

Remarks on Large N Coherent States

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Abstract

Analogs of ordinary Gaussian coherent states on bosonic Fock spaces are constructed for the case of free Fock spaces, which appear to be natural mathematical structures suitable for description of large N matrix models.

1. Basic idea: Gaussian coherent states (for a clear discussion see for example [1]) describe extremely well ground states of large N theories with fields in the fundamental representation of the symmetry group (so called vector models). On the other hand the same Gaussian coherent states fail to describe ground states of large N theories with fields in the adjoint representation (so called matrix models). (This was clearly emphasized by Yaffe and co-workers [2]). The question arises if there is a natural analog to Gaussian coherent states even for matrix models. Recall that these states are inherent to the structure of the bosonic Fock spaces. Recently it has become apparent that another algebraic structure seems to be natural for the large N matrix models, the so called free Fock space. This concept naturally appears within the context of non-commutative probability theory, developed by Voiculescu and collaborators [3], and it has been used in the physics literature in the analysis of large N matrix models by Haan [4] and Cvitanović and co-workers [5], and more recently by Douglas [6],[7] and Gopakumar and Gross [8]. (Similar ideas have been exploited in the study of so called infinite statistics by Greenberg [9]). It thus seems reasonable to ask if one can find analogous coherent states related to the structure of the free Fock space, which would hopefully play a role similar to ordinary Gaussian coherent states. In this short note we examine this question from a rather elementary point of view. Nevertheless this simple exercise seems to be needed if we are to get more familiar with the physics contained in the free Fock spaces. The note is organized as follows: First we review the construction of familiar Gaussian coherent states on a bosonic Fock space and then following the same steps we construct their analogs in the case of a free Fock space. We also comment on the uniqueness of the construction.

2. Coherent states on a bosonic Fock space: Let us briefly review some well-known facts about Gaussian coherent states. Start with the usual bosonic algebra $[a, a^\dagger] = 1$,

$a|0\rangle = 0$. (The bosonic Fock space being defined as $|k\rangle: |0\rangle, a^\dagger|0\rangle, \frac{1}{\sqrt{2!}}(a^\dagger)^2|0\rangle, \dots, \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \dots; \langle n|m\rangle = \delta_{n,m}$). Coherent states are then generated by the following formally unitary operator (whose form is related to the phase-space structure arising in the classical limit)

$$U(z, \bar{z}) = \exp(za^\dagger - \bar{z}a),$$

or more specifically

$$|z\rangle = U(z, \bar{z})|0\rangle.$$

The states have the following familiar form

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle.$$

It follows by construction that $\langle z|z\rangle = 1$. The property of the "resolution of unity" is true as well

$$\frac{1}{\pi} \int d^2z |z\rangle \langle z| = 1$$

implying the self-reproducing property of the Gaussian kernel

$$\langle z'|z\rangle = \exp\left(-\frac{|z|^2}{2} + \bar{z}'z - \frac{|z'|^2}{2}\right)$$

or in other words

$$\langle z_1|z_2\rangle = \frac{1}{\pi} \int d^2z \langle z_1|z\rangle \langle z|z_2\rangle.$$

The creation and annihilation operators have the following simple actions: $\langle z|a^\dagger = \bar{z} \langle z|$ and $a|z\rangle = z|z\rangle$. These relations follow from the fact that

$$0 = U(z, \bar{z})a|0\rangle = U(z, \bar{z})aU^{-1}(z, \bar{z})U(z, \bar{z})|0\rangle$$

and

$$U(z, \bar{z})aU^{-1}(z, \bar{z}) = a - z$$

by the use of the Baker-Hausdorff formula.

3. Coherent states on a free Fock space: Let us try to emulate above procedure in the case of the so called Cuntz algebra [3] $aa^\dagger = 1$, $a|0\rangle = 0$ (The corresponding free Fock space being defined as $|k\rangle: |0\rangle, a^\dagger|0\rangle, (a^\dagger)^2|0\rangle, \dots, (a^\dagger)^n|0\rangle, \dots; \langle n|m\rangle = \delta_{n,m}$). From completeness $a^\dagger a = 1 - |0\rangle\langle 0|$ follows $[a, a^\dagger] = |0\rangle\langle 0|$. As in the previous case look at the states

