaction - but they are not uniquely determined by the S-matrix.

ii) Construction of interpolating field operators from a given S-matrix.

The question is now whether such a scheme is sufficient. This seems not to be the case for the following reasons:

- so far it has not been possible to formulate the principle of causality in the frame of pure S-matrix theory;
- we believe that local commutativity of field operators is equivalent to causality; it is called "micro-causality" and represents up to now the only known means to build in causality;
- we are therefore forced to construct (interacting) field operators in order to be able to add to our scheme the principle of (micro-) causality.

Whether such interpolating field operators really exist is an open question.

We shall construct them formally as follows:

a) we express $A_{\text{out}}$ as a function of $A_{\text{in}}$

b) we generalise this expression to a definition of $A$, the interpolating field.

a) The S-operator is defined by $A_{\text{out}}(x) = S^+ A_{\text{in}}(x) S$ to this we add

$$A_{\text{in}}(x) - S^+ S A_{\text{in}}(x) (=0)$$

and obtain

$$A_{\text{out}} = A_{\text{in}} + S^+ [A_{\text{in}}, S].$$

We have to evaluate $[A_{\text{in}}, S]$. The calculation is done in Appendix 1. The result is: [For the function $\Delta(z - \xi)$ see Appendix 2].
\[
\left[ A_{\text{in}}(\bar{z}), S \right] = \int d^4 \xi \Delta (\bar{z} - \xi) \sum_{n} \frac{(-i)^n}{n!} \times \\
\times \left\{ d^4x_1 \cdots d^4x_n \left( \Gamma_{n+1}^{t} (\xi, x_1, \ldots, x_n) : A_{\text{in}}(x_1) \cdots A_{\text{in}}(x_n) : \right) \right\}
\]

or in short
\[
\left[ A_{\text{in}}(\bar{z}), S \right] = \int d^4 \xi \Delta (\bar{z} - \xi) i \frac{\delta S}{\delta A_{\text{in}}(\xi)}
\]  \hspace{1cm} (8)

therefore
\[
A_{\text{out}}(\bar{z}) = A_{\text{in}}(\bar{z}) + \int d^4 \xi \Delta (\bar{z} - \xi) i S + \frac{\delta S}{\delta A_{\text{in}}(\xi)}
\]  \hspace{1cm} (9)

It must be noted that the expression \( \frac{\delta S}{\delta A_{\text{in}}(\xi)} \) itself is not well defined, however, in the integral together with the \( \Delta \)-function it makes sense.

We introduce now the definition
\[
j(\xi) \equiv i S + \frac{\delta S}{\delta A_{\text{in}}(\xi)}
\]  \hspace{1cm} (10)

and obtain
\[
A_{\text{out}}(x) = A_{\text{in}}(x) + \int d^4 \xi \Delta (x - \xi) j(\xi)
\]  \hspace{1cm} (11)

Also \( j(\xi) \) is not uniquely defined, since it contains \( \frac{\delta S}{\delta A_{\text{in}}(\xi)} \) and from the formula preceding (8) one sees that this is not defined since in contra-distinction to \( S \) itself it depends on how \( \Gamma_{n+1}^{t} (\xi, x_1, \ldots, x_n) \) is extrapolated off the energy shell.

However, the integral is well defined since \( \square \) see Appendix [3]
\[ \Delta(x-\xi) = \frac{i}{(2\pi)^{3/2}} \int e^{ik(x-\xi)} \delta(k^2 + m^2) \varepsilon(k) \, d^4k \quad \varepsilon(k) = \begin{cases} 1 & k_o > 0 \\ -1 & k_o < 0 \end{cases} \]

By this a further \( \delta(k^2 + m^2) \) is introduced and the integral in (11) becomes again independent of the extrapolation of the \( \mathcal{T} \)-functions. Hereby we have achieved the first step a).

b) We now introduce the interpolating field \( A(x) \) by

\[ A(x) = A_{\text{in}}(x) - \int d^4\xi \Delta_R(x-\xi) j(\xi) \quad \Delta_R(x) = -\Theta(x_o)\Delta(x) \quad j(\Theta(x_o)) = \begin{cases} 1 & x_o > 0 \\ 0 & x_o < 0 \end{cases} \]

We must show two things:

a) that the asymptotic condition holds

b) that this definition is related to the old-fashioned Hamiltonian field theory.

\[ \lim_{x_o \to -\infty} A(x) = A_{\text{in}}(x) \]

since the integral vanishes because of \( \Delta_R \).

\[ \lim_{x_o \to +\infty} A(x) = A_{\text{in}}(x) + \int d^4\xi \Delta(x-\xi) j(\xi) = A_{\text{out}} + A_{\text{out} - A_{\text{in}}} = A_{\text{out}}(x) \]

because of (11).

This is an intuitive reasoning, since we have assumed that
\[ \lim_{x_0 \to \pm \infty} \int d^4 \xi \Delta_R(x - \xi) \hat{j}(\xi) = \int d^4 \xi \lim_{x_0 \to \pm \infty} \Delta_R(x - \xi) \hat{j}(\xi) = \begin{cases} 0 & \text{for } x_0 \to -\infty \\ A_{out} - A_{in} & \text{for } x_0 \to +\infty \end{cases} \]

That is not obvious and in fact it holds only in a restricted sense, namely that of "weak convergence" i.e. for matrix elements of the operators, after these have been folded with suitable normalizable functions [see p. 22].

We look for a differential equation for \( A(x) \):
\[
(\Box - m^2) A(x) = (\Box - m^2) A_{in}(x) - \int d^4 \xi (\Box - m^2) \Delta_R(x - \xi) \hat{j}(\xi)
\]
\[
= 0
\]
\[
(\Box - m^2) \Delta_R(x - \xi) = -\delta(x - \xi) \]
[see (A3,13)]

hence
\[
(\Box - m^2) A(x) = \hat{j}(x)
\] (13)

That is what one would write in the old formalism. The difference is, however, that here we have this equation not as an equation of motion which has to be solved but as an identity following from the definitions.

Furthermore, it is clear that neither \( A(x) \) nor \( j(x) \) are uniquely determined in contradistinction to the integral in (11) which was well defined. The reason is that if \( j(x) \) is not uniquely determined \( A(x) \) cannot be either because of the differential equation (13). This can be seen more directly:

Whereas the \( \Delta(x) \)-function in (11) can be written
\[
\Delta(x) = \frac{-i}{(2\pi)^3} \int e^{ikx} \frac{\delta(k^2 + m^2)}{k^2 + m^2} \epsilon(k) \, d^4k
\]
and the \( \delta(k^2 + m^2) \) makes the integral in (11) unique, this is no longer true in (12), since the \( \Delta_R(x) \) can be written
\[
\Delta_R(x) = \lim_{\varepsilon \to 0} \left( \frac{i}{2\pi} \right)^4 \int \frac{d^4 k}{(k+\varepsilon)^2 + m^2} \; \varepsilon = (\vec{\varepsilon}, i\varepsilon_0); \; \varepsilon_0 > 0
\]

There is no longer a \( \delta \)-function which makes the extrapolation of the \( \mathcal{T}' \) off the energy shell irrelevant. Hence, the non-uniqueness of \( j \) is carried over to the interpolating field \( A(x) \). In other words:

Even if the S-matrix were completely given, then still \( j(x) \) as well as \( A(x) \) would not be uniquely determined. There exist infinitely many such \( j(x) \) and \( A(x) \) satisfying the same asymptotic conditions and yielding the same S-matrix elements.

**Causality.**

We now impose the condition

\[
\left[ A(x), A(x') \right] = 0 \quad \text{for} \quad x-x' \quad \text{spacelike}
\]

Whether this is really causality in the sense one would like it, is an open question, since "the sense one would like it" is extremely difficult to define. Presently it seems at least the only causality condition from which one really is able to draw conclusions on the structure of the S-matrix.

In drawing conclusions on the structure of the S-matrix using (14) one has to assume that such an S-matrix does exist. That is not evident and has so far not been proven. The point is:

a) we started from a given S-matrix and wrote it with the \( \mathcal{T}' \)-functions as a functional of the set \( \{ A_{\text{in}}(x) \} \);
b) we found that the \( \mathcal{T}' \)-functions are not uniquely defined;
c) we defined operators \( A(x) \) by means of the \( \mathcal{T}' \)-functions;
d) we impose by (14) a very strong condition on the \( A(x) \). This condition will

a) restrict the possible extrapolations of \( \mathcal{T}' \) off the energy shell
b) possibly (and probably) restrict the \( \mathcal{T}' \) on the energy shell.
It may therefore happen that a given $S$-matrix cannot be represented by $\mathcal{C}'$-functions if causality is imposed. If the worst happens, then it could be that there is no $S$-matrix at all which can be written by means of the $\mathcal{C}'$-functions with causality imposed. Nothing is known presently about that:

...the compatibility of the principles has not yet been proven...

Of course, $S$ has to be unitary too, but that is a simpler condition which, however, has not yet been exhausted completely in modern field theory.

C) Consequences of the principles in terms of properties of the field operators.

We have seen that we can formally construct interpolating field operators from pure $S$-matrix theory. However, the question remains entirely open, whether for a given non-trivial $S$-matrix causal interpolating fields exist according to the principles used above. The proof of the existence (or non-existence) seems so difficult that it may be a long time coming. Therefore it may be more fruitful for the moment to assume the existence of interpolating fields, to state the axioms they should follow and to deduce their further properties. This is the aim of the following section.

We write down now the principles which we used above together with some others following from physical common sense. We shall not attempt to answer the question whether these principles are compatible with each other and/or whether they are independent.

For simplicity, we still consider a neutral scalar particle of mass $m$.

1) Principle of invariance:

There exists a unitary representation $U(L,a)$ of the orthochronous Lorentz group (i.e. no time-inversion; otherwise $U$ is antiunitary) such that
It may therefore happen that a given S-matrix cannot be represented by $\zeta'$-functions if causality is imposed. If the worst happens, then it could be that there is no S-matrix at all which can be written by means of the $\zeta'$-functions with causality imposed. Nothing is known presently about that:

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1) Principle of invariance:

There exists a unitary representation $U(L,a)$ of the orthochronous Lorentz group (i.e. no time-inversion; otherwise $U$ is antiunitary) such that
\[
\phi' = U(L, a) \phi \\
A(x + a) = U^*(L, a) A(x) U(L, a) \equiv A'(x) \\
(\phi', A' \Psi') = (\phi, A \Psi)
\]

(15)

Taking $L$ and $a$ as infinitesimal Lorentz transformations and translations respectively, one finds that there are hermitian operators $M_{\mu \nu}$ and $P_\mu$ respectively, which represent angular and linear four-momenta (see Jauch and Rohrlich, Theory of photons and electrons, p. 10-14). In particular one has for translations

\[
U(a) = e^{i P_\mu a_\mu}; \quad \frac{\partial A}{\partial x_\mu} = -i \left[ P_\mu, A \right]
\]

(16)

We shall frequently use this equation in the form

\[
A(x+y) = e^{-i P_y} A(x) e^{i P_y} \quad \left( P_y = P_\rho y_\rho \right)
\]

(17)

2) Spectrum conditions:

a) there exists one single invariant state $\Omega$ describing the vacuum:

\[
U(L, a) \Omega = \Omega
\]

(18)

from $U(a) = 1 + \sum \frac{i n}{n!} P_\mu a_\mu$ and $U(a) \Omega = \Omega$ for any $a$ follows

\[
P_\mu \Omega = 0
\]

(19)

b) the eigenvalues of $P_0$ and of $-P^2$ are positive or zero.

c) the discrete eigenvalues of $-P^2$ are $0$ and $m^2$, where the latter belongs to one-particle states.

3) Causality:

\[
[A(x), A(y)] = 0 \quad \text{for} \quad x - y \quad \text{spacelike.}
\]
4) Completeness:

\[
\{ A(x) \} \text{ forms a complete set of operators, i.e. any operator must be a functional of the } A(x) \text{ and any state } \emptyset \text{ can be reached from any other state:}
\]

\[
\phi = \sum \int f(x_1, \ldots, x_n) A(x_1) \ldots A(x_n) \, dx_1 \ldots dx_n \cdot \psi
\]

by a suitable choice of the functions \( f(x_1, \ldots, x_n) \).

5) Asymptotic condition:

a) \( A_{\text{in}}(x) = A(x) + \int A_R(x-x')j(x')dx' \)
\( A_{\text{out}}(x) = A(x) + \int A_L(x-x')j(x')dx' \)
\( j(x) = (\Box - m^2)A(x) \) (definition of \( j(x)!! \))

b) Completeness: \( \{ A_{\text{in}}(x) \} \) and \( \{ A_{\text{out}}(x) \} \) form each a complete set of operators as in 4). This guarantees that \( A(x), A_{\text{in}}(x), A_{\text{out}}(x) \) can be developed each with respect to each other.

c) Commutation relation:

\[
\left[ A_{\text{in}}(x), A_{\text{out}}(y) \right] = i \Delta(x-y)
\]

Discussion of these principles

i) Conditions on \( j(k) \).

From 5) it follows that \( (\Box - m^2)A_{\text{in}}(x) = 0 \). If one writes 5a) in momentum space, then with

\[
A_{\text{in}}(x) = \left( \frac{i}{2\pi} \right)^{3/2} \int \delta(k^2 + m^2) \tilde{A}_{\text{in}}(k) e^{ikx} \, dk
\]
\[
A(x) = \left( \frac{i}{2\pi} \right)^{3/2} \int \tilde{A}(k) e^{ikx} \, dk
\]
\[
\tilde{j}(x') = \left( \frac{i}{2\pi} \right)^{3/2} \int \tilde{f}(p) e^{ipx'} \, dp
\]
and (A3,11) one obtains

\[ \delta \left( k^2 - m^2 \right) \tilde{A}_m (k) = \tilde{A}(k) + \left\{ \mathcal{P} \frac{1}{k^2 + m^2} - i \pi \delta \left( k^2 + m^2 \right) \right\} \tilde{\mathcal{F}}(k) . \]  

(20)

\[ \mathcal{F}(k) \] has to have some properties in order that the product of it with the curled bracket exists: It must be finite on the energy shell because of the \( \delta \)-function and the principal value must exist also (it does if e.g. \( \partial \mathcal{F} / \partial k^2 \) is continuous on \( k^2 = m^2 \)).

ii) Uniqueness of the vacuum state, one particle states.

From 5a) follows in Fourier space

\[ \tilde{\mathcal{F}}(k) = - \left( k^2 + m^2 \right) \tilde{A}(k) . \]

The energy-momentum operator has (by the definition of translation invariance) the property:

\[ \left[ P_\mu, \tilde{A}(p) \right] = p_\mu \tilde{A}(p) . \]

Therefore

\[ \left[ P_\mu, \mathcal{F}(p) \right] = p_\mu \mathcal{F}(p) \]

and with (20)

\[ \left[ P_\mu, \tilde{A}_m (p) \right] = p_\mu \tilde{A}_m (p) \]

That is: the operator \( P_\mu \) has the same effect on the interpolating as on the asymptotic fields. Therefore it has throughout the same physical meaning and there exists nothing like a \( P_\mu \text{in} \) or \( P_\mu \text{out} \) besides it — there is only the one four-vector (operator) \( P_\mu \).

It is therefore possible to define the vacuum \( \Omega \) uniquely by \( P_\mu \cdot \Omega = 0 \) and there is then no sense in distinguishing \( \Omega \text{in} \) and \( \Omega \text{out} \).
The same applies now to the one-particle states. The operator
\[-p^2 = -p^2_{\mu}\]
has according to the spectrum condition eigenvalues 0, \(m^2\) and then a continuum from \((2n)^2\) to \(\infty\). By postulate, there is only one state \(\Omega\) with \(-p^2\Omega = 0\). Since the eigenvalue \(m^2\) is discrete, there is a subspace \(H_1 \subset H\) of states
\[-p^2 = m^2\] belonging to the eigenvalue \(m^2\). This space is Lorentz invariant since \(p^2\) commutes with \(U(L,a)\) as an operator which is invariant by definition. Let us take such states \(\phi_{in} = a^*_{in}(-p)\Omega\) \(= \tilde{A}_{in}(p)\Omega\) (since \(a(p)\Omega = 0\))
\[
\phi_{out}^b = b^*_{out}(-p)\Omega = \tilde{A}_{out}(p)\Omega.
\]

Then
\[
P_{\mu} \phi_{in}^b = P_{\mu} \tilde{A}_{in}(p)\Omega = \tilde{A}_{in}(p) P_{\mu} \Omega + \left[ P_{\mu}, \tilde{A}_{in}(p) \right] \Omega = 0 + P_{\mu} \tilde{A}_{in}(p) \Omega = P_{\mu} \phi_{out}^b.
\]

Since we have assumed scalar neutral particles there is no other degree of freedom for one-particle states except the four momentum. This is seen to be the same for the \(c^p\) and \(b^p\), hence (apart from a phase factor) these two are equal:
\[
\phi_{in}^p = \phi_{out}^p = \phi^p
\]
(i.e. the subscripts will be omitted as meaningless).

Creation and annihilation operators, complete systems of states.

We take a complete set of solutions \(f_\alpha(x)\) of the homogeneous Klein-Gordon equation (for this and the following see Appendix 2) and write
\[ A_m(x) = \sum A_{m,\alpha} f_{\alpha}(x) + A_{m,\alpha}^* f_{\alpha}^*(x) \]  

where the operator properties are in the \( A_{\alpha} \) and \( A_{\alpha}^* \) and the \( f \)'s are ordinary functions. The operator coefficients are found to be

\[ A_{m,\alpha} = i \int dx^e \left[ f_{\alpha}^*(x) \frac{\partial A_m(x)}{\partial x^e} - A_m(x) \frac{\partial f_{\alpha}^*(x)}{\partial x^e} \right] \equiv i \int dx^e f_{\alpha}^*(x) \frac{\partial}{\partial x^e} A_m(x) \]  

\[ \text{See (A2,5), (A2,9).} \] The \( A_{m,\alpha} \) are constant in time. The commutation relations (5c) imply then [see (A3,20), (A3,22)]

\[ [A_{m,\alpha}, A_{m,\beta}^*] = \delta_{\alpha\beta}; \quad [A_{m,\alpha}, A_{m,\beta}] = [A_{m,\alpha}^*, A_{m,\beta}^*] = 0 \]

from which follows

\[ A_{m,\alpha}^* \text{ creates a particle in a state described by } f_{\alpha}(x) \]

\[ A_{m,\alpha} \text{ annihilates } f_{\alpha}(x) \]

Therefore: \( \emptyset = A_{m,\alpha}^* \Omega \) etc. The complete system of states is then given by

\[ \{ \phi^{(m)}_{\alpha_1,\alpha_2,\ldots,\alpha_h} \} = \left\{ [n_1! \cdots n_h!]^{\frac{1}{2}} A_{m_1,\alpha}^* A_{m_2,\alpha_1}^* \cdots A_{m_h,\alpha_h}^* \Omega \right\} \]

where the factor in front of the \( A^* \)'s is for normalizing: \( n_1, \ldots, n_h \) are the numbers of particles in identical states.

With (22), (24) and (26) it follows that

\[ (\Omega, A_m(x) A_{m,\alpha}^* \Omega) = \sum \rho \left( \Omega, (A_{m,\rho} f_{\rho} + A_{m,\rho}^* f_{\rho}^2) A_{m,\alpha}^* \Omega \right) = f_{\alpha}(x) \]
or shorter

\[
\left( \mathcal{Q}, A_{\text{in}}(x) \phi_{\text{in}}^{(\alpha)} \right) = \ell_\alpha(x)
\]

(27)

Since the system \( \{ A_{\text{in}}(x) \} \) is supposed to be a complete operator system, the system of states (26) is also complete.

The same holds then for the out-states.

iv) **The S-matrix**

Remember that we are in the Heisenberg picture, where the states are fixed and the operators change. Therefore \( \mathcal{Q}_{\text{in}} \) and \( \mathcal{Q}_{\text{out}} \) are not states which go over one into the other as time proceeds from \(-\infty\) to \(+\infty\); rather the system is described once and for ever either by \( \mathcal{Q}_{\text{in}} \)-states or by \( \mathcal{Q}_{\text{out}} \)-states. These are the eigenstates of asymptotic limits of changing operators. Since these operators are different for \( t \to +\infty \) and \( t \to -\infty \), the corresponding states \( \mathcal{Q}_{\text{out}} \) and \( \mathcal{Q}_{\text{in}} \) differ also.

The S-matrix describes the entity of all possible results of measurements at \( t \to +\infty \) when the state at \( t \to -\infty \) has been given. Let \( \{ \mathcal{Q}_{\text{out}}(\alpha) \} \) and \( \{ \mathcal{Q}_{\text{in}}(\alpha) \} \) be the corresponding complete systems *) , then according to the rules of quantum mechanics one has to expand the given state \( \mathcal{Q}_{\text{in}}(\alpha) \) with respect to the eigenstates of the operators which belong to the next measurement, namely the out-operators:

\[
\phi_{\text{in}}(\alpha) = \sum_{(\beta)} S(\beta|\alpha) \phi_{\text{out}}(\beta)
\]

(28)

*) \((\alpha) \equiv (\alpha_1, \alpha_2, \ldots, \alpha_r) \) with arbitrary \( r=0,1,2,\ldots, \infty \). The \( \{ \mathcal{Q}_{\text{out}}(\alpha) \} \) are a countable set, as they are by construction a countable set of countable sets.

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and the absolute squares of the coefficients $|S_{(i)}(\alpha)|^2$ give the probability to find the system in the state $\psi_{\text{out}(\lambda)}$ when a measurement "$A_{\text{out}}$" is performed and the system was in the state $\psi_{\text{in}(\alpha)}$ before that measurement.

From (28) follows

$$S_{(j)}(\omega) = (\phi_{\text{out}(\rho)} | \phi_{\text{in}(\omega)})$$

(29)

and (28) can be written in abstract form

$$\psi_{\text{in}} = S \psi_{\text{out}}$$

from which follows

$$A_{\text{in}} = S A_{\text{out}} S^*$$

(28')

With (28') we may rewrite (29) as follows

$$S_{(j)}(\omega) = (\phi_{\text{out}(\rho)} , S \phi_{\text{out}(\omega)}) = (\phi_{\text{in}(\rho)} , S \phi_{\text{in}(\omega)})$$

(30)

This form is more familiar. In the usual formulation the $S$-matrix element is written

$$S_{\phi_i} = (\phi_f , S \phi_i)$$

where $f$ is the final state, $i$ the initial state. The underlying picture is however different, since one usually formulates the $S$-matrix in the interaction picture where the states are not constant in time. Then $\psi_1$ is a state at (loosely speaking) $t = -\infty$ and $\psi_1(+\infty) = S \psi_1$ is that state which has developed out of $\psi_1$ in the course of time for $t \rightarrow +\infty$. The quantum mechanical rule is then: develop $\psi(+\infty)$ into a series with respect to the eigenstates of the observables at $t=\infty$, the coefficients will then be the probability amplitudes. Now the observables obey in the interaction picture the free field equations, so they are essentially the same as our in-field, consequently, their eigenstates at $t=\infty$ are the same as at $t=-\infty$ and in fact they are essentially our $\psi_{\text{in}}$-states. That all holds only in a loose sense but the point is that the states $\psi_1$ and the states $\phi_f$ belong to one and the same orthogonal system. Hence one has

$$\psi(+\infty) = \sum S_{\phi_i} \phi_f$$

with

$$S_{\phi_i} = (\phi_f , S \phi_i)$$
which therefore is strongly analogous to the above formulation. One has to avoid one confusion: one must keep in mind that if one wants to relate our in-states to the \( \phi_1 \) in the usual formulation, then the states \( \phi_1(\pm \infty) \) have nothing to do with our out-states. It should be mentioned, however, that the interaction picture does not exist, since it leads to internal contradictions.

v) Details on the asymptotic condition.

The asymptotic conditions 5a) say that – if the integrals exist – the limit of \( \mathcal{A}(x) \) for \( t \to \pm \infty \) is \( \mathcal{A}_{\text{out}}(x) \).

This can be stated much more rigorously by writing

\[
\begin{align*}
\mathcal{A}_{\text{in}}(x) &= \sum \mathcal{A}_{\text{in}, \alpha} f_{\alpha}^* + \mathcal{A}_{\text{in}, \alpha} f_{\alpha}^* \\
\mathcal{A}(x) &= \sum \mathcal{A}_\alpha(x_0) f_{\alpha}^* + \mathcal{A}_\alpha(x_0) f_{\alpha}^*
\end{align*}
\]

(31)

Whereas the \( \mathcal{A}_{\text{in}, \alpha} \) are constant, the \( \mathcal{A}_\alpha(x_0) \) depend on time, and the coefficients are calculated (see (A2,13) and (A2,9)) by

\[
\mathcal{A}_\alpha(x_0) = \frac{i}{\hbar} \int_{x_0}^{x_0} f_{\alpha}^*(x) \frac{\partial}{\partial x_0} \mathcal{A}(x) \, dx_0 \equiv (f_{\alpha}, \mathcal{A})
\]

We may now multiply in this sense the whole asymptotic condition by a normalized solution \( f(x) \) of the homogeneous Klein–Gordon equation: 5a) leads then to

\[
(f, \mathcal{A}_{\text{in}}) = (f, \mathcal{A}) + \int d^s x' \, j(x') (f, \Delta_R(x-x'))
\]

We may replace \( \Delta_R \) by \( \Delta \), if we write in the last integral

\[
\int d^s x' \, j(x') (f, \Delta_R(x-x')) \equiv \int d^s x' \, j(x') (f, \Delta(x-x')) \, dx'
\]

The scalar product \( (f, \Delta(x-x')) \) refers to the \( x \)-coordinate only, it is calculated in (A2,18):
\[ (\ell, \Delta(x-x')) = i \int \int \left( \frac{\partial}{\partial x_0} \Delta(x-x') \right) dx^+ = -i f^*(x') . \] (32)

We obtain
\[ (\ell, A_m) = (\ell, A) - i \int_{-\infty}^{x_0} dx_0' dx^+ f^*(x') \cdot j(x') . \] (33)

Since \( f(x) \) was supposed to be a normalized solution of the homogeneous Klein-Gordon equation, the integral must vanish for \( x_0 \rightarrow -\infty \).

Hence we can write the precise form of the asymptotic condition:

\[
\lim_{x_0 \to -\infty} A_f(x_0) = A_{\ell, \infty} \quad (A_{\ell, \infty} \text{ is time independent})
\]

with
\[
A_f(x_0) = (\ell, A) = i \int_{x_0 = \text{const}} \left\{ \ell^+(x) \frac{\partial A(x)}{\partial x_0} - A(x) \frac{\partial \ell^+(x)}{\partial x_0} \right\} dx^+ \quad (34)
\]

the limit is in the weak sense
\[
\lim_{x_0 \to -\infty} (\Psi, A_f(x_0) \phi) = (\Psi, A_{\ell, \infty} \phi) \quad \text{for any two states } \Psi \text{ and } \phi.
\]

\[ v) \quad \text{Further consequences of the asymptotic condition} \]

We shall prove for one-particle states \( \phi \)

\[
\left\{ \begin{align*}
(\Omega, A_{\infty}(x)^\phi (x)) &= (\Omega, A(x)^\phi (x)) = (\Omega, A_{\infty}(x)^\phi (x)) = f(x) \\
(\Omega, A_{\infty}(x_0) \phi_1) &= (\Omega, A_d(x_0) \phi_1) = const.
\end{align*} \right. \] (35)
That is: the matrix element does not in fact depend on \( x_0 \).

**Proof:**

We first consider \( A(x) \) and its matrix elements:

\[
(\Omega, A(x) \phi_1) = (\Omega, \frac{\partial A(x)}{\partial x_0} \phi_1) = -i \left( \Omega, [p_\mu, A(x)] \phi_1 \right)
\]

\[\text{[invariance principle (16)]}\]

\[
i \left( \Omega, A(x) p_\mu \phi_1 \right).
\]

\[\text{[since } p_\mu \Omega = 0, \text{ (19)]}\]. Hence

\[
\frac{\partial}{\partial x_\mu} (\Omega, A(x) \phi_1) = -i \left( \Omega, A(x) p_\mu \phi_1 \right) = \Delta^2 \left( \Omega, A(x) \phi_1 \right)
\]

\[\text{[spectrum condition 2a)]}\].

Hence the matrix element is a solution of \((\Box - m^2)F(x) = 0\). From the asymptotic condition and (27) follows the first line of (35). For the proof of the second line we use (31) and the fact that for any solution of \((\Box - m^2)F(x) = 0\) we can write

\[
F(x) = \sum \alpha_\alpha f_\alpha(x) + \bar{\alpha}_\alpha \bar{f}_\alpha^*(x)
\]

with constant coefficients. We obtain

\[
F(x) = (\Omega, A(x) \phi_1) = \sum \left( \Omega, A_\alpha(x) \phi_1 \right) f_\alpha(x) + \left( \Omega, A_\alpha^*(x) \phi_1 \right) \bar{f}_\alpha^*(x).
\]

Since \( F(x) \) has been shown to be a solution of \((\Box - m^2)F = 0\), the coefficients must be constant. This proves the second line of (35). Formula (35) has been used for carrying through renormalization without perturbation theory (Källén).
vii) **Compatibility and independence of the principles.**

These questions have never been thoroughly investigated and only some facts are known:

α) one may ask whether there are at all solutions \( \neq A_{\text{free}} \) with an S-matrix \( \neq 1 \) (compatibility). Certainly there are such solutions, if causality is not required, which shows that causality is independent from the other principles. Namely, we may extrapolate \( A \) off the energy shell and define \( j(x) \) and \( A(x) \) along the lines given on pages 9 - 13. It can be shown that then all the other conditions are satisfied.

β) suppose that the operator system \( \{ A(x) \} \) exists (with or without causality condition). Suppose then that \( \{ A_{\text{in}}(x) \} \) and \( \{ A_{\text{out}}(x) \} \) are complete (one of them suffices); it follows that \( \{ A(x) \} \) is complete. The proof has been found by Borchers. It goes as follows: according to p. 6 suppose that there were an operator \( C \) commuting with all \( A(x) \):

\[
[A(x), C] = 0.
\]

From \((\Box - m^2)A(x) = j(x)\) follows that \([j(x), C] = 0\) and with the asymptotic condition

\[
[A_{\text{in}}(x), C] = [A_{\text{out}}(x), C] = 0.
\]

Since \( \{ A_{\text{out}}(x) \} \) is complete, \( C \) must be a multiple of the unit operator, hence \( \{ A(x) \} \) is complete. Note that the argument does not work in the opposite sense. This is not surprising, since \( \{ A_{\text{in}}(x) \} \) is equivalent to \( \{ A_{\text{in}}^*, A_{\text{in}} \} \) [see (31)] whereas \( \{ A(x) \} \) is equivalent to \( \{ A_{\text{in}}^*(x_0); A_{\text{in}}(x_0) \} \) which is a much larger set (not countable) as it depends on \( x_0 \).

Since the question of compatibility and completeness is unsolved, one may rather look at the principles from the other side and assume \( A(x) \) exists:

viii) **Consequences of the existence of a causal \( A(x) \).**

We list here only a few remarks without going into the proofs and detailed explanations.
a) We have to suppose for $A(x)$

(1) Invariance
(2a) Existence of vacuum
(2b) Positive definite $P_0$ and $-P^2_{\rho}$
(3) Causality
(4) Completeness
(5) Asymptotic condition, and existence of 1-particle states with mass $m$ such that

$$(\Omega, A(x) \phi_1^m) = e^{-i(x-y)P_{\rho}} \phi_1 = m^2 \phi_1$$

b) This being given, we can define $A_{\text{in}}(x)$ and $A_{\text{out}}(x)$ by the asymptotic condition 5a). Then the canonical commutation relations

$$[A_{\text{in}}(x), A_{\text{in}}(x')] = i\Delta(x-x')$$

can be proved by using the reduction technique (see below). Completeness of $\{ A_{\text{in}}(x) \}$ cannot be shown.

c) One can imagine that there exist field operators describing two different particles, i.e. there are then one-particle states $\phi^{(m_1)}$ and $\phi^{(m_2)}$. One has then

$$A_{\text{in}}^{(m_1)}(x) = C_1 \left[ A(x) + \int \Delta_{\rho}(m_1, x-y) (\Box - m_1^2) A(y) \, dy \right]$$

$$A_{\text{in}}^{(m_2)}(x) = C_2 \left[ A(x) + \int \Delta_{\rho}(m_2, x-y) (\Box - m_2^2) A(y) \, dy \right]$$

and one would normalize the one-particle states by

$$C_1 \left( \Omega, A(x) \phi^{(m_1)} \right) = C_2 \left( \Omega, A(x) \phi^{(m_2)} \right) = \left( \frac{i}{2\pi} \right)^{3/2} e^{i k x}.$$ 

The in and out fields of different masses commute.

d) Bound states may exist: assume two causal fields $\Psi(x)$ and $A(x)$ being given, where $\Psi$ may be called "nucleon and $A$ "pion" -
but for the moment assume them both scalar and uncharged. Then there will be eigenvalues of \(-P^2\) equal to 0, \(\mu^2, m^2, M^2, \ldots\) and the continuum. The discrete states (\(\mu =\) meson mass, \(m =\) nucleon mass, \(M =\) deuteron mass) are partly isolated, partly embedded in the continuum. The one-particle states are

\[ \phi_k, \phi_N, \phi_D \]

and the matrix elements of the "elementary fields" \(\psi\) and \(A\) with \(\Omega\) and \(\varphi_D\) and other bound states vanish:

\[ (\Omega, A(x) \phi_D) = (\Omega, \psi(x) \phi_D) = 0. \]

It is however possible to construct a causal operator

\[ B(x) = \lim_{\xi \to 0} \frac{\psi(x+\xi)\psi(x-\xi)}{F(\xi)} \]

where \(F(\xi)\) is some function.

For this \(B(x)\) one can show the existence of \(B_{\text{in}}\) \(B_{\text{out}}\) and the causality of \(B(x)\). If the above expression does not exist, one can define

\[ B(x, \xi) = \psi(x+\xi)\psi(x-\xi) \quad \text{and} \quad B_{\text{in}}(x, \xi) \]

and then take

\[ \lim_{\xi \to 0} \left[ \frac{B_{\text{in}}(x, \xi)}{F(\xi)} \right]. \]

This operator has non-vanishing matrix elements \((\Omega, B(x) \varphi_D)\) and it represents a "deuteron field".

e) One may be able to eliminate fields: assume that \(A(x)\) and \(\psi(x)\) represent pion and nucleon respectively. If (and only if) \(\{\psi(x)\}\) is a complete set of operators, then one can eliminate \(A(x)\) by replacing it (in the above limit sense) by a "nucleon-antinucleon bound state" \(A(x) \rightarrow c \cdot \psi(x) \bar{\psi}(x)\).
f) One may have unstable fields; these are fields to which no asymptotic field belongs. Precisely: if \( B(x) \) is such a field and \( \sum_{\text{see } (A2,13)} \)

\[
B(x) = \sum B_\alpha(x) f_\alpha(x) + B^*_\alpha(x) f^*_\alpha(x)
\]

then

\[
\lim_{x_0 \to \pm \infty} (\psi, B_\alpha(x_0) \phi) = 0
\]

whatever the mass \( m \) may be which the \( f_\alpha(x) \) belong to. Hence we are not able to define creation and annihilation operators for a particle related to \( B(x) \). (Glaser-Kfillèn).

g) One can try to use such unstable fields to describe resonances or unstable particles (Salam).

h) The Gursey model uses only unstable fields \( A_1, A_2 \ldots \) and tries to keep their number to a minimum. All observable particles are then stable bound states of the unstable fields (in the above sense).

D) The Reduction Technique.

i) Matrix elements.

We assume now that a causal field \( A(x) \) with one single mass \( m \) exists. We then express the S-matrix by means of \( A(x) \).

We list again what we can use:

\[
\begin{align*}
A_{\in\in}^{\mu\nu}(x) &= A(x) + \sum_{A_v} \Delta_{\in\in}^{\mu\nu}(x-x') \int (x') d^nx' \\
\int (x) &= (\Box - m^2) A(x) \\
\{ A_{\in\in}^{\mu\nu}(x) \} \quad \text{and} \quad \{ A_{\in\in}^{\mu\nu}(x) \} \quad \text{are complete} \\
[ A_{\in\in}^{\mu\nu}(x), A_{\in\in}^{\mu\nu}(x') ] &= i \Delta(x-x')
\end{align*}
\]

Instead of this we may use a complete set of normalized positive frequency functions \( f_\alpha(x) \) and write
f) One may have unstable fields; these are fields to which no asymptotic field belongs. Precisely: if \( B(x) \) is such a field and \[ B(x) = \sum B_\alpha(x_0) f_\alpha(x) + B_\alpha^*(x_0) f_\alpha^*(x) \]
then
\[ \lim_{x_0 \to \pm \infty} \left( \psi, B_\alpha(x_0) \phi \right) = 0 \]
whatever the mass \( m \) may be which the \( f_\alpha(x) \) belong to. Hence we are not able to define creation and annihilation operators for a particle related to \( B(x) \). (Glasner-Källén).

g) One can try to use such unstable fields to describe resonances or unstable particles (Salam).

h) The Gursely model uses only unstable fields \( A_1, A_2 \ldots \) and tries to keep their number to a minimum. All observable particles are then stable bound states of the unstable fields (in the above sense).

D) The Reduction Technique.

i) Matrix elements.

We assume now that a causal field \( A(x) \) with one single mass \( m \) exists. We then express the S-matrix by means of \( A(x) \).

We list again what we can use:

\[ A_{\mu}(x) = A(x) + \sum_{\nu} \Delta_{\mu \nu} (x-x') \int (x') d^nx' \]

\[ \psi(x) = (\not \! x - m^2) A(x) \]

\[ \{ A_{\mu}(x) \} \quad \text{and} \quad \{ A_{\nu}(x) \} \quad \text{are complete} \]

\[ [A_{\mu}(x), A_{\nu}(x')] = i \Delta(x-x') \]

Instead of this we may use a complete set of normalized positive frequency functions \( f_\alpha(x) \) and write
\( A_\alpha(x_0) = i \int \frac{dx}{x_0} f^*_\alpha(x) \frac{\partial}{\partial x_0} A(x) \)

\[
\begin{align*}
A_\alpha^{\text{in}} &= \lim_{\chi_0 \to \infty} \int \frac{dx}{x_0} f^*_\alpha(x) \frac{2}{\partial x_0} A(x) \\
(\Box - m^2) f^*_\alpha(x) &= 0
\end{align*}
\]

\[\text{See (A2,5), (A2,9) and (A2,13)}\]

\[\text{asymptotic condition}\]

\[\text{are complete}\]

\[\begin{align*}
[A_\alpha^{\text{in}}, A^*_\beta^{\text{in}}] &= \delta_{\alpha\beta} \\
[A_\alpha^{\text{in}}, A^*_\beta^{\text{out}}] &= 0
\end{align*}\]

\[\text{See (A3,20)}\]

Let us consider a scattering of two particles (see figure). We have

\[\phi_\alpha^{\text{in}} = N_{\alpha\beta} A_\alpha^{\text{in}} A^*_\beta^{\text{in}} \Omega\]

\[N_{\alpha\beta} = \frac{1}{\sqrt{1 + \delta_{\alpha\beta}}}\]

Since we do not know anything about the commutator of \( A_\alpha^{\text{in}} \) and \( A_\alpha^{\text{out}} \) we cannot directly evaluate these matrix elements. We can, however, reduce them with the help of the asymptotic condition to formulae containing only products of \( A(x), A(y) \ldots \). We still do not know very much about the matrix elements of these products, but at least we can apply causality. So we write

\[\phi_\alpha^{\text{out}} = \phi_\alpha^{\text{in}} = \phi_\alpha^{\text{in}}\], see (21)
\[
(\phi^{a'}_{\nu}, \phi^{\alpha'}_{\nu}) = \lim_{x_0 \to \infty} (\phi^{a'}_{\nu}, A^{(x_0)}_{\alpha}(x_0) \phi^{\alpha'}_{\nu}) = \\
= \int_{-\infty}^{\infty} dx_0 \frac{\partial}{\partial x_0} (\phi^{a'}_{\nu}, A^{(x_0)}_{\alpha}(x_0) \phi^{\alpha'}_{\nu}) + \lim_{x_0 \to -\infty} (\phi^{a'}_{\nu}, A^{(x_0)}_{\alpha}(x_0) \phi^{\alpha'}_{\nu})
\]

The second term is equal to
\[
(\phi^{a'}_{\nu}, A^{(x_0)}_{\alpha}(x_0) \phi^{\alpha'}_{\nu}) = (\phi^{a'}_{\nu}, \phi^{\alpha'}_{\nu}) = \frac{\delta_{\alpha' a} \delta_{\nu \rho} + \delta_{\alpha' a} \delta_{\nu \rho'}}{\sqrt{1+\delta_{\rho \rho'}} - \sqrt{1+\delta_{\rho' \rho'}}} = I_{a' \rho' a \rho},
\]
by which we define the quantity \(I_{a' \rho' a \rho}\), which just stays for "nothing happens" and/or "exchange".

We now insert the explicit form of \(A^{(x_0)}_{\alpha}(x_0)\) from (39) and obtain
\[
(\phi^{a'}_{\nu}, \phi^{\alpha'}_{\nu}) = i \int d^4 x \left[ \frac{\partial}{\partial x_0} \left( \phi^{a'}_{\nu}, A^{(x)}_{\alpha}(x) \phi^{\alpha'}_{\nu} \right) \right] + I_{a' \rho' a \rho}
\]

Now the corresponding integral with space derivatives vanishes:
\[
\int d^4 x \left[ \frac{\partial}{\partial x_k} \left( \phi^{a'}_{\nu}, A^{(x)}_{\alpha}(x) \phi^{\alpha'}_{\nu} \right) \right] = 0,
\]
because it is the divergence of a vector field integrated over the whole space. The corresponding surface integral is zero for normalizable \(f_{\nu}^{*}\). Hence
\[
(\phi^{a'}_{\nu}, \phi^{\alpha'}_{\nu}) = -i \int d^4 x \left[ \frac{\partial}{\partial x_\mu} \left( \phi^{a'}_{\nu}, A^{(x)}_{\alpha}(x) \phi^{\alpha'}_{\nu} \right) \right] + I_{a' \rho' a \rho} \quad \text{or}
\]
\[
(\phi^{a'}_{\nu}, \phi^{\alpha'}_{\nu}) = -i \int d^4 x f_{\rho'}^{*}(x) (\Box - m^2) (\phi^{a'}_{\nu}, A^{(x)}_{\alpha}(x) \phi^{\alpha'}_{\nu}) + I_{a' \rho' a \rho} \quad (40)
\]

We now proceed in the same way to take out the operator \(A^{*}_{\in \nu} \) and express it by a field operator: consider the remaining matrix element in (40):
\[
\left( \phi^*, A(x) A^*_{p\mu}, \phi^x \right) = \lim_{y_0 \to -\infty} \left( \phi^{x'}, A(x) A^*_{p\mu}(y_0) \phi^x \right)
\]
\[
= -i \lim_{y_0 \to -\infty} \int d^4y \left[ i \frac{\partial}{\partial y^0} \left( \phi^{x'}, A(x) A(y) \phi^x \right) \right]
\]
\[
= i \int d^4y \frac{\partial}{\partial y^0} \left[ i \frac{\partial}{\partial y^0} \left( \phi^{x'}, A(x) A(y) \phi^x \right) \right]
\]
\[
- i \lim_{y_0 \to \infty} \int d^4y \left[ i \frac{\partial}{\partial y^0} \left( \phi^{x'}, A(x) A(y) \phi^x \right) \right]
\]
\[
\text{(i)}
\]

The last term could be written
\[
\left( \phi^{x'}, A(x) A^*_{p\mu}, \phi^x \right) = \left( \phi^{x'}, A(x) \phi^{x'\mu} \right)
\]

This is as useless as the term we started from. The situation changes, however, if we use in the integrals (i) not \( A(x)A(y) \) but instead either
\[
T \left( A(x)A(y) \right) = \begin{cases} A(x)A(y) & \text{for } x_0 > y_0 \\ A(y)A(x) & \text{for } x_0 < y_0 \end{cases}
\]

or
\[
R \left( A(x)A(y) \right) = -i \Theta(x-y) \left[ A(x), A(y) \right]
\]

(41)

(41')

Since in
\[
- i \lim_{y_0 \to -\infty} \int d^4y \left[ i \frac{\partial}{\partial y^0} \left( \phi^{x'}, A(x) A(y) \phi^x \right) \right]
\]

for \( x \) fixed always \( x_0 > y_0 \), we can replace \( A(x)A(y) \) by \( TA(x)A(y) \). In this case the boundary term at \( y_0 \to +\infty \) becomes
\[
\left( \phi^{x'}, A^*_{p\mu}, A(x) \phi^x \right) = \delta^{x'}_{\mu \rho} \left( \Omega, A(x) \phi^x \right) = \delta^{x'}_{\mu \rho} f_\alpha(x)
\]

(see (35))
and this vanishes if inserted into (40), since \((\Box - m^2)f_\alpha = 0\). But the \(R\)-product also works: for \(y_0 \to -\infty\) \(q(x-y)\) is +1 and one has in fact replaced \(A(x)A(y)\) by the commutator. The second term of the commutator vanishes for the same reasons as above:

\[
\lim_{y_0 \to -\infty} \int \frac{d^4 q}{(2\pi)^4} \left( \phi^{\alpha'}(x) A(y) A(x) \phi^{\alpha} \right) = \delta_{\alpha\alpha'} \int \phi^{\alpha}(x)
\]

\((\Box - m^2) \int \phi^{\alpha}(x) = 0\).

In this case there is no boundary term at \(y_0 \to +\infty\) since there \(q(x-y) = 0\).

We can now proceed just as in the first reduction and obtain \((K_x = \Box - m^2)\)

\[
(\phi^{\alpha'}_{\text{out}}, \phi^{\alpha}_{\text{in}}) = \int \phi^{\alpha'}_{\text{in}} \phi^{\alpha}_{\text{in}} \left( \int d^4 x d^4 y \int \phi^{\alpha'}_{\text{in}}(x) A(x)A(y) \phi^{\alpha}_{\text{in}} \right) \tag{42}\]

or

\[
(\phi^{\alpha'}_{\text{out}}, \phi^{\alpha}_{\text{out}}) = \int \phi^{\alpha'}_{\text{out}} \phi^{\alpha}_{\text{out}} + i \left( \int d^4 x d^4 y \int \phi^{\alpha'}_{\text{out}}(x) A(x)A(y) \phi^{\alpha}_{\text{out}} \right) \tag{42'}
\]

We may now eliminate \(A_{\text{out}}\) and \(A_{\text{in}}\) and reduce the whole matrix element to a vacuum expectation value of a four fold \(T\) or \(R\) product. For the latter case we define

\[
R A(x_1)...A(x_n) = (-i)^{n-1} \sum_{\text{perm.}} \Theta(x_1-x_2) ... \Theta(x_{n-1}-x_n) \left[ A(x_1) A(x_2) ... A(x_n) \right] \tag{43}
\]

The final result is

\[
(\phi^{\alpha'}_{\text{out}}, \phi^{\alpha}_{\text{out}}) = \int \phi^{\alpha'}_{\text{out}} \phi^{\alpha}_{\text{out}} + \left( \frac{1}{i} \right) \left\{ \left[ \int d^4 x d^4 y d^4 x' d^4 y' \right] \times \right.
\]

\[
\left. \left[ \int \phi^{\alpha}(x) \phi^{\alpha'}_{\text{in}}(y) \phi^{\alpha'}_{\text{out}}(x') \phi^{\alpha}_{\text{out}}(y') K_x K_y K_{x'} K_{y'} < \left( \frac{T}{R} \right) A^0 A^0 A^0 A^0 >_o \right] \right\} \tag{44}
\]
We finally use plane waves

\[ \phi_k(x) \to \phi_R^k(x) = \left( \frac{\lambda}{2\pi} \right) \frac{1}{2} e^{i(\mathbf{R}_k \cdot \mathbf{x} - \omega t)} ; \quad \omega = \sqrt{\mathbf{k}^2 + m^2} \]

and define an operator \( T \) by

\[ S = T + 2\pi i \delta(p - k - p' - k') \cdot T \]

for the transition matrix elements. As will be seen later (p. 54) the \( \delta \)-function is implicitly present on the right hand side of (45) and therefore it is convenient to write it explicitly in the definition of \( T \). This gives

\[ 2\pi i \delta(p - k - p' - k') \cdot (\phi^k_{m'}, \ T \ \phi^p_{m}) = \]

\[ = \left( \frac{\lambda}{2\pi} \right)^2 \left( \frac{-i}{\lambda} \right) \int d^3 k x' e^{i(\mathbf{k} \cdot \mathbf{x}' - \omega t)} K_x K_x' \left( \phi^{p'}_p \left( \frac{\lambda}{2\pi} \right) A(k) A(k') \Phi^p \right) \]

\[ = \left( \frac{\lambda}{2\pi} \right)^6 \left( \frac{-i}{\lambda} \right) \int d^3 k d^3 k' d^3 y d^3 y' e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} K_x K_x' K_y K_y' \left( \left( \frac{\lambda}{2\pi} \right) A(k) A(k') A(y) A(y') \right). \]

Here, of course, \( k^2 = p^2 = k'^2 = p'^2 = m^2 \) since we have only one kind of particle.

(45) shows that the \( T \)-matrix element is essentially the Fourier transform of the \( T \) or \( R \) product of the field operators, taken between vacuum states.

So far not much is known on the four fold \( T \) or \( R \) product and for practical work the first form has always been used, though the second form would yield much more information.
ii) The S-matrix in terms of vacuum expectation values of T-products.

We go back to formula (7) for the S-matrix and take the $\delta (\Sigma k_i)$ inside the $\tilde{C}$-function, calling it $c$ from now on:

$$S = \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int d^4 k_1 \ldots d^4 k_n \ C(k_1 \ldots k_n) \ \delta(k_1^2 - m^2) \ldots \delta(k_n^2 - m^2) \ A_{m_1}(k_1) \ldots A_{m_n}(k_n):$$

(46)

We try to solve for the coefficients. Since we know that commuting with an $A_{in}^*(k')$ would eliminate one of the operators and replace it by a $\delta$-function, we try this. We have

$$A_{in}(k) = \theta(k) A(k) + \theta(-k) A^*(k)$$

$$A_{in}^*(k') = \theta(k') A^*(k') + \theta(-k') A(k')$$

$$[a(k), a^*(k')] = 2\omega \delta(k-k') \ ; \ \omega = \sqrt{k^2 + m^2}$$

(see A2,23 and A2,26)

Hence

$$[A_{in}(k), A_{in}^*(k')] = 2\omega \delta(k-k') \left[ \theta(k) \theta(k') - \theta(-k) \theta(-k') \right]$$

(47)

Writing $S = \sum_n S_n$ and taking only $S_n$:

$$[S_n, A_{in}^*(k)] = \frac{(-i)^n}{n!} \int d^4 k_1 \ldots d^4 k_n \ C(k_1 \ldots k_n) \ \delta(k_1^2 + \omega_1) \ldots \delta(k_n^2 - \omega_n) \ \prod_{i=1}^{n} \left[ \theta(k_i) \theta(k_i) - \theta(-k_i) \theta(-k_i) \right]: A_{in}(k_1) \ldots A_{in}^*(k_n):$$

Since everything is symmetric in $k_1 \ldots k_n$, the sum gives n-times the same and we may take n-times $k_i = k_1$. The result is after integration $\int d^4 k_1$ and with $\theta(1) - \theta(-1) = \epsilon(1)$.

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\[
[S_m, A^*_m (l)] = \frac{(-i)^n}{(n-1)!} \varepsilon (\ell) \int d^4 k_2 \cdots d^4 k_n \ C (l, k_2 \cdots k_n) \ \delta (k_2^2 + m^2) \cdots : A_m (k_2) \cdots :
\]

Repeating this \( n \) times with \( l_1, l_2 \cdots l_n \), we obtain

\[
\left[ \cdots \left[ [S_m, A^*_m (l_1)], A^*_m (l_2) \right] \cdots A^*_m (l_n) \right] = (-i)^n C (l_1, \cdots l_n) \ \varepsilon (l_1) \cdots \varepsilon (l_n)
\]

In the \( n \)-fold commutator with \( S_m; \ m > n \) there will remain a Wick-product of \( m-n \) operators. The \( n \)-fold commutator with \( S_m; \ m < n \) is zero. Hence we must arrange that the higher terms vanish and that is easy because the vacuum expectation value of any Wick product of in (and out) operators is zero

\[
C (k_1 \cdots k_n) = i^n \varepsilon (k_1) \cdots \varepsilon (k_n) \langle \cdots \left[ [S_m, A^*_m (k_1)], A^*_m (k_2) \right] \cdots A^*_m (k_n) \rangle_0
\]

We now wish to express \( c \) by \( T \)-products rather than by the formula just derived. By this we are then able to eliminate formally the \( S \)-matrix itself from the coefficients, whereas in (48) it is explicitly present.

To this end we need a generalised reduction formula:

\[
[S \cdot T (x_1, \ldots x_n), A_{\alpha m}] = i \int d^4 z \ f_{\alpha}^4 (z) K_2 \ S \cdot T (x_1, \ldots x_n, z) \]

\[
T (x_1, \ldots x_n) \equiv \ T A (x_1) \cdots A (x_n)
\]

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Proof:

\[ A_{\text{out}} \cdot T(x_1, \ldots, x_n) \rightarrow T(x_1, \ldots, x_n) A_{\text{in}} \]

\[ = \lim_{z_0 \to \infty} A_{\alpha}(z_0) \cdot T(x_1, \ldots, x_n) \rightarrow \lim_{z_0 \to \infty} T(x_1, \ldots, x_n) A_{\alpha}(z_0) \]

\[ = \lim_{z_0 \to \infty} I \left\{ \int \frac{d^2 z}{2} f^*(z) \frac{\partial}{\partial z_0} A(z) \cdot T(x_1, \ldots, x_n) \right\} \]  

\[ = \left[ \int \frac{d^2 z}{2} \frac{\partial}{\partial z_0} \left[ f^*(z) \frac{\partial}{\partial z} T(x_1, \ldots, x_n, z) \right] \right. \]

\[ = \left. \int \frac{d^2 z}{2} \frac{\partial}{\partial z_0} \left[ f^*(z) \frac{\partial}{\partial z} T(x_1, \ldots, x_n, z) \right] \right\} \]

The space terms vanish, since they are equal to a surface integral at infinity. Since \( \partial_\rho \partial_\rho f^*(z) = m^2 f^*(z) \), we obtain finally

\[ = \int \frac{d^2 z}{2} f^*(z) K_z T(x_1, \ldots, x_n, z) = A_{\text{out}} \cdot T(x_1, \ldots, x_n) \rightarrow T(x_1, \ldots, x_n) A_{\text{in}} \]

We multiply now by \( S \) from the left:

\[ \text{SA}_{\text{out}} = A_{\text{in}} S \]  

\[ A_{\text{in}} \cdot S T - S T A_{\text{in}} = \left[ S T, A_{\text{in}} \right] = \int \frac{d^2 z}{2} f^*(z) K_z S T(x_1, \ldots, x_n, z) \]

\[ \text{q.e.d.} \]

In the same way one obtains

\[ \left[ S T(x_1, \ldots, x_n), A_{\text{in}}^* \right] = -i \int \frac{d^2 z}{2} f^*(z) K_z S T(x_1, \ldots, x_n, z) \]

(491)

For the corresponding formula with \( A_{\text{in}}^*(k) \) one must be a bit careful if one wishes it not to be restricted to \( k_0 = + \sqrt{k^2 + m^2} \). For it to be valid for any \( k \) one must take (A2, 25):

\[ A_{\text{in}}^*(k) = -i \frac{\epsilon(k)}{(2\pi)^{3/2}} \int e^{ikx} \frac{\partial}{\partial x_0} A_{\text{in}}(x) \, dx \]
and obtains

$$\left[ ST(x_1, \ldots, x_n), A_{\mu}^a(x) \right] = -\frac{i\varepsilon(k)}{(2\pi)^{3/2}} \int d^4z \, e^{ikz} \, \sigma \, T(x_1, \ldots, x_n, z)$$  \hspace{1cm} (49'')

putting $A_{\mu}^a(x) = \sum A_{\mu}^a(x)$, one obtains with (49), (49') and (A2,16) immediately

$$\left[ ST(x_1, \ldots, x_n), A_{\mu}^a(x) \right] = \int \Delta(k-z) \, K_2 \, ST(x_1, \ldots, x_n, z) \, d^4z$$  \hspace{1cm} (49'')

Repeated application of (49'') serves to build up the n-fold commutator of (48):

$$\langle \ldots [S, A_{\mu}^a(k_1)] \ldots A_{\mu}^a(k_n) \rangle > = \frac{(-i)^n}{(2\pi)^{3n/2}} \varepsilon(k_1) \ldots \varepsilon(k_n) \int d^4x_1 \ldots d^4x_n \, e^{i \sum k_j x_j} \, K_{x_1} \ldots K_{x_n} \langle ST(x_1, \ldots, x_n) \rangle >$$

Since the vacuum state is unique: $S|0\rangle = S^+|0\rangle = |0\rangle$ we may omit $S$ and obtain

$$C(k_1, \ldots, k_n) = \frac{1}{(2\pi)^{3n/2}} \int d^4x_1 \ldots d^4x_n \, e^{i \sum k_j x_j} \, K_{x_1} \ldots K_{x_n} \langle T(k_1, \ldots, k_n) \rangle >$$  \hspace{1cm} (50)

This defines the $c(k_1, \ldots, k_n)$ also outside the mass shell $k_i^2 = -m^2$ but this is arbitrary. One may restrict (50) to the mass shell and define $c(k_1, \ldots, k_n)$ arbitrary outside. \[ \text{See discussion p. 7 - 2/7} \]

If one inserts this into (46):

$$S = \sum \frac{(-i)^n}{n!} \int d^4k_1 \ldots d^4k_n \, C(k_1, \ldots, k_n) \delta(k_1^2 + m^2) \ldots \delta(k_n^2 + m^2) \, A_{\mu}^a(k_1) \ldots A_{\mu}^a(k_n)$$
one obtains

\[ S = \sum \left( \frac{1}{2\pi} \right)^{\frac{3n}{2}} \int d^{3}x_{1} \ldots d^{3}x_{n} \, d^{3}p_{1} \ldots d^{3}p_{n} \, K_{1} \ldots K_{n} \left< T(x_{1}, \ldots, x_{n}) \right> \times \]

\[ \times e^{i(\sum_{a} p_{a} x_{a} - \sum_{a} p_{a} x_{a})} \delta(p_{1}^{2} + m_{1}^{2}) \ldots \delta(p_{n}^{2} + m_{n}^{2}) \right| A_{m_{1}}(x_{1}) \ldots A_{m_{n}}(x_{n}) \right> \]

With

\[ A_{m}(x) = \frac{4}{(2\pi)^{3/2}} \int \delta(p^{2} - m^{2}) \, e^{ikx} \, A_{m}(k) \, d^{3}k \]

one can write alternatively

\[ S = \sum \left( \frac{1}{2\pi} \right)^{3n/2} \int d^{3}x_{1} \ldots d^{3}x_{n} \, A_{m_{1}}(x_{1}) \ldots A_{m_{n}}(x_{n}) \left< T(x_{1}, \ldots, x_{n}) \right> \right| \]

For practical calculations, however, the first form (51) is more convenient, since one usually takes matrix elements between plane wave states.

In fact, if one calculates for example

\[ \left( \Phi_{\ell m_{1}}, \Phi_{\ell' m_{2}} \right) = \left( \Phi_{\ell m_{1}}, \Phi_{\ell m_{2}} \right) \]

and uses (51) with (A2,22')

\[ \int d^{3}k \, e^{i\mathbf{k} \cdot \mathbf{x}} \, \delta(k^{2} - m^{2}) \, A_{m}(k) = \left\{ \frac{2\pi}{2\omega} \left[ a(k) e^{i(k \cdot x - \omega t)} + a^{\ast}(k) e^{-i(k \cdot x - \omega t)} \right] \right\} \]

then one sees that only the zero order and 4th order term of \( S \) can contribute.

Writing \( \Lambda = 1 + \varepsilon \Delta i \, \delta(\mathbf{p} \cdot \mathbf{k} - \mathbf{p}' \cdot \mathbf{k}') \, T \) one finds for the 4th order term exactly the expression (45).

It must be mentioned that one cannot generally replace the T-product by an R-product. This is only possible if there are just two incoming and any number of outgoing particles.
iii) \( A(x) \) in terms of \( A_{1n}(x) \); R-product.

In this case we need the reduction formula for the general R-product.

It is defined by

\[
R(x, x_1, \ldots, x_n) = (-i)^n \sum_{\text{perm.}} \Theta(x-x_1) \cdots \Theta(x_{n-1}-x_n) \cdots \left[ [A(x), A(x_1)] [A(x_1), A(x_2)] \cdots [A(x_{n-1}), A(x_n)] \right]
\]  

(52)

of all permutations only one survives (if at all), namely that one

\[
P\left( x_1, \ldots, x_n \right) \text{ for which } t \geq t_1 \geq t_2 \geq \cdots \geq t_n
\]

all others contain a \( \Theta(x_j-x_k) \) which vanishes. One may therefore define \( R \) also

\[
R(x, x_1, \ldots, x_n) = \begin{cases} 
(-i)^n \left[ ... \left[ A(x), A(x_1) \right] \cdots A(x_n) \right] & \text{for } x_0 > x_1 \geq \cdots \geq x_n \\
0 & \text{if any } x_{k_0} > x_0 \\
R(x, x_1, \ldots, x_n) & \text{symmetric in } x_1, \ldots, x_n
\end{cases}
\]  

(52')

In other words: \( R \) equals \((-i)^n\) times the n-fold commutator of the points \( x, x_1, \ldots, x_n \) taken in time ordered sequence and it vanishes unless all points \( x_1, \ldots, x_n \) lie in the backward cone of \( x \). Suppose \( x_i \) is the first point to lie spacelike to \( x \), then it lies also spacelike to \( x_{i-1}, \ldots, x_1 \) and then

\[
\left[ ... \left[ A(x), A(x_1) \right] \cdots A(x_1) \right] = \cdots \left[ A(x), A(x_1) \right] \cdots A(x_n) = 0
\]

since \( A(x_i) \) commutes with all \( A(x_{i-1}), \ldots, A(x_1) \).
If, however, all \( x_1 \ldots x_n \) lie inside the backward light cone of \( x \), then \( x_i \) and \( x_k \) may well lie spacelike to each other without \( R \) being zero.

We prove now a formula similar to (49).

Consider \( R(x, x_1 \ldots x_n) \) and reorder, if necessary, the labels such that \( x^0 > x^1 > \ldots > x^n \). Then

\[
\left[ R(x, x_1 \ldots x_n), \gamma^\mu_{\nu\alpha} \right] = \sum \text{see (52)} \]

\[
= \lim_{z_0 \to -\infty} (-i)^n \left[ \left[ A(x), A(x_1) \right] \ldots A(x_n) \right] = \sum \text{see (39)}
\]

\[
= \lim_{z_0 \to -\infty} i \int \frac{\partial}{\partial z_0} \left[ \left[ A(x), A(x_1) \right] \ldots A(z) \right] d^2z
\]

\[
= \lim_{z_0 \to -\infty} i \int \frac{\partial}{\partial z_0} \left[ \frac{\partial}{\partial z_0} R(x, x_1 \ldots x_n, z) d^2z \right]
\]

\[
= \int \frac{\partial}{\partial z_0} \left[ \frac{\partial}{\partial z_0} R(x, x_1 \ldots x_n, z) \right] d^2z
\]

since \( R \) is zero for \( z_0 = +\infty \)

\[
= -\int \frac{\partial}{\partial \mu} \left[ \frac{\partial}{\partial \mu} R(x, x_1 \ldots x_n, z) \right] d^2z
\]

(surface terms added)

\[
= -\int f^*_\alpha(z) K_\mu R(x, x_1 \ldots x_n, z) d^2z
\]

hence

\[
\left[ R(x, x_1 \ldots x_n), \gamma^\mu_{\nu\alpha} \right] = -\int d^2z f^*_\alpha(z) K_\mu R(x, x_1 \ldots x_n, z) \]

\[
= -\int d^2z f^*_\alpha(z) K_\mu R(x, x_1 \ldots x_n, z) \] \quad (53)
Similarly

\[ [R(x, x', \ldots, x_n), A^\dagger_{in}(z)] = + \int d^4z \int f_\alpha(z) K_\alpha \{ R(x, x', \ldots, x_n, z') \] (53')

and with \( A_{in}(z) = \sum_\alpha A_{in}^\dagger f_\alpha(z) + A_{in} f_\alpha^*(z) \) and \((\Lambda 2, 16)\)

\[ [R(x, x', \ldots, x_n), A_{in}(z)] = -i \int d^4z' \Delta(z-z') K_{z'} \{ R(x, x', \ldots, x_n, z') \] (53'')

Finally, in the special case of plane waves, using \((\Lambda 2, 25)\) one finds

\[ [R(x, x', \ldots, x_n), A_{in, k}] = -\frac{\xi(k)}{(2\pi)^{3/2}} \int d^4z \frac{i^2k^2}{z^2} K_z \{ R(x, x', \ldots, x_n, z) \] (53''')

\[ [R(x, x', \ldots, x_n), A^\dagger_{in, k}] = \frac{\xi(k)}{(2\pi)^{3/2}} \int d^4z \frac{k^2}{z^2} K_z \{ R(x, x', \ldots, x_n, z) \} \] (53''''

With the help of these formulae we may now express \( A(x) \) by an expansion over Wick products of \( A_{in} \)'s, and the coefficients will be essentially vacuum expectation values of \( R \)-products. As we know already, the field operators \( A(x) \) are not uniquely determined by giving \( A_{in}(x) \) and \( A_{out}(x) \), therefore we cannot expect to find the \( A(x) \) expressed by the \( A_{in} \) without finding the \( A \) somehow in the coefficients. It is just the same as if we expand an arbitrary state \( \phi = \sum c_i \psi_i \) - the \( \phi \) is then still present in the coefficients: \( c_i = \langle \phi, \psi_i \rangle \).

We put for instance in the plane wave representation

\[ A(x) = A_{in}(x) + \sum_{k_1, \ldots, k_n} \frac{1}{\hbar^2} \int d^4k_1 \ldots d^4k_n C(x, k_1, \ldots, k_n) \delta(k_1^2, m^2) \ldots \delta(k_n^2, m^2) \{ A_{in}(k_1) \ldots A_{in}(k_n) \} \] (54)

Exactly as in the S-matrix case on p. 34 – 36 it follows that
\[ C(x, p_1, ..., p_n) = \mathcal{E}(k_1) ... \mathcal{E}(k_n) \langle \cdots [A(x), A^*(k_1)] ... A^*(k_n) \rangle \geq 0 \quad (55) \]

Repeated application of \((53')\) gives then

\[ C(x, p_1, ..., p_n) = \left( \frac{i}{2\hbar} \right)^{3n/2} \int d^3 p_1 ... d^3 p_n e^{i \sum p_j z_j} k_1 ... k_n \langle R(x, z_1, ..., z_n) \rangle \geq 0 \quad (56) \]

One may either insert this in \((54)\) and leave it as it stands e.g. for working out matrix elements in plane wave states or one may use

\[ A_{in}(x) = \left( \frac{i}{2\hbar} \right)^{3/2} \delta(k^2, m^2) e^{ikx} A_{in}(k) d^4 k \]

to obtain at once the formula in coordinate space:

\[ A(x) = A_{in}(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^n z_1 ... d^n z_n K_1 ... K_n \langle R(x, z_1, ..., z_n) \rangle : A_{in}(z_1) ... A_{in}(z_n) : \quad (57) \]

If there are more fields, they have of course to appear all in the expansion.

E) Dispersion relations.

i) Introduction.

The formalism developed so far has not much practical advantage as compared to the old Hamiltonian one — however, it permits the derivation of the dispersion relations, which we shall study now. Whereas the new formalism is much clearer and more physical in its principles, it leaves more or less the same questions open as did the old-fashioned Lagrange-Hamilton form in its latest state of renormalization — and perturbation — theory; non-trivial solutions are
\[ C(x_1, \ldots, x_n) = E(k_1) \cdots E(k_n) \left< \ldots \left[ A(x), A^*_{m_1}(k_1) \right] \ldots A^*_{m_n}(k_n) \right> \] (55)

Repeated application of (55') gives then

\[ C(x_1, \ldots, k_n) = \left( \frac{1}{2\pi} \right)^{3n/2} \int d^3z_1 \ldots d^3z_n e^{i \sum K_j z_j} K_1 \ldots K_n \left< R(x, z_1, \ldots, z_n) \right> \] (56)

One may either insert this in (54) and leave it as it stands e.g. for working out matrix elements in plane wave states or one may use

\[ A_{m_1}(x) = (\frac{1}{2\pi})^{3/2} \delta(k_1^2 - m_1^2) e^{ikx} A_{m_1}(k) d^3k \]

to obtain at once the formula in coordinate space:

\[ A(x) = A_{m_1}(x) + \sum_{2} \frac{1}{n!} \int d^3z_1 \ldots d^3z_n K_1 \ldots K_n \left< R(x, z_1, \ldots, z_n) \right> : A_{m_1}(z_1) \ldots A_{m_n}(z_n) : \]

If there are more fields, they have of course to appear all in the expansion.

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i) **Introduction.**

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known neither in the one nor in the other formulation. In fact the present formalism adds nothing new physically or mathematically - it only states the old idea in a much better way. If one wants to obtain solutions - then still the perturbation theory is the only means to calculate something. Whether this something has anything to do with the wanted solutions and whether solutions exist at all, that nobody knows. Strangely enough, in quantum electrodynamics one can produce very detailed results which agree within our best technical limits with experiment. Yet in principle in quantum electrodynamics also the above mentioned questions remain open.

Under these circumstances it seems to be the greatest merit of the new formalism to provide a tool for the discussion of the general properties of the transition amplitudes or more generally of vacuum expectation values of products of field operators. The point is that here the interacting field operators and not the free field operators or some perturbation theoretical approximations are discussed.

The main source of information seems to be the local commutativity. Of course, all principles of the theory enter more or less in the discussion, however local commutativity - which one believes to represent causality - has most weight. In the dispersion relations, which are but another expression of local commutativity, one has for the first time in quantum field theory exact relations between continuous variables (angles, energies, cross-sections) which can be tested experimentally, whereas so far the only exact relations coming from field theory have been concerned with symmetry properties and conservation laws.

The reader who is not familiar with the idea of dispersion relations and causality should now first read Appendix 5, where the general concept is outlined in abstract form and illustrated by an example.

We shall use henceforward the metric \( x^2 = x_0^2 - k^2 \)
ii) The Dyson representation.

Turning back to (45) one sees that what we would like to know is the structure of the vacuum expectation value of a time ordered or retarded product of four field operators. This would give much information on all scattering processes with two incoming and two outgoing particles. This information might then be confronted with experiments and thus show whether our basic principles (above all the local commutativity) are right.

Unfortunately very little is known about these "four-point" functions. Therefore one has to be content for a while with what one can conclude if only two operators are written explicitly, the two others remaining in the states as in the first line of (45).

We shall presently study the simplest expression of this kind, namely the matrix element

$$\langle p, \alpha | [A(x), B(y)] | q, \beta \rangle$$

(58)

where $p$ and $q$ are four momenta, $\alpha$ and $\beta$ the remaining quantum numbers, $A(x)$ and $B(y)$ any two scalar neutral fields obeying all postulates.

We use the translation invariance, Eq. (17) which in the present metric becomes

$$A(x + \xi) = e^{i P \xi} A(x) e^{-i P \xi} ; \quad P \xi = P_0 \xi_0 - P \xi_1 \xi_2 \xi_3$$

(17')

to obtain

$$\langle p, \alpha | [A(x), B(y)] | q, \beta \rangle = e^{\frac{i}{2}(P \cdot Q)(x-y)} \langle p, \alpha | [A(\frac{x-y}{2}), B(\frac{y-x}{2})] | q, \beta \rangle$$

(59)
Hence it suffices to consider instead of (58)
\[
\langle \mathcal{P}_x | [A(\xi), B(-\xi)] | \Omega_\beta \rangle \equiv \tilde{f}(\chi)
\]  
(60)

What do we know about this function \( \tilde{f}(\chi) \) and its Fourier transform \( f(q) \)?

a) Local commutativity says : \( \tilde{f}(\chi) \equiv 0 \) for \( x \) spacelike.

b) Define \( f(q) = \int e^{i q \cdot x} \tilde{f}(\chi) d^4 x = f_1(q) + f_2(q) \) (where \( f_1(q) \) corresponds to \( AB \) and \( f_2(q) \) to \( -BA \) of the commutator). Introduce a complete set of states to evaluate

\[
\tilde{f}_n(q) = \sum_n \left\langle \mathcal{P}_x | A(\xi) | n \right\rangle \left\langle n | B(-\xi) | \Omega_\beta \right\rangle
\]

take \( | n \rangle = | l, \gamma \rangle \) where \( l \) means an eigenstate of four momentum \( l \) and use again translation invariance (17') to obtain

\[
\tilde{f}_n(q) = \sum_{\delta} \left\langle \mathcal{P}_x | A(0) | \ell, \xi \right\rangle \left\langle \ell, \xi | B(0) | l_\delta \right\rangle
\]

\[
\tilde{f}_n(q) = (2\pi)^4 \sum_{\delta} \left\langle \mathcal{P}_x | A(0) | \frac{p+q}{2} + q, \delta \right\rangle \left\langle \frac{p+q}{2} + q, \delta | B(0) | l_\delta \right\rangle
\]

(61)

Thus the intermediate states \( | \frac{p+q}{2} + q, \delta \rangle \) have to be physical states with total four momentum \( \frac{p+q}{2} + q \) and other quantum numbers \( \gamma \). They must fulfill two conditions : \( \frac{p+q}{2} + q \) must be in the forward light cone and its square must be greater than or equal to the square of the lowest possible mass value in the state \( | \ell, \gamma \rangle \) such that \( \langle \mathcal{P}, \alpha | A(0) | l, \gamma \rangle \neq 0 \) and \( \langle 1, \gamma | B(0) | \Omega, \beta \rangle \neq 0 \). With a similar consideration on \( f_2(q) \) we arrive then at the two conditions
a) \( f(x) = 0 \) for \( x \) spacelike \( (x^2 < 0 \text{ in our present metric}) \)

\[
\left\{ \begin{array}{c}
\frac{P+Q}{2} + q \in L^+ \quad \text{and} \quad \frac{(P+Q+q)^2}{2} > m_1^2 \\
\frac{P+Q}{2} - q \in L^+ \quad \text{and} \quad \frac{(P+Q-q)^2}{2} > m_2^2
\end{array} \right. 
\]

b) \( f(q) = 0 \) unless

\[
\left\{ \begin{array}{c}
\frac{P+Q}{2} + q \in L^+ \quad \text{or} \quad \frac{(P+Q+q)^2}{2} > m_1^2 \\
\frac{P+Q}{2} - q \in L^+ \quad \text{and} \quad \frac{(P+Q-q)^2}{2} > m_2^2
\end{array} \right.
\]

where *) \( m_1^2 = \text{Min}(l^2) \) such that

\[
\begin{align*}
\langle P, \alpha | A(0) | 1, \gamma > & \neq 0 \quad \text{and} \\
\langle 1, \delta | B(0) | q, \beta > & \neq 0
\end{align*}
\]

\( m_2^2 = \text{Min}(l^2) \) such that

\[
\begin{align*}
\langle P, \alpha | B(0) | 1, \gamma > & \neq 0 \quad \text{and} \\
\langle 1, \delta | A(0) | q, \beta > & \neq 0
\end{align*}
\]

The problem is to find a general representation of the class of functions obeying (62). The solution to this problem is the Dyson representation, which will be derived on the following pages. The result is found in (73) and (74) on p. 51-52.