$$|z\rangle = U(z, \bar{z})|0\rangle$$

with

$$U(z, \bar{z}) = \exp(za^\dagger - \bar{z}a).$$

By expanding the exponential and using the basic properties of the annihilation and creation operators, namely $a|n\rangle = |n-1\rangle$ and $a^\dagger|n\rangle = |n+1\rangle$, we are led to

$$|z\rangle = \sum_{n=0}^{\infty} (n+1)J_{n+1}(2|z|)|z|^{-(n+1)}z^n|n\rangle$$

where J_n denotes the Bessel function of order n. We have used the well-known expansion

$$J_n(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+n}}{2^{2i+n} i!(i+n)!}.$$

By construction (and by virtue of the identity $\sum_{n=0}^{\infty} (n+1)^2 J_{n+1}(2|z|)^2 = |z|^2$; special case of Watson's 2.72(1) [10]), $\langle z|z\rangle = 1$. Unfortunately, it turns out that

$$\frac{1}{\pi} \int d^2z \langle n|z\rangle\langle z|m\rangle = (n+1)\delta_{nm}.$$

(Here we have made use of $\int_0^\infty dr J_m^2(2r)r^{-1} = \frac{1}{2m}$; special case of Watson's 13.41(2) [10].)

We therefore define a new set of states with the automatic property of the "resolution of unity"

$$|z\rangle = \sum_{n=0}^{\infty} \sqrt{n+1} J_{n+1}(2|z|)|z|^{-(n+1)}z^n|n\rangle. \quad (1)$$

The self-reproducing kernel is given by

$$K(z', z) \equiv \langle z' | z \rangle = \sum_{n=0}^{\infty} (n+1) J_{n+1}(2|z|) J_{n+1}(2|z'|) (|z||z'|)^{-(n+1)} (zz')^n. \quad (2)$$

Using the multiplication formula for the Bessel functions (with definitions: $z = r_1 e^{i\theta_1}$ and $z' = r_2 e^{i\theta_2}$), to wit

$$J_{n-k}(r_1) J_k(r_2) = \frac{1}{2\pi} \int_0^{2\pi} d\beta \exp(in\alpha - ik\beta) J_n(r)$$

(where $r^2 = r_1^2 + 2r_1 r_2 \cos \beta + r_2^2$ and $r \exp(i\alpha) = r_1 + r_2 \exp(i\beta)$; Watson's 11.3(3) [10];

for a beautiful derivation of this formula consult [12]) as well as the integral representation

$$J_n(r) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp(ir \sin \phi - in\phi)$$

and the recursion formula $2n J_n(r) = r(J_{n-1}(r) + J_{n+1}(r))$ we get ($\theta \equiv \theta_1 - \theta_2$)

$$K(z', z) = \frac{e^{-i\theta}}{8\pi^2 r_1 r_2} \int_0^{2\pi} d\beta \int_0^{2\pi} d\phi \cos \phi \exp(ir \sin \phi) \sum_{n=1}^{\infty} e^{in(2\alpha - \beta + \theta - 2\phi)}. \quad (3)$$

If $z = z'$ ($r_1 = r_2 = R$, $2\alpha = \beta$ and $\theta = 0$) we get, after doing the β -integral first and then the ϕ -integral,

$$F(R) \equiv \langle z | z \rangle = \sum_{k=0}^{\infty} \frac{(-1)^k R^{2k} (2k)!}{[(k+1)!]^2 [k!]^2}. \quad (4)$$

It is apparent that the behavior of $\langle z | z \rangle$ around $R = 0$ is regular. The sum is evidently convergent and by construction $\langle z | z \rangle$ is positive definite (as it should be if $|z \rangle$'s are to form a Hilbert space, which they do by construction).

We quote a number of other representations of $K(z', z)$ and $F(R)$. Another integral representation implied by the multiplication formula and $J_{-n} = (-1)^n J_n$ is

$$K(z', z) = -\frac{1}{2\pi r_1 r_2} \int_0^{2\pi} d\beta \frac{e^{i\beta} J_0(2r)}{[1 + e^\theta e^\beta]^2}. \quad (5)$$

Apparently the most explicit expression for $K(z', z)$ is given by the following double-sum representation

$$K(z', z) = \frac{e^{-i\theta}}{r_1 r_2} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} e^{in\theta} (-1)^{n+k} \frac{n!^n}{k!(k+n)!} P_k^n(r_1^2 + r_2^2) \quad (6)$$

where $P_k^n(x)$ denotes the associated Legendre functions. (Here we have made use of the following well-known integral representation of $P_k^n(x)$; Whittaker and Watson's 15.61 [11]),

$$P_k^n(\cos \gamma) = \frac{i^n(n+k)!}{2\pi k!} \int_0^{2\pi} d\varphi (\cos \gamma + i \sin \gamma \cos \varphi)^k e^{in\varphi}.$$

The above double sum reduces to the single sum formula for $\langle z|z \rangle$ which of course follows from the integral representation

$$F(R) = -\frac{1}{2\pi R^2} \int_0^{2\pi} d\beta \frac{J_0(4R \cos \frac{\beta}{2})}{4[\cos \frac{\beta}{2}]^2}, \quad (7)$$

which is a special case of (5). By utilizing the following remarkable property of Bessel functions; Watson's 5.4(5) [10]

$$J_\nu(z)J_\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{\mu+\nu+m} \Gamma(\mu+\nu+2m+1)}{m! \Gamma(\mu+m+1) \Gamma(\nu+m+1) \Gamma(\mu+\nu+m+1)}$$

we arrive at

$$R^2 \frac{dF(R)}{dR} + RF(R) = J_0(2R)J_1(2R)$$

implying yet another integral representation

$$F(R) = \frac{1}{R} \left(\int \frac{1}{R} J_0(2R)J_1(2R) dR + c \right) \quad (8)$$

c being a constant of integration such that $F(0) = 1$.

Any of the above expressions for $K(z', z)$ satisfy by construction the self-reproducing property

$$K(z_1, z_2) = \frac{1}{\pi} \int d^2 z K(z_1, z) K(z, z_2).$$

It is also possible to find actions of the operators a and a^\dagger but the expressions are a bit unwieldy and we do not state them explicitly. One fact is evident though, the new coherent states are not the eigenstates of the annihilation operator a .

4. Comments on the uniqueness of new coherent states: One could ask how robust our construction is. In particular one could ask if one could get more manageable expressions if one examined the eigenstates of the annihilation operator a as appropriate coherent states. This question can of course be easily answered. Let us look at $|z\rangle$ such that $a|z\rangle = z|z\rangle$. By expanding

$$|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

and acting with the operator a on the left we obtain

$$|z\rangle = \sum_{n=0}^{\infty} z^n |n\rangle.$$

By examining $\langle z'|z\rangle$ we see that for the reasons of convergence we have to constrain $|z|$ inside the unit circle. Again, the property of the "resolution of unity" is not satisfied. So we redefine the states so that this condition is automatically fulfilled (the new states will thus cease to be eigenstates of a)

$$|z\rangle = \sum_{n=0}^{\infty} \sqrt{n+1} z^n |n\rangle. \quad (9)$$

The self-reproducing kernel is now

$$\langle z'|z\rangle = \frac{1}{(1-zz')^2} \quad (10)$$

and the normalization condition reads

$$\langle z|z\rangle = \frac{1}{(1-z\bar{z})^2}.$$

Unfortunately this construction, unlike the previous one, is valid only for $|z| < 1$.

The more important question is: why have we at all decided to use the exponential representation for the unitary operator $U(z, \bar{z})$, namely $U(z, \bar{z}) = \exp(za^\dagger - \bar{z}a)$ (which

indeed is the appropriate one for the Gaussian coherent states)? We do not have a convincing answer to this question. We have examined, by trial and error, a number of different representations for $U(z, \bar{z})$ and only in the case of the familiar exponential one have we found an interesting result. It is quite possible that there is a more natural form for $U(z, \bar{z})$ which would be more suitable for coherent states on free Fock spaces.

5. Conclusion: Analogs of familiar Gaussian coherent states are constructed for the case when the annihilation and creation operators satisfy the Cuntz algebra. The result turns out to be more complicated, but the essential features of ordinary Gaussian coherent states are preserved. By doing this fairly simple exercise some familiarity with unusual free Fock spaces is gained, albeit no real insight into relevant physical questions concerning large N matrix models. It would therefore be extremely interesting to see if these new states have any practical physical applications.

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