We show the meaning of condition b) in Figs. 1 and 2, choosing the Lorentz system in which \( \mathbf{P} = 0 \). We call \( (P+Q)/2 = a > 0 \) and choose axes labelled by \( |\mathbf{q}| \) and \( q_o \). The above conditions read then

\[
a + q_o > 0 \quad \text{and} \quad (a+q_o)^2 - |\mathbf{q}|^2 > m_1^2 \quad \text{or} \\
a - q_o > 0 \quad \text{and} \quad (a-q_o)^2 - |\mathbf{q}|^2 > m_2^2
\]

The first yields a hyperbola 1 opened to \( q_o > 0 \), the second a hyperbola 2 opened to \( q_o < 0 \). In both cases the asymptotes are given by \( q_o = \pm |\mathbf{q}| - a \) and \( q_o = \pm |\mathbf{q}| + a \) respectively, whereas the vertices of the hyperbolas lie at \( m_1-a \) and \( -m_1 + a \) respectively. Thus they overlap if \( m_1-a < -m_2+a \) or \( (m_1+m_2)/2 < a \), otherwise they do not. This is shown in the following Figs. 1 and 2.

*) See the discussion following Eq. (92).
Fig. 1 \[ \frac{m_1 + m_2}{2} < a \]

Fig. 2 \[ \frac{m_1 + m_2}{2} > a \]
We now first satisfy condition a) by writing
\[ \tilde{f}(x) = \int_0^\infty \delta(\mu^2 - x^2) \tilde{\phi}(x) \, d\mu^2 \] (63)

Here \( \phi(x) = \tilde{f}(x) \) for \( x^2 \geq 0 \); otherwise it is arbitrary.

In Appendix 3, part IV, we prove a relation (A3,30) which in our present metric reads
\[ \delta(\mu^2 - x^2) = 16\pi^2 \int_0^\infty dk^2 \Delta_P(x^2, k^2) \Delta_P(k^2, \mu^2) \] (64)

Hence
\[ \tilde{f}(x) = 16\pi^2 \int d\mu^2 dk^2 \tilde{\phi}(x) \Delta_P(x^2, k^2) \Delta_P(k^2, \mu^2) \]

Using \( \Delta(x^2, k^2) = -2 \varepsilon(x) \Delta_P(x^2, k^2) \) and integrating over \( \mu^2 \) gives
\[ \tilde{f}(x) = \int_0^\infty dk^2 \Delta(x^2, k^2) \tilde{\phi}(x, k^2) \] (65)

where several factors and the result of the \( \mu^2 \)-integration have been absorbed in a new function \( \tilde{\phi}(x, k^2) \). It is evident that to any given \( \tilde{f}(x) \) vanishing for \( x \) spacelike, a \( \tilde{\phi}(x, k^2) \) can be found such that (65) holds — simply by performing the steps from (63) to (65) in an actual case. On the other hand, (65) does not determine \( \tilde{\phi} \) uniquely.

The reason we have preferred (65) to (63) becomes obvious when we now pass to the Fourier representation in order to satisfy condition b): Eq. (64) serves to obtain a 4-dimensional Fourier representation of \( \delta(\mu^2 - x^2) \) by means of the \( \Delta \)-function. Let us define
\[ \tilde{f}(p) = \int \tilde{f}(x) e^{iqx} \, d^4x \]
\[ \tilde{\phi}(x, k^2) = \int \phi(u, k^2) e^{-iu^x} \, d^4u \] (furthermore from (A3,6) (metric changed!))
\[ \Delta(x, k^2) = -\frac{i}{(2\pi)^3} \int \varepsilon(p^2, k^2) \varepsilon(p) e^{-ipx} \, d^4p \]
and transfer \((65)\) into the Fourier space. The result is

\[
\hat{f}(q) = -2\pi i \int_0^{\infty} dq^2 \int d^2 k \varphi(u, k^2) \delta\left((q-u)^2 - k^2\right) \varepsilon\left(q_o - u_o\right)
\]  

(67)

We shall now satisfy conditions \((62b)\) by restricting the support of \(\varphi(u, k^2)\), that is the region in \(u\) and \(k^2\) where \(\varphi(u, k^2)\) is different from zero.

To \(f(q)\) apparently only such points \(u, k^2\) contribute which lie on the hyperboloid \((q-u)^2 = k^2\).

Keep \(u\) and \(k^2\) fixed for a moment and let \(q\) vary over both shells of the hyperboloid \((q-u)^2 = k^2\). If \(q\) could reach in this way the forbidden \(q\) region of conditions \(b)\) \(\int\text{see Figs. 1 and 2}\), then the integral \((67)\) would contain a contribution to \(f(q)\) for non-allowed \(q\)-values. It would, of course, be possible to restrict the integration in \((67)\) such that only the allowed parts of the hyperboloid \((q-u)^2 = k^2\) contribute. This would ensure that every pair of functions \(\tilde{f}(x), f(q)\) obeying \((62a, b)\) can be represented. Unfortunately that leads to very complicated conditions on the weight function \(\varphi(u, k^2)\), which effectively would become explicitly \(q\)-dependent by restricting its support to the allowed region. It is much simpler to demand that

\[
\varphi(u, k^2) = 0, \quad \text{unless both shells of the hyperboloid } (q-u)^2 = k^2 \quad (u, k^2 \text{ fixed}) \quad \text{lie in the allowed } q \text{-region}
\]

(68)

Dyson has proved that this condition is not too restrictive (Phys. Rev. 110, 1460 (1958)) : in fact every pair of functions \(\tilde{f}(x), f(q)\) obeying \((62a, b)\) can be represented by weight functions \(\varphi(u, k^2)\) following condition \((66)\).

We have to make the implicit definition \((68)\) of the region \(S\) in \(u, k^2\), where \(\varphi(u, k^2)\) may be different from zero, more explicit.

1) \((68)\) says that the upper (lower) shell of \((q-u)^2 = k^2\) must lie above (below) the upper (lower) shells drawn in Figs. 1 and 2.
2) A necessary condition is that the upper (lower) asymptotic half cone lies above (below) the asymptotic half cone of shell 1
\[ \text{shell } 2 \text{ of Figs. 1 and 2, since otherwise the hyperboloid } (q-u)^2 = k^2 \text{ must cross one of the shells 1 or 2, which are the boundaries of the allowed q-region}. \]
In other words, the vertex \( q = u \) of the double cone \((q-u)^2 = 0\) must lie at the same time in the upper half cone of shell 1 and in the lower half cone of shell 2. That means
\[ u + \frac{P+Q}{2} \in L^+ \quad \quad u - \frac{P-Q}{2} \in L^- \tag{69} \]

3) We now ask for the condition that the upper shell of \((q-u)^2 = k^2\) just touches shell 1 in one point. We have for 1 the equation
\[ (q + A)^2 = \omega_4^2 \tag{70} \]
There must then be a point \( \vec{q} \), where both hyperboloids have the same tangential plane, hence in \((\vec{q} - u)(q - \vec{q}) = 0\) and \((\vec{q} + A)(q - \vec{q}) = 0\) the coefficients must be proportional
\[ (\vec{q} - u) = \alpha (\vec{q} + A) \quad \quad \vec{q} = \frac{c_4 \lambda + \mu}{\lambda - \alpha} \tag{70} \]
Squaring (70) yields
\[ \frac{(\vec{q} - u)^2}{(\vec{q} + A)^2} = \alpha^2 = \frac{k^2}{\omega_4^2} \tag{71} \]
since \( \vec{q} \) must satisfy the equations of both hyperboloids. Inserting \( \vec{q} \) from (70) into \((q-u)^2 = k^2\) gives
\[ \alpha^2 (A + u)^2 = (1 - \alpha)^2 k^2 \quad \text{or} \quad \alpha \sqrt{(A + u)^2} = \pm (1 - \alpha) k \]
This inserted into (71) gives
\[ k_1 = \omega_4 \pm \sqrt{(\frac{P + Q}{2} + u)^2} \tag{72} \]
We are concerned with shell 1 and the upper shell of \((q-u)^2 = k^2\). Since \(u\) lies inside the asymptotic half cone of \(1\), the smaller of the values \(k_-\) of (72) is the right one [see Fig. 3]. If \(k_-\) is negative, then that means that \(u\) lies above shell 1 and there is no touching point.

The upper shell of \((q-u)^2 = k^2\) will then lie always above shell 1 of \((q+\frac{P+Q}{2})^2 = m_1^2\) if \(k \geq k_-\). Similarly we find a condition for the lower shell lying always below shell 2. The two conditions together give then

\[
\begin{align*}
 k &\geq \max \left\{ 0 ; \ m_1 - \sqrt{\frac{(P+Q+u)^2}{2}} ; \ m_2 - \sqrt{\frac{(P+Q-u)^2}{2}} \right\} \\
\end{align*}
\]

to which adds (69):

\[
\frac{P+Q}{2} + u \in L^+ \quad \text{and} \quad \frac{P+Q}{2} - u \in L^+
\]

This defines the region \(S(u, k^2)\) where \(\varphi(u, k^2)\) may differ from zero.

We should write \(\varphi(u, k^2; Pa Q \lambda)\), since we obtain a new function for each set \(Pa, Q \lambda\). Sometimes we shall use the shorthand notation \(\varphi(u, k^2)\). The Dyson representation reads then:
\( \tilde{f}(x) = \left< P, \alpha \left| \left[ A\left( \frac{x}{2} \right), B\left( -\frac{x}{2} \right) \right] \right| Q, \beta \right> \) has a Fourier transform

\[
f(q) = \int e^{i q x} \tilde{f}(x) d^4 x,
\]

which can always be written

\[
f(q) = \int \int d^4 u \varphi(u, k^2; P, Q, \alpha Q, \beta) \delta\left( (q - u)^2 - k^2 \right) \xi(q_0^2 - u_0^2),
\]

where \( \varphi(u, k^2; P, Q, \alpha Q, \beta) \) is a weight function, which is zero for \((u, k^2) \notin \frac{1}{2} S\) \(\overline{\text{defined in (73)}}\).

In applications the retarded commutator is used rather than the commutator proper \(\overline{\text{see (45)}}\). Hence we wish to represent the function

\[
\tilde{f}_R(x) = -i \Theta(x_0) \left< P, \alpha \left| \left[ A\left( \frac{x}{2} \right), B\left( -\frac{x}{2} \right) \right] \right| Q, \beta \right>
\]

\[
= -i \Theta(x_0) \tilde{f}(x);
\]

put

\[
-i \Theta(x_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i q_0' x_0}}{q_0' + i \epsilon} dq_0'.
\]

This gives

\[
\tilde{f}_R(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i q_0' x_0}}{q_0' + i \epsilon} \left( \frac{1}{2\pi} \right)^4 f(q) e^{-i q x} d^4 q d^4 q_0'.
\]

The Fourier transform becomes

\[
f_R(p) = \int e^{ipx} \tilde{f}_R(x) d^4 x = \left( \frac{1}{2\pi} \right)^5 \int \frac{e^{-i q_0' x_0}}{q_0' + i \epsilon} e^{i (p - q)x} f(q) d^4 q d^4 q_0'.
\]

\[
= \frac{1}{2\pi} \int \frac{\delta(p - q)}{q_0' + i \epsilon} \delta(p_0 - q_0 - q_0') f(q) d^4 q d^4 q_0'.
\]
\[ f_R(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(p, p_o - q_o')}{q_o' + i\varepsilon} \, dq_o' = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(p, p_o')}{p_o' - p_o - i\varepsilon} \, dp_o' \] (75)

Inserting \( f(p) \) from (74) gives with \( p_o' - u_o = z_o \)

\[ f_R(p) = -\frac{1}{2\pi} \int d^4u \, dk^2 \, \varphi(u, k^2) \int_{-\infty}^{\infty} \delta(z_o - (\vec{p} - \vec{u})^2 - k^2) \frac{\varepsilon(z_o)}{u_o + z_o - p_o - i\varepsilon} \, dz_o \]

\[ \varepsilon(z_o) \delta(z_o^2 - \omega^2) = \frac{\delta(z_o - \omega) - \delta(z_o + \omega)}{2\omega} \; ; \; \omega = \sqrt{(\vec{p} - \vec{u})^2 + k^2} \]

gives

\[ \int_{-\infty}^{\infty} dz_o \ldots = \frac{1}{2\omega} \left( \frac{1}{\omega - p_o + u_o - i\varepsilon} - \frac{1}{\omega - p_o + u_o + i\varepsilon} \right) = \frac{-2\omega}{2\omega \left[ -(p_o + u_o - i\varepsilon)^2 - \omega^2 \right]} \]

\[ = \frac{1}{(p_o - u_o + i\varepsilon)^2 - (\vec{p} - \vec{u})^2 - k^2} = \frac{1}{(p + i\varepsilon - \mu)^2 - k^2} \]

where \( \varepsilon \) is a real, infinitesimal four vector \( \in L^+ \). Thus we find for the retarded commutator:

\[ \tilde{f}_R(x) = \langle P, \alpha \mid RA(\vec{x}) B(-\vec{x}) \mid q, \beta \rangle \] has a Fourier transform

\[ f_R(q) = \int \frac{e^{iqx}}{2\pi} \tilde{f}_R(x) \, d^4x \, , \]

which can always be written

\[ f_R(q) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int d^4u \int dk^2 \varphi(u, k^2; P\alpha q, \beta) \frac{\omega}{(q + i\varepsilon - u)^2 - k^2} \; ; \; \varepsilon \in L^+ \text{ real} \] (74')

where \( \varphi(u, k^2; P\alpha q, \beta) \) is a weight function, which is zero for \( (u, k^2) \in \mathcal{S} \), defined in (73). In fact \( \varphi(u, k^2; P\alpha q, \beta) \) is the same as that for the ordinary commutator (74).
Note: In (74) and (74') the integrals may not converge. In such cases \( \Phi(u, k^2) \) has to be divided by a polynomial in \( k^2 \) of sufficient degree, say \( n \). Then \( f_R(q) \) and \( f(q) \) resp. are defined only up to an arbitrary polynomial in \( q \) of degree \( 2(n-1) \).

iii) Some properties of vacuum amplitudes in two-particle processes.

We shall consider a scattering process

\[ p + k \rightarrow p' + k' \]  

(76)

of two scalar neutral particles of masses \( p^2 = p'^2 = m^2 \) ("nucleons") and \( k^2 = k'^2 = \mu^2 \) ("mesons") with fields \( \Psi(x) \) and \( A(x) \) respectively. From (45) we have for the scattering amplitude

\[ 2\pi i \delta(p + k - p' - k'). T(p', k'; p, k) = -\frac{i}{(2\pi)^3} \int d^4x \int d^4x' \tilde{e}^{(k'x - k'x')} \langle p'\mid R'A(x')A(x)\mid p \rangle \]

(77)

where \( R' \) is defined by

\[ R'A(x')A(x) = -i\left(\partial'\partial'\gamma_5\partial\cdot\partial\mu\right)\delta(x' - x)\left[A(x'), A(x)\right] = (\partial'\partial'\gamma_5\partial\cdot\partial\mu)RA(x')A(x) \]

Using translation invariance as in (59) gives

\[ 2\pi i \delta(p + k - p' - k'). T' = -\frac{i}{(2\pi)^3} \int d^4x \int d^4x' \tilde{e}^{(k'x - k'x')} \tilde{e}^{(p'x - px)} \frac{x + x'}{2} \langle p'\mid R'A(\frac{x'}{2})A(-\frac{x}{2})\mid p \rangle \]

Introducing \( \frac{x + x'}{2} = y \) and \( x' - x = z (\rightarrow x) \), we can take (the \( \delta \)-function "cancels")

\[ T'(p', k'; p, k) = -\int d^4x \tilde{e}^{ikx} \frac{b + b'}{2} \langle p'\mid R'A(\frac{x}{2})A(-\frac{x}{2})\mid p \rangle \]  

(78)

This formula is already sufficient for the derivation of forward dispersion relations under the condition that the mass spectrum has a particular form:
\( \mu \) is the lowest mass, followed by \( M \). The continuum begins at \( 2\mu \). If two-meson states are excluded (as is the case in some sums over intermediate states) then the "meson-nucleon" system has possibly discrete bound states with \( N \leq m^* \leq M + \mu \) and a continuum extending from \( M + \mu \) to \( \infty \).

How under these conditions a forward dispersion relation can be derived is shown in detail in Appendix 6. For that derivation one needs an explicit form of the imaginary part of \( T \), which will prove to be useful also for our present purposes. Therefore, we shall derive it here:

\[
\text{Im } T = \frac{1}{2} (T - T^*)
\]

For the construction of \( T^* \) we observe \( \sqrt{\text{proof below}} \):

a) that \( R''A(\frac{x}{2})A(-\frac{x}{2}) \) is hermitian,

b) that \( T \) remains invariant under the exchange of \( p \) and \( p' \).

From (78)

\[
T^* = -\int d^4x \ e^{i\frac{p + p'}{2}} \langle p \mid (R''A(\frac{x}{2})A(-\frac{x}{2}))^* \mid p' \rangle
\]

With a) and b) and changing \(-x\) into \(x\) we obtain

\[
T^* = -\int d^4x \ e^{i\frac{p + p'}{2}} \langle p' \mid R''A(\frac{x}{2})A(\frac{x}{2}) \mid p \rangle
\]

hence

\[
T^* - T^* = -\int d^4x \ e^{i\frac{p + p'}{2}} \langle p' \mid R''A(\frac{x}{2})A(-\frac{x}{2}) - R''A(-\frac{x}{2})A(\frac{x}{2}) \mid p \rangle
\]

Furthermore, we shall show below that

c) \( R''A(\frac{x}{2})A(-\frac{x}{2}) - R''A(-\frac{x}{2})A(\frac{x}{2}) = -i \left[ j(\frac{x}{2}), j(-\frac{x}{2}) \right] \)

Therefore we obtain

\[
\text{Im } T(p'; p) = \frac{1}{2} \int d^4x \ e^{i\frac{p + p'}{2}} \langle p' \mid \left[ j(\frac{x}{2}), j(-\frac{x}{2}) \right] \mid p \rangle
\]

(79)
Proof of a), b) and c):

We first prove b). From the explicit form of \( T \) in (76) we read off that \( T \) depends on the four vectors

\[ k + k', \ p \ \text{and} \ p' \]

Since it is relativistically invariant, it can depend only on the invariant products

\[ (k + k')^2; \ (k + k')_p; \ (k + k')_{p'}; \ pp'; \ p^2 = p'^2 = M^2 \]

However, from \( p + k = p' + k' \) follows \( p - p' = k' - k \) and by multiplication by \( k + k' \) \((k + k')(p - p') = k'^2 - k^2 = 0\). Therefore the second and third of the above products are equal. Hence all possible variables on which \( T \) can depend rest invariant under the exchange of \( p \) and \( p' \) and so will \( T \).

We now prove a) and c). The exact definition of \( R' A(\frac{\pi}{2}) A(-\frac{\pi}{2}) \) is the following:

\[ R'(\frac{\pi}{2}) A(-\frac{\pi}{2}) = -i \left\{ \left( \partial_{\mu} \mu^2 \right)(\partial_{\nu} \mu^2) \Theta(u_t \nu_t) \left[ A(u), A(v) \right] \right\} u \Rightarrow \frac{\pi}{2}, \ v \Rightarrow -\frac{\pi}{2} \]

\[ R'(\frac{\pi}{2}) A(-\frac{\pi}{2}) \]

restoring the old position inside the commutator gives back the - sign and thus \( R'(\frac{\pi}{2}) A(-\frac{\pi}{2}) \).

\[ R'(\frac{\pi}{2}) A(-\frac{\pi}{2}) A(\frac{\pi}{2}) - R'(\frac{\pi}{2}) A(-\frac{\pi}{2}) A(\frac{\pi}{2}) = \]

\[ -i \left\{ \left( \partial_{\mu} \mu^2 \right)(\partial_{\nu} \mu^2) \Theta(u_t \nu_t) \left[ A(u), A(v) \right] - \Theta(u_t \nu_t) \left[ A(u), A(v) \right] \right\} u \Rightarrow \frac{\pi}{2}, \ v \Rightarrow -\frac{\pi}{2} \]

\[ = -i \left\{ \left( \partial_{\mu} \mu^2 \right)(\partial_{\nu} \mu^2) \Theta(u_t \nu_t) + \Theta(u_t \nu_t) \left[ A(u), A(v) \right] \right\} u \Rightarrow \frac{\pi}{2}, \ v \Rightarrow -\frac{\pi}{2} \]

\[ = -i \left[ \gamma(\frac{\pi}{2}), \gamma(-\frac{\pi}{2}) \right]. \]
It is convenient to define another function $M(p'k'; pk)$ by

$$M(p'k', pk) = \int \frac{d^4x}{2} e^{ikx} <p' | \hat{f}(\frac{x}{2}) \hat{f}(\frac{-x}{2}) | pk>$$  \hspace{1cm} (80)$$

This gives with (79)

$$\text{Im} \ T(p'h', pk) = \frac{1}{2} \left[ M(p'h', pk) - M(p', k, p, -k) \right]$$  \hspace{1cm} (81)$$

Formulæ (78), (79), (80) and (81) together with the Dyson representation (74), (74') will be the basis of the following derivation of dispersion relations for non-forward scattering.

iv) Dispersion relations for non-forward scattering.

In what follows we shall frequently change the variables of our functions $T, M$ or $\text{Im} \ T$ but always retain the same symbols for the functions. The variables we shall deal with, are those treated in detail in Appendix 4, and listed there together in (A4.10).

As we know that $T$ is the Fourier transform of a retarded commutator (78), we have a situation similar to that in the Titchmarsh theorem (A7) or in the proof of classical dispersion relations (A5):

The Fourier transform $F(\omega)$ of a function $f(t)$, which latter vanishes for $t < 0$ is the boundary value of a function $\Phi(\omega)$, which is analytic in the upper half $\omega$-plane. Furthermore $\text{Re}F(\omega)$ and $\text{Im} F(\omega)$ are Hilbert transforms of each other, i.e. they obey dispersion relations.

However, the situation is only similar and it is not possible to apply directly Titchmarsh's theorem. The whole trouble comes from the fact that we deal with a four dimensional Fourier transform instead of a one dimensional one:
\[ T(p'k',pk) = -\int d^4x \ e^{ix \ \frac{p+k}{2}} \langle p^i | R'A(\frac{x}{2})A(-\frac{x}{2}) | p \rangle \]

\[ = -\int_0^\infty dt t^{\frac{k_0+k'_0}{2}} \int d^4x e^{-ix \ \frac{k_0+k'_0}{2}} \langle p^i | R'A(\frac{x}{2})A(-\frac{x}{2}) | p \rangle \]

Though it is true that, with \( \omega = \frac{k_0+k'_0}{2} \) and putting the second integral = \( f(t) \), one has

\[ T = -\int_{-\infty}^\infty f(t)e^{i\omega t} dt ; \quad f(t) = 0 \quad \text{for} \quad t < 0 \]

- it is not true that \( f(t) \) depends apart from \( t \) only on irrelevant variables (parameters); it depends in fact explicitly on \( \omega \) because of the properties of the relativistic four momentum which relate \( k+k' \) to \( k_0+k'_0 = 2\omega \). It is this fact, namely, that

\[ T = -\int_{-\infty}^\infty f(t,\omega)e^{i\omega t} dt \]

which makes it impossible to conclude from \( f(t,\omega) = 0 \) for \( t < 0 \) that

\[ T = -\int_0^\infty f(t,\omega_1+i\omega_2)e^{i\omega_1 t - i\omega_2 t} dt \]

must be analytic in \( \omega_2 > 0 \). We shall see that \( f(t,\omega) \) depends exponentially on a square root involving \( \omega_2^2 \) and thus the damping factor \( e^{-\omega_2 t} \) may be insufficient. This is not just because we have non-forward scattering; the difficulty comes from the mass of the particles and shows up already in the case of forward scattering (see e.g. Appendix 6, after Eq. (64,1)); the above \( f(t,\omega) \) is there equal to

\[ 2 \int_0^\infty dt e^{i\omega(\sqrt{\omega^2 - \mu^2})} \langle M \ | R'A(\frac{x}{2})A(-\frac{x}{2}) | M \rangle \]

In the case of non-forward scattering there arise, however, additional difficulties which we shall discuss now. The reader who is not yet familiar with forward dispersion relations should perhaps now turn first to Appendix 6 and afterwards continue to follow the present development.
The result of a quite long and somewhat complicated investigation will be that in spite of these difficulties $T$ obeys dispersion relations with respect to the variable $\omega = \frac{1}{2}(k_0 + k'_0)$.

As (82) suggests, we should choose $\omega = \frac{1}{2}(k_0 + k'_0)$ as one of our variables. Consequently, we look for a coordinate system, in which this has a simple meaning. Furthermore, we wish to express $\omega$ in an invariant form. All this is done in Appendix 4; the system is the Breit-system or brickwall system [see Fig. A4.4.] and the invariant variables which we may use are

$$\Delta^2 = -\frac{1}{4}(p-p')^2 = -\frac{1}{4}(k-k')^2$$

$$\omega = \frac{(p+p')(k+k')}{2\sqrt{(p+p')^2}}$$

The invariant formulas (83) reduce in the brick-wall system to $\Delta^2 = \frac{1}{4} \left| \Delta_p \right|^2$ where $\Delta_p$ is the change of the nucleon momentum. $\omega$ = energy of the meson before and after the collision $= \frac{1}{2}(k_0 + k'_0) = k_0 = k'_0$. With an unit-three-vector $\vec{e} \perp \vec{p}$ we have [see (A4.41)]

$$p = \left( \vec{p}, \sqrt{\Delta^2 + m^2} \right) \quad ; \quad k = \left( -\vec{p} + \vec{e} \sqrt{\omega^2 - \mu^2 - \Delta^2} , \omega \right)$$

$$p' = \left( -\vec{p}, \sqrt{\Delta^2 + m^2} \right) \quad ; \quad k' = \left( \vec{p} + \vec{e} \sqrt{\omega^2 - \mu^2 - \Delta^2} , \omega \right)$$

with these variables (82) becomes

$$T(\omega, \Delta^2) = -\left( d^4 x e^{i(\omega x - \vec{e} \times \sqrt{\omega^2 - \mu^2 - \Delta^2})} \right) \left< -\vec{p}, \Delta^2 \right| R' A(\xi) A(-\xi) \left| \vec{p}, \Delta^2 \right>$$

We may compare this with the corresponding expressions (A6,1) in Appendix 6, we see that there are two differences: the square root in $k x = \vec{e} \times \sqrt{\omega^2 - \mu^2}$ has here the $-\Delta^2$, which makes it imaginary even for $\omega^2 > \mu^2$ and the matrix element has still a direction $\vec{p}$ in the states, therefore we cannot just integrate over $\xi$. 

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However the main idea, namely to remove the difficulty of the imaginary square root by introducing a convergence factor, is nearly the same: there we introduced a further factor $e^{-\varepsilon |\tau|}$, here we shall make $\mu^2$ negative. There we had to go to $\varepsilon \to 0$ afterwards, here we must go back to the physical value $\mu^2 > 0$.

We shall consider always $\Delta^2$ as a fixed parameter and look for the behaviour of $T$ with respect to $\omega$.

We introduce a new variable $\zeta$ (which replaces $\mu^2 = k^2 = k'^2$) and define a new function $T_1$ of three variables

$$T_1(\omega, \Delta^2; \zeta) = -\int d^4x \ e^{i(\omega x - \tilde{\omega} x \sqrt{\omega^2 - \zeta - \Delta^2})} \langle -\frac{\rho}{\bar{p}} \Delta^2 | R' \bar{A}(\xi) A(-\xi) | \bar{p}, \Delta^2 \rangle$$

(86)

So that $T_1(\omega, \Delta^2, \zeta = \mu^2) \equiv T(\omega, \Delta^2)$.

This new function is analytic in the upper $\omega$ half plane provided we make $\zeta$ negative and $< -\Delta^2$ (of course the "positive" branch of the square root is always understood).

This argument of course implies that the matrix element and the integration do not make the integral behave much worse than the integrand. It is assumed henceforward that $T_1(\omega, \Delta^2, \zeta)$ is bounded by a polynomial. This has not been proven so far, though it seems physically plausible (for one-dimensional scattering Symanzik proved it). If $T_1(\omega, \Delta^2, \zeta)$ is bounded by a polynomial of order $n$, then one has to consider in all what follows not $T$ itself but $T$ divided by a polynomial of order $n+1$ with roots only in the lower $\omega$-half plane. We shall not do it here, since it complicates the formulae without adding anything essential. It has been carried through for illustration in the proof of forward dispersion relations, in Appendix 6.

For unphysical values smaller than $-\Delta^2$ of $\zeta$ we have a dispersion relation and the problem is to show that by analytical continuation we can pass to $\zeta = +\mu^2$.
For this we shall need the explicit knowledge of the domain of analyticity of \( T_1(\omega, \Delta^2, \zeta) \) which shall be derived now:

From (86) it follows that \( T_1 \) is analytic at least in the domain

\[
\mathcal{R}_1 : \quad \text{Im} \, \omega > \left| \text{Im} \sqrt{\omega^2 - \zeta - \Delta^2} \right|.
\]

(87)

This is obvious, since in (86) the matrix element vanishes because of local commutativity whenever \( |\vec{x}| > x_0 \). Hence in the exponent the "damping" term

\[
- x_0 \text{Im} \, \omega + |\vec{x}| \omega \Delta \cdot \text{Im} \sqrt{\quad}
\]

is always negative in \( \mathcal{R}_1 \).

The explicit form of \( \mathcal{R}_1 \) is found in an elementary calculation (see Appendix 5/):

\[
\omega = \omega_1 + i \omega_2 \quad \text{and} \quad \zeta = \zeta_1 + i \zeta_2
\]

\[
\mathcal{R}_1 : \begin{cases}
\omega_2 > 0 & ; \quad \omega_2^2 > \zeta_1 + \Delta^2 \\
2\omega_2(\omega_2 - \sqrt{\omega_2^2 - \zeta_2 - \Delta^2}) < \zeta_2 < 2\omega_2(\omega_2 + \sqrt{\omega_2^2 - \zeta_2 - \Delta^2})
\end{cases}
\]

(871)

Hence, if we make \( \zeta_2 < -\Delta^2 \), then \( \omega_2 > 0 \); \( \omega_2^2 > 0 \); and \( \zeta \) may lie inside a small strip containing the real axis.

For such values of \( \zeta \) and \( \omega \) \( T_1(\omega, \Delta^2, \zeta) \) is analytic in the upper \( \omega \)-half plane. If it is also sufficiently bounded, we may write

\[
T_1(\omega, \Delta^2, \zeta) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im} \, T_2(\omega', \Delta^2, \zeta)}{\omega' - \omega} \, d\omega'; \quad \text{Im} \, \omega > 0 \quad \zeta_1 < -\Delta^2
\]

(88)

This relation is valid in the whole upper half plane and it yields, in the limit \( \omega_2 \to 0 \) the usual dispersion relations (for the proof of these statements see Appendix 7, IV).
v) **Detailed proof of dispersion relations.**

The whole problem is now to show that we may make \( \zeta \rightarrow \mu^2 \) without violating Eq. (88). For this we must write the integrand in such a way that \( \zeta \) appears explicitly. Fortunately, we need only statements about the integrand, i.e. \( \text{Im } T_1 \), and to make those is somewhat simpler than to make statements on \( T_1 \).

a) **Properties of the imaginary part of the scattering amplitude.**

We shall use the Dyson representation of the function \( M \) defined by (80). Also with \( M \) we may pass to the variables \( \omega, \Delta^2 \) define in complete analogy to (86) a function \( M_1(\omega, \Delta^2, \zeta) \) and find from (81) and (83) that

\[
\text{Im } T_1(\omega, \Delta^2, \zeta) = \frac{1}{2} \left[ M_1(\omega, \Delta^2, \zeta) - M_1(-\omega, \Delta^2, \zeta) \right] \tag{89}
\]

From which follows that \( \text{Im } T_1(\omega, \Delta^2, \zeta) \) is odd in \( \omega \).

(88) and (89) together give

\[
T_1(\omega, \Delta^2, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} M_1(\omega', \Delta^2, \zeta) \left[ \frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right] d\omega' \tag{90}
\]

Before we can proceed, we must look a bit more into the structure of \( M_1(\omega', \Delta^2, \zeta) \). According to (80)

\[
M_1(\omega, \Delta^2, \zeta) = \int d^4 x \ e^{i k' \cdot x} \frac{k \cdot k'}{2} \langle p'| j(\frac{\zeta}{2}) j(-\frac{\zeta}{2}) | p \rangle
\]

We consider the general behaviour of \( M_1 \) as a function of \( \omega', \Delta^2, \zeta \) or sometimes as function of \( W^2, K^2, \zeta \) or any other set of the invariant variables (44,10). Consequently \( k \) and \( k' \) lie no longer on the mass-shell \( (k^2 + k'^2 = \zeta) \). Furthermore \( \omega' \) (or sometimes \( W^2 \)) is an integration variable since we are interested in \( M_1 \) under the integral sign in (90). Therefore \( k, k', p \) and \( p' \) no longer refer to the actual process, they are general variables and in order to avoid confusion, we shall rename them by
\[ k \rightarrow Q \]
\[ k' \rightarrow Q' \]
\[ p \rightarrow P \]
\[ p' \rightarrow P' \]

and obtain

\[ M_2(\omega', \Delta^2, \xi) = \int d^nx \ e^{i k \cdot x} \frac{Q + Q'}{2} <P' \mid \hat{3}(\xi) \hat{3}(-\xi) \mid P> \]  \hspace{1cm} (91)

From (44, 10) follows that the variables are

\[ W^2 = (P + Q)^2 = 2\omega' \sqrt{\Delta^2 + M^2 + M^2 + \xi + \Delta^2} \]
\[ K^2 = \frac{(W^2 + M^2 + \xi)^2 - 4M^2W^2}{4W'^2} \]  \hspace{1cm} \text{(not always > 0)}
\[ \cos \Theta = 1 - \frac{2\Delta^2}{K^2} \]  \hspace{1cm} \text{(|cos \Theta| can be > 1)}
\[ \omega' = \frac{(P + P')(Q + Q')}{2 \sqrt{(P + P')^2}} \]
\[ \Delta^2 = -\frac{1}{4} (P - P')^2 = -\frac{1}{4} (Q - Q')^2 = -\frac{1}{4} (p - p')^2 = \frac{1}{2} k^2 (1 - \omega \Theta) \]  \hspace{1cm} \text{(91')}

Since \( \Delta^2 \) is the same on both sides of (90) we keep it fixed with the consequence that \((P - P')^2 = (p - p')^2\). Furthermore, we require \( P^2 = P'^2 = M^2 \) (as already implicitly contained in the relation between \( \omega' \) and \( W'^2 \)).

The expression (91) does not contain a commutator, so the Dyson representation cannot yet be applied. Furthermore, the states contain still \( P \) and \( P' \) and the matrix element depends then of course on the angles between \( \vec{x}, \vec{P}, \) and \( \vec{P}' \).

All this can be removed as follows:
We introduce a complete system of intermediate states $|l, \gamma \rangle \langle l, \gamma |$ such that $\Gamma$ means an eigenstate of the four momentum:

$$M_\Gamma = \sum_{l, \gamma} \int d^4x \ e^{ix \cdot Q + iQ'} e^{iQ \cdot \frac{\gamma}{2}} P l \langle l \gamma | \langle \gamma \rangle \langle l \gamma | P \rangle$$

and use translation invariance \( \Xi(17) \) on p.457 to obtain

$$M_\Gamma = \sum_{l, \gamma} \int d^4x \ e^{ix \cdot Q + iQ'} e^{iQ \cdot \frac{\gamma}{2}} P l \langle l \gamma | \langle \gamma \rangle \langle l \gamma | P \rangle$$

Since $Q + P = Q' + P'$ we obtain from the integration a factor $(2\pi)^d \delta(Q + P - l)$, hence

$$M_\Gamma = (2\pi)^d \sum_{\gamma} \langle P | \langle \gamma \rangle \langle Q + P, \gamma | Q + P, \gamma | P \rangle$$

(92)

Of course $Q + P$ has to be a physical four momentum, i.e. $(Q + P)^2 = M'^2 \geq M^2$, since $P$ and $P'$ denote one-nucleon states (see the arguments below). In fact there is this discrete value, \( \Xi \) there may be bound states $n_1^*, ..., (see Appendix 6) \Xi$ and a continuum beginning at $M + \mu$.

Since $\langle P |$ and $| P \rangle$ are one-particle states, they are identical with the corresponding in- and out-states \( \Xi \) see Eq.(21) \( \Xi \) and we may replace each of the factors in (92) by a commutator, using in- (or out-) fields for the $P$ and $P'$ particle:

Let $\Psi_{in}(\vec{P})$ and $\Psi_{in}^*(\vec{P})$ denote absorption and creation operators, then

$$\langle P | \langle \gamma \rangle \langle Q + P, \gamma | Q + P, \gamma | P \rangle = \langle 0 | \Psi_{in}(\vec{P}) \langle \gamma \rangle \langle l \gamma | P \rangle$$

$$= \langle 0 | \left[ \Psi_{in}(\vec{P}), \langle \gamma \rangle \langle l \gamma | P \rangle \right] + \langle 0 | \langle \gamma \rangle \langle l \gamma | P \rangle \Psi_{in}(\vec{P}) | Q + P, \gamma \rangle$$

The second term is equal to (since $P + Q = P' + Q'$)

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\[ \langle 0 | \hat{j}(0) | Q+P-P', \gamma' \rangle = \langle 0 | \hat{j}(0) | Q', \gamma' \rangle \]

where \( \gamma' \) is some set of quantum numbers, which in general is different from \( \gamma \). The important point, however, is that this term vanishes for \( Q^2 < m^2 \), where \( m \) is the lowest mass for which \( \langle 0 | j(0) | Q', \gamma' \rangle \neq 0 \). If there are no selection rules from elsewhere, which fix this lowest mass \( m \), we may find it from Eq. (57) similar arguments apply to other cases where spectrum properties enter (e.g. below (92)):

In our case, where we have two fields \( A \) and \( \psi \), we will find in the expansion of (57) every kind of product of \( A_{\text{in}} \) and \( \psi_{\text{in}} \) operators in the Wick product. Applying then \( \Box - \mu^2 \) on (57), we see that \( j(0) \) is a sum over Wick products of \( A_{\text{in}} \) and \( \psi_{\text{in}} \) with at least two factors. The lowest term of \( j(0) \) contains already the products of two destruction operators, namely \( A_{\text{in}} A_{\text{in}} \) and \( A_{\text{in}} \psi_{\text{in}} \) and \( \psi_{\text{in}} \psi_{\text{in}} \). The corresponding minimum masses are \( 2 \mu, M+\mu \) and \( 2M \) and the lowest mass is \( m=\text{Min}(2 \mu, M+\mu, 2M) \). Selection rules may even exclude this minimum mass. In \( \pi^-N \) scattering one has not \( m=2 \mu \) but \( m=\mu \) since the mesons are pseudoscalar. In our present model it is of course \( 2 \mu \).

We can write therefore

\[ \langle P' | j(0) | Q+P, \gamma \rangle = \langle 0 | [\gamma_{\text{in}}(P'), j(0)] | P+Q, \gamma \rangle \] (93)

and since this is true for \( Q^2 < (2 \mu)^2 \), it remains valid if we go with \( \xi \rightarrow \mu^2 \). This would not have been the case if we had used the reduction formula in order to eliminate the nucleons \( P \) and \( P' \) from the states and had left the mesons there. One obviously should eliminate always the particles with the lowest mass from the states, then (93) is always fulfilled for physical values of \( Q^2 \) and \( Q' \).

*) One can go to the centre of mass system and, if parity is a good quantum number, one has for two mesons \( \langle 0 | j | E \rangle = 0 \) since the states \( \langle 0 | \) and \( | E \rangle \) do not change under space reflection, but \( j(x)=(\Box - \mu^2)A(x) \) does, as \( A(x) \) is pseudoscalar.
We finally arrive at the form in which Dyson's representation can be applied, by using Eq. (53") and
\[ \begin{align*}
\langle j(0) | \rho(0) \rangle &= \left( \frac{\alpha}{2\pi} \right)^3 \left[ R(x) , \Psi_{\gamma}(\vec{P}) \right] \bigg|_{x=0} \\
\end{align*} \]
From (52) or (52') we have \( R(x) \equiv A(x) \), hence
\[ \left[ \Psi_{\gamma}(\vec{P}) , \rho(0) \right] = \left( \frac{\alpha}{2\pi} \right)^3 \left[ R(x) , \Psi_{\gamma}(\vec{P}) \right] \bigg|_{x=0} \]
From (53") follows then (\( P_o > 0 \) always)
\[ \left[ \Psi_{\gamma}(\vec{P}) , \rho(0) \right] = \left( \frac{\alpha}{2\pi} \right)^3 \int d^4 z \ e^{i\vec{P}\vec{Z}} K_x K_z R A(x) \Psi(z) \bigg|_{x=0} \]
\[ = \left( \frac{\alpha}{2\pi} \right)^3 \int d^4 z \ e^{i\vec{P}\vec{Z}} R A(o) \Psi(z) \bigg|_{x=0} \]
by definition of \( R \) (Eq. (77)). With a corresponding operation \[\int d^4 z \ e^{i\vec{P}\vec{Z}} R A(x) \Psi(z) \bigg|_{x=0} \] applied on \( \langle \varphi_+ P, \varphi | j(0) | 0 \rangle \) one obtains from (92), (93) and (94)
\[ M = (2\pi)^4 \sum_{\gamma} \langle \varphi_+ | \Psi_{\gamma}(\vec{P}) , j(0) | P+Q, \gamma \rangle \langle P+Q, \gamma | \Psi_{\gamma}(\vec{P}) \rangle | 0 \rangle \]
\[ = 2\pi \sum_{\gamma} \int d^4 y \ d^4 z \ e^{i(P_y - P_y)} \langle 0 | R A(o) \Psi(z) | P+Q, \gamma \rangle \langle P+Q, \gamma | R A(o) \Psi(y) | 0 \rangle . \]
Changing \( z \rightarrow -z, \ y \rightarrow -y \), using translation invariance and \( P+Q=P'+Q' \)
we obtain
\[ M = 2\pi \sum_{\gamma} \left[ \int d^4 y \ d^4 z \ e^{i\frac{1}{2}(\vec{P} - \vec{P}')} - \frac{1}{2}(\vec{P} - \vec{P}) \right] \times \]
\[ \times \langle 0 | R A(o) \Psi(-\frac{1}{2}) | P+Q, \gamma \rangle \langle P+Q, \gamma | R A(o) \Psi(-\frac{1}{2}) | 0 \rangle . \]

The Dyson representation may be applied to this form. First, however, we shall introduce another variable which exhibits somewhat more explicitly the role of \( \zeta \).
We shall use \( \sqrt{\text{see } (91)} \)

\[
W' = 2\omega \sqrt{\Delta^2 + M^2 + \zeta + 2\Delta^2}
\]

instead of \( \omega' \) in (90) and obtain

\[
T_1(\omega, \Delta^2, \zeta) = \frac{1}{2\pi} \int \frac{dW'}{M^2} \cdot M_1(W', \Delta^2, \zeta) \times \left\{ \frac{1}{W'^2 M^2 - \zeta - 2\Delta^2 - 2\omega \sqrt{\Delta^2 + M^2}} + \frac{1}{W'^2 M^2 - \zeta - 2\Delta^2 + 2\omega \sqrt{\Delta^2 + M^2}} \right\}
\]

(96)

Actually, the integration should have gone from \(-\infty\) to \(+\infty\), but \(\omega' = (q+p)^2 \gg M^2\) and that is the mass of the intermediate states in (92). Thus \(M_1(W', \Delta^2, \zeta) = 0\) for \(W' < M^2\), so that the integration from \(-\infty\) to \(M^2\) yields nothing. This is no contradiction to Eq. (89); it simply follows that for a given real value of \(\omega\) either \(M_1(\omega)\) or \(M_1(-\omega)\) vanishes and that for positive values of \(\omega\) (in particular for physical values of \(\omega\) and \(\zeta\)) it holds that

\[
\text{Im} T_1(\omega, \Delta^2, \zeta) = \frac{1}{2} M_1(\omega, \Delta^2, \zeta)
\]

(89')

with the consequence that \(M_1(\omega, \Delta^2, \zeta)\) is real.

Now comes the important argument:

b) Analytical continuation to \(\zeta \to \mu^2\)

Let us disregard in (96) for a moment the analytic properties of \(M_1(W', \Delta^2, \zeta)\) and consider only the denominators. The condition that they never vanish is

\[
|\zeta_2| < 2\omega \sqrt{(\Delta^2 + M^2)}
\]

(97)
If \( M_1(W^2, \Delta^2, \zeta) \) were analytic everywhere, then (96) would define a function analytic in (97) which - loosely speaking - is a larger domain than \( R_1 \) (as there is no restriction on \( \omega_1 \)) and which contains part of \( R_1 \). This new function would be equal to the old one in \( R_1 \) and therefore be in fact identically the same. We then could pass to \( \zeta_1 = \mu^2, \zeta_2 \to 0 \) and afterwards \( \omega_2 \to 0 \), thus obtain the physical scattering amplitude on the left hand side and an integral representation on the right hand side, which is equivalent to the desired dispersion relation.

We now make the argument more precise. First we note that what we must require of \( M_1(W^2, \Delta^2, \zeta) \) is only

\[
M_1(W^2, \Delta^2, \zeta) \text{ must be analytic in } \zeta_1 < \mu^2; \quad |\zeta_2| < \delta \]

(with small \( \delta \)) for \( N^2 \leq W^2 \leq \infty \) and \( 0 \leq \Delta^2 \leq \Delta_{\text{max}}^2 \)

In \( W^2 \) we must require all values \( \geq N^2 \), since we have to integrate; in \( \Delta^2 \) there may be a limitation, and we shall see that there is one.

Now assume (98) to be fulfilled, and define by (96) a new function \( T_2(\omega, \Delta^2, \zeta) \) which is then analytic in

\[
\begin{align*}
\zeta_1 & \leq \mu^2 \\
|\zeta_2| & < M_{\text{min}} \{ \delta; 2\omega_2 \sqrt{\Delta^2 + M^2} \} \quad \text{if } \omega_2 > 0
\end{align*}
\]

(99)

Remember that (96) was derived for \( T_1(\omega, \Delta^2, \zeta) \) in \( R_1 \) with the additional restriction \( \zeta_1 < -\Delta^2 \).

The situation now is:
1) We have a function $T_1(\omega, \Delta^2, \zeta)$ which is analytic in $R_1$.

2) $T_1(\omega, \Delta^2, \mu^2) = T(\omega, \Delta^2)$ is the physical scattering amplitude. In this physical amplitude always $\omega = \omega_1 \geq \sqrt{\Delta^2 + \mu^2}$ [see (84), (85)].

3) If one restricts $R_1$ further to $R'_1$ by putting $\zeta_1 \leq -\Delta^2$, then $T_1(\omega, \Delta^2, \zeta)$ obeys an integral representation (96) which is equivalent to a dispersion relation, if $\omega_2 \to 0$.

4) Supposing some properties (98) of $M_1(\mathbf{W}^2, \Delta^2, \zeta)$, we find that one can define a function $T_2(\omega, \Delta^2, \zeta)$ by the same formula (96) which is analytic in $\zeta$ and $\omega$ inside $R_2$.

5) Since $R_1$ and $R_2$ have a finite intersection in which $T_2 = T_1$, both functions must be equal everywhere.

6) Since $T_2 = T_1$ was defined in $R_2$ by the integral (96) it follows that $T_1$ is defined by the same integral not only in $R_1$ but also in $R_2$.

7) $R_2$ contains $\zeta = \mu^2$ (and $\omega_1 \geq \sqrt{\Delta^2 + \mu^2}$) and there $T_1$ tends to the physical scattering amplitude $T$ for $\omega_2 \to 0$. Hence the physical scattering amplitude $T(\omega)$ obeys the dispersion relation which follows if in (88) one puts formally $\zeta = \mu^2$ and lets $\omega_2 \to 0$ [see Appendix 7, IV].

c) **Proof of the required analytical behaviour of $M_1(\mathbf{W}^2, \Delta^2, \zeta)$**.

It remains to prove that $M_1(\mathbf{W}^2, \Delta^2, \zeta)$ has the properties (98) used in the above proof. For this we shall apply the Dyson representation to $M$.

According to (95) we may write

$$M = 2\pi \sum_q \left( d^4x \ e^{ix A \frac{q^P}{2}} \langle 0 | R'(\xi) \psi(-\frac{\xi}{2}) | P + Q, \gamma \rangle \right) \times$$

$$\times \left[ \left( d^4x \ e^{ix A \frac{q^P}{2}} \langle 0 | R'(\xi) \psi(-\frac{\xi}{2}) | P + Q, \gamma \rangle \right)^* \right]$$
where the hermiticity of $R' A(\frac{\lambda}{2}) \psi(\frac{-\lambda}{2})$ has been used (p. 56). We now can copy down directly from (74') the following representation of $M$:

\[
M = \frac{1}{2\pi} \sum_\gamma \left\{ \frac{\phi(u', \lambda^2, p^' + q'^', \gamma')}{(\frac{p^' + q'^'}{2} + i\epsilon - u')^2 - \lambda^2} \right\} \frac{\phi^*(u, \lambda^2, p + q, \gamma)}{(\frac{p + q}{2} - i\epsilon - u)^2 - \lambda^2} d^4u d^4u' d^2\lambda d^2\lambda'^2
\]

where $\epsilon \in L^+$ is meant to tend to zero. We may carry out the sum over $\gamma'$ by defining a new weight function

\[
\phi(u, \lambda^2, u', \lambda'^2, q + p) = \sum_\gamma \phi(u, \lambda^2, q + p, \gamma'). \phi^*(u, \lambda^2, q + p, \gamma')
\]

\[
M = \frac{1}{2\pi} \int \left\{ \frac{d^4u d^4u' d^2\lambda d^2\lambda'}{(\frac{p^' + q'^'}{2} + i\epsilon - u')^2 - \lambda^2} \frac{\phi(u, \lambda^2, u', \lambda'^2, p + q)}{(\frac{p + q}{2} - i\epsilon - u)^2 - \lambda^2} \right\}
\]

The support of $\phi$ in all four variables is given by

\[
\frac{p + q}{2} + u \in L^+; \quad \frac{p + q}{2} - u \in L^+
\]

\[
\lambda > \max \left\{ 0; \frac{u}{2} - \sqrt{(\frac{p + q}{2} + u)^2}; \frac{u}{2} - \sqrt{(\frac{p + q}{2} - u)^2} \right\}
\]

(the same formulae are valid for $u'$ and $\lambda'^2$)

*) Strictly speaking this is not so. In fact we deal here not with simple $R$-products as is supposed in the Dyson representation (74'), but rather with $R'$-products, which are defined in (77). They contain additional Klein-Gordon operators. If we could push them through the step functions, everything would be alright. One may in fact do so, since it can be shown that the effect of differentiating the step functions can be compensated by a further polynomial in the denominator [see Appendix 57].
Notice the remark following (74'). \( m_1 \) and \( m_2 \) are now given by (62), which reads here

\[
m_1^2 = \text{Min} (l^2) \quad \text{such that} \quad \langle 0 | K_{\mu} A | l_{\xi'}, l_{\xi''} > \neq 0 \quad \text{and} \quad \langle l_{\xi'} | K_{\mu} A | P + Q, l_{\xi''} > \neq 0
\]

\[
m_2^2 = \text{Min} (l^2) \quad \text{such that} \quad \langle 0 | K_{\mu} A | l_{\xi'} > \neq 0 \quad \text{and} \quad \langle l_{\xi'} | K_{\mu} A | P + Q, l_{\xi''} > \neq 0
\]

where \( K_{M} \equiv 0 - M^2 \), \( K_{A} \equiv 0 - \mu^2 \). Since \( K_{M} \Psi_{\text{in}} = K_{A} \Psi_{\text{in}} = 0 \) one has \( m_1 = 2\mu; \) \( m_2 = M + \mu \) in our neutral scalar model [see p. 657]. In real \( \pi - N \) scattering one has \( m_1 = 3\mu; \) \( m_2 = M + \mu \).

We shall now eliminate as many as possible of the redundant variables and exhibit explicitly the dependence on \( \zeta \) and \( \Delta^2 \). To this end we go to the centre of mass system and have

\[
\frac{\vec{Q} - \vec{P}}{2} = \vec{Q}; \quad \frac{Q_0 - P_0}{2} = \frac{Q_0 - P_0}{2} = \frac{\zeta - M^2}{2W'}
\]

\[
M(\mathcal{W}', \Delta^2, \zeta) = \frac{1}{2\pi} \int \frac{d\lambda^1 d\lambda^2 d\lambda'}{d^3 u d^3 u'} \psi(u, \mu, \xi, \lambda^1, \lambda^2, \lambda', \mathcal{W}') \left[ \left( \frac{M^2 + \mu^2 - i\epsilon}{2W} \right)^2 - (\vec{Q} - \vec{u} - i\epsilon)^2 - \lambda^2 \right] \left[ \left( \frac{M^2 + \mu^2 - i\epsilon}{2W} \right)^2 - (\vec{Q} - \vec{u})^2 - \lambda'^2 \right]
\]

Since we know that \( M \) can (apart from the "artificial" variable \( \zeta \)) only depend on two invariants, for which we choose \( \mathcal{W}' \) and \( \Delta^2 \) [see (91')], we conclude from

\[
\Delta^2 = \frac{1}{2} K^2 (1 - \cos \theta)
\]

with \( K^2 \cos \theta = \frac{\zeta}{\zeta'} \); that \( M \) can depend on \( \vec{Q} \) and \( \vec{Q}' \) only in the form \( \zeta \zeta' \).

In (101) we see explicitly the \( \Delta^2 \) and \( \zeta \) dependence, since both appear in the denominators whereas the \( \mathcal{W}' \) dependence is hidden in the weight function.
We introduce polar coordinates
\[ \bar{\xi} = \xi(1,0,0) \quad \bar{\eta} = \eta(0,1,0) \]
\[ \bar{\xi} = \xi(0,1,0) \quad \bar{\eta} = \eta(0,1,0) \]
\[ \bar{u} = u'(\sin\theta \cos\phi', \sin\theta' \sin\phi', \cos\phi') \]
and obtain
\[ M = \frac{1}{8\pi K^2} \int \frac{d\lambda' \lambda' d\lambda d\lambda' du du' du'd\eta d\phi d\phi'}{\left[ x'(\xi) - \cos(\phi - \phi') \right] [x(\xi) - \cos \phi]} \]
with
\[ x'(\xi) = \varphi(\ldots) = \varphi(u_0, u_0', \lambda', \lambda, u_0, u_0', \ldots) \]
\[ x = \frac{\lambda^2 u_0^2 + u_0'^2 K^2 - \left( \frac{M^2}{2\lambda} + u_0 + i\epsilon \right)^2}{2 \lambda K u_0 \sin \theta} \]
Next we introduce the new variables
\[ \alpha = \phi' - \phi \]
\[ \beta = \phi' \]
and
\[ u \sin\theta = u_1 \quad \text{later on again called } u \]
\[ u' \sin\theta' = u_1' \quad \text{later on again called } u' \]
and
\[ \lambda_1 \quad \text{and} \quad \lambda_1' \quad \text{(later on called again } \lambda \text{ and } \lambda') \]
such that
\[ \lambda^2 + u^2 = \lambda_1^2 + u_1^2 \]
\[ \lambda_1'^2 + u_1'^2 = \lambda_1'^2 + u_1'^2 \]
The \( \sin \Theta \) and \( \sin \Theta' \) have disappeared and the \( \Theta \) and \( \Theta' \) integrations give only a new weight function \( \tilde{\varphi} \); we obtain

\[
M = \frac{1}{8 \pi k^2} \int d\lambda^2 d\lambda'^2 du du' du'' du''' d\lambda_0 d\lambda_0' \varphi(\lambda, \lambda', u, u', \cos \alpha, W)
\]

\[
\frac{[\tilde{x}'(\xi) - \omega(\beta - \beta')][\tilde{x}(\xi) - \omega(\beta - \alpha)]}{[\tilde{x}'(\xi) - \omega(\beta - \beta')][\tilde{x}(\xi) - \omega(\beta - \alpha)]}
\]

\( \tilde{x} \) and \( \tilde{x}' \) are the same as \( x \) and \( x' \) without the \( \sin \Theta \) and \( \sin \Theta' \) in the denominators. Since \( \beta \) does not appear in \( \tilde{\varphi} \), we may use (see Appendix 8, II)

\[
2\pi \int_0^1 d\beta \frac{1}{x - \omega(\beta - \beta')} \cdot \frac{1}{x - \omega(\beta - \alpha)} = 2\pi \frac{\tilde{x}'/\sqrt{\tilde{x}'^2 - 1} + \tilde{x}/\sqrt{\tilde{x}^2 - 1}}{\tilde{x}' + \sqrt{\tilde{x}'^2 - 1} - \omega(\beta - \alpha)}
\]

and find

\[
M = \int d\lambda^2 d\lambda'^2 du du' du'' du''' d\lambda_0 d\lambda_0' \tilde{\varphi}(u, u', \cos \alpha, W'). \frac{y}{\sqrt{y^2 - K^2}} + \frac{y'}{\sqrt{y'^2 - K^2}}
\]

with

\[
y = \frac{K^2 + \lambda^2 + \nu^2 - (\mu_0 + \lambda_0)}{2 \mu} \quad y' = \frac{K^2 + \lambda'^2 + \nu'^2 - (\mu_0' + \lambda_0')}{2 \mu'}
\]

A somewhat lengthy but elementary discussion shows that the infinitesimal imaginaries \( \pm i \varepsilon \) in \( y \) and \( y' \) have no relevance to the present singularities in the integrand of \( M \) if all variables vary within their prescribed range. We therefore drop the \( \pm i \varepsilon \) entirely and find out under what conditions no singularities are connected with \( \zeta \to \mu^2 \). We must avoid

\[
\begin{align*}
\text{i)} & \quad y = k^2 \quad \text{and} \quad y'^2 = k^2 \\
\text{ii)} & \quad y y' + \sqrt{(y^2 - K^2)(y'^2 - K^2)} - K^2 \omega(\beta - \alpha) = 0
\end{align*}
\]

(103)

This can be decided by inspection of the range of the variables which is given by (100): \( \frac{P + Q}{2} = \frac{W'}{2} \) yields

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\[ 0 \leq \mu \leq \frac{W'}{2} \quad \text{and} \quad |\mu_0| \leq \frac{W'}{2} - \mu \]

\[ \lambda > \max \left\{ 0; \mu_1 - \sqrt{(\frac{W'}{2} + \mu_0)^2 - \mu^2}; \mu_2 - \sqrt{(\frac{W'}{2} - \mu_0)^2 - \mu^2} \right\} \]

(100')

the same for \( u' \) and \( \lambda' \).

This is true for the old variables \( u \) before the substitution on p.72. But for the new variables \( u_1 \) and \( u_1' \) which here again are called \( u \) and \( u' \), the ranges are not larger than those given by (100').

First one has to calculate the minimum of \( y^2 - K^2 \) for \( K^2 = K^2(w', \xi) \geq 0 \) and \( -\Delta^2 \leq \xi \leq \mu^2 \) real, the other variables staying inside their allowed region (100'). The result is for \( y \) and \( y' \)

\[ \min (y^2 - K^2) = \frac{(\mu_1^2 - \mu^2)(\mu_2^2 - \mu^2)}{W'^2 - (\mu_1 - \mu_2)^2} \]

(104)

Since \( m_1 = 2\mu \) (in real \( T-N \) scattering \( m_1 = 3\mu \)) and \( n_2 = M/\mu \) we see that this minimum is always \( > 0 \).

We now solve ii) for \( \cos \delta \) and obtain

\[ K^2 \cos \delta = Y \cos \alpha \pm i \sqrt{Y^2 - K^2}. \sin \alpha \]

\[ Y = y y' + \sqrt{(y^2 K^2)(y'^2 K^2)} \]

(105)

*) The proof of this statement is long and complicated, though elementary. It is found that the minimum is assumed on the boundary given by (100'). The proof is carried through explicitly in Okubo's appendix to Salam's lectures: "Analytic properties of expectation values of products of field operators"; University of Rochester NYO-6796 (1958). The minimum is determined with respect to all varying variables including \( \xi \).
Since from (91'), definition of $K^2$, it follows that as $K^2$ and $\omega S$ are both real, either $\sin \alpha = 0$ or $y^2 < K_A$ is necessary. We shall see below that $y > 0$ and $y' > 0$ are necessary for the existence of dispersion relations. In that case $Y > 0$ and $Y^2 > K^2$, hence only for $\sin \alpha = 0$ a singularity is possible. That yields

$$K^2 \omega S = \pm Y = K^2 \left(1 - \frac{2\Delta^2}{K^2}\right)$$

Since $Y > |K^2|$ we need the + sign for $Y$. Solving for $\Delta^2$ gives

$$\Delta^2 = \frac{1}{2} \left(K^2 + 2y^2 + \sqrt{(y^2-K^2)(y^2-K^2)}\right)$$ (106)

If all variables vary, the r.h.s. varies between a minimum value and $\infty$. Taking $\Delta^2$ smaller than this minimum ensures the non-existence of singularities in $M$ and therefore we have dispersion relations.

We obtain

$$\Delta^2_{\text{max}} = \text{Min} \left\{ \frac{1}{2} \left(K^2 + 2y^2 - K^2\right) \right\} = \text{Min} \left\{ \frac{1}{2} \left(K^2 + 2y^2 - K^2\right) \right\} = \text{Min} \left\{ y^2 \right\}$$

One finds $\text{Min}(y^2) = K^2(W, \xi = \mu^2) + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2}$.

Note that this has nothing to do with $\text{Min}(y^2 - K^2)$ where $K(W, \xi)$ is also varied - here only $y^2(W, \xi, u, \lambda)$ is varied, with the above result, which holds independently of the sign of $K^2(W, \mu^2)$.

$$\Delta^2_{\text{max}} = \text{Min} \left[ K^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2} \right] ; \quad K^2 = \frac{\left[W^2 - (M + \mu)^2\right] \left[W^2 - (M - \mu)^2\right]}{4 W^2}$$

The minimum has to be taken with respect to $W^2$. Its value depends on the process considered, it has to be $\geq 0$. 

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For our model we have $m_1 = 2\mu$; $m_2 = M + \mu$, whereas for real $\pi-N$ scattering $m_1 = 3\mu$; $m_2 = M + \mu$. In that case

$$\Delta_{\text{max}}^2 \approx \frac{\delta \mu}{3} \cdot \frac{2M + \mu}{2M - \mu}.$$

If $N-N$-scattering is taken, one finds that a dispersion relation could be proved only if

$$\frac{\mu}{\lambda} > \sqrt{2} - 1$$

which is not true.

In fact dispersion relations are not necessarily false if the above conditions are not satisfied. It may turn out that the stated limitations have their origin not in nature, but in the way the proofs are done.

vi) One-particle contribution and final form of dispersion relations.

Now we have proven that formula (96) is correct for $\zeta = \mu^2$ we can omit the subscripts $1$ on $T$ and $M$. We have

$$T(\omega, \Delta^2) = \frac{1}{2\pi} \int_{m^2}^{\infty} \frac{dW^{12}{ }^2}{M(W^{12}{ }^2, \Delta^2)} \left[ \frac{1}{W^2 - M^2 - 2\Delta^2 - 2\omega \sqrt{\Delta^4 + M^2}} + \frac{1}{W^2 - M^2 - 2\Delta^2 + 2\omega \sqrt{\Delta^4 + M^2}} \right]$$

The integration can be split into two parts, since there is one contribution from $W^2 = m^2$ (one-particle states), and the contribution from $W^2 \gg (M + \mu)^2$. We shall separate off the single particle contribution. To this end we take the expression for $M(W^2, \Delta^2)$ which we find above Eq. (92) and write the $\delta$-function explicitly:

$$M(W^{12}{ }^2, \Delta^2) = (2\pi)^4 \sum_{\delta \not= \delta} \delta(P + Q - \ell) \langle \phi | \delta | \ell \rangle \langle \ell \delta | \phi(0) | P \rangle.$$
According to \((A2,28')\) we may replace the symbolical \(\sum_{\ell',\delta'} \) by its explicit meaning:

\[
\sum_{\ell',\delta'} |\ell',\delta' \times \ell,\delta| \rightarrow \int_0^{\infty} dE \int \frac{d^4 e}{2\sqrt{E^2 + M^2}} |\ell,\mu \times \ell,\mu|
\]

where \(E^2 = m^2\) and \(\int \) means: sum over discrete masses, and integrate over the continuum. Hence

\[
M(W^2, \Delta^2) = (2\pi)^4 \int_0^{\infty} dE \int \frac{d^4 e}{2\sqrt{E^2 + M^2}} \delta(P+Q-E) <\ell,\mu|j(0)|\ell,\mu|j(0)|P>
\]

\[= M_a(W^2, \Delta^2) + M_b(W^2, \Delta^2)\]

Here \(M_a\) is the contribution from discrete mass values, in fact from \(m=M\). The other term \(M_b\) is the contribution of the continuum. Since \(P'\) and \(P\) are states with at least one nucleon, the continuum begins at \(M_{\mu}\).

Thus we find

\[
M_a(W^2, \Delta^2) = (2\pi)^4 \int \frac{d^4 e}{2\sqrt{E^2 + M^2}} \delta(P+Q-E) <\cdots \times \cdots>
\]

\[= (2\pi)^4 \delta(P+Q_0 - E_0) \int \frac{d^4 e}{2\sqrt{E^2 + M^2}} \delta(P+Q-E) <\cdots \times \cdots>
\]

\[= (2\pi)^4 \frac{\delta(P+Q_0 - \sqrt{(P+Q)^2 + M^2})}{2\sqrt{(P+Q)^2 + M^2}} <\cdots \times \cdots>
\]

Now

\[
\frac{\delta(P+Q_0 - \sqrt{(P+Q)^2 + M^2})}{2\sqrt{(P+Q)^2 + M^2}} = \delta((P+Q)^2 - M^2)
\]

since \(P_0 + Q_0 > 0\).
Hence:

\[ M_a(W, \Delta^2) = (2\pi)^4 \delta((P+Q)^2 - M^2) \langle P' | \bar{\psi}(0) | P+Q \rangle \langle P+Q | \psi(0) | P \rangle \]

\[ = (2\pi)^4 \delta(W - M^2) \langle P' | \bar{\psi}(0) | P+Q \rangle \langle P+Q | \psi(0) | P \rangle \]

The Dyson representation becomes here very simple:

\[ M_a = (2\pi)^4 \delta(W - M^2) \left\{ \int \frac{d^4u \, d^4u' \, d^2 \lambda \, d^2 \lambda' \, \phi(u, \lambda; W') \, \phi^*(u', \lambda'; W')} {[(y^2 - \lambda^2) \, [(y'^2 - \lambda'^2)^2]} \right\] \]

The integral shows no singularities since \( y^2 - \lambda^2 > 0 \). We can pass to \( \zeta = \mu^2 \) and find

\[ M_a = g \cdot \delta(W - M^2) \]

(107)

where \( g \) is the integral for \( W^2 = M^2 \) times \( (2\pi)^4 \). With this \( M_a \) we obtain from (96) the final form of the dispersion relation:

\[ \text{Re} \, T(W, \Delta^2) = \frac{g}{4} \left\{ \frac{1}{W^2 + 4\Delta^2 - M^2 - 2\mu^2} - \frac{1}{W^2 - M^2} \right\} + \frac{1}{\pi} P \int \frac{dW'^2 \, \text{Im} \, T(W', \Delta^2)} {(M+\mu)^2} \left\{ \frac{1}{W'^2 + 4\Delta^2 - 2(M+\mu)^2 + W'^2} + \frac{1}{W'^2 - W^2} \right\} \]

(106)

It is left to the reader to show that for \( \Delta^2 = 0 \) and after the transformation from the centre of mass system to the rest system of the nucleon, this formula is identical to (A6, 19).
vii) Analyticity in $\cos 2\alpha$

The above integral (108) contains for $\Delta^2 \neq 0$ still unphysical contributions. This is seen immediately by inspection of (A4.10):

$$K^2 = \frac{(W^2 + M^2 - \mu^2)}{4W^2} - \frac{4M^2W^2}{4W^2} \quad \text{which can be written} \quad \frac{[W^2 + (M+\mu)^2][W^2 + (M-\mu)^2]}{4W^2} $$

and

$$\Delta^2 = \frac{1}{2} K^2 (1 - \cos 2\alpha)$$

From the latter we conclude $K^2 > \Delta^2$, whereas from the first relation it follows that in the integral in (108), where $W^2$ begins at $(M+\mu)^2$, the values $0 \leq K^2 < \Delta^2$ contribute.

Therefore there are parts of the integral where $\text{Im} T(W', \Delta^2)$ cannot be taken from experiments. In that case one either may find its value there from other processes or - if in the forward dispersion relation ($\Delta^2 = 0$) no unphysical region contributes - one may find it by an analytical continuation from physical regions. That goes as follows:

Write formula (102) with $\xi = \mu^2$ in the form

$$H(W', \Delta^2) = \int d\lambda' d\lambda'' \ldots \overline{\Phi}(u, u', \ldots, \omega, \omega') \cdot \frac{1}{K} \left[ \frac{\sqrt{y^2 - K^2}}{y^2 + \sqrt{(y^2 - K^2)(y^2 + K^2)}} \right] \left(102'\right)$$

The dependence on $\Delta^2$ appears explicitly in $\cos (2\alpha - \alpha)$ and we may find out how $M$ behaves as a function of $\cos \alpha$ in the complex $\cos \alpha$-plane.
Introduce a new variable
\[ z = \frac{y'y^1 + \sqrt{(y^1 - k^1)(y^2 - k^2)}}{\kappa^2} \]
then
\[ M(W', \Delta^2) = \int_{z_0}^{\infty} \int_{0}^{2\pi} \frac{\Phi(z, \omega \alpha, W')}{z - \omega \alpha (\Delta - \kappa)} \, d\omega \, d\alpha \tag{109} \]
and here the lower limit \( z_0 \) follows from p.74, namely
\[ z_0 = 1 + \frac{2(w_z^2 - \mu^2)(w_z^2 - M^2)}{K^2 \left[ W_z^2 - (w_z^2 - \mu_z^2)^2 \right]} \tag{110} \]
For given \( z \) and \( \alpha \), one finds that the denominator vanishes if
\[ x = \Re \cos \delta; \quad y = \Im \cos \delta \]
\[ x = 2 \omega \alpha \]
\[ y = \sqrt{2z^2 - 1} \sin \alpha \]
The singularities lie, if \( \alpha \) varies from 0 to \( 2\pi \), on the ellipse
\[ \frac{x^2}{2z^2} + \frac{y^2}{2z^2 - 1} = 1 \]
with foci at \( x = \pm 1 \) and major semi axis \( z \).

The smallest ellipse is that with \( z = z_0 \). Thus \( M(W', \Delta^2) \) is an analytic function of \( \cos \delta \) inside an ellipse in the \( \cos \delta \)-plane with foci at \( \pm 1 \) and major semi axis \( z_0 \). \([\text{See (110)}]\\)
Since the value of \( M(W, \Delta^2) \) may be measured for the physical values \(-1 \leq \cos \delta \leq 1\), one can continue to the non-physical values:
\[ \Im T(W, \Delta^2) = \frac{1}{2} M(W, \Delta^2) = \frac{W}{2\pi \kappa^2} \sum_{\ell = 0}^{\infty} (2\ell + 1) \Im C^\ell (W) \cdot P^\ell (\cos \delta) \tag{111} \]
is the partial wave expansion of \( M(W, \Delta^2) \). The coefficients \( \Im C^\ell (W) \) may be obtained for physical values of \( \cos \delta \) and (111) gives then the extension to non-physical ones.
APPENDIX 1

Commuting a field operator with a Wick-product

\[ S = 1 + \sum \frac{(-i)^n}{n!} \int \cdots \int \mathcal{U}_n(x_1 \ldots x_n) : A_{in}(x_1) \ldots A_{in}(x_n) : \]

Calculate \( \begin{bmatrix} A_{in}(z), S \end{bmatrix} \).

Using the notation \( A_{in}(x_i) \equiv A_i \) we see that we need

\[ \begin{bmatrix} A_z, :A_1 \ldots A_n: \end{bmatrix} \]

The Wick-product is defined by the rule:

Write \( A_1 \ldots A_n = (A_1^+ + A_1^-)(A_2^+ + A_2^-) \ldots (A_n^+ + A_n^-) \), multiply out \( (2^n \text{ terms}) \) and bring in each term all destruction operators \( A_i^+ \) to the right hand side.

In carrying out this operation one ignores the existence of commutation rules. The result is \( :A_1 A_2 \ldots A_n: \).

Consider one particular \( A_i^+ = A_1^+ + A_i^- \). Obviously one half of all terms will contain the \( A_i^+ \) and the other half contains the \( A_i^- \). Since the \( A_i^- \) commute among themselves as well as the \( A_i^+ \) do, one can write for any \( i \):

\[ :A_1 \ldots A_n: = A_i^- :A_1 \ldots A_{i-1} A_{i+1} \ldots A_n: + :A_1 \ldots A_{i-1} A_{i+1} \ldots A_n: A_i^+ \quad (A1,1) \]

We introduce a shorthand notation

\[ :A_1 \ldots A_n: \equiv W_n^{-1}; \quad :A_{i-1} A_{i+1} \ldots A_n: \equiv \frac{\partial W_n}{\partial A_i} \quad (A1,2) \]
The latter one being the Wick-product of \( n-1 \) operators \( A_1 \) missing. \( W_n \) is of course a function of all operators \( A_1^+ \cdots A_n^+, A_1^- \cdots A_n^- \).

Now, \( [A_z, A_1^\pm] \) is a c-number. Therefore, according to a general rule

\[
[A_z, W] = \sum_{i=1}^{n} \left[ A_z, A_i^- \right] \frac{\partial W}{\partial A_i^-} + \left[ A_z, A_i^+ \right] \frac{\partial W}{\partial A_i^+}.
\]

But from (A1,1) and (A1,2) follows

\[
\frac{\partial W}{\partial A_i^+} = \frac{\partial W}{\partial A_i^-} = \frac{\partial W}{\partial A_i},
\]

hence

\[
[A_z, W] = \sum_{i=1}^{n} \left[ A_z, A_i \right] \frac{\partial W}{\partial A_i} = i \sum_{i=1}^{n} \Delta(z-x_i) \frac{\partial W_n}{\partial A_i} \quad \text{(A1,3)}
\]

Therefore

\[
\left[ A_{in}(z), S \right] = \sum_n \frac{i(-i)^n}{n!} \sum_k d_x^1 \cdots d_x^n \Delta(z-x) \tau_n(x_1 \cdots x_n) \frac{\partial W_n}{\partial A_k}
\]

Now, \( \tau_n(x_1 \cdots x_n) \) and \( W_n \) are symmetric in all variables. Therefore we obtain \( n \) times the same integral as \( k \) runs from 1 to \( n \) and we may change the names of the variables such that always \( x_n \rightarrow x_1 \) and finally call \( x_1 = \xi \).

Then

\[
\left[ A_{in}(z), S \right] = \sum \frac{(-i)^{n-1}}{(n-1)!} \int \Delta(z-\xi) d_x^1 d_x^2 \cdots d_x^n \tau_n(x_2 \cdots x_n) \frac{\partial W_n}{\partial A_{in}(\xi)}
\]

By changing the labels to \( x_1 \cdots x_{n-1} \) and going from \( n-1 \rightarrow n \) we obtain
\[
\left[ A_\text{in}(z), S \right] = \int d^4 \xi \, \Delta(z-\xi) \cdot \sum \frac{(-i)^n}{n!} \int d^q \xi_1 \ldots d^q \xi_n \, \tau_{n+1}^{\dagger}(\xi_1 \ldots \xi_n) : A_\text{in}(x_1) \ldots A_\text{in}(x_n) :
\]

The curled bracket may be written in shorthand notation as

\[
i \frac{\delta S}{\delta A_\text{in}(\xi)}
\]

and we have then

\[
\left[ A_\text{in}(z), S \right] = \int d^4 \xi \, \Delta(z-\xi) \, i \frac{\delta S}{\delta A_\text{in}(\xi)}
\]
APPENDIX 2

Complete orthonormal systems for the Klein-Gordon equation.

(metric \( x^2 = x^2 - x_0^2 \))

I. Construction of a complete system of the homogeneous equation.

We want a complete system by which every solution of \((\Box - m^2)f = 0\) can be built up. In order to find a complete set of solutions we consider the most general solution of the homogeneous equation: \((\Box - m^2)f(x) = 0\). We can write

\[
\mathcal{F}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \Delta(k^2 + m^2) \mathcal{\tilde{f}}(k) e^{i k \cdot x}
\]

(A2,1)

by which of course \(\mathcal{\tilde{f}}(k)\) is uniquely defined only on the mass shell. Using

\[
\delta\left( f(x) \right) = \sum_i \frac{\delta(x - x_i)}{|f'(x)|_{x = x_i}}
\]

where \(f(x_i) = 0\) we obtain with \(\omega \rightarrow \sqrt{k^2 + m^2}\)

\[
\mathcal{F}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\delta(k^2 + \omega) + \delta(k^2 - \omega)}{2\omega} \mathcal{\tilde{f}}(k, k_0) e^{i(k^2 - k_0^2)/2} dk \text{d}k_0
\]

\[
= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{2\omega} e^{i\vec{k} \cdot \vec{x}} \left\{ \mathcal{\tilde{f}}(\vec{k}, \omega) e^{-i\omega x_0} + \mathcal{\tilde{f}}(\vec{k}, -\omega) e^{i\omega x_0} \right\}
\]

The solution splits into

\[
\mathcal{F}(x) = \mathcal{F}^+(x) + \mathcal{F}^-(x)
\]
\[ f^+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{i\vec{k} \cdot \vec{x} - \omega t} \tilde{f}(\vec{k}, \omega) = \frac{1}{(2\pi)^3} \int \delta(k^2 + \omega^2) \Theta(x) e^{ikx} \tilde{f}(k) \, dk \]

(A2,2)

\[ f(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{i\vec{k} \cdot \vec{x} + \omega t} \tilde{f}(\vec{k}, \omega) = \frac{1}{(2\pi)^3} \int \delta(k^2 + \omega^2) \Theta(x) e^{ikx} \tilde{f}(k) \, dk \]

with

\[ \Theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} \]

We call \( f^+ \) and \( f^- \) functions of the positive and negative class respectively.

As the second representation shows, this splitting is invariant under all Lorentz transformations without time reflection. Each of these two functions is expressed as a three-dimensional integral. The fact that the spectral functions \( \tilde{f}(k, \omega) \) and \( \tilde{f}(k, -\omega) \) can be developed with respect to any three-dimensional complete system, serves now to construct the complete set of solutions of the homogeneous equation. Consider first the positive component \( f^+(x) \) of \( f(x) \). Put

\[ \tilde{f}(\vec{k}, \omega) = \sum_\alpha c_\alpha \varphi_\alpha(\vec{k}) \]

where

\[ \frac{1}{2\omega} \sum_\alpha \varphi^*_\alpha(\vec{k}) \varphi_\alpha(\vec{k'}) = \delta(\vec{k} - \vec{k'}) \]

(A2,2')

\[ \int \frac{\varphi^*_\alpha(\vec{k}) \varphi_\beta(\vec{k})}{2\omega} \, \frac{d^3k}{2\omega} = \delta_{\alpha \beta} \]

Thus

\[ \sum_\alpha \frac{\varphi_\alpha(\vec{k})}{\sqrt{2\omega}} = \Psi(\vec{k}) \]

is a complete system in the usual sense. Then

\[ f^+(x) = \sum_c [\frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{i\vec{k} \cdot \vec{x} - \omega t} \varphi_\alpha(\vec{k})] = \sum_\alpha c_\alpha f^+_\alpha(x) \]
The square bracket defines \( f_\alpha^+(x) \). The system \( \{ f_\alpha^+(x) \} \) is obviously sufficient to construct any given \( f^+(x) \), i.e. it is complete with respect to solutions of the positive class.

We could now do the same for \( f^-(x) \) but is is simpler to observe that

\[
\tilde{f}_\alpha^+(x) = \frac{4}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{2\omega} e^{-i(\vec{k} \cdot \vec{x} - \omega t)} \varphi_\alpha^*(-\vec{k}) = \frac{4}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{2\omega} e^{i(\vec{k} \cdot \vec{x} + \omega t)} \varphi_\alpha^*(\vec{k})
\]

which means that

\[
\text{if } f_\alpha^+(x) \text{ is of the positive class, then } f_\alpha^{+*}(x) \text{ is of the negative class.}
\]

Therefore, if \( \{ f_\alpha^+(x) \} \) is complete with respect to the positive class solutions, the system \( \{ f_\alpha^{+*}(x) \} \) is complete with respect to the negative class solutions.

We may omit the + of \( f_\alpha^+(x) \) and simply call it \( f_\alpha(x) \); then we can state:

\[
\text{If } \left\{ \frac{\varphi_\alpha(\vec{k})}{\sqrt{2\omega}} \right\} \text{ is a complete orthonormal system in the usual sense and}
\]

\[
f_\alpha(x) = \frac{4}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{2\omega} \varphi_\alpha(\vec{k}) \ e^{-i(\vec{k} \cdot \vec{x} - \omega t)} = \frac{4}{(2\pi)^{3/2}} \int d^3 \vec{k} \delta(\vec{k}^2 - \omega^2) \Theta(\omega) \varphi_\alpha(\vec{k}) \ e^{i\vec{k} \cdot \vec{x}}
\]

then \( \{ f_\alpha(x) \} \) and \( \{ f_\alpha^*(x) \} \) together form a complete system with respect to the solutions of the homogeneous Klein-Gordon equation.

(A2,4)
II. Two kinds of scalar products and orthonormality.

Since the system \( \{ f^\alpha(x), f^\beta(x) \} \) is complete with respect to the solutions of the homogeneous Klein-Gordon equation, we may write for any such solution

\[
f(x) = \sum \alpha_\alpha f^\alpha(x) + b_\alpha f^\beta(x)
\]

But now we need a method to calculate \( a_\alpha \) and \( b_\alpha \). This can be done if the \( f^\alpha \) and \( f^\beta \) obey some orthogonality relation. To this end we must define a scalar product and see whether the functions are orthogonal.

The scalar product must of course be constant in time, since the coefficients \( a_\alpha \) and \( b_\alpha \) are constant.

From elementary quantum mechanics we remember that the expression

\[
\int \left( f^\beta \frac{\partial g}{\partial x_\rho} - g \frac{\partial f^\beta}{\partial x_\rho} \right) d^3 x
\]

is time independent if \( f \) and \( g \) obey the homogeneous Klein-Gordon equation. With a normalizing constant \( K \) and using (A2.4), we obtain

\[
K \int \left( f^\beta \frac{\partial g}{\partial x_\rho} - g \frac{\partial f^\beta}{\partial x_\rho} \right) d^3 x =
\]

\[
= K \frac{i}{(2\pi)^3} \int \frac{d\vec{\kappa}}{2\omega} \frac{d\vec{\rho}}{2\omega} \varphi^\alpha(\vec{\kappa}) \varphi^\beta(\vec{\rho}) e^{i(\vec{\kappa} - \vec{\rho}) \cdot x} e^{i(\omega_\kappa - \omega_\rho) x_\rho} \left( -i \omega_\kappa \cdot i \omega_\rho \right) d^3 x
\]

The \( x \)-integration yields \( \delta(\vec{\kappa} - \vec{\rho}) \) and then the \( \vec{\rho} \)-integration gives

\[
= -i K \int \frac{d\vec{\kappa}}{2\omega} \varphi^\alpha(\vec{\kappa}) \varphi^\beta(\vec{\kappa}) = i K \delta_{\alpha\beta}
\]

In one special case one prefers another normalization, namely if one lets the \( f^\alpha \) degenerate to plane waves.
According to (A2,2') and the following formula one should put then

\[ \Phi_{\tilde{\mathbf{k}}}(\tilde{\mathbf{k}}') = \Phi_{\tilde{\mathbf{k}}'}(\tilde{\mathbf{k}}) : \]

\[ \Phi_{\tilde{\mathbf{k}}'}(\tilde{\mathbf{k}}') = \sqrt{2\omega} \delta(\tilde{\mathbf{k}} - \tilde{\mathbf{k}}') \]

and obtain

\[ \int \Phi_{\tilde{\mathbf{k}}'}^*(\tilde{\mathbf{k}}) \Phi_{\tilde{\mathbf{k}}''}(\tilde{\mathbf{k}}) \frac{d\tilde{\mathbf{k}}}{2\omega} = \delta(\tilde{\mathbf{k}}' - \tilde{\mathbf{k}}'') \]

\[ \int \Phi_{\tilde{\mathbf{k}}'}^*(\tilde{\mathbf{k}}') \Phi_{\tilde{\mathbf{k}}'''}(\tilde{\mathbf{k}}') \frac{d\tilde{\mathbf{k}}'}{2\omega} = \delta(\tilde{\mathbf{k}}' - \tilde{\mathbf{k}}''') \]

One would obtain then

\[ f_{\tilde{\mathbf{k}}}(x) = f_{\tilde{\mathbf{k}}'}{\tilde{\mathbf{k}}'}(x) : \]

\[ f_{\tilde{\mathbf{k}}'}{\tilde{\mathbf{k}}'}(x) = \frac{e^{i((\tilde{\mathbf{k}} \cdot \tilde{\mathbf{x}}) - \omega t)}}{\sqrt{2\omega} \cdot (2\pi)^{3/2}} \]

and this is not an invariant function. That does not exclude its use, but an invariant set is sometimes more convenient, hence one puts

\[ f_{\tilde{\mathbf{k}}'}{\tilde{\mathbf{k}}'}(x) = \frac{e^{i((\tilde{\mathbf{k}} \cdot \tilde{\mathbf{x}}) - \omega t)}}{(2\pi)^{3/2}} \]

which amounts to

\[ \Phi_{\tilde{\mathbf{k}}'}{\tilde{\mathbf{k}}'}(\tilde{\mathbf{x}}) = 2\omega \delta(\tilde{\mathbf{k}} - \tilde{\mathbf{k}}') . \]

The orthogonality and completeness relations thus obtain an additional factor 2\omega on the right hand side. Putting \( K = i \) we may therefore define a scalar product by

\[ (f, q) = i \left( \int f^* \frac{\partial q}{\partial x_0} - q \frac{\partial f^*}{\partial x_0} \right) d\tilde{x} \equiv i \int f^* \frac{\partial}{\partial x_0} q \; d\tilde{x} \quad (A2,5) \]
With this definition we obtain

\[
\begin{align*}
(f_\alpha, f_\rho) &= \delta_{\alpha\rho} \\
(f_\alpha^*, f_\rho^*) &= 2\omega\delta(\vec{k} - \vec{k}') \\
(f_\alpha^*, f_\rho) &= 0 \\
(f_\alpha, f_\rho^*) &= -\delta_{\alpha\rho} \\
(f_\alpha^*, f_\rho^*) &= -2\omega\delta(\vec{k} - \vec{k}')
\end{align*}
\] (A2.6)

This scalar product has the usual properties:

\[
(f, g) = (g, f)^* \quad \text{and} \quad (\lambda f, g) = \lambda (f, g)
\] (A2.7)

but it is not positive definite as follows from (A2.6). We see furthermore from (A2.6):

The space of functions of positive class is orthogonal to the space of functions of negative class.

\] (A2.8)

It is clear now that if \( f(x) \) is a solution of the homogeneous equation, we have with constant coefficients

\[
\begin{align*}
f(x) &= \sum a_\alpha f_\alpha(x) + b_\alpha f_\alpha^*(x) \equiv f_+(x) + f_-(x) \\
a_\alpha &= (f_\alpha, f) \\
b_\alpha &= -(f_\alpha^*, f)
\end{align*}
\] (A2.9)

\[
(f, g) = \sum a_\alpha^{(f)} a_\alpha^{(g)} - b_\alpha^{(f)} b_\alpha^{(g)} = (f_+, g_+) + (f_-, g_-)
\]

The fact that the positive solutions are orthogonal to the negative ones, enables us to define another scalar product

\[
\begin{align*}
\langle f, g \rangle &= (f_+, g_+) - (f_-, g_-) \\
\langle f_\alpha, f_\rho \rangle &= (f_\alpha, f_\rho) = \delta_{\alpha\rho} \\
\langle f_\alpha^*, f_\rho^* \rangle &= -(f_\alpha^*, f_\rho^*) = \delta_{\alpha\rho}
\end{align*}
\] (A2.10)
Hence

\[ \hat{f}(\mathbf{x}) = \sum q_x \hat{f}_x + b_x \hat{f}_x^* = \hat{f}_+ + \hat{f}_- \]

\[ a_x = \langle \hat{f}_x, \hat{f} \rangle \quad b_x = \langle \hat{f}_x^*, \hat{f} \rangle \]

\[ \langle \hat{f}, \hat{q} \rangle = \sum a_x^* \langle \hat{q} \rangle \quad b_x^* \langle \hat{q} \rangle \]

Here \( \langle \hat{f}, \hat{f} \rangle > 0 \) for \( \hat{f} \neq 0 \).

Since the Fourier representation is always of particular interest, we write the two scalar products explicitly for that case: With (A2,2) we find easily

\[ (\hat{f}, \hat{q}) = (\hat{f}_+, \hat{q}_+) + (\hat{f}_-, \hat{q}_-) = \int \frac{d \mathbf{k}}{2 \omega} \left[ \hat{f}(\mathbf{k}, \omega) \hat{q}(\mathbf{k}, \omega) - \hat{f}^*(\mathbf{k}, \omega) \hat{q}^*(\mathbf{k}, \omega) \right] \]

\[ \langle \hat{f}, \hat{q} \rangle = (\hat{f}_+, \hat{q}_+) - (\hat{f}_-, \hat{q}_-) = \int \frac{d \mathbf{k}}{2 \omega} \left[ \hat{f}(\mathbf{k}, \omega) \hat{q}(\mathbf{k}, \omega) - \hat{f}^*(\mathbf{k}, \omega) \hat{q}^*(\mathbf{k}, \omega) \right] \]

One can also write

\[ (\hat{f}, \hat{q}) = \int d^4 \mathbf{k} \delta(\mathbf{k}^2 + m^2) \mathcal{E}(\mathbf{k}) \hat{f}^*(\mathbf{k}) \hat{q}(\mathbf{k}) \]

\[ \langle \hat{f}, \hat{q} \rangle = \int d^4 \mathbf{k} \delta(\mathbf{k}^2 + m^2) \hat{f}^*(\mathbf{k}) \hat{q}(\mathbf{k}) \]

\[ \mathcal{E}(\mathbf{k}) = \begin{cases} 1 & \text{if } \mathbf{k}_0 > 0 \\ -1 & \text{if } \mathbf{k}_0 < 0 \end{cases} \]

III. Representation of an arbitrary function by \( \{ f_x, f_x^* \} \)

Suppose we have an arbitrary function \( F(x) \). We may represent it first at a fixed time \( t = t_0 \); all that we need is to take that solution \( f(x) \) of the homogeneous Klein-Gordon equation, which fulfills
\[ F(x)_{t=t_0} = \dot{f}(x)_{t=t_0} \]

\[ \ddot{F}(x)_{t=t_0} = \ddot{f}(x)_{t=t_0} \]

We call this solution \( f(x, t_0) \) and expand it:

\[ f(x, t_0) = \sum a_\alpha(t_0) \hat{f}_\alpha + b_\alpha(t_0) \hat{f}_\alpha^* \]

We have then from above:

\[ \dot{f}(x, t_0) = \sum a_\alpha(t_0) \dot{\hat{f}}_\alpha + b_\alpha(t_0) \dot{\hat{f}}_\alpha^* \]

The coefficients are not differentiated since \( t_0 \) is a fixed parameter labelling the solution. But this may now be done at any time \( t \); thus any function \( F(x) \) has the representation:

\[ F(x) = \sum a_\alpha(x_0) \hat{f}_\alpha + b_\alpha(x_0) \hat{f}_\alpha^* \equiv F_+ + F_- \]

\[ \frac{\partial F(x)}{\partial x_0} = \sum a_\alpha(x_0) \frac{\partial \hat{f}_\alpha}{\partial x_0} + b_\alpha(x_0) \frac{\partial \hat{f}_\alpha^*}{\partial x_0} \]  \( \text{(A2,13)} \)

Since the coefficients are not differentiated, we still have orthogonality of \( F_+ \) and \( G_- \), hence we retain the formulae (A2,9) and (A2,11) with the difference that the coefficients are time-dependent.

Note that \( F = F_+ + F_- \) in (A2,13) means here a definition of \( F_+ \) and \( F_- \) and no longer implies that e.g. \( F_+ \) contains positive frequencies only.
IV. Completeness and orthonormality relations of \( \{ f_\alpha(x), f_\alpha^*(x) \} \).

For a three-dimensional complete systems one has the relations

\[
\begin{align*}
\int \psi_\alpha^*(\vec{x}) \psi_\rho(\vec{x}) \, d\vec{x} &= \delta_{\alpha\rho} \quad \text{(orthonormality)} \\
\sum_\alpha \psi_\alpha^*(\vec{x}') \psi_\alpha(\vec{x}) &= \delta(\vec{x}-\vec{x}') \quad \text{(completeness)}
\end{align*}
\]  

(A2,14)

Our set \( \{ f_\alpha, f_\alpha^* \} \) is not complete in a strict sense with respect to all (reasonable) four-dimensional functions, but only with respect to the solutions of the homogeneous Klein–Gordon equation since only these can be represented by constant coefficients, whereas in the general case the coefficients are time dependent. We therefore cannot expect the same kind of relations as (A2,14) to hold for \( \{ f_\alpha, f_\alpha^* \} \). In fact, the analogon to the first of the above relations was in our case (A2,5-6)

\[
i \int \left( f_\alpha^* \frac{\partial f_\rho}{\partial x_0} - f_\rho \frac{\partial f_\alpha^*}{\partial x_0} \right) d\vec{x} = \delta_{\alpha\rho}
\]

In analogy to that, we may try for the "completeness relation"

\[
i \sum_\alpha \left[ f_\alpha^*(x') \frac{\partial f_\alpha(x)}{\partial x'_0} - f_\alpha(x') \frac{\partial f_\alpha^*(x)}{\partial x'_0} \right] = \frac{\partial}{\partial x'_0} \sum_\alpha \left[ f_\alpha^*(x') f_\alpha(x) - f_\alpha(x') f_\alpha^*(x) \right]
\]

Now we take (A2,4) and calculate

\[
i \sum_\alpha f_\alpha^*(x') f_\alpha(x) = \frac{i}{(2\pi)^3} \int \frac{d\vec{k} d\vec{k}'}{2\omega} e^{i(\vec{k}x' - \vec{k}' \vec{x})} e^{i(\omega x_0 - \omega' x'_0)} \cdot \frac{1}{2\omega'} \sum_\alpha \psi_\alpha^*(\vec{k}') \psi_\alpha(\vec{k})
\]

\[= \frac{i}{(2\pi)^3} \int \frac{d\vec{k}}{2\omega} e^{i(\vec{k}(\vec{x} - \vec{x}')) - \omega(\vec{x}_0 - \vec{x}_0')} \]

Remember (A2,2').
Since
\[ \delta(k_0^2, \omega^2) = \frac{\delta(k_0 + \omega) + \delta(k_0 - \omega)}{2\omega} \]

one may write this

\[ i \sum_\alpha f_\alpha^*(x') f_\alpha(x) = i \frac{1}{(2\pi)^3} \int d^3 k \, \delta(k_0^2, \omega^2) \Theta(k_0) e^{ik \cdot (x-x')} = -\Delta_+ (x-x') \]

as a glance at (A3.7) shows.

One treats similarly the second term of the sum. We have

\[ \sum_\alpha \bar{f}_\alpha^* (x') \bar{f}_\alpha(x) = \Delta_+ (x-x') \]
\[ \sum_\alpha \bar{f}_\alpha (x') f_\alpha^* (x) = -i \Delta_- (x-x') \]

Therefore the sum gives

\[ \sum_\alpha \bar{f}_\alpha^* (x') \bar{f}_\alpha(x) - \bar{f}_\alpha (x') f_\alpha^* (x) = \Delta (x-x') \]  \hspace{1cm} (A2.15)

Our above expression becomes for \( x' = x_0 \) using (A3.3)

\[ i \frac{\partial}{\partial x_0} \sum_\alpha \left[ \bar{f}_\alpha^* (x') f_\alpha (x) - \bar{f}_\alpha (x') f_\alpha^* (x) \right]_{x'_0 = x_0} = -\frac{\partial}{\partial x_0} \Delta (x-x')_{x'_0 = x_0} = \delta(x-x') \]  \hspace{1cm} (A2.17)

This is the analogy to the second equation (A2.14).
V. The Cauchy boundary value problem for $x_0=\text{const.}$

Equation (A2,16) has a very useful consequence. We take an arbitrary solution \( f(x) \) of the homogeneous equation and calculate the scalar product

\[
(f, \Delta(x-x')) = i \int \left[ \frac{\partial \Delta(x-x')}{\partial x_0} \right] \frac{\partial f^*(x')}{\partial x_0} \, dx^2
\]

Inserting (A2,16) for \( \Delta(x-x') \) and putting

\[
f(x) = \sum_{\rho} q_{\rho} f_{\rho} + b_{\rho} f_{\rho}^*
\]

gives

\[
(f, \Delta) = \sum_{\rho} -i \left( a_{\rho} f_{\rho} + b_{\rho} f_{\rho}^* \right) \left( f_{\rho}^* f_{\rho} - f_{\rho} f_{\rho}^* \right)
\]

\[
= -i \sum_{\rho} a_{\rho}^* f_{\rho}^* \delta_{\rho} + b_{\rho}^* f_{\rho} \delta_{\rho} = -i \sum_{\rho} b_{\rho}^* f_{\rho} (x') + a_{\rho}^* f_{\rho}^* (x') = -i f^* (x')
\]

where (A2,6) has been used. Hence

\[
(f(x), \Delta(x-x')) = -i f^* (x')
\]

\[
\left\{ \begin{array}{l}
\int \left[ \frac{\partial \Delta(x-x')}{\partial x_0} - \Delta(x-x') \frac{\partial f(x)}{\partial x_0} \right] \, dx^2 = - f(x') \\
\end{array} \right. \quad (A2,16)
\]

The value of \( t \) is arbitrary. This can be used now to construct a solution of the homogeneous Klein-Gordon equation, which at \( x_0=t \) has given boundary values \( f(x') \) by definition a solution and at \( x_0=t \) it has the value \( f(x,t) \) and the normal derivative \( \frac{\partial}{\partial x_0} f(x,t) \). Let these boundary conditions be prescribed, then (A2,18) defines a function which is a solution and has these boundary values:

\[
\text{Given: } F(x,t) = q(x) \quad \hat{F}(x,t) = \hat{h}(x) \quad (A2,19)
\]

Find: Solution \( F(x) \) of the homogeneous Klein-Gordon equation having these boundary values:

\[
F(x) = \int \left[ \hat{h}(y) \Delta(y-x) - q(y) \frac{\partial \Delta(y-x)}{\partial y_0} \right] \, dy^2
\]
Since we know how to write a special solution of inhomogeneous Klein-Gordon equation \( A_{3,15} \) and since the general solution differs from it only by a solution of the homogeneous equation, we can choose the boundary values of the latter \( A_{2,19} \) always such that we obtain a solution of the inhomogeneous equation with prescribed Cauchy boundary values at \( x_0 = t \).

Once having derived \( A_{2,19} \) we can prove it also directly:

\[
F(x, t) = \int_{y_0 = t} \left[ k(y) \Delta(y-x) - g(y) \frac{\partial \Delta(y-x)}{\partial y_0} \right] \delta(x_0 - y_0) \, dy = g(x)
\]

From \( A_{3,3} \):

\[
\Delta(y-x) = 0 \quad \text{if} \quad \frac{\partial}{\partial y_0} \Delta(y-x) = - \delta(y-y_0)
\]

\[
\hat{F}(x, t) = \int_{y_0 = t} \left[ k(y) \frac{\partial \Delta(y-x)}{\partial x_0} - g(y) \frac{\partial}{\partial x_0} \frac{\partial}{\partial y_0} \Delta(y-x) \right] \delta(x_0 - y_0) \, dy = k(x)
\]

where

\[
\frac{\partial}{\partial x_0} \frac{\partial}{\partial y_0} \Delta(y-x) = - \frac{\partial^2}{\partial x_0^2} \Delta(y-x) = \left( - \nabla_x^2 + m^2 \right) \Delta(y-x)
\]

because \( (\Box - m^2) \Delta = 0 \). Since no time derivatives occur on the right hand side, we may now pass to \( x_0 = y_0 \), but \( \Delta(x-y) \big|_{x_0=y_0=0} = 0 \).

VI. Plane wave decomposition.

In II. it was mentioned already that the most convenient system for plane waves is the invariant one:
\[ \mathcal{F}_R(x) \equiv \frac{1}{(2\pi)^{3/2}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} ; \quad \omega = \sqrt{k_0^2 + \mathbf{k}^2} \]

\[ (\mathcal{F}_R, \mathcal{F}_R') = \frac{1}{(2\pi)^3} \int e^{-ik'x} \frac{\partial}{\partial x_0} e^{ik'x} d^3x = 2\omega \delta(\mathbf{k} - \mathbf{k}') \]

For the in and out field we have

\[ A_{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}} \int \delta(k_0^2 + \mathbf{k}^2) e^{ik'x} A_{\text{in}}(k') d^3k' \]  \hspace{1cm} (A2, 21)

Since \[ A_{\text{out}}^* = A_{\text{in}} \] we obtain

\[ A_{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega} \left[ A_{\text{in}}(k, \omega) e^{i(k \cdot x - \omega t)} + A_{\text{in}}^*(k, \omega) e^{-i(k \cdot x - \omega t)} \right] \]  \hspace{1cm} (A2, 22)

It is customary to introduce another notation:

\[ A_{\text{out}}(k, \omega) = A_{\text{in}}(k') ; \quad \omega > 0 \]  \hspace{1cm} (A2, 23)

Then one can write

\[ A_{\text{out}}(k) = \Theta(k_0) A_{\text{in}}(k) + \Theta(-k_0) A_{\text{in}}^*(-k) \]  \hspace{1cm} (A2, 23')

and obtains

\[ A_{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega} \left[ A_{\text{in}}(k) e^{i(k \cdot x - \omega t)} + A_{\text{in}}^*(k) e^{-i(k \cdot x - \omega t)} \right] \]  \hspace{1cm} (A2, 22')
With the help of (A2,20) one finds
\[ A_{\text{in}}(\vec{k}) = i \int \frac{e^{-i(\vec{k} \cdot \vec{x} - \omega t)}}{(2\pi)^{3/2}} \frac{\partial}{\partial x_0} A_{\text{in}}(x) \, d\vec{x} \]  
\[ (A2,24) \]

Since \( A_{\text{in}}(x) \) may be defined arbitrarily outside the mass shell, we may use the same formula with \( k_o \neq \omega \) for defining \( A_{\text{in}}(k) \) for all \( k \).

We obtain for \( k_o > 0 \)
\[ A_{\text{in}}(k) = \frac{i}{(2\pi)^{3/2}} \int e^{-i\vec{k} \cdot \vec{x}} \frac{\partial}{\partial x_0} A_{\text{in}}(x) \, d\vec{x} \] ; \( k_o > 0 \)

For \( k_o < 0 \) we use the fact that \( A^*(k, \omega) = A_{\text{in}}(-k, -\omega) \) and define \( A_{\text{in}}(-k) = A^*_\text{out}(k) \). In \( A_{\text{in}}(-k) \) we have then \(-k_o < 0\). Thus
\[ A_{\text{in}}(-k) = -\frac{i}{(2\pi)^{3/2}} \int e^{i\vec{k} \cdot \vec{x}} \frac{\partial}{\partial x_0} A_{\text{in}}(x) \, d\vec{x} \]

Putting \(-k = k'\) one obtains the first formula with the difference that \( k'_o < 0 \) and a minus sign in front. Hence
\[ A_{\text{in}}(k) = \frac{i \varepsilon(k)}{(2\pi)^{3/2}} \int e^{-i\vec{k} \cdot \vec{x}} \frac{\partial}{\partial x_0} A_{\text{in}}(x) \, d\vec{x} \] for all \( (k, k'_o) \)  
\[ (A2,25) \]

Since according to (A2,20) the orthogonality relation gives not \( \delta(\vec{k} - \vec{k}') \) but \( 2 \omega \delta(\vec{k} - \vec{k}') \) on the right hand side, also the commutation relations for the \( a(\vec{k}) \) will contain such a factor. This is easily seen in going through the derivation in A3, II. One obtains
\[ \left[ a_{\text{in}}(\vec{k}), a^*_\text{in}(\vec{k}') \right] = 2\omega \delta(\vec{k} - \vec{k}') \]  
\[ (A2.26) \]
VII. Sum over intermediate states.

Frequently one encounters matrix elements of a product of operators. One prefers to write this as a product of matrix elements of single operators. This is achieved by a sum over intermediate states:

\[ \sum_{\gamma} |\gamma \rangle \langle \gamma| \]

\( \gamma \) = all relevant quantum numbers of a state

This is the unit operator if the sum \( \gamma \) goes over a complete orthonormal system of states. We shall make this a little more explicit. The states

\[ |\alpha_1 \alpha_2 \ldots \alpha_n \rangle = A_{j_1}^{\dagger} A_{j_2}^{\dagger} \ldots A_{j_n}^{\dagger} \frac{1}{\sqrt{\Pi (z_1 \ldots z_n)}} |0\rangle \]

are orthonormal ( \( \Pi (z_1 \ldots z_n) = z_1^* z_2^* \ldots z_k^* \) if \( z_1, z_2, \ldots z_k \) are equal) because of the commutation relations:

\[ \langle \alpha_1 \ldots \alpha_n | \alpha'_1 \ldots \alpha'_n \rangle = \delta_{mn} \cdot \begin{cases} 1 & \text{if (apart from permutations)} \alpha_1 \ldots \alpha_n = \alpha'_1 \ldots \alpha'_n \\ 0 & \text{otherwise} \end{cases} \]

(A2,27)

*) We have namely

\[ A_{i \mu \alpha} A_{j \mu \alpha}^{\dagger} A_{k \mu \alpha}^{\dagger} \ldots A_{l \mu \alpha}^{\dagger} = \sum_{j=1}^{n} \delta_{dd_j} \delta_{dd'_j} A_{i \mu \alpha}^{\dagger} \ldots A_{l \mu \alpha}^{\dagger} A_{i \mu \alpha} \ldots A_{l \mu \alpha} \ldots A_{k \mu \alpha} A_{l \mu \alpha} A_{i \mu \alpha} \]

repeated application yields

\[ \langle \alpha_1 \ldots \alpha_n | \alpha'_1 \ldots \alpha'_n \rangle = \sum_{j_1 = 1}^{n} \delta_{\alpha_1 \alpha'_1} \ldots \delta_{\alpha_n \alpha'_n} = \sum_{P(w)} \delta_{\alpha_1 \alpha'_1} \ldots \delta_{\alpha_n \alpha'_n} \]

The sum goes over \( j_1, j_2, \ldots j_n \) from 1 to \( n \) but with \( j_1 \neq j_2 \neq \ldots \neq j_n \). In other words, it goes over all permutations of the \( \alpha' \). If all \( \alpha' \)'s are different, then there is exactly one permutation where the sum is equal to one: \( \alpha_1 = \alpha'_1 \ldots \) etc. If some are equal, the square root with the factorials corrects that.
The unit operator is here

\[ \sum_{n=0}^{\infty} \frac{1}{n!} |\alpha_1 \ldots \alpha_n \rangle \langle \alpha_1 \ldots \alpha_n | = 1 \]  

(A2,26)

and if we take a fixed \( n \), that part will be the operator which projects into the \( n \)-particle subspace. That (A2,29) is true follows immediately by applying it to any state \( |\alpha_1 \ldots \alpha_k \rangle \), which is reproduced by means of (A2,27). The most frequently used system is, however, that of plane waves and then the sum in (A2,28) over the \( \alpha \)'s has to be replaced by an integral. The way this has to be done is prescribed uniquely by the commutation relations again: For reasons of invariance we defined the operators \( a(p) \) and \( a^*(p) \) such that

\[ [a_{\mu}(p), a_{\mu}^*(p')] = 2\omega \delta(p-p') \]

\[ \omega = \sqrt{p^2 + \alpha^2} \]

\( \sum(A2,26) / \)

A state of \( n \)-particles may be written

\[ |p_1 \ldots p_n \rangle = a^*(p_1) \ldots a^*(p_n) |0 \rangle \]

From the commutation relation it follows that

\[ \langle p_1 \ldots p_n | \xi_1 \ldots \xi_m \rangle = \delta_{\mu \nu} \sum_{P_{\text{perm}(\mu)}} \delta(p_1 - \xi_1) \ldots \delta(p_n - \xi_m) \cdot 2\omega_1 \ldots 2\omega_n \]  

(A2,27)'

where all \( n! \) permutations of the \( \xi \)'s are taken. We consider one particular \( n \) and write the condition down in formal analogy to (A2,28)

\[ \sum_{p_1 \ldots p_n} \frac{1}{n!} |p_1 \ldots p_n \rangle \langle p_1 \ldots p_n | \xi_1 \ldots \xi_m \rangle = |\xi_1 \ldots \xi_m \rangle \]
From (A2,27') this gives

\[ \sum_{p_1 \ldots p_n \text{ Permut}(q)} 2\omega_1 \ldots 2\omega_n \delta(p_i - \vec{k}_i) \ldots \delta(p_n - \vec{k}_n) | p_1 \ldots p_n \rangle = | k_1 \ldots k_n \rangle \]

Therefore we must now replace

\[ \sum_{p_1 \ldots p_n \text{ Permut}(q)} \longrightarrow \int \frac{dp_1}{2\omega_1} \ldots \frac{dp_n}{2\omega_n} \ldots \]

Summing over all \( n \) we find

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{dp_1}{2\omega_1} \ldots \frac{dp_n}{2\omega_n} | p_1 \ldots p_n \rangle \langle p_1 \ldots p_n | = 1 \] (A2,28')

(In the present formalism of one single scalar neutral field, there are no other quantum numbers. Otherwise one has to sum over all other quantum numbers also).

We may rewrite (A2,28') in a more compact form. A state \( | p_1 \ldots p_n \rangle \) is a state of \( n \)-particles of mass \( m \) with total energy momentum \( P = \sum p_i \) \((\vec{P} = \sum \vec{p}_i; \quad E = \sum \sqrt{p_i^2 + m^2} \) and a fictitious mass \( M = \sqrt{E^2 - P^2} \)) and we may characterize the state by \( P \). But then it is highly degenerate, since there are many states with four momentum \( P \) coming from all possible partitions \( \sum \vec{p}_i = P \) with \( n=1,2 \ldots \). We shall now carry out the integration and sum in (A2,28') in two steps:

1) integration and sum such that \( P \) and \( M \) are the resultant four momentum and mass;

2) integration over \( M \) and \( P \).

ad 1) we multiply (A2,28') under the integral by \( \delta(\sum p_i - P) \delta(P^2 + M^2) \)

ad 2) we integrate over \( P \) and \( M^2 \)
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \delta(\sum_{i=1}^{n} p_i - p) \left| p_1 \cdots p_n \right| \frac{d^4 p_1}{2\omega_1} \cdots \frac{d^4 p_n}{2\omega_n} = 1 \]

Put
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \delta(\sum_{i=1}^{n} p_i - p) \left| p_1 \cdots p_n \right| \frac{d^4 p_1}{2\omega_1} \cdots \frac{d^4 p_n}{2\omega_n} = \left| p, m > < p, m \right| \]

Here \( \sum \) means: sum over discrete and integrate over continuous mass values. The last integral we call by definition \( \left| p, m > < p, m \right| \). Splitting

\[ \delta(p^2 + m^2) = \frac{\delta(E + \Omega) + \delta(E - \Omega)}{2\Omega} ; \quad \Omega = \sqrt{p^2 + m^2} \]

and observing the spectrum condition \( E > 0 \) gives

\[ \sum_{n=0}^{\infty} \frac{d^4 m}{2\Omega} \left| p, m > < p, m \right| = 1 \quad (A2, 23') \]

Of course, \( \left| p, m > < p, m \right| \) is different from zero only if \( m \) lies in the non-vanishing part of the mass spectrum (discrete or continuous). We shall sometimes write the sum over states in a form less explicit than \( (A2, 23') \) but more explicit than \( \sum \left| \gamma > < \gamma \right| \), namely as

\[ \sum_{p, \gamma} \left| p, \gamma > < p, \gamma \right| \quad (A2, 23'') \]

where eigenstates of the four momentum are meant and \( \gamma \) designates the remaining quantum numbers. For scalar real fields this is identical to \( (A2, 23') \).
APPENDIX 3

The commutation relations and invariant functions

(metric \( x^2 = x^2 - x_0^2 \))

1. The invariant functions (Green's functions).

We start with the well-known commutation relations of free fields \( A_{\text{in}}(x) \) and \( A_{\text{out}}(x) \) respectively:

\[
\begin{align*}
\left[ \dot{A}_{\text{in}}(x,t), A_{\text{in}}(x',t) \right] &= -i \delta(x - x') \\
\left[ A_{\text{in}}(x,t), \dot{A}_{\text{in}}(x',t) \right] &= 0
\end{align*}
\]

(A3,1)

We now consider the more general commutator by which we define a function \( i \Delta(x,x') \):

\[
\left[ A_{\text{in}}(x), A_{\text{in}}(x') \right] \equiv i \Delta(x,x')
\]

By applying \( e^{iPx} \) from the left and \( e^{-iPx} \) from the right, we obtain

\[
\left[ A_{\text{in}}(x-x'), A_{\text{in}}(0) \right] = i \Delta(x,x')
\]

hence \( \Delta(x,x') \) is a function of \( x-x' \) only. We may replace \( x-x' \) by \( x \) and have

\[
\left[ A_{\text{in}}(x), A_{\text{in}}(0) \right] = i \Delta(x)
\]

(A3,2)
This $\Delta(x)$ is a solution of the homogeneous Klein-Gordon equation and its boundary values for $x_0=t=0$ are given by (A3,1). Thus $\Delta(x)$ is uniquely defined:

$$\left\{ \begin{align*}
\Box - m^2 \Delta(x) &= 0 \\
\Delta(x, x_0=0) &= 0 \\
\frac{\partial}{\partial x_0} \Delta(x, x_0=0) &= -\delta(x)
\end{align*} \right. \quad (A3,3)$$

To solve this equation, we consider first the more general problem of solving formally the equation

$$\Box F(x) = \Phi(x)$$

In Fourier space we obtain with

$$F(k) = \frac{1}{(2\pi)^4} \int \hat{\Phi}(k) e^{ikx} d^4k \quad ; \quad \Phi(x) = \frac{1}{(2\pi)^4} \int \hat{\Phi}(k) e^{ikx} d^4k$$

$$-(k^2 + m^2) \hat{F}(k) = \hat{\Phi}(k) \quad (A5,4)$$

and

$$F(x) = -\frac{1}{(2\pi)^4} \int \frac{\hat{\Phi}(k)}{k^2 + m^2} e^{ikx} d^4k$$

The denominator has zeros at $k_0 = \pm \omega, \quad \omega = \sqrt{k^2 + m^2}$. Therefore the value of the integral depends still on the path of integration in the complex $k_0$-plane. In order to fulfill now equations (A3,3) we may choose a suitable path of integration such that $\hat{\Phi}(k) \neq 0$, since $\hat{\Phi}(k) \equiv 0$ would yield the trivial solution $F(x) \equiv 0$. From (A3,3) it follows, when we replace in (A3,4) $F(x)$ by $\Delta(x)$ that we must find a $\hat{\Phi}(k)$ and a path of integration such that:
\[ \Delta(x) = -\frac{i}{(2\pi)^4} \int \frac{\tilde{\phi}(k)}{k^2 + \omega^2} e^{ikx} d^4k \]

\[ (\Box - \omega^2) \Delta(x) = -\frac{i}{(2\pi)^4} \int \frac{\tilde{\phi}(k)}{k^2 + \omega^2} e^{ikx} d^4k = 0 \]

\[ \Delta(x, x_0=0) = \frac{i}{(2\pi)^4} \int \frac{\tilde{\phi}(k) e^{ikx}}{(k^2 + \omega)(k^2 - \omega)} \, dk \, d^3k = 0 \]

\[ \frac{\partial}{\partial x_0} \Delta(x, x_0=0) = \frac{i}{(2\pi)^4} \int \frac{\tilde{\phi}(k) k_0 e^{ikx}}{(k^2 + \omega)(k^2 - \omega)} \, dk \, d^3k = -\delta(x) \]

We integrate the last equation over the whole x-space and obtain with

\[ \frac{i}{(2\pi)^3} \int e^{ikx} \, dx = \delta(k) \]

\[ -\frac{i}{2\pi} \int \frac{\tilde{\phi}(k) k_0 \, dk \, d^3k}{(k^2 + \omega)(k^2 - \omega)} \delta(k) \, d^3k = \frac{i}{2\pi} \int \frac{\tilde{\phi}(0, k_0) k_0 \, dk_0}{(k_0^2 + \omega)(k_0^2 - \omega)} = -1 \]

The integration path is not yet defined, but the form of the integrand suggests to try the way C \[\text{\text{see Fig. (A3-1)}}\]

![Fig. (A3-1)](image)

\[ -\frac{i}{2\pi} \int_C \frac{\tilde{\phi}(0, k_0) k_0 \, dk_0}{(k_0^2 + \omega)(k_0^2 - \omega)} = -\frac{i}{2\pi} \cdot 2\pi i \left[ \frac{1}{2} \tilde{\phi}(0, \omega) + \frac{1}{2} \tilde{\phi}(0, -\omega) \right] = -1 \]
This is achieved if we put $\tilde{f}(k) = \text{const.} = -1$. One sees now that also the other conditions are fulfilled:

$$\Delta(\vec{x}, x_0=0) = -\frac{1}{(2\pi)^4} \int_C \frac{d\vec{k}_0}{(k_0^2 + \omega)(k_0^2 - \omega)} \int e^{i\vec{k}\vec{x}} d\vec{k} = 0$$

$$(\Box - m^2) \Delta(x) = \frac{1}{(2\pi)^4} \int_C e^{-ik_0x_0} d\vec{k}_0 \int e^{i\vec{k}\vec{x}} d\vec{k} = 0$$

since the contour integrals vanish. Hence

$$\Delta(x) = \frac{1}{(2\pi)^4} \int_C \frac{e^{i\vec{k}\vec{x}}}{k^2 + m^2} d^4k$$

(A3,5)

One sees at once that this function is invariant. Hence it is zero for all spacelike $x$, since it is true for one special spacelike $x : (\vec{x}, x_0=0)$. It is an odd function of $x$, since

$$\Delta(-x) = \frac{1}{(2\pi)^4} \int_C \frac{e^{-i\vec{k}\vec{x}}}{k^2 + m^2} d^4k = \frac{1}{(2\pi)^4} \int_C \frac{e^{-i\vec{k}\vec{x}}}{k^2 + m^2} d^4k$$

By changing $k \rightarrow -k$ one inverts the direction along $C$, hence $\Delta(-x) = -\Delta(x)$.

Another useful form of the $\Delta$-function is obtained by performing the $k_0$-integration along $C$:

$$\Delta(x) = -\frac{i}{(2\pi)^3} \int d\vec{k} e^{i\vec{k}\vec{x}} \left[ \frac{e^{-i\omega x_0} - e^{i\omega x_0}}{2\omega} \right]$$

We reintroduce a $k_0$-integration from $-\infty$ to $\infty$ using $\delta$-functions:

$$\Delta(x) = -\frac{i}{(2\pi)^3} \int d\vec{k} e^{i\vec{k}\vec{x}} d\vec{k}_0 e^{-ik_0x_0} \left[ \frac{\delta(k_0^2 - \omega) - \delta(k_0^2 + \omega)}{2\omega} \right]$$

which leads to the compact form

$$\Delta(x) = -\frac{i}{(2\pi)^3} \int d\vec{k} \delta(k^2 + m^2) \lambda(k) e^{i\vec{k}\vec{x}} \quad \lambda(k) = \begin{cases} 1; k_0 > 0 \\ -1; k_0 < 0 \end{cases}$$

(A3,6)
It is often useful to split the $\Delta$-function into a positive and negative frequency part [see Fig. (A3-1)]

\[
\Delta_+(x) = -\frac{i}{(2\pi)^3} \int d^4k_2 \delta(k_2^2 - m^2) \Theta(k^0) e^{i k x} = \frac{1}{(2\pi)^4} \int_{C_+} \frac{e^{i k x}}{k^2 + m^2} d^4k_2
\]
\[
\Delta_-(x) = \frac{i}{(2\pi)^3} \int d^4k_2 \delta(k_2^2 - m^2) \Theta(-k) e^{i k x} = \frac{1}{(2\pi)^4} \int_{C_-} \frac{e^{i k x}}{k^2 + m^2} d^4k_2
\]

\[
\Theta(k) = \begin{cases} 1 & ; k_0 > 0 \\ 0 & ; k_0 < 0 \end{cases}
\]

We finally investigate the consequences of choosing other paths in the complex $k_0$-plane. We first restrict ourselves to two special cases [see Fig. (A3-2)]

![Fig. (A3-2)](image)

\[
\Delta_R(x) \equiv -\frac{1}{(2\pi)^4} \int_{C_R} \frac{dk_0 e^{-i k_0 x_0}}{(k_0 + \omega)(k_0 - \omega)} e^{i k_\perp x} d^2k_\perp
\]

As one sees, one may close the path in the upper half plane if $x_0 < 0$. One obtains zero. If $x_0 > 0$ one must close the path in the lower half plane and the result is $-\Delta(x)$, as a glance at Fig. (A3-1) shows. Similarly for $\Delta_A(x)$. We have therefore
\[ \Delta_R(x) = -\Theta(x) \Delta(x) = \frac{1}{(2\pi)^n} \left\{ \int_{C_R} \frac{e^{ikx}}{k^2 + \mu^2} \, d^nk \right\} \]
\[ \Delta_A(x) = \Theta(-x) \Delta(x) = \frac{1}{(2\pi)^n} \left\{ \int_{C_A} \frac{e^{ikx}}{k^2 + \mu^2} \, d^nk \right\} \]

The situation is illustrated in Fig. (A3-3), where the shadowed regions are those where the functions are \( \neq 0 \).

![Diagram showing shadowed regions](image)

**Fig. (A3-3)**

It follows

\[ \Delta(x) = \Delta_A(x) - \Delta_R(x) \]  

(A3, 9)

Since these are also frequently used, we give some other representations.

Taking \( \Delta_R \), we may write the identity [see Fig. (A3-2')]

![Diagram showing contour](image)

**Fig. (A3-2')**
\[
\mathcal{S} = \frac{1}{2} \int_{\mathcal{C}_R} + \frac{1}{2} \int_{-\mathcal{C}_R} + \frac{1}{2} \int_{\mathcal{C}_A} + \frac{1}{2} \int_{-\mathcal{C}_A}
\]

The two in the middle are just the principal value integral, whereas the first and the left give together the path \( \gamma \). Hence

\[
\Delta_R(x) = \frac{1}{(2\pi)^4} \text{P} \int \frac{e^{ikx}}{k^2 + m^2} d^4k - \frac{1}{2} \Delta(x)
\]

\[
\Delta_A(x) = \frac{1}{(2\pi)^4} \text{P} \int \frac{e^{ikx}}{k^2 + m^2} d^4k + \frac{1}{2} \Delta(x)
\]

\[
\begin{align*}
\Delta_R(x) &= \frac{1}{(2\pi)^4} \int d^4k e^{ikx} \left[ P \frac{1}{k^2 + m^2} + i\pi \epsilon(k) \delta(k^2 + m^2) \right] \\
\Delta_A(x) &= \frac{1}{(2\pi)^4} \int d^4k e^{ikx} \left[ P \frac{1}{k^2 + m^2} - i\pi \epsilon(k) \delta(k^2 + m^2) \right]
\end{align*}
\]

We obtain still another expression if we shift the two poles away from the real \( k_o \)-axis and then integrate along the real axis:

\[
\frac{1}{k^2 + m^2} = -\frac{1}{(k_o + \omega)(k_o - \omega)}
\]

\[
\Delta_p = \frac{1}{2} (\Delta_A + \Delta_R) \quad \text{is frequently called} \quad \overline{\Delta}.
\]
In order to have both poles shifted to the lower half plane, we may replace
\( k_0 \rightarrow k_0 + i \varepsilon_0; \quad \varepsilon_0 > 0 \). We can write this more elegantly by choosing a four-
vector \((\vec{\varepsilon}, i \varepsilon_0) \equiv \varepsilon\) and write then

\[
\Delta_R(x) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^4} \int \frac{e^{ikx}}{(k^2 + \varepsilon)^2 + m^2} d^4k
\]

\[
\Delta_A(x) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^4} \int \frac{e^{ikx}}{(k^2 - \varepsilon)^2 + m^2} d^4k
\]

\( \varepsilon \equiv (\vec{\varepsilon}, i \varepsilon_0); \quad \varepsilon_0 > 0 \)

We determine now the properties of \( \Delta_R(x) \) and \( \Delta_A(x) \) which correspond to
\( (A3,3) \):

From the integral representation \( (A3,8) \) follows at once

\[
(\Box - m^2) \Delta_R(x) = -\frac{1}{(2\pi)^4} \int e^{ikx} d^4k = -\delta(x)
\]

whereas

\[
\Delta_A(\vec{x}, x_0 = 0) = 0
\]

follows from the first half of \( (A3,8) \) and the corresponding property of \( \Delta(x) \).

For the evaluation of \( \frac{\partial}{\partial x_0} \Delta_R(\vec{x}, x_0 = 0) \) we use \( (A3,10) \).

\[
\frac{\partial}{\partial x_0} \frac{1}{(2\pi)^4} \int \frac{e^{ikx}}{k^2 + m^2} d^4k = \frac{i}{(2\pi)^4} \int \frac{k_0 dk_0}{(k^0 \omega)(k^0 \omega)} e^{ikx} d^4k
\]

This principal value integral is zero, since the integrand is odd. One can see
this also by complex integration, Fig. \( (A3-4) \):
The principal value in the left figure is $\frac{1}{2}$ the integral along the upper path, plus one half along the lower. This can be deformed into the paths in the right figure. Put $k_o = ke^{i\epsilon}$ and neglect $\omega$ for $K \to \infty$:

$$P\int = i \int_{-\pi}^{0} d\varphi + i \int_{\pi}^{0} d\varphi = 0$$

Hence from (A3,10) remains with the help of (A3,3)

$$\frac{\partial}{\partial x_o} \Delta_R (x',0) = \frac{1}{2} \delta(x')$$

$$\frac{\partial}{\partial x_o} \Delta_A (x',0) = -\frac{1}{2} \delta(x')$$

The functions $\Delta_R$ and $\Delta_A$ are the Green's functions needed for the solution of the inhomogeneous Klein-Gordon equation with prescribed asymptotic behaviour for $t \to +\infty$ or $t \to -\infty$. Suppose we want the solution of the (classical!) equation

$$(\Box - m^2) F(x) = \delta(x)$$
We put
\[ \bar{q}(x) = \int \delta(x - x') \bar{q}(x') \, d^4x' = - (\Box - \mu^2) \int \Delta_R(x - x') \bar{q}(x') \, d^4x' \]

where (A3,13) has been used. Hence
\[ (\Box - \mu^2) \left\{ F(x) + \int \Delta_R(x - x') \bar{q}(x') d^4x' \right\} = 0 \]

The curly bracket is therefore a solution of the homogeneous equation, which we may call \( F_{\text{out}}(x) \). Thus we arrive at two descriptions:

\[ F(x) = F_{\text{in}}(x) - \int \Delta_R(x - x') \bar{q}(x') \, d^4x' \tag{A3,15} \]

Since \( \Delta_R(x) \) is \( \neq 0 \) only in the forward light cone, one has contributions to \( F(x) \) only from points \( x' \) lying in the backward light cone of \( x \). \[ \text{Fig. (A3-5)} \]

That is the reason for the subscript \( R \): retarded contributions. It follows from the property \( \Delta_R(\vec{x}, x_0 < 0) = 0 \), that \( \lim_{x_0 \to -\infty} F(x) = F_{\text{in}}(x) \).
i.e. \( x_0 \to -\infty \) the function \( F \) and its time derivative (in fact all derivatives) coincide with the prescribed values \( F_{\text{in}}(x), \dot{F}_{\text{in}}(x) \).

The other description is

\[
F(x) = F_{\text{out}} - \int \Delta_{A}(x-x') \rho(x') \, d^4x'
\]

(A3,16)

Only the points \( x' \) in the forward light cone of \( x \) contribute.

\[
\lim_{x_0 \to +\infty} F(x) = F_{\text{out}}(x) \quad \text{since} \quad \Delta_{A}(x, x_0 > 0) = 0
\]

One can prescribe either \( \lim_{x_0 \to -\infty} F(x) \) or \( \lim_{x_0 \to +\infty} F(x) \) but not both of them, unless \( F_{\text{in}}(x) \) and \( F_{\text{out}}(x) \) satisfy

\[
F_{\text{out}}(x) = F_{\text{in}}(x) + \int \Delta(x-x') \rho(x') \, d^4x'
\]

(A3,17)

(to obtain this subtract (16) from (15) and remember \( \Delta(x) = \Delta_{A}(x) - \Delta_{R}(x) \)).

In this case one has

\[
\lim_{x_0 \to -\infty} F(x) = F_{\text{in}}(x) \quad ; \quad \lim_{x_0 \to +\infty} F(x) = F_{\text{out}}(x)
\]

The next path we consider is that in Fig. (A3-6)
We define
\[ \Delta_F(x) = -i \left( \frac{i}{2} \Delta - \frac{i}{2} \Delta^+ + \Delta_P \right) ; \quad \Delta_P = \frac{1}{(2\pi)^4} P \int \frac{d^4k}{k^2 + m^2} \]  
(A3, 18')

If one considers where one can close the path \( \mathcal{F} \), then one sees that
\[ \Delta_F(x) = \begin{cases} 
  i \Delta^+(x) & \text{for } x_o > 0 \\
  -i \Delta^-(x) & \text{for } x_o < 0 
\end{cases} 
\]  
(A3, 19)
II. Further commutation relations: creation and absorption operators.

If we expand an operator \( A_{\text{out}}^{\text{in}}(x) \) with respect to \( \{ f_\alpha, f_\alpha^* \} \), then (omitting the subscripts in and out sometimes)

\[
A_{\text{in}}^{\text{out}}(x) = \sum A_{\mu\nu} f_\nu(x) + A_{\mu\nu}^* f_\nu^*(x) = A_{\mu\nu}(x) + A_{\mu\nu}^*(x)
\]

Denoting \( f_\alpha(x) = f_\alpha^* ; f_\alpha(x') = f_\alpha^* ; A(x') = A' \) etc., one obtains

\[
[A(x), A(x')] = [A_+, A_+] + [A_-, A_+] + [A_-, A_+] + [A_+, A_+'] = (\text{in the same order})
\]

\[
= \sum_{\alpha\beta} \left( [A_\alpha, A_\beta] f_\alpha f_\beta + [A_\alpha, A_\beta^*] f_\alpha^* f_\beta + [A_\alpha^*, A_\beta] f_\alpha^* f_\beta + [A_\alpha^*, A_\beta^*] f_\alpha f_\beta^* \right)
\]

\[
= i \Delta (x-x') = \sum_\alpha f_\alpha^* f_\alpha - f_\alpha f_\alpha^*
\]

because of \((A3,2)\) and \((A2,16)\).

We now use the orthogonality relations \((A2,6)\). By multiplying \( f_\delta \) in the sense of \((A2,5)\) from the left by \( f_\delta^\ast \)

- \( f_\delta f^\ast_\delta \) we obtain \([A_\delta, A_\delta^*] = 0\)
- \( f_\delta^\ast f^\ast_\delta \) we obtain \([A_\delta^*, A_\delta^*] = 0\)
- \( f_\delta f^\ast_\delta \) we obtain \([A_\delta, A_\delta^*] = \delta_{\gamma\delta}\)
- \( f_\delta^\ast f^\ast_\delta \) we obtain \([A_\delta^*, A_\delta] = - \delta_{\gamma\delta}\)

\*) We mean, e.g.:

\[
(f_\delta f_\delta^\ast, \sum f_\alpha f_\alpha^* - f_\alpha^* f_\alpha) = \sum (f_\delta f_\delta^\ast)(f_\delta f_\delta^\ast) - (f_\delta f_\delta^\ast)(f_\delta f_\delta^\ast) = \sum \delta_{\gamma\delta} 0 + 0 \delta_{\gamma\delta} =
\]
Hence

\[ [A^+_{\text{in}}(\kappa), A^+_{\text{in}}(\kappa')] = [A^-_{\text{in}}(\kappa), A^-_{\text{in}}(\kappa')] = [A^\text{in}_{\text{out}}, A^\text{in}_{\text{out}}] = [A^\text{out}_{\text{in}}, A^\text{out}_{\text{in}}] = 0 \]

(A3,20)

\[ [A^\text{in}_{\text{out}}, A^\text{in}_{\text{out}}] = \delta_{\kappa,\kappa'} ; \quad [a(\vec{k})^\dagger_{\text{in}}, a(\vec{k}')^\dagger_{\text{in}}] = 2\omega \delta(\vec{k} - \vec{k}') \]

\[ \text{See (A2,6)} \]

This in turn gives with (A2,15):

\[
\begin{align*}
[A^+_{\text{in}}(\kappa), A^-_{\text{in}}(\kappa')] &= \sum_\alpha f^\alpha(\kappa) f^\alpha_*(\kappa') = i \Delta_+(\kappa - \kappa') \\
\end{align*}
\]

(A3,21)

\[
\begin{align*}
[A^-_{\text{in}}(\kappa), A^+_{\text{in}}(\kappa')] &= -\sum_\alpha f^\alpha_*(\kappa) f^\alpha(\kappa') = i \Delta_-(\kappa - \kappa') \\
\end{align*}
\]

Since for \( t=\infty \) the particle number becomes constant, we may define operators

\[ Z_{\text{in}} = \sum A^\text{in}_{\text{in}} \quad ; \quad Z_{\text{out}} = \sum A^\text{out}_{\text{in}} \]

which measure the number of particles. Let \( \phi_N \) be an eigenstate of these operators with eigenvalue \( N \) (integer):

\[ Z \phi_N = N \phi_N \]

From (A3.20) follows then

\[ A^\text{out}_{\rho} Z \phi_N = N (A^\text{out}_{\rho} \phi_N) = \sum A^\text{out}_{\rho} A^\text{in}_{\kappa} (A^\text{out}_{\rho} \phi_N) - (A^\text{out}_{\rho} \phi_N) \]

or

\[ Z (A^\text{out}_{\rho} \phi_N) = (N+1) (A^\text{out}_{\rho} \phi_N) \quad ; \quad \text{similar} \quad Z (A^\text{out}_{\rho} \phi_N) = (N-1) (A^\text{out}_{\rho} \phi_N) \]
Hence

\[ A_{\alpha}^{\mu} \text{ and } A_{\mu}^{\nu} (x) \text{ are destruction operators} \quad (A3,22) \]

\[ A_{\alpha}^{\mu} \text{ and } A_{\mu}^{\nu} (x) \text{ are creation operators.} \]

### III. Vacuum expectation values of products of free-field operators.

The vacuum state \( \Omega \) is defined as the state with the lowest eigenvalue of \( -F^2 \), namely \( -F^2 \Omega = 0 \). This definition is consistent also for interacting fields. For free-fields we could define also \( \Omega \) by \( Z = 0 \), with \( Z \) the above particle number operator. Evidently then

\[ A_{\alpha}^{\mu} \Omega = A_{\mu}^{\nu} (x) \Omega = \langle \Omega | A_{\alpha}^{\mu} \rangle = \langle \Omega | A_{\mu}^{\nu} (x) \rangle = 0 \quad (A3,23) \]

For the Wick product [see Appendix 1] it follows that

\[ \langle \Omega, : A_{\alpha}^{\mu} (x_1) \ldots A_{\mu}^{\nu} (x_n) : \Omega \rangle = 0 \quad (A3,24) \]

If one defines the T- and R-product by

\[ T \ A(x) A(x') = \begin{cases} A(x) A(x') & x_0 > x_0' \\ A(x') A(x) & x_0 < x_0' \end{cases} \]

\[ R \ A(x) A(x') = -i \Theta(x-x') \left[ A(x), A(x') \right] \quad (A3,25) \]
then one finds easily the following relations

\[
\langle \Omega | A_{\text{in}}(x) A_{\text{in}}(x') | \Omega \rangle = i \Delta_+(x-x')
\]

\[
\langle \Omega | A_{\text{in}}(x') A_{\text{in}}(x) | \Omega \rangle = -i \Delta_-(x-x')
\]

\[
\langle \Omega | [A_{\text{in}}(x), A_{\text{in}}(x')] | \Omega \rangle = i \Delta(x-x')
\]

\[
\langle \Omega | \{A_{\text{in}}(x), A_{\text{in}}(x')\} | \Omega \rangle = i \left( \Delta_+(x-x') - \Delta_-(x-x') \right) \equiv \Delta_\eta(x-x')
\]

\[
\langle \Omega | T[A_{\text{in}}(x) A_{\text{in}}(x')] | \Omega \rangle = \begin{cases} 
 i \Delta_+(x-x') & x_0 > x_0' \\
 -i \Delta_-(x-x') & x_0 < x_0'
\end{cases} \equiv \Delta_F(x-x')
\]

\[
\langle \Omega | R A_{\text{in}}(x) A_{\text{in}}(x') | \Omega \rangle = \Theta(x-x') \Delta(x-x') = -\Delta_R(x-x')
\]

IV. An orthogonality relation.

From (A3,6) we have ( \( \Delta_\rho \) is often called \( \bar{\Delta} \) )

\[
\frac{1}{2} \left( \Delta_+ + \Delta_R \right) = \Delta_\rho = -\frac{1}{2} \varepsilon(x_0) \Delta(x)
\]  

(A3,27)

We shall write the mass value explicitly. (A3,6) gives

\[
\Delta(x^2, m^2) = -\frac{i}{(2\pi)^3} \int d^4k \varepsilon(k_0) \delta(k_0^2 + m^2) e^{ikx}
\]
with the inversion

$$\varepsilon(k_0) \delta(k^2, m^2) = \frac{i}{2\pi} \int \Delta(x^2, m^2) e^{ikx} d^4x$$  \hspace{1cm} (A3.28)$$

Further we have, since \( \Delta(x^2, m^2) \equiv 0 \) for \( x^2 > 0 \), the identity

$$\Delta(x^2, m^2) = \int_0^\infty \delta(\lambda^2 + x^2) \Delta(-\lambda^2, m^2) d\lambda^2$$  \hspace{1cm} (A3.29)$$

$$= -2 \int_0^\infty \varepsilon(x_0) \delta(\lambda^2 + x^2) \Delta_p(-\lambda^2, m^2) d\lambda^2$$ \hspace{1cm} (By (A3.27))$$

This we insert into (A3.28) and find

$$\varepsilon(k_0) \delta(k^2, m^2) = -2 \frac{i}{2\pi} \int_0^\infty \varepsilon(x_0) \delta(\lambda^2 + x^2) \Delta_p(-\lambda^2, m^2) d\lambda^2 e^{ikx} d^4x$$

Again using (A3.28) inside the integral yields

$$\varepsilon(k_0) \delta(k^2, m^2) = -2 \cdot \frac{1}{4\pi^2} \int_0^\infty d\lambda^2 \Delta(u^2, \lambda^2) e^{ixy} \Delta_p(-\lambda^2, m^2) e^{ikx} d^4x d^4y$$

$$= \frac{1}{2\pi^2} (2\pi)^4 \int_0^\infty d\lambda^2 \Delta(k^2, \lambda^2) \Delta_p(-\lambda^2, m^2).$$

Multiplication by \( \varepsilon(k_0) \) and using (A3.27) again gives

$$\delta(m^2, k^2) = 16\pi^2 \int_0^\infty d\lambda^2 \Delta_p(-k^2, \lambda^2) \Delta_p(-\lambda^2, m^2)$$  \hspace{1cm} (A3.30)$$

\begin{align*}
\left( k^2 > 0 \ , \ m^2 > 0 \right)
\end{align*}
APENDIX 4

Coordinate systems and variables

(metric \( x^2 = x_0^2 - x^2 \))

I. The number of independent variables.

Consider a scattering process with two incoming and two outgoing particles and assume only two different masses (the formulae can easily be generalized):

\[
\begin{align*}
\mathbf{k}_1 + \mathbf{p}_1 &= \mathbf{k}_2 + \mathbf{p}_2 \\
\mathbf{k}_1^2 &= \mathbf{k}_2^2 = \mu^2 \\
\mathbf{p}_1^2 &= \mathbf{p}_2^2 = M^2
\end{align*}
\]

\[\{ \text{(A4,1)} \} \]

Fig. (A4-1)

The scattering amplitude \( T(\mathbf{k}_1, \mathbf{p}_1, \mathbf{k}_2, \mathbf{p}_2) \) depends on four four-vectors, i.e. on 16 variables which however are not independent and in fact reduce to two. Because of (A4,1), first equation, only three four-vectors are arbitrary. Since \( T \) must be L-invariant, it can only depend on the invariants, which can be built from three four-vectors, e.g. from \( \mathbf{k}_1, \mathbf{p}_1 \) and \( \mathbf{p}_2 \):

\[
\mathbf{k}_1^2, \mathbf{p}_1^2, \mathbf{p}_2^2, \mathbf{k}_1\mathbf{p}_1, \mathbf{k}_1\mathbf{p}_2, \mathbf{p}_1\mathbf{p}_2
\]
Since no invariant tensor of rank 3 exists, one cannot form an invariant containing all three four-vectors at once. Because of the two lower lines of (A4,1), which constitute four equations, we are left with only two of the six invariants and they have to be taken from the last three, since the first are fixed constants. Of course we could have chosen any other two independent invariants, e.g. any scalar product

\[ \sum_{i=1}^{2} \left( a_i k_i + b_i p_i \right) \left( \sum_{i=1}^{2} a'_i k'_i + b'_i p'_i \right) \]

or any set of two functions of the above invariants. Which two variables are the most convenient ones, depends on what one wants to do. In the following we shall exhibit some particularly useful ones.

II. Two useful coordinate systems and related invariants.

a) The centre-of-mass system.

Here we put \( k_1 + p_1 = 0 \); from (A4,1) follows then

\[ \begin{align*}
\sqrt{k_1^2 + \mu^2} &= \sqrt{k_2^2 + \mu^2} = \sqrt{p_2^2} = K \\
\sqrt{k_1^2 + \mu^2} + \sqrt{k_2^2 + \mu^2} &= \mathcal{W} \end{align*} \]

(only the angle changes) (A4,2)
\( W \) is the total CM-energy and \( K \) the magnitude of the momentum of any one of the particles.

\[ W^2 = (k_1 + p_1)^2 = (k_2 + p_2)^2 = (k_1 + p_1)(k_2 + p_2) = \quad \text{(total CM-energy squared)} \]

\[ K^2 = \frac{(W^2 + M^2 - \mu^2)^2 - 4M^2W^2}{4W^2} = \quad \text{(magnitude of momentum in CM)} \]

where the first equation is obvious and the second follows by solving (A4,2) for \( K^2 \).

b) The Breit system (brick wall).

In the CM-system \( \vec{p}_1 + \vec{p}_2 = -(\vec{k}_1 + \vec{k}_2) \) \( \checkmark \text{see Fig. (A4-3)} \)

Fig. (A4-3)

Suppose one applies a L-transformation such that \( \vec{p}_1 + \vec{p}_2 = 0 \). The result is given in Fig. (A4-4):

Fig. (A4-4)
The behaviour of the particles is then as if they were reflected at a hard wall such that the p-particle hits it perpendicularly. With (A4,1) we find

\[ p_i = (\vec{p}_i, E) \quad ; \quad k_i = (\vec{k}_i, \omega) \]

\[ p_2 = (-\vec{p}, E) \quad ; \quad k_2 = (\vec{k}_2, \omega) \]

These equations suggest to use the following quantities to characterize the dynamics:

\[ \alpha \] the magnitude \( |\vec{p}| \) of the momentum of the p-particle:

\( \frac{2\vec{p}}{\omega} \) is the momentum transfer.

\[ \beta \] the energy \( \omega \) of the k-particle.

We now should write these quantities in an invariant form. We see that

\( p_1 - p_2 = (2\vec{p}, 0) \), hence \( |\vec{p}|^2 \text{Breit} = -\frac{1}{4} (p_1 - p_2)^2 \) is the invariant formulation. We shall call this quantity henceforward \( \Delta^2 \). For the second quantity we observe that \( p_1 + p_2 = 2(\vec{p}, E) \), hence \( (p_1 + p_2)(k_1 + k_2) = 4E \omega \). We have also 

\( (p_1 + p_2)^2 = 4E^2 \), hence

\[ \omega = \frac{(p_1 + p_2)(k_1 + k_2)}{2 \sqrt{(p_1 + p_2)^2}} \]

is the invariant formulation.

Thus we have found two other variables:

\[
\Delta^2 = -\frac{1}{4} (p_1 - p_2)^2 = -\frac{1}{4} (\vec{k}_1 - \vec{k}_2)^2 \quad (\frac{1}{\Delta^2} \text{ in the Breit system})
\]

\[ \omega = \frac{(p_1 + p_2)(\vec{k}_1 + \vec{k}_2)}{2 \sqrt{(p_1 + p_2)^2}} \quad \text{(energy of the k-particle in the Breit system)} \]

With these variables we may write

\[ E = \sqrt{\Delta^2 + \omega^2} \quad \text{and} \quad \vec{k}_1 = \frac{\vec{p} + \omega \vec{E}}{\sqrt{\omega^2 - \mu^2 - \Delta^2}} \]

(see Fig. (A4-4)), so that
\[ p_1 = \left( \vec{p}, \sqrt{\Delta^2 + M^2} \right) ; \quad k_2 = \left( -\vec{p} + \vec{e} \sqrt{\omega^2 - \mu^2 - \Delta^2}, \omega \right) ; \quad \vec{e} \perp \vec{p} \]
\[ p_2 = \left( -\vec{p}, \sqrt{\Delta^2 + m^2} \right) ; \quad k_2 = \left( \vec{p} + \vec{e} \sqrt{\omega^2 - \mu^2 - \Delta^2}, \omega \right) ; \quad \vec{e} = 1 \]

\(III\). Further relations.

We first write down some relations following from (A4,1). \( p_1 - p_2 = k_2 - k_1 \)
gives

\[ (k_1 + k_2)(p_1 - p_2) = k_2^2 - k_1^2 = \mu^2 - \mu^2 = 0 \]
\[ (k_1 - k_2)(p_1 + p_2) = 0 \]

Multiplying out these equations and subtracting them leads to \( k_2 p_1 = k_1 p_2 \),
adding leads to \( k_1 p_1 = k_2 p_2 \). We have therefore (the 3rd equation follows from
the 2nd with (A4,1))

\[ (k_1 + k_2)(p_1 - p_2) = (k_1 - k_2)(p_1 + p_2) = 0 \]
\[ k_1 p_2 - k_2 p_1 = k_1 p_1 - k_2 p_2 = 0 \]
\[ p_1 (k_1 - k_2) = k_1 k_2 - \mu^2 \]

(and similar relations)

Now we express our variables in another few forms:

\[ \Delta^2 = \frac{A}{2} (k_1 k_2 - \mu^2) = \frac{A}{2} (p_1 p_2 - M^2) \]
which follows directly from \((A4,4)\). If we multiply out \(\omega\), we find with \(P_1P_2\) taken from \((A4,6)\)

\[
\omega = \frac{k_1(p_1 + p_2)}{2\sqrt{\Delta^2 + m^2}} = \frac{p_1(k_1 + k_2)}{2\sqrt{\Delta^2 + m^2}} \tag{A4,7}
\]

From this follows

\[
2\omega\sqrt{\Delta^2 + m^2} = k_1p_1 + k_2p_2
\]

From \((A4,1)\) follows, by multiplication by \(p_2\)

\[
k_1p_2 = k_2p_2 + m^2 - p_1p_2
\]

Hence with \((A4,5)\)

\[
2\omega\sqrt{\Delta^2 + m^2} = 2k_1p_1 + m^2 - p_1p_2
\]

Now

\[
2k_1p_1 = (p_1 + k_1)^2 - p_1^2 - k_1^2 = W^2 - m^2 - \mu^2
\]

\[
p_1p_2 = 2\Delta^2 + m^2
\]

gives

\[
W^2 = 2\omega\sqrt{\Delta^2 + m^2} + m^2 + \mu^2 + 2\Delta^2 \tag{A4,8}
\]

Finally we work out \(\Delta^2\) in the CM-system using \((A4,4)\) and \((A4,2)\)

\[
\Delta^2 = -\frac{1}{4} \left[ (k_{10} - k_{20})^2 - k_1^2 - k_2^2 + 2 |\vec{k_1}||\vec{k_2}| \cos \Theta \right] = \frac{1}{2} \frac{K^2(1 - \omega \Theta)}{\omega^2} \tag{A4,9}
\]

\[
\omega = 1 - \frac{2\Delta^2}{K^2}
\]
One can write down of course many more relations and use still other variables, but the present formulae are the most useful ones. As one sees, one may use for instance $W$ and $\cos\theta$ or $\Delta^2$ and $\omega$ or any other independent pair.

For convenience we list all derived invariant variables together:

\[
W^2 = (k_1 \cdot p_1)^2 = (k_2 \cdot p_2)^2 = (k_1 \cdot p_1)(k_2 \cdot p_2) = \text{square of the total CM-energy}
\]

\[
K^2 = \frac{(W^2 + M^2 - \mu^2)^2 - 4M^2W^2}{4W^2} = \text{square of the three-momentum of one of the incoming particles in CM-system}
\]

\[
\Delta^2 = -\frac{1}{4}(p_1 \cdot p_2)^2 = -\frac{1}{4}(k_1 \cdot k_2)^2 = \frac{1}{2}(k_1 \cdot k_2 - \mu^2)
\]

\[
= \frac{\lambda}{2}(p_1 \cdot p_2 - M^2) = \frac{\lambda}{4}\Delta^2\beta^2 = \frac{\lambda}{4}(\text{momentum transfer})^2 \text{ in the Breit-system}
\]

\[
\omega = \frac{(p_1 \cdot p_2)(k_1 \cdot k_2)}{2\sqrt{(p_1 \cdot p_2)^2}} = \frac{p_1 \cdot (k_1 \cdot k_2)}{2\sqrt{\Delta^2 + M^2}} = \frac{p_1 \cdot (k_1 \cdot k_2)}{2\sqrt{\Delta^2 + M^2}} = \text{energy of the k-particle in the Breit system}
\]

\[
\omega \Theta_{cn} = 1 - \frac{2\Delta^2}{K^2} = \text{scattering angle in the CM-system}
\]

\[
W^2 = 2\omega \sqrt{\Delta^2 + M^2} + M^2 + \mu^2 + 2\Delta^2
\]
APPENDIX 5

The basic idea of dispersion relations

The most far-reaching conclusions which nowadays can be drawn from the field theoretical formalism, are the dispersion relations. In fact they express mainly the structural properties of the scattering amplitudes which originate in local commutativity — the latter being assumed to be the adequate formulation of causality, at least of microcausality.

Dispersion relations as a consequence of causality have been known for a long time in classical physics and in order to make the main idea clear, we shall give here an abstract derivation of classical dispersion relations.

Assume that one has a physical system, which suffers an action from outside, called input \( a(t) \) and answers to this by producing an output \( b(t) \). The internal properties of the system are not specified except that we assume:

1. \( \alpha \) the internal properties of the system are constant in time;
2. \( \beta \) the system relates input to output in a causal manner.

It is well-known from systems obeying non-linear equations of motion that they may (but must not) show self-excitation. Therefore it seems natural to restrict oneself because of \( \beta \) to systems which relate input and output linearly. For a general non-linear system nothing can be proven because of the possible non-causal connection between input and output and those non-linear systems which are causal seem not to have been investigated. We shall therefore add:

3. \( \gamma \) the output shall be a linear functional of the input.
Condition \( \gamma \) says that at a given time \( t \) the output \( b(t) \) must be a linear superposition of inputs at other times \( t' \) and condition \( \beta \) says that only the inputs of times \( t' \leq t \) may contribute to \( b(t) \). Hence the general form is

\[
b(t) = \int_{-\infty}^{+\infty} L(t, t') a(t') dt' \quad \text{where} \quad L(t, t') = 0 \quad \text{for} \quad t' > t
\]

Condition \( \alpha \) says now that the function \( L(t, t') \), which describes the properties of the system, should be a function of \( t-t' \) only:

\[
b(t) = \int_{-\infty}^{+\infty} L(t-t') a(t') dt' ; \quad L(t-t') = 0 \quad \text{for} \quad t-t' < 0 \quad (A5,1)
\]

We take the Fourier transform of this equation, which reads

\[
\tilde{b}(\omega) = \tilde{L}(\omega) \cdot \tilde{a}(\omega) \quad (A5,2)
\]

and says how the system responds to a monochromatic input. A simple example — in fact the starting point of Kramers and Kronig in their pioneer work on classical dispersion relations — is the dielectric polarization: Let \( a(t) \) and \( \tilde{a}(\omega) \) represent an electric field strength, \( b(t) \) and \( \tilde{b}(\omega) \) the polarization of a piece of matter, then \( L = \frac{\varepsilon - 1}{4\pi} \) will be the (tensor of) dielectric permeability, directly related to the refractive index.

Now comes the main point:

From the Fourier representation

\[
\tilde{L}(\omega) = \int_{-\infty}^{+\infty} L(\tau) e^{i\omega \tau} d\tau \quad (A5,3)
\]
and since \( L(\tau) \equiv 0 \) for \( \tau < 0 \) it follows that \( \tilde{L}(\omega) \) is an analytic function regular in the upper half of the \( \omega \)-plane, since only positive \( \tau \) contribute to the integral.

Therefore the value of \( \tilde{L}(\omega) \) for \( \text{Im} \omega > 0 \) is given by \( \int \) see Fig. \( (A5-1) \)

\[
\tilde{L}(\omega) = \frac{i}{2\pi i} \int_C \frac{L(\omega')}{\omega' - \omega} \, d\omega' + \frac{i}{2\pi i} \int_{C'} \frac{L(\omega')}{\omega' - \omega} \, d\omega'. \tag{A5, 4}
\]

The second integral contributes nothing — it has been added just for fun. Assume that the contribution of the large half circles vanish if they extend to an infinite radius (if not, then one considers \( \tilde{L}(\omega) \) divided by a polynomial instead of \( \tilde{L}(\omega) \) itself — see Appendix 6, for an explicit use of that trick). Let the radius go to \( \infty \) and allow \( \omega \) and the straight paths which go parallel to the real axis approach the real axis, but keep always \( \omega \) between them. The limit of this yields (for a more detailed proof see Appendix 7)

\[
\tilde{L}(\omega) = \frac{i}{2\pi i} \cdot 2P \int_{-\infty}^{\infty} \frac{L(\omega')}{\omega' - \omega} \, d\omega' \tag{\text{\( \omega \) real}} \tag{A5, 5}
\]

where \( P \) means the principal value of the integral. Write now
\[ \tilde{I}(\omega) = \text{Re} \tilde{I}(\omega) + i \text{Im} \tilde{I}(\omega) \]

denotes

\[
\begin{align*}
\text{Re} \tilde{I}(\omega) &= \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} \tilde{I}(\omega')}{\omega' - \omega} d\omega' \\
\text{Im} \tilde{I}(\omega) &= -\frac{P}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} \tilde{I}(\omega')}{\omega' - \omega} d\omega'
\end{align*}
\tag{A5,6}
\]

Thus \( \text{Re} \tilde{I}(\omega) \) and \( \text{Im} \tilde{I}(\omega) \) are Hilbert transforms of each other. Note that this is a direct consequence of \( \tilde{I}(\omega) \) being analytic for \( \text{Im} \omega > 0 \) and this again followed immediately from \( I(t-t') \) being causal. Thus the so-called "dispersion relations" (A5,6) are only another formulation of the principle of causality. They give nothing new but they relate the real and imaginary parts of the properties of the considered system. We have just proven part of the Titchmarsh theorem (see Appendix 7). Though in the field of theoretical applications it appears in a somewhat hidden form, the basic idea remains the same as here.

There are chiefly three applications of the dispersion relations:

\( \alpha \) The real and imaginary part of \( \tilde{I}(\omega) \) have different physical meaning and the dispersion relations allow to calculate one of them once the other one is known.

\( \beta \) We may use them to test mathematical models \( \sqrt{I(t)} \) whether these models are causal or not.

\( \gamma \) We may use them in order to test experimentally whether a system behaves causally or not.
We shall presently consider very briefly the case of the optical refractive index as an example.

Assume a plane monochromatic wave penetrating through matter with complex refractive index \( n(\omega) = n_1(\omega) + i n_2(\omega) \):

\[
\hat{f}(k, t) = e^{-i\omega(t - \frac{n\chi}{c})} = e^{-\frac{\omega}{c} \chi x} e^{-i\omega(t - \frac{n\chi}{c})}
\]

This wave behaves as if \( n_1(\omega) \) were the refractive index but it is damped exponentially by \( n_2(\omega) \). If one formulates causality by writing \( n^c(\omega) = \epsilon(\omega) \) and putting \( \tilde{\omega}(\omega) = \frac{1}{4\pi} (\epsilon(\omega) - 1) \), then one obtains immediately by (A5,6)

\[
\begin{align*}
\text{Re} \left[ n^2(\omega) - 1 \right] &= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} \ n^2(\omega')}{\omega' - \omega} \, d\omega' \\
\text{Im} \ n^2(\omega) &= -\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} \left[ n^2(\omega') - 1 \right]}{\omega' - \omega} \, d\omega'
\end{align*}
\]

(A5,7)

One finds the same result if one demands that no wave should travel faster than light. In gases of low density one can write \( 2(n-1) \) instead of \( n^2 - 1 \) and transform (A5,7) into

\[
\begin{align*}
n_1(\omega) - 1 &= \frac{2}{\pi} \int_{0}^{\infty} \frac{\omega' n_1(\omega')}{\omega'^2 - \omega^2} \, d\omega' \\
n_2(\omega) &= -\frac{2\omega}{\pi} \int_{0}^{\infty} \frac{n_2(\omega') - 1}{\omega'^2 - \omega^2} \, d\omega'
\end{align*}
\]

(A5,7')

Use has here been made of the fact that \( n_1(\omega) \) is even, \( n_2(\omega) \) is odd. We may illustrate with these formulae the above mentioned three main applications.

*) If \( \rho \) is the density then \( \sigma = \epsilon + f(\rho) \); \( n \sim 1 + \frac{1}{2} f(\rho) \)
α) Measuring the absorptive part $n_2(\omega)$ over all frequencies enables one to calculate the refractive part $n_1(\omega)$ at any frequency (and vice versa).

β) We may take the damped linear electron oscillator as a model and test, whether it is causal. Let $P = N \cdot e \cdot x$ be the polarization ($N =$ number of electrons per unit volume, $e =$ electron charge, $x =$ the dislocation of the electron under an external force). The equation of motion becomes, if $E(\omega)e^{-i\omega t}$ is the external field:

$$\ddot{x} + \kappa \dot{x} + \omega_0^2 x = \frac{e}{m} E(\omega) e^{-i\omega t}$$

with the solution

$$P(\omega) = N e x = \frac{N e^2}{m} E(\omega) \frac{1}{\omega_0^2 - \omega^2 - i\omega \gamma}$$

For homogeneous matter (and cubic crystals) one has $P = \frac{\varepsilon - 1}{4\pi} E$ from which follows

$$\frac{\varepsilon(\omega) - 1}{4\pi} = \frac{N e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega \gamma} \equiv \tilde{\varepsilon}(\omega)$$

For a causal system $\tilde{\varepsilon}(\omega)$ should be regular in the upper $\omega$ half plane. In fact, the only poles of $\tilde{\varepsilon}(\omega)$ lie at

$$\omega = -\frac{i}{2} \gamma \pm \sqrt{\omega_0^2 - \frac{1}{4} \gamma^2}$$

in the lower half plane. Hence (A5,7) is valid and the model is admissible from the point of view of causality. This model shows even more: replace in the equation of motion $E(\omega)e^{-i\omega t}$ by $E(t) = \delta(t)$ and solve it. The result is

$$P(t) = \frac{N e^2}{m \sqrt{\omega_0^2 - \gamma^2/4}} e^{-\frac{1}{2} \gamma t} \sin \left( t \sqrt{\omega_0^2 - \gamma^2/4} \right)$$

After such a $\delta$-push, the electrons oscillate with a frequency

$$\omega = \sqrt{\omega_0^2 - \gamma^2/4}$$

and a "lifetime" of $(\delta/2)^{-1}$ — just the real and imaginary part of the poles.
If one tries to invent a better classical model and replaces the phenomenonological damping term \( \gamma \dot{x} \) by the electron's radiation acting back on the electron, this yields a term proportional to \( \dot{x} \). Now \( \frac{\varepsilon - 1}{4\pi} \) is no longer regular in the upper half plane. Hence this model cannot be accepted since it cannot be causal. Indeed, it is well-known that these equations predict self-acceleration of the electron.

\[ f \]

Everybody believes that coloured sunglasses will behave causally—no light will come out of it before light has arrived. Therefore nobody will doubt that \((A5,7)\) will be verified by an experiment. There may be systems where the causality is not so obvious and where \((A5,7)\) is the appropriate means to prove it—this seems to be part of the situation in field theory.

But even the green glass may cause some trouble, if looked at from another point of view. Suppose there were no connection between the refractive and absorptive part of the refractive index. The problem is then: suppose you are in a dark room where some time later a flashlight will be on. Idealize the light as a function of time by a \( \delta(t) \)-function:

\[
\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} d\omega
\]

says that in it all frequencies are contained (each as an \( \cos \) or \( \sin \) wave) in such a way that they cancel each other except at the instant \( t=0 \). If you put on your spectacles with green glasses they let through only some of the everlasting green light and the other waves which are necessary for the mutual cancellation are absorbed. Why then do you not see in the dark with your green glasses? The dispersion relations are the answer: the refractive index depends on the frequency via the integral over the absorptive part in such a way that there is no green (or any) glass which simply can absorb a part of the spectrum. The dispersion
relations show how the refractive index must depend on $\omega$ in order that the remaining waves in the green region obtain just the right phase shifts as to cancel again everywhere except for $t > 0$. We must admit the $\gg$ sign, since the glass may contain systems which emit some radiation still at $t > 0$ (see the classical example of the damped oscillator, p. 134).

From the dependence of the refractive index on the frequency $\omega$ (dispersion) comes the name of the dispersion relations. But as the above quite general derivation may show, it has a much wider application in classical physics and even particularly in electrical engineering. Its applications in field theory are the main subject of these lectures.
APPENDIX 6

Proof of forward dispersion relations

(metric $x^2 = x_0^2 - x^2$)

(This unpublished proof is due to V. Glaser and H. Joss)

Assume a simple model: scalar neutral nucleons and mesons. The scattering amplitude and its imaginary part are see (78) p. 54 and (79) p. 55/

$$T = -\int d^4x \, e^{i \frac{R+R'}{2} \cdot x} \, \langle p' | R' A(\frac{x}{2}) A(-\frac{x}{2}) | p \rangle$$

$$\text{Im} \, T = \frac{4}{2} \int d^4x \, e^{i \frac{R+R'}{2} \cdot x} \, \langle p' | [j(\frac{x}{2}), j(-\frac{x}{2})] | p \rangle$$

$|p\rangle$ is a one-nucleon state; $A(x)$ is the meson operator, $j(x)$ the meson current, $k$ the meson momentum: $p+k = p'+k'$. In the nucleon rest system:

$$p = p' = (M, 0)$$
$$k = k' = (\omega, \vec{k})$$

which gives

$$T = -\int d^4x \, e^{i(\omega t - \vec{k} \cdot \vec{x})} \, \langle M | R' A(\frac{x}{2}) A(-\frac{x}{2}) | M \rangle$$

$$\text{Im} \, T = \frac{4}{2} \int d^4x \, e^{i(\omega t - \vec{k} \cdot \vec{x})} \, \langle M | [j(\frac{x}{2}), j(-\frac{x}{2})] | M \rangle$$

(A6,1)
The matrix elements do not contain any direction in the states and therefore are independent on the direction of \( \vec{r} \). We integrate over the directions of \( \vec{r} \) (without changing the notation of the matrix elements):

\[
T = -2\pi \int_0^\infty \int_0^{\infty} \frac{\sin \frac{kr}{\kappa}}{kr} \int_{-\infty}^{+\infty} dt \, e^{i\omega t} \langle M \mid R' A(\frac{x}{2}) A(-\frac{x}{2}) \mid M \rangle
\]

The retarded product is \( \neq 0 \) only in the forward light cone, hence in fact the \( t \)-integration from \(-\infty\) to \( r \) yields nothing. Since on the light cone we may expect singularities, we write with an arbitrarily small \( \delta > 0 \)

\[
T = -4\pi \int_0^\infty \int_0^{\infty} \frac{\sin \frac{kr}{\kappa}}{kr} \int_{r-\delta}^{+\infty} dt \, e^{i\omega t} \langle M \mid R' A(\frac{x}{2}) A(-\frac{x}{2}) \mid M \rangle
\]

The matrix element contains only linear combinations of \( \delta^2(x^2) \) and derivatives thereof, and is certainly a smooth function inside the light cone. For \( t \to \infty \) the matrix element tends to

\[
\langle M \mid [K_x A_{\text{out}}(\frac{x}{2}), K_x A_{\text{in}}(-\frac{x}{2})] \mid M \rangle = 0
\]

since \( \frac{K_x A_{\text{in}}}{\text{d}t} \to 0 \) \( K_x \equiv \iota K_{\mu} \mu^2 \mathcal{J} \). Therefore there are no contributions from \( \omega \), but there are contributions from the lower limit of the form \( e^{i\omega r} P(\omega, r) \) with \( P \) being a polynomial of finite order in \( \omega \) (coming from the \( \delta^{(n)}(x^2) \)).

Now \( \kappa = \sqrt{\omega^2 - \mu^2} \) and \( T \) has the form
\[ T = -4 \kappa \int_0^\infty \frac{e^{i \omega r \sqrt{\omega^2 - \mu^2}}}{\sqrt{\omega^2 - \mu^2}} e^{i \omega r} g(\omega, r) \equiv \int_0^\infty F(r, \omega) \, dr. \]

This expression is defined for all \( \omega \); the \( \omega \) plane is cut from \(-\mu\) to \(+\mu\) and the integrand \( F(r, \omega) \) is regular in the upper half plane \( (*) \) and (assumed to be) bounded by a polynomial \( \prod (\omega) = \prod \left[ (\omega + i a_v)^2 - a_v^2 \right] \).

For a first orientation the reader may disregard this polynomial by putting it \( = 1 \) and skip all calculations concerning it \( \int \).

\[ \omega \]
\[ -\mu \quad +\mu \]

Fig. (A6-1)

\( (*) \) The exponent is for large \( \omega \)

\[ \pm i \omega r \left( 1 - \frac{\mu^2}{2 \omega^2} \right) + i \omega r = \pm i \omega r + i \omega r \mp i \omega \frac{\mu^2}{2 \omega} \]

where because of the \( sin \) both signs occur. If \( \omega \) has a large positive imaginary part, the \( - \) sign is the worst. We have then a behaviour

\[ e^{ir \frac{\mu^2}{2 \omega}} \longrightarrow 1 \quad \text{for} \quad \omega \to \infty. \]

So for fixed \( r \) the integrand is bounded if \( g(r, \omega) \) is bounded. Now we neither know so far that \( F(r, \omega) \) the contribution from the light cone, is a polynomial of finite order, nor do we know that the whole integrand is bounded by a polynomial of finite order. This has, however, been proven by Symanzik in one-dimensional scattering. It seems very plausible also in the present case and shall be assumed to be the case henceforward.
Since $F(r, \omega)$ has these properties, there exists (see Appendix 7) the relation (integrated along the upper cut)

$$\text{Re} \left[ \frac{F(r, \omega)}{\Pi(\omega)} \right] = \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im} \left[ \frac{F(r, \omega')}{\Pi(\omega')} \right]}{\omega' - \omega} \, d\omega' \quad (A6,2)$$

We now investigate what the symmetry properties of our functions are:

$$\langle M | R' | A(\frac{x}{2}) A(-\frac{x}{2}) | M \rangle =$$

$$= (\mathcal{O}_\omega - \mu^2)(\mathcal{Q}_v - \mu^2) \left\{ \langle M | (i) \Theta(u_v - v) [A(u), A(v)] | M \rangle \right\}$$

$$\langle M | R' | M \rangle^* = K_{\mu} K_{v'} \langle M | (i) \Theta(u_v - v) [A(u), A(v)] | M \rangle_{u = \frac{x}{2}; v = -\frac{x}{2}}$$

$$= K_{\mu} K_{v'} \langle M | (i) \Theta(u_v - v) [A(u), A(v)] | M \rangle_{u = \frac{x}{2}; v = -\frac{x}{2}}$$

$$= \langle M | R' | M \rangle = \text{real}$$

Therefore

$$\mathcal{F}(r, \omega) = \mathcal{F}(r, -\omega)$$

$$\text{Re } \mathcal{F}(r, \omega) \quad \text{even}$$

$$\text{Im } \mathcal{F}(r, \omega) \quad \text{odd}$$

(A6,3)

the same holds for $T(\omega)$.  

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The reader who only wants information concerning the main idea may omit the following calculation concerning the polynomial $\prod (\omega)$ and jump directly to p.145, Eq. (A6,5').

The polynomial which we have inserted for convergence reasons, has roots $\omega = a_v - i b_v$ and $\omega = -a_v - i b_v$, i.e. below the real axis and symmetric to the imaginary axis. We choose $a_v > \mu$.

![Fig. (A6-2)](image)

Furthermore

$$\prod^* (\omega) = \prod \left[ (\omega^*-i b_v)^2 - a_v^2 \right] = \prod (-\omega^*)$$

hence on the real axis

$$\prod^* (\omega) = \prod (-\omega)$$

$$\text{Re} \prod (\omega) = \text{even}$$

$$\text{Im} \prod (\omega) = \text{odd}$$

We calculate

$$\frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega - \omega'} \text{Im} \left[ \frac{F(\omega, \omega')}{\prod (\omega')} \right]$$

$$\text{Im} \frac{F}{\prod} = \text{Re} F \cdot \text{Im} \frac{1}{\prod} + \text{Im} F \cdot \text{Re} \frac{1}{\prod}$$
In order to separate $\text{Re}\frac{1}{\Pi}$ and $\text{Im}\frac{1}{\Pi}$ we multiply by the cc. and expand the numerator with respect to the small $b_v$:

$$\Pi^*(\omega) = \prod_v \left[(\omega - i b_v)^2 - \lambda_v^2 \right] \approx \prod_v (\omega^2 - \lambda_v^2) - \sum_{\mu \neq \nu} 2i \omega b_{\mu} \prod_{\mu \neq \nu} (\omega^2 - \lambda_{\mu}^2) + O(\epsilon^4)$$

Our integral becomes

$$\frac{d}{\epsilon^2} \int_{-\infty}^{+\infty} d\omega' \frac{\text{Im} F(\omega')}{(\omega' - \omega) \prod_v (\omega' + i b_v + a_v)(\omega' - i b_v + a_v)(\omega' + a_v)(\omega' - a_v)}.$$ 

This is a principal value integral with respect to the pole $\omega' = \omega$. There are the other poles

$$\omega' = \left\{ \begin{array}{c}
  a_v + i b_v \\
  a_v - i b_v \\
  -a_v + i b_v \\
  -a_v - i b_v
\end{array} \right\}$$

those above the real axis have been created artificially by multiplying by $\Pi^*$. The integrand vanishes in the upper half plane for $|\omega| \rightarrow \infty$. The situation is now given in Fig. (A6-3a):

![Fig. (A6-3a)](image)

The situation is also given in Fig. (A6-3b):

![Fig. (A6-3b)](image)
If we let \( b_\nu \to 0 \), then the poles move between the two paths of the principal value integral and what then happens is shown in the Fig. (A6-3b).

1) There remains a principal value integral with respect to all poles, that part which contains \( \Re F(\omega')2 \omega \sum b_\nu \ldots \) goes to zero with \( b_\nu \), the denominator becomes \( \prod (\omega^2 - a_\nu^2)^2 \) which partly cancels with the numerator. Hence with \( b_\nu \to 0 \) we obtain as one contribution

\[
\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega'}{(\omega' - \omega) \prod_{\nu} (\omega^2 - a_\nu^2)} \Im \frac{F(\omega')}{F(\omega')}
\]

2) There remain \( \frac{1}{2} \) the residues from all the poles except \( \omega' = \omega/2 \).

The small circles have opposite sense above and below the real axis. In inserting the values at the poles we only retain again terms proportional to \( b_\nu \) and neglect higher ones. For one pair of poles \( \omega' = a_\nu^2 \pm i b_\nu \) we find:

\[
\pi i \frac{\Im F(a_\nu)}{2} \frac{2i b_\nu a_\nu \prod_{\nu \neq \nu} (a_\nu^2 - a_\mu^2) - \Re F(a_\nu) 2a_\nu \sum b_\nu \prod_{\nu \neq \nu} (a_\nu^2 - a_\mu^2)}{(a_\nu^2 - \omega) 2a_\nu 2i b_\nu 2a_\nu \prod_{\nu \neq \nu} (a_\nu^2 - a_\mu^2)^2} = \pi i \times \text{the corresponding expression with } (b_\nu \to -b_\nu)
\]

The first term with \( \Im F(a_\nu) \ldots \) drops out by \( b_\nu \to -b_\nu \) and subtraction, since \( b_\nu \) cancels in numerator and denominator. In the second term the \( b_\nu \) in the sum are fixed, so that \( b_\nu \to -b_\nu \) plus subtracting gives for the pole pair

\[
2\pi i \frac{-\Re F(a_\nu) \sum_{\nu} b_\nu \prod_{\mu \neq \nu} (a_\nu^2 - a_\mu^2)}{2i b_\nu 2a_\nu (a_\nu^2 - \omega) \prod_{\nu \neq \nu} (a_\nu^2 - a_\mu^2)^2}
\]

but

\[
\sum_{\nu} b_\nu \prod_{\mu \neq \nu} (a_\nu^2 - a_\mu^2) = b_\nu \prod_{\mu \neq \nu} (a_\nu^2 - a_\mu^2)
\]

since all other terms contain \( a_\nu^2 - a_\mu^2 = 0 \).
This pair of poles gives therefore
\[-\frac{\text{Re } F(a_s)}{2a_s (a_s^2 - \omega^2) \prod_{\nu \neq s} (a_s^2 - a_\nu^2)}\]

For each pole with $a_s$, there is another one with $-a_s$. This has to be added. This gives
\[-\frac{\text{Re } F(a_s)}{2a_s \prod_{\nu \neq s} (a_s^2 - a_\nu^2)} \left\{ \frac{\text{Re } F(a_s)}{a_s - \omega} - \frac{\text{Re } F(-a_s)}{-a_s - \omega} \right\}\]

\[\Rightarrow -\sum_{s} \frac{\pi \text{Re } F(a_s)}{2a_s \prod_{\nu \neq s} (a_s^2 - a_\nu^2)} \left\{ \frac{1}{a_s - \omega} + \frac{1}{a_s + \omega} \right\}\]

since $\text{Re } F(\omega)$ is even and summed over all poles. The principal value contribution is, since $\text{Im } F(\omega')$ is odd

\[\frac{\text{P}}{\pi} \left\{ \int_{-\infty}^{0} + \int_{0}^{\infty} \right\} = \frac{\text{P}}{\pi} \int_{0}^{\infty} \frac{d\omega' \text{ Im } F(h, \omega')}{(\omega' - \omega) \prod (\omega'^2 - a_\nu^2)} + \frac{\text{P}}{\pi} \int_{0}^{\infty} \frac{d\omega' \text{ Im } F(h, \omega')}{(\omega' + \omega) \prod (\omega'^2 - a_\nu^2)}\]

Altogether:

\[\frac{\text{P}}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega' - \omega} \text{ Im } \left\{ \frac{F(h, \omega)}{\prod (\omega')} \right\} = \]

\[= -\sum_{s} \frac{\text{Re } F(h, a_s)}{2a_s \prod_{\nu \neq s} (a_s^2 - a_\nu^2)} \left[ \frac{1}{a_s - \omega} + \frac{1}{a_s + \omega} \right] + \frac{\text{P}}{\pi} \int_{0}^{\infty} \frac{d\omega' \text{ Im } F(h, \omega')}{\prod (\omega')} \left[ \frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right]\]

\[\prod (\omega') = \prod (\omega'^2 - a_\nu^2)\]
With (A6,2) we obtain

$$\text{Re } F(r, \omega_0) + \Pi(\omega_0) \sum_i \frac{\text{Re } F(r, a_i)}{2a_i \Pi_i (a_i^2 \cdot a_i^2)} \left[ \frac{1}{a_i^2 - \omega_0} + \frac{1}{a_i^2 + \omega_0} \right] =$$

$$= \Pi(\omega_0) \frac{P}{\pi} \int_0^\infty d\omega \frac{\text{Im } F(r, \omega)}{\Pi(\omega)} \left[ \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right] \quad (A6,5)$$

If we disregard the polynomial \( \Pi(\omega) \) by putting it equal to one, the formula reads as follows directly from (A6,2) and (A6,3): \( A6,5' \)

$$\text{Re } F(r, \omega_0) = \frac{P}{\pi} \int_0^\infty d\omega \frac{F(r, \omega)}{\Pi(\omega)} \left[ \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right]$$

Now we want a dispersion relation for

$$\mathcal{T} = \int_0^\infty F(r, \omega) \, dr$$

On the left-hand-side we may carry out this integration, it will change \( \mathcal{F} \) into \( T(\omega) \), where we choose \( \omega \) as well as all \( a_\mu \) in the physical region \( > \mu \). On the right-hand-side however we have to interchange the integrations \( \int dr \int d\omega \rightarrow \int d\omega \int dr \) in order to obtain \( T \) instead of \( \mathcal{F} \). Here lies the difficulty, since the \( \omega \)-integration includes the unphysical region \( 0 < \omega < \mu \) where \( \sin h \sqrt{\omega^2 - \mu^2} \) increases exponentially with \( r \).

We split the integral into

$$\frac{P}{\pi} \int_0^\mu + \frac{P}{\pi} \int_\mu^\infty$$

where in the latter the \( \lambda \)-integration meets no difficulty and it simply means replacing \( F(r, \omega) \) by \( T(\omega) \).
Consider now the first integral. We know that
\[
\int_0^\infty \frac{d\varepsilon}{\pi} \int_0^\infty \frac{d\omega}{\Pi(\omega)} \Im F(\varepsilon, \omega) \left[ \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right] = \Phi(\varepsilon)
\]  
(A6,7)
also exists.

Considered as function of (complex) $\varepsilon$, $\Phi(\varepsilon)$ is analytic in $\Re \varepsilon > 0$ and the wanted integral is the limit of this function for $\varepsilon \to +0$.

From (A6,1) follows
\[
\Im F(\varepsilon, \omega) = 2\pi \varepsilon \frac{\sin \omega}{\omega} \int_{-\infty}^{+\infty} e^{i\omega t} dt \left< M \left| \begin{pmatrix} i \left( \frac{\varepsilon}{2} \right) \\ i \left( \frac{-\varepsilon}{2} \right) \end{pmatrix} \right| n \right> \]  
(A6,8)
where the matrix element does not depend on the direction of $\varepsilon$. We consider the integral
\[
I = \int_{-\infty}^{+\infty} e^{i\omega t} dt \left< M \left| \begin{pmatrix} i \left( \frac{\varepsilon}{2} \right) \\ i \left( \frac{-\varepsilon}{2} \right) \end{pmatrix} \right| n \right> \equiv J(\varepsilon) - J(-\varepsilon)
\]
and introduce a complete set of states (A2,28) $J_{\Omega} = \sqrt{\frac{p^2 + m^2}{E}} \sum \Omega$.
$J \sum_{\Omega} dm^2$ means: sum over discrete and integrate over continuous mass values.

\[
J(\varepsilon) = \sum_{\Omega} dm^2 \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{2\Omega} \left< M \left| \begin{pmatrix} i \left( \frac{\varepsilon}{2} \right) \end{pmatrix} \right| P'm \cdot \left< \begin{pmatrix} i \left( \frac{-\varepsilon}{2} \right) \end{pmatrix} \right| n \right>
\]
With
\[ j_\frac{\chi}{2}(\chi) = e^{iP\frac{\chi}{2}} j(0) e^{-iP\frac{\chi}{2}} \]
and \( P' = (\Omega, \mathbf{p}') \)
we obtain
\[ J(x) = \delta \int d\mu \int \frac{dP}{2\Omega} \sum_{t} e^{i\omega t} \left| \frac{d\mathbf{p}}{2\Omega} \right| e^{i\mathbf{m} \cdot \mathbf{p}} e^{-i\Omega t} e^{i\mathbf{p}' \cdot \mathbf{x}} \left| \langle M | j(0) | P' \rangle \right|^2 \]
\((A6, 9')\)

Since the matrix element does not depend on the direction of \( P' \), we integrate over all directions \((\mathbf{p}' \cdot \mathbf{x} = p' r; \ |\mathbf{p}'| = p')\). Furthermore the \( t \)-integration yields a \( \delta \) function:
\[ J(x) = (2\pi)^2 \int d\mu \int \frac{dP}{2\Omega} \delta(M + \omega - \Omega) \frac{2i \sin p r}{i p r} \left| \langle M | j(0) | P' \rangle \right|^2 = \]
\[ = (2\pi)^2 \int d\mu \int \frac{dP}{2\Omega} \delta(M + \omega - \Omega) \frac{2i \sin p r}{i p r} \left| \langle M | j(0) | P' \rangle \right|^2 \]
\[ p^2 + m^2 = \Omega^2 \quad ; \quad 2p dP = 2\Omega d\Omega \]

gives
\[ J(x) = (2\pi)^2 \int d\mu \int \frac{2i \sin (2\sqrt{(M + \omega)^2 - m^2})}{i} \left| \langle M | j(0) | E = M + \omega \rangle \right|^2 \left| p = \sqrt{(M + \omega)^2 - m^2} \right|^2 \]
\((A6, 9')\)

This is an integral over all physical \( \text{intermediate states of energy} \ M + \omega \) and mass \( m \).

Formally the vacuum belongs also to the complete set of states, but
\[ \langle M | j(0) | 0 \rangle = 0. \]
For the mass spectrum we assume for illustration that there is a stable bound state with mass $m^*$. The figure shows where contributions to the integral can come.

Consider now the integral $\mathcal{J}(x)$. The range of the $m^2$-integration is limited by two conditions ($\omega < \mu$ is fixed for the moment):

a) $m \geq M$ since the matrix element vanishes if the intermediate state does not contain at least one nucleon.

b) $m \leq (M+\omega)$ since otherwise the intermediate state has imaginary momentum (unphysical — the sum was over physical states only).

Introducing the abbreviations

$$\sqrt{(M+\omega)^2 - m^2} = \rho(m)$$

$$\langle M|\hat{\jmath}(0)|M+\omega; \rho(m)\rangle^2 = M(\omega,m)$$

We obtain

$$\mathcal{J}(x) = \frac{(2\pi)^2}{\mathcal{R}} \left[ M(\omega,M) \cdot \sin x \rho(M) + \Theta(\omega+M-m^*) \cdot M(\omega,m^*) \sin x \rho(m^*) \right]$$

(A6,11)

where the step function $\Theta(\omega+M-m^*)$ takes care of the condition b) above.

Since $m \leq M+\omega < M+\mu$, we have only contributions coming from the $\sum$-sign in $\int \, dm^2$, the continuum is not reached.
The other term \( J(-x) \) of the commutator in the integral \( I(p) \) gives no contribution, since, as one sees from (A6.9) changing \( x \) into \(-x\) affects only \( t \) but not \( \hat{x} \) (because of the integration over all directions of \( \hat{F}' \)) and \( t \to -t \) has the same effect as changing \( M \to -M \) and \( \Omega \to -\Omega \) inside \( \delta(M+\omega-\Omega) \) which is the same as changing \( \omega \to -\omega \). With negative \( \omega \) conditions a) and b) on the range of the \( m^2 \)-integration cannot be fulfilled.

Inserting now (A6.11) into (A6.8) and this into (A6.7) we obtain

\[
\Phi(E) = 8\pi^2 \int_0^\infty e^{-\frac{E}{\omega}} \rho M \left[ \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right] \frac{\text{sinh} \ K \omega}{K} \times \\
\times \left\{ M(M,M) \delta(M,\rho(M)) + \Theta(M+M^*) M(M,M^*) \delta(M,\rho(M^*) \right\}
\]

where \( K = \sqrt{\mu^2 - \omega^2} \).

We now may interchange the \( r \) and \( \omega \) integration, if \( \text{Re} \ E > \mu \), since then the exponential increase of \( \text{sinh} \ K \omega \) is compensated. This is permitted because \( \Phi(E) \) is an analytic function of \( E \) for \( \text{Re} \ E \geq 0 \). The \( r \)-integration yields

\[
\Phi(E) = 2\pi^2 \int_0^\mu \frac{d\omega}{K\Pi(\omega)} \left[ \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right] \times \\
\times \left\{ M(M,M) \left[ \frac{1}{\varepsilon + K + i \rho(M)} - \frac{1}{\varepsilon + K - i \rho(M)} - \frac{1}{\varepsilon - K + i \rho(M)} + \frac{1}{\varepsilon - K - i \rho(M)} \right] \\
+ \Theta(M+M^*) M(M,M^*) \left[ \frac{1}{\varepsilon + K + i \rho(M^*)} - \frac{1}{\varepsilon + K - i \rho(M^*)} - \frac{1}{\varepsilon - K + i \rho(M^*)} + \frac{1}{\varepsilon - K - i \rho(M^*)} \right] \right\}
\]

(A6.12)

\[
K = \sqrt{\mu^2 - \omega^2} \\
\rho(M) = \sqrt{(M + \omega^2)^2 - \omega^2}
\]
Now comes the main point: we know that $\mathcal{G}(\varepsilon)$ is analytic in $\varepsilon$ for $\text{Re}\varepsilon > 0$. Though we have calculated it for $\text{Re}\varepsilon > \mu$ the continuation to $\text{Re}\varepsilon < \mu$ exists and as we shall now see this tells us also something about the continuation of the functions $\mathcal{M}_\varepsilon(\omega, m)$ and $\mathcal{M}_\varepsilon(\omega, m^*)$.

First we observe that putting $\varepsilon = 0$ under the integral makes everything zero. But the meaning is of course that one should first integrate over $\omega$ and then let $\varepsilon \to 0$. To this end we consider the behaviour of the denominators when $\omega$ goes from $0$ to $\mu$ (or from $m^*-M$ to $\mu$ in the second term). All denominators contain $\pm K \pm ip(m)$ with $m=M, m^*$. We draw the $K+ip$ plane in Fig. (A6-4)

![Diagram](image)

Fig. (A6-4)

The paths of all four combinations $\pm K \pm ip(m)$ are shown, the outer curves belong to $m=M$, the inner to $m=m^*$. All of them start on the real axis if $\omega = 0$ and $m^*-M$ respectively and end on the imaginary axis for $\omega = \mu$.

Obviously, if $\varepsilon$ is to go to zero, it has to cross over these curves and one obtains the situation of Fig. (A6-5)

*) For $m^* \to M+\mu$ the inner curves shrink to one single point: the origin.
We now introduce the variable
\[ z = K(\omega) + i \rho(\omega, \mu) \]

There are in fact two such variables, one with \( p(\omega, \mu) \) the other with \( p(\omega, \mu^*) \) but the main effect is that they lead to integrals over different paths, and of course the inverse \( \omega(z) \) is calculated differently. We shall write therefore \( \omega_M(z) \) and \( \omega_m^*(z) \), but consider only one variable \( z \). Transforming the integrals gives then a new integrand

\[
F_{M^*}(z) \equiv \left\{ \frac{d\omega}{dz} \frac{\Pi(\omega(z))}{K(\omega(z))} \left[ \frac{1}{\omega(z) - \omega_0} + \frac{1}{\omega(z) + \omega_0} \right] \right\}_{M^*}
\]  

(A6, 13)

where the subscripts \( M \) and \( m^* \) designate which function \( \omega(z) \) is meant.

We obtain now according to Fig. (A6-5):

\[
\Phi(\xi) = \frac{2\pi^2}{i} \left\{ \sum_{C_4} - \sum_{C_3} + \sum_{C_2} \right\} F_M(z) \frac{dz}{\xi - z} + \\
+ \frac{2\pi^2}{i} \left\{ \sum_{C_4'} - \sum_{C_3'} + \sum_{C_2'} \right\} F_{m^*}(z) \frac{dz}{\xi' - z} + \\
+ 4\pi^3 \left[ F_M(\xi) + F_{m^*}(\xi) \right]
\]  

(A6, 14)
As one sees, all the integrals cancel for $\varepsilon = 0$ and only $4\pi^3 \left[ F_M(0) + F_M^*(0) \right]$ from the poles remains. But (A6,14) says even more: it says that the function $F_M(z)$ has an analytic continuation inside the half ring between $C_4$ and $C_4'$, and $F_M(z) + F_M^*(z)$ have an analytic continuation inside the domain enclosed by $C_4'$ and the imaginary axis. This follows from the fact that $\phi(\varepsilon)$ is regular for $\text{Re}\varepsilon > 0$. Note that these functions were defined originally only on $C_1$ and $C_1'$ (the other paths bring nothing new).

![Diagram](image)

**Fig. (A6-6)**

We consider now roughly what are the corresponding domains in the $\omega$-plane. We find by inverting

$$z = \sqrt{\mu^2 - \omega^2} + i \sqrt{(M + \omega)^2 - \mu^2}$$

$$\omega = \frac{-M \left( \frac{\varepsilon^2 + \mu^2 + n^2 - \omega^2}{2 \varepsilon^2 n^2} \right) \pm \frac{z}{2 \varepsilon^2 n^2} \sqrt{4\mu^2 \omega^2 - (\varepsilon^2 - \mu^2 + M^2 - \omega^2)^2}}$$

(A6,15)

Now we follow in Fig. (A6-6) the path $C_1' - D' - C_4'$. $C_1'$ and $C_4'$ lie on the real $\omega$ axis and correspond to $m - M \leq \omega \leq \mu$ (dotted). If $z$ follows the path $D'$, $\omega$ first stays on the real axis and then leaves it to go along a ring-shaped curve. This is shown in Figs. (A6-7) and (A6-7'). (The first shows the path for $m = M$, the second for $m > M$).
The points \((\alpha)\), \((\beta)\) and \((\gamma')\) correspond to

\[(\alpha) \quad z = \pm i \sqrt{M^2 - (m-\mu)^2} \quad ; \quad \omega = \frac{M \mu}{m-\mu}\]

\[(\beta) \quad z = 0 \quad ; \quad \omega = \frac{m^2 - M^2}{2m} \quad ( > m-M)\]

\[(\gamma') \quad z = \pm i \sqrt{(M+\mu)^2 - m^2} \quad ; \quad \omega = \mu\]

The path corresponding to \(C_1'\) and \(C_4'\) is dotted, the path corresponding to \(D\) is a full line. The larger \(m\), the smaller the region inside the curves becomes and for \(m \to M+\mu\) this region shrinks to one single point : \(\omega = \mu\). The point \((\beta)\) lies exactly on the origin for \(m = \sqrt{N^2 + \mu^2}\).
We now take up again the argument following Eq. (A6,14). It now reads

\[ H(\omega, M) \mathcal{M}(\omega, m) + H(\omega, \bar{m}) \mathcal{N}(\omega, \bar{m}) \]  

is analytic, where

\[ H(\omega, m) = \frac{\sqrt{(M+\omega)^2 - m^2}}{M \sqrt{\mu^2 - \omega^2} + \omega \left[ \sqrt{\mu^2 - \omega^2} + i \sqrt{(M+\omega)^2 - m^2} \right]} \]

If there is no stable bound state with mass \( M < m < M + \mu \), then \( \mathcal{M}(\omega, M) \) is analytic inside the whole of \( D \).

This comes out by writing \( F(z) \) in (A6,13) as a function of \( \omega \). Although one cannot conclude that inside \( D' \) \( \mathcal{M}(\omega, m) \) and \( \mathcal{M}(\omega, M) \) are separately analytic, this seems at least plausible.
One cannot conclude: let \( m \) go to \( M + \mu \), then the curve \( D' \) shrinks to the point \( \omega = \mu \) and \( \mathcal{M}(\omega, M) \) is analytic everywhere inside \( D \), hence it is always analytic inside \( D \). This is wrong, since the mass spectrum is determined in principle by the field theory and hence one cannot consider \( \mathcal{M}(\omega, M) \) as independent of the mass value \( m \) of the bound state. In fact: if a certain interaction of fields yields a bound state with mass \( m \), and another one leads to a mass \( m' \), then also the structure of the function \( \mathcal{M}(\omega, M) \) will be different for the two different theories.

We finally evaluate (A6, 14) putting \( m = m^* \)

\[
\phi(0) = 4\pi^3 \left[ F_M(0) + F_{M'}(0) \right]
\]

This corresponds to the points \((\beta)\) and \((\beta')\) in Fig. (A6–3).

\[
\frac{d^2}{d\omega} \left|_{\omega = K + i\mu} \right. = \frac{i\omega (K + i\mu) + iKM}{Kp} = \frac{M}{K}
\]

With this into (A6, 13) we find

\[
\phi(0) = \frac{4\pi^3}{M} \left\{ \frac{\mathcal{M}\left(-\frac{\mu^2}{2M}, M\right)}{\prod\left(-\frac{\mu^2}{2M}\right)} \left( \frac{1}{\omega - \frac{\mu^2}{2M}} - \frac{1}{\omega + \frac{\mu^2}{2M}} \right) + \frac{\mathcal{M}(\omega^*, M^*)}{\prod(\omega^*)} \left( \frac{1}{\omega^* - \omega_0} + \frac{1}{\omega^* + \omega_0} \right) \right\}
\]

We now collect the results:

i) We perform the \( r \)-integration on (A6, 5). This changes \( P(r, \omega_0) \) into \( T(\omega_0) \) on the left-hand-side and yields \( \prod(\omega_0) \cdot \phi(0) \) on the right-hand-side (see (A6, 7)).

ii) Insert the expression for \( \phi(0) \) from (A6, 17).
Thus we arrive at the complete forward dispersion relation:

\[
\frac{\text{Re } T(\omega)}{\Pi(\omega)} = \sum_i \frac{\text{Re } T(a_i)}{2 a_{\phi} \Pi(a^2_{\phi} - a^2_{\phi_\omega})} \left[ \frac{1}{a_{\phi} - \omega} + \frac{1}{a_{\phi} + \omega} \right] + \\
+ \frac{P}{\pi} \int_\mu^\infty d\omega \frac{\text{Im } T(\omega)}{\Pi(\omega)} \left[ \frac{1}{\omega - \omega_\omega} + \frac{1}{\omega + \omega_\omega} \right] + \\
+ \frac{4\pi^3}{M} \left\{ \frac{\mathcal{M}(\frac{\mu^2}{2M}; M)}{\Pi(-\frac{\mu^2}{2m})} \left[ \frac{1}{\mu^2 - \omega_\omega} + \frac{1}{\mu^2 + \omega_\omega} \right] + \\
+ \frac{\mathcal{M}(\omega^*_\omega, m^*_\omega)}{\Pi(\omega^*_\omega)} \left[ \frac{1}{\omega^*_\omega - \omega_\omega} + \frac{1}{\omega^*_\omega + \omega_\omega} \right] \right\}
\]

with

\[
\Pi(\omega) = \prod_v (\omega^2 - a^2_v)
\]

\(\omega \text{ real } > \mu\) , otherwise arbitrary. This polynomial warrants convergence of the integrals involved. If it is not necessary, one may put it = 1 and omit \(\sum_i\) in (A6,18). 

\(\mathcal{M}(\omega, m) = \left| \langle M | \frac{1}{2} (0) | E = M + \omega \rangle \right|^2 \), \(p = \sqrt{(M+\omega)^2 - m^2}\)

\(m^*_\omega = \text{mass of bound state } M \leq m^*_\omega \leq M + \mu\)

\(\omega^*_\omega = \frac{m^2 - M^2 - \mu^2}{2M} = - \frac{\mu^2}{2M} \text{ for } m^*_\omega = M\)

Of course, \(\mathcal{M}(\frac{\mu^2}{2M}; M)\) and \(\mathcal{M}(\omega^*_\omega, m^*_\omega)\) are both well defined constants — which however no one can calculate.
The first of them, $\mathcal{H} \left( -\frac{\kappa^2}{2M} + iM \right)$ has a very simple physical meaning: let us calculate this matrix element squared in the first perturbation approximation: for a real scalar meson and nucleon we have

$$(\Box - \mu^2) A(x) = \frac{g}{\kappa} \Psi^2(x) = j(x)$$

as the meson field equation. For $\Psi(x)$ we have to take the free field as first approximation, thus (A2,22')

$$\Psi(x) = \frac{i}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega} \left[ a(k) e^{ikx} + a^*(\vec{k}) e^{-ikx} \right]$$

$$j^\mu(0) = \frac{g}{(2\pi)^3} \int \frac{d^3p}{4\omega_k \omega_p} \left[ a(\vec{k}) a(\vec{p}) + a(\vec{k}) a^*(\vec{p}) + a^*(\vec{k}) a(\vec{p}) + a^*(\vec{k}) a^*(\vec{p}) \right]$$

We calculate the element for physical states

$$\langle M | j^\mu(0) | k' \rangle$$

(where $M$ means the nucleon at rest)

$$\langle M | = \langle \Omega | a(0) ; | k' \rangle = a^*(\vec{k}') | \Omega \rangle$$

gives

$$\langle M | j^\mu(0) | k' \rangle \bigg|_{\text{perturb.}} = \frac{g}{(2\pi)^3} \int \frac{d^3p}{4\omega_k \omega_p} \langle \Omega | a(0) a^*(\vec{k}) a(\vec{p}) a^*(\vec{k}') | \Omega \rangle$$

whereas the other three terms of $j(0)$ do not contribute. The commutation relations (A2,26) yield

$$\langle M | j^\mu(0) | k' \rangle \bigg|_{\text{perturb.}} = \frac{g}{(2\pi)^3} \int \frac{d^3p}{4\omega_k \omega_p} \delta(\vec{k}) \delta(\vec{p} - \vec{k}') 4\omega_p \omega_k = \frac{g}{(2\pi)^3}$$
Thus the perturbative matrix element is constant for physical states. The analytic continuation of a constant is trivial: the function is constant everywhere. Hence the perturbation matrix element, taken at the unphysical four momentum which occurs in $\mathcal{M}(-\frac{\mu^2}{2M}; M)$, is also

$$
\langle M | j(0) | E = M - \frac{\mu^2}{2M}; p = i\mu \sqrt{1 - \frac{\mu^2}{2M}} \rangle_{\text{perturb}} = \frac{g}{(2\pi)^3}
$$

The only difference between the true matrix element and the perturbation matrix element is that $j(0)$ is the current of the interacting fields rather than $j_{\text{pert}}(0)$, which is the current of the incoming (free) fields. Thus in the true matrix element we should replace $g$ by the renormalized coupling constant:

$$
\mathcal{M}(-\frac{\mu^2}{2M}; M) = \frac{g_\pi}{(2\pi)^3} \tag{A6,19}
$$

$g_\pi$ = renormalized coupling constant, by which we have not only a definition of the renormalized coupling constant but also by inserting it into the dispersion relation (A6,18) a method to determine it experimentally.

If we omit the terms coming from the polynomial $\prod (\omega)$ and the term corresponding to the "bound state" with mass $m^*$, we obtain with (A6,19) the most familiar form

$$
\text{Re } T'(\omega_0) = \frac{g_\pi}{2M} \left[ \frac{1}{\omega_0 - \frac{\mu^2}{2M}} - \frac{1}{\omega_0 + \frac{\mu^2}{2M}} \right] + \left\{ \frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0} \right\} \tag{A6,20}
$$

$$
+ \frac{p}{\pi} \int d\omega \Im \kappa T(\omega) \left[ \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right]
$$
We shall add one remark: in (A6,16) it was stated that $\mathcal{M}(\omega,M)$ is analytic inside the region $D$ in the $\omega$-plane, if there are no stable bound states with mass between $M$ and $M+\mu$. This is actually the case in the meson-nucleon system. Hence for the meson-nucleon system

$$\mathcal{M}(\omega,M) = \left| \langle M \mid \hat{J}(0) \mid M+\omega \ ; \ p = \sqrt{\omega^2 + 2M\omega} \right|^2$$

is analytic in $D$.

Now from CTP-invariance it follows that the matrix element is real and furthermore it does not vanish for $0 \leq \omega \leq \mu$. Therefore the square root of the analytic continuation of $\mathcal{M}$ inside $D$ is the analytic continuation of the square root of $\mathcal{M}$ on the real $\omega$-axis ($0 \leq \omega \leq \mu$). We thus know that the function

$$f(\omega) = \langle M \mid \hat{J}(0) \mid M+\omega \ ; \ \sqrt{\omega^2 + 2M\omega} \rangle$$

is itself analytic inside $D$ [see Figs. (A6-6) to (A6-8) and Eq. (A6,15)].
APPENDIX 7

Titchmarsh's theorem and some useful related formulae

I. We first prove the familiar relation

\[ \lim_{\varepsilon \to 0} \frac{1}{x - i\varepsilon} = P \frac{1}{x} + \frac{i}{\pi} \delta(x) \quad (A7,1) \]

We suppose a test function \( f(z) \) which is analytic in an infinitesimal region \( R \) near the origin \( z = x + iy \), namely \( R : |z - i\varepsilon| \leq \varepsilon \) and \( y > 0 \) (i.e. on the closed domain given by the cut circle of the figure).

We call \( \Gamma \) a path consisting of the real axis from \(-\infty\) to \(-\sqrt{\varepsilon^2 - \varepsilon^2} \) plus the part of the circle with radius \( \varepsilon > \varepsilon \) from \(-\sqrt{\varepsilon^2 - \varepsilon^2} \) to \(\sqrt{\varepsilon^2 - \varepsilon^2} \) plus the real axis from \(\sqrt{\varepsilon^2 - \varepsilon^2} \) to \(+\infty\). Since \( f(z) \) is analytic in \( z \in R \), we have

\[ + \int_{-\infty}^{\infty} \frac{f(x)}{x - i\varepsilon} \, dx - \int_{\Gamma} \frac{f(z)}{z - i\varepsilon} \, dz = 2\pi i f(i\varepsilon) \quad (A7,2) \]
Consider now the integral along $\Gamma$:

$$\int_{\Gamma} = \left\{ \int_{-\infty}^{-\sqrt{\frac{\xi}{\varrho}} \xi} + \int_{\sqrt{\frac{\xi}{\varrho}} \xi}^{\infty} \right\} \frac{f(x)}{x - i \xi} \, dx + \int_{\pi - \alpha(\xi, \varrho)} \frac{f(\varrho e^{i \varphi}) \, i \varrho e^{i \varphi}}{\varrho e^{i \varphi}} \, d\varphi$$

where

$$\alpha(\xi, \varrho) = \arctan \frac{\xi}{\varrho}.$$

We can now pass to the limit $\varepsilon \to 0$ and afterwards (since $\varrho > \varepsilon$) with $\varrho \to 0$ we obtain

$$\int_{\Gamma} = \lim_{\varrho \to 0} \left\{ \int_{-\infty}^{-\varrho} + \int_{\varrho}^{\infty} \right\} \frac{f(x)}{x} \, dx - i \pi f(0)$$

The first term is the definition of the Cauchy principal value. Inserting this into (A7.2) we obtain

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{f(x)}{x - i \varepsilon} \, dx = \mathcal{P} \int_{-\infty}^{+\infty} \frac{f(x)}{x} \, dx + i \pi f(0) \quad (A7.3)$$

Note that for this derivation nothing has been supposed except that $f(z)$ be analytic in an infinitesimal region near $z=0$ and of course tacitly the existence of the principal value integral. This is sufficient to write (A7.3) in the symbolical form (A7.1).

II. Secondly we prove another familiar relation:

$$\frac{1}{2} \left\{ \int_{\Gamma_1} + \int_{\Gamma_2} \right\} \frac{f(z)}{z} \, dz = \mathcal{P} \int_{-\infty}^{+\infty} \frac{f(x)}{x} \, dx \quad (A7.4)$$
where $\Gamma_1$ and $\Gamma_2$ are the paths shown in the figure:

Again, $f(z)$ need not be analytic except in a $\rho$-neighbourhood of $z=0$ but (A7,4) is of course most useful in cases where $\Gamma_1$ and $\Gamma_2$ may be shifted around in some finite regions of the complex plane.

The proof is obvious

$$\frac{1}{2} \left\{ \int_{\Gamma_1} + \int_{\Gamma_1} \right\} \frac{f(z)}{z} \, dz = \left\{ \int_{-\infty}^{0} + \int_{0}^{\infty} \right\} \frac{f(x)}{x} \, dx + \left\{ \int_{-\pi}^{0} + \int_{0}^{\pi} \right\} \frac{f(\rho e^{i\varphi})i\rho e^{i\varphi}}{\rho e^{i\varphi}} \, d\varphi$$

The first expression is the principal value and the second vanishes if $\rho \to 0$ and if the straight parts of $\Gamma_1$ and $\Gamma_2$ approach the real axis.

III. Titchmarsh's theorem.

We shall not try here a too rigorous proof which can be found in Titchmarsh's book on Fourier Integrals, but rather make the main idea clear.

In most cases of the derivation of dispersion relations (except classical - see Appendix 5) the theorem is not directly applicable. Nevertheless it states in a simple and clear form the basic idea which - sometimes hidden under lengthy calculations - governs all dispersions relations. Loosely speaking it says that for a function $F(\omega)$ the three properties:
(α) obeying dispersion relations,
(β) having a Fourier transform vanishing for \( t < 0 \),
(γ) being analytic in the upper half plane,

are in fact only one single property described three times in different words.

A very suggestive proof of a part of the theorem, based on physical arguments and showing the physical meaning of it, is given in Appendix 5. The proof given there is for the sake of clearness not detailed. We formulate in the following the theorem again in a more precise form and give its proof:

The following three statements are fully equivalent, i.e. each of them separately is necessary and sufficient for the two others being true:

(α) there exists a complex function \( F(\omega) = \Re F(\omega) + i \Im F(\omega) \) of the real variable \( \omega \) such that the real and imaginary parts are Hilbert transforms of each other:

\[
\Re F(\omega) = \frac{p}{\pi} \int_{-\infty}^{+\infty} \frac{\Im F(\omega')}{\omega' - \omega} d\omega' \\
\Im F(\omega) = -\frac{p}{\pi} \int_{-\infty}^{+\infty} \frac{\Re F(\omega')}{\omega' - \omega} d\omega'
\]

(these relations hold except perhaps on a set of measure zero).

(β) the Fourier transform \( f(t) \) of \( F(\omega) \) has the property

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega t} d\omega = 0 \quad \text{for } t < 0
\]

(γ) \( F(\omega) \) is the boundary value of a complex function \( \phi(\omega) \) when \( \omega = \omega + i\omega_2 \) approaches the real axis and \( \phi(\omega) \) is analytic for \( \omega_2 > 0 \). \( \phi(\omega) \) is uniquely determined by \( F(\omega) \) and \( F(\omega) \) is uniquely determined by \( \phi(\omega) \) except perhaps on a set of measure zero.
Before we prove the theorem we note that we have not mentioned the conditions under which the considered integrals converge. We suppose these conditions fulfilled and furthermore that an integral of \( \phi(\omega) \) along an infinite half-circle in the upper \( \omega \)-plane gives zero. This may be achieved by considering \( F(\omega_1) \) and \( \phi(\omega) \) divided by a polynomial instead of \( F(\omega_1) \) and \( \phi(\omega) \) proper. The polynomial must have no zeros in \( \omega_2 > 0 \).

We prove the logical chain \((\alpha) \rightarrow (\beta) \rightarrow (\gamma) \rightarrow (\alpha)\):

\((\alpha) \rightarrow (\beta)\): We take the two formulae together in the form

\[
F(\omega_1) = -\frac{i}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{F(\omega_1')}{\omega_1'-\omega_1} \, d\omega_1'
\]

The Fourier transform becomes

\[
\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega_1' \int_{-\infty}^{\infty} \frac{F(\omega_1')}{\omega_1'-\omega_1} \, d\omega_1' \, e^{-it\omega_1'} d\omega_1
\]

we assume that the integrations can be interchanged:

\[
\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega_1') d\omega_1' \left( -\frac{1}{i\pi} \right) \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{-it\omega_1'}}{\omega_1'-\omega_1'} \, d\omega_1
\]

Now, with (A7,4)

\[
-\frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{e^{-it\omega_1'}}{\omega_1'-\omega_1'} \, d\omega_1' = \begin{cases} e^{-it\omega_1'} & \text{for } t > 0 \\ -e^{-it\omega_1'} & \text{for } t < 0 \end{cases}
\]

hence

\[
\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega_1') e^{-it\omega_1'} \, d\omega_1' \quad \text{for } t > 0
\]

\[
\hat{f}(t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega_1') e^{-it\omega_1'} \, d\omega_1' = -\hat{f}(t) \quad \text{for } t < 0
\]

therefore \( f(t) = 0 \) for \( t < 0 \).
\((\beta) \rightarrow (\gamma)\): We invert \((\beta)\):

\[
F(\omega_1) = \int_{-\infty}^{\infty} f(t) e^{i\omega_1 t} \, dt
\]

and define a complex function

\[
\phi(\omega) = \int_{-\infty}^{\infty} f(t) e^{i(\omega_1 + i\omega_2) t} \, dt
\]

i) For \(\omega_2 \rightarrow 0\) this function has the same Fourier representation as \(F(\omega_1)\) and thus \(\phi(\omega) \rightarrow F(\omega_1)\) except on a set of measure zero.

ii) \(\phi(\omega)\) is analytic in \(\omega_2 > 0\) since \(f(t)\) is zero for \(t < 0\) and only terms \(e^{-\omega_2 t}\) contribute to \(\phi(\omega)\).

iii) Assume there were another function \(\phi'(\omega)\), also analytic in \(\omega_2 > 0\) and with boundary values \(F(\omega_1)\).

From Cauchy's formula follows for \(\omega_2 > 0\):

\[
\phi(\omega) = \frac{1}{2\pi i} \oint \frac{\phi'(\omega')}{\omega' - \omega} \, d\omega'
\]

\[
\phi'(\omega) = \frac{1}{2\pi i} \oint \frac{\phi'(\omega')}{\omega' - \omega} \, d\omega'
\]

where the path \(\gamma\) goes from \(-\infty + i\xi\) to \(+\infty + i\xi\) and then along an infinite half-circle through \(+\omega i\) back to \(-\infty + i\xi\). We suppose that the contributions of the half-circle vanish and may then go with \(\xi \rightarrow 0\):
\[
\phi(\omega) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty - i\varepsilon}^{\infty + i\varepsilon} \frac{\Phi(\omega')}{\omega' - \omega} d\omega' = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\omega')}{\omega' - \omega} d\omega' \quad (\omega_2 > 0)
\]

\[
\Phi(\omega) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty - i\varepsilon}^{\infty + i\varepsilon} \frac{\Phi'(\omega')}{\omega' - \omega} d\omega' = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\omega')}{\omega' - \omega} d\omega' \quad (\omega_2 > 0)
\]

Therefore both functions are equal. (Note that this is not a trivial consequence of the theorem that two analytic functions which are equal on an infinite set with a limiting point inside the domain of analyticity, are equal everywhere. Here they only have the same boundary values on the real axis which, however, does not belong to the domain of analyticity).

\[(\gamma) \to (\alpha) : \quad \text{From Cauchy's formula we have}
\]

\[
\int_C \frac{\Phi(\omega')}{\omega' - (\omega_1 + i\varepsilon)} d\omega' = 0
\]

where \(C\) is shown in the figure, the radius of the half-circle \(\gamma\) is \(\varepsilon\), say \(\gamma = \sqrt{\varepsilon}\); the distance of the straight parts of \(C\) from the real axis is \(\varepsilon\), the contribution of the infinite half-circle in the upper plane is supposed to vanish. We have then
\[
\int_\mathcal{C} \frac{\phi(\omega')d\omega'}{\omega'-(\omega+i\varepsilon)} = \left\{ \int_{-\infty}^{\omega-\varepsilon} \frac{\phi(\omega'+i\varepsilon)}{\omega'-\omega} d\omega' + i \int_{0}^{\pi} \frac{\phi(\omega'+i\varepsilon+\xi e^{i\varphi})\xi e^{i\varphi}}{\pi} d\varphi \right\}
\]

We now go to \( \varepsilon \to 0 \) and use the fact that \( \phi(\omega) \to F(\omega) \):

\[
P\int_{-\infty}^{+\infty} \frac{F(\omega')}{\omega'-\omega} d\omega' - i\pi F(\omega) = 0
\]

which gives the two relations of \((\alpha)\) if separated into real and imaginary part. It must be noted that since \( \phi(\omega) \) must not tend to \( F(\omega) \) pointwise, the Hilbert relations are true except on a set of measure zero.

IV. We finally show that for a complex function \( \phi(\omega) \) obeying Titchmarsh's theorem, the relation holds

\[
\phi(\omega) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im} \phi(\omega')}{\omega'-\omega} d\omega'
\]

for all \( \omega > 0 \)

i) We first consider the integral as the definition of a function \( \phi'(\omega) \), analytic in \( \omega > 0 \):

\[
\phi'(\omega) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im} \phi(\omega')}{\omega'-\omega} d\omega'
\]
ii) We put $\omega = \omega_1 + i\varepsilon$ and use (A7,1) to find

$$\lim_{\varepsilon \to 0} \Phi'(\omega_1 + i\varepsilon) = \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im} F(\omega'_1)}{\omega'_1 - \omega_1} d\omega'_1 + \text{Im} F(\omega_1)$$

(except on a set of measure zero)

iii) From Titchmarsh's theorem (α) follows that the first integral is the real part of $F(\omega_1)$. Hence

$$\lim_{\varepsilon \to 0} \Phi'(\omega_1 + i\varepsilon) = F(\omega_1)$$

(almost everywhere).

iv) Because now

$$\lim_{\varepsilon \to 0} \Phi'(\omega_1 + i\varepsilon) = \lim_{\varepsilon \to 0} \phi(\omega_1 + i\varepsilon) = F(\omega_1)$$

the functions $\phi'(\omega)$ and $\phi(\omega)$ are equal as shown in the proof of Titchmarsh's theorem.
Some detailed calculations suppressed in the text

I. The analyticity domain $R_1$ of $T_1(\omega, \Delta^2, \zeta)$.

By (67) $R_1$ is given by $\omega_2 > \left| \text{Im} \sqrt{\omega_1^2 - \omega_2 - \Delta^2} \right|$. We have to express $\text{Im} \sqrt{\omega_1^2 - \omega_2 - \Delta^2}$ by $\omega_1, \omega_2, \zeta_1, \zeta_2$

\[
( \omega = \omega_1 + i \omega_2 ; \quad \zeta = \zeta_1 + i \zeta_2 )
\]

Consider $\sqrt{z} = x + iy$; $z = u + iv$ and express $y$ by $u$ and $v$: Squaring gives

\[
\begin{align*}
    u &= x^2 - y^2 \\
    v &= 2xy \\
    x &= v / 2y
\end{align*}
\]

Insert above

\[
\begin{align*}
    u &= \frac{v^2}{4y^2} - y^2 \quad \text{or} \quad y^4 + uy^2 - \frac{1}{4}v^2 = 0
\end{align*}
\]

Since $y^2 > 0$ we have the unique solution

\[
y^2 = \frac{1}{2} \left( -u + \sqrt{u^2 + 4v^2} \right) \quad (A8,1)
\]

Now back to (67):

\[
\omega_2 > \left| \frac{\text{Im} \sqrt{\omega_1^2 - \omega_2^2 - \Delta^2 - \zeta_1} + i \left( 2\omega_1 \omega_2 - \zeta_2 \right)}{u} \right|
\]
Applying (A8,1) gives (since \( \omega_2 > 0 \) anyway)

\[
\omega_2^2 > \frac{1}{2} \left( -\omega_1^2 + \omega_2^2 + \Delta^2 + \xi_1 + \sqrt{(\omega_1^2 - \omega_2^2 - \Delta^2 - \xi_1)^2 + (2\omega_2 \omega_2 - \xi_2)^2} \right)
\]

Squaring is permitted since \( \sqrt{\cdot} > 0 \) by definition, hence

\[
(\omega_1^2 + \omega_2^2 - \Delta^2 - \xi_1)^2 > (\omega_1^2 - \omega_2^2 - \Delta^2 - \xi_1)^2 + (2\omega_2 \omega_2 - \xi_2)^2
\]

\[
4\omega_2^2 (\omega_1^2 - \Delta^2 - \xi_1) > (2\omega_2 \omega_2 - \xi_2)^2
\]

Now follows \( \omega_1^2 > \xi_1 + \Delta^2 \) which is the first condition of (87') in the text. Extracting the root leads to

\[
2\omega_2 (\omega_1 - \sqrt{\omega_1^2 - \Delta^2 - \xi_1}) < \xi_2 < 2\omega_2 (\omega_1 + \sqrt{\omega_1^2 - \Delta^2 - \xi_1})
\]

which is the second condition of (87') in the text.

II. Calculation of \( I = \int_0^\frac{\pi}{2} \frac{d\beta}{[x - \omega_\alpha (\beta - \delta)][y - \omega \beta (\rho - \omega)]} \)

Put

\[
\omega_\alpha (\beta - \delta) = \frac{1}{2} (z + \frac{1}{z}) \quad z = e^{i(\beta - \delta)}
\]

\[
\omega \beta (\rho - \omega) = \frac{1}{2} (z' + \frac{1}{z'}) \quad z' = \bar{z} \cdot e^{i(\delta - \alpha)} = \bar{z} e^{i\varphi}
\]

One obtains an integral over the unit circle:

\[
I = -4i \oint \frac{z'dz}{[z^2 - 2xz + 1][z^2 - 2y z' + 1]}
\]
The roots are

\[ z_{1,2} = x \pm \sqrt{x^2 - 1} \quad ; \quad z_1 \cdot z_2 = 1 \]

\[ z_{3,4} = y \pm \sqrt{y^2 - 1} \quad ; \quad z_3 \cdot z_4 = 1 \]

Hence only one root of each pair lies inside the unit circle. Assume \( x > 1 \) and \( y > 1 \), then \( z_2 \) and \( z_4^* \) are inside. Put

\[ x = \Sh \lambda \quad ; \quad \sqrt{x^2 - 1} = \Sh \lambda \]

\[ y = \Sh \mu \quad ; \quad \sqrt{y^2 - 1} = \Sh \mu \]

\[ I = -4i \oint \frac{z e^{-i\varphi} \, dz}{(z - e^\lambda)(z - e^{-\lambda})(z - e^{i\mu - i\varphi})(z - e^{-i\mu + i\varphi})} \]

The poles inside the unit circle are \( e^{-\lambda} \) and \( e^{i\mu - i\varphi} \); this gives

\[ I = 4\pi \left[ \frac{e^{-\mu - i\varphi}}{\Sh \mu (e^{-\lambda + i\varphi} - e^{2\mu - 2i\varphi} + e^{-\mu - i\varphi})} + \frac{e^{-\lambda - i\varphi}}{\Sh \lambda (e^{\mu - i\varphi} - e^{2\lambda - 2i\varphi} + e^{-\mu - i\varphi})} \right] \]

Multiplying the first and second term respectively by \( e^{i\mu + i\varphi} / e^{\lambda + i\varphi} \) and \( e^{\lambda + i\varphi} / e^{\lambda + i\varphi} \) gives

\[ I = 2\pi \left[ \frac{1}{\Sh \mu (\Cosh \lambda - \Cosh (\mu + i\varphi))} + \frac{1}{\Sh \lambda (\Cosh \mu - \Cosh (\lambda - i\varphi))} \right] \]

Multiply numerators and denominators by the complex conjugate of the denominators to obtain

\[ I = 2\pi \frac{\Sh \lambda \Cosh \lambda + \Sh \mu \Cosh \mu - (\Sh \lambda \Cosh \mu + \Cosh \lambda \Sh \mu) \cos \varphi}{\Sh \lambda \Sh \mu (\Cosh^2 \mu - 2 \Cosh \lambda \Sh \mu \cos \varphi + \Cosh^2 \lambda + \cos^2 \varphi - 1)} \]
Find the roots of the denominator:

\[ \omega \psi_1 = \text{Ch}(\lambda + \mu) \]

\[ \omega \psi_2 = \text{Ch}(\lambda - \mu) \]

and use

\[ \text{Sh} \lambda \text{Ch} \lambda + \text{Sh} \mu \text{Ch} \mu = \frac{1}{2} \text{Sh} 2\lambda + \frac{1}{2} \text{Sh} 2\mu = \text{Sh}(\lambda + \mu) \text{Ch}(\lambda - \mu) \]

to obtain

\[ I = 2\pi \frac{\text{Sh}(\lambda + \mu) \left[ \text{Ch}(\lambda - \mu) - \omega \phi \right]}{\text{Sh} \lambda \text{Sh} \mu \left[ \text{Ch}(\lambda - \mu) - \omega \phi \right] \left[ \text{Ch}(\lambda + \mu) - \omega \phi \right]} = 2\pi \frac{\text{Ch} \mu}{\text{Sh} \mu} + \frac{\text{Ch} \lambda}{\text{Sh} \lambda} \]

or in the old variables

\[ 2\pi \int_0^{\alpha} \frac{d\beta}{\left[ x - \cos(\beta - \delta) \right] \left[ y - \cos(\beta - \alpha) \right]} = 2\pi \frac{x/\sqrt{x^2 - 1} + y/\sqrt{y^2 - 1}}{xy + \sqrt{(x^2 - 1)(y^2 - 1)} - \cos(\delta - \alpha)} \]