A Method for Non-Invasive Measurements of Sextupole Resonance Driving Terms using Wobbling Diagnostics with Applications to LEP

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Abstract
We propose a method to determine the coefficients of the hamiltonian that represents the cumulative effect of all sextupoles in a beam line to first order. The method is based on the low-frequency sinusoidal excitation of orbit corrector magnets and detecting BPM signals at mixed harmonics of the exciting signals.

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1 Introduction

In a recent note [1] we discussed a method that allows to determine both phases of the sum and difference coupling resonances non-invasively by exciting orbit correction magnets and observing the response of the beam at selected beam position monitors (BPM) at the exciting frequencies. The beam line used in ref. 1 was purely linear which entails that only the exciting signals are present at the BPMs. In the presence of non-linear elements such as sextupoles, octupoles, and decapoles this will no longer be the case. The non-linearities will produce a mixture of the exciting frequencies such as twice or thrice the exciting frequency and sums of different frequencies.

This effect can most easily be understood with the aid of a simple model of a driven harmonic oscillator with an added non-linearity [2]. The equation of motion of such a model is given by

\[ x'' + \omega_0^2 x + \alpha x^2 = F \cos \omega t \]  

(1)

where \( \omega_0 \) is the natural frequency of the system, \( \omega \) is the exciting frequency and \( F \) is the force that is applied and \( \alpha \) is the strength of the non-linearity which will generate mixing terms such as \( \cos^2(\omega t) \). Throughout this report we will assume that \( \omega \) is much smaller than the natural frequency \( \omega_0 \), i.e. we are dealing with the non-resonant case. Thus the frequency response of the system is very flat, because it is far away from the eigen-frequencies and mainly unperturbed by those.

In a circular accelerator the unperturbed beam oscillates with the betatron frequencies and the wobbling orbit corrector magnets excite this system. Sextupoles in the machine will then produce the mixing frequencies. Our task is now defined by the requirement to develop an algorithm that allows to determine the hamiltonian from observed frequencies at a given set of BPM and thus diagnose aberrations which are generated by the cumulative effect of all non-linearities in the machine. The wobbling frequencies are far away from the tune and we thus perform measurements which are non-resonant with the beam's betatronic motion. Therefore we are not limited to the measurement of aberrations which cause resonances that are close to that of the beam such as discussed in ref. 3. All aberrations are detected with similar accuracy, whether they are resonant with the beam or not.

We will start with a very brief overview of non-linear dynamics in storage rings with hamiltonians and then discuss the experimental setup and the technique to measure the mixed frequencies. Then we deal with the reconstruction of the hamiltonian from the spectra and the hardware requirements followed by concluding remarks.

2 Hamiltonians

We assume that we are dealing with a linear beam line in which point-like non-linear elements are situated. Each non-linearity is represented by its hamiltonian, e.g. an upright sextupole is described by \( H_3 = k_3/6(x^3 - 3xy^2) \) which describes the kick effected by that element with the aid of the Poisson-bracket [1]. The hamiltonian of each non-linear element is then mapped to the end of the beam line (the reference point). This procedure only requires the linear map between the position of the element and the reference point. Having accumulated all elements at the reference point we can concatenate them using the Campbell-Baker-Haussdorff (CBH) formula and obtain the following representation.
of the map $\mathcal{M}$ through the beam line

$$\mathcal{M} = e^{-iH: R}$$

where $R$ is the linear transfer matrix through the beam line and $H$ is a polynomial in the variable $(x, x', y, y')$ that describes the cumulative effect of all non-linear elements.

In what follows we assume that the reference point is chosen to be in normalized phase space, i.e. to be at $\beta_x = \beta_y = 1$ m and $\alpha_x = \alpha_y = 0$. In that case $R$ is the direct sum of two rotation matrices and the coefficients of the polynomial $H$ can be interpreted as simple linear combination of resonance driving terms which we are about to determine.

In this report we will only deal with sextupolar aberrations of which there are 20, namely the number of independent monomials of order three in the four variables $x, x', y, y'$. Thus we need to determine the coefficients $h_n$ as defined in eq. 3

$$H = h_1x^3 + h_2x^2x' + h_3x^2y + h_4x^2y' + h_5x^2z$$

$$+ h_6xx'y + h_7xx'y' + h_8xy^2 + h_9xyy + h_{10}xy^2$$

$$+ h_{11}x'^2 + h_{12}x'^2y + h_{13}xy'^2 + h_{14}x'y^2 + h_{15}x'y'$$

$$+ h_{16}x'y'^2 + h_{17}y^3 + h_{18}y'y^2 + h_{19}y'y'^2 + h_{20}y'^3$$

with the implied ordering of the monomials. The hamiltonian coefficients $h_n$ are related to resonance driving terms by simple linear relations which can be found by substituting action and angle variables, such as $x = \sqrt{2J_x} \cos \psi_x$, etc. in eq. 3 and collecting terms proportional to $\cos(n\psi_x + m\psi_y)$.

### 3 Diagnostic Setup and Filtering

In order to measure the 20 coefficients of the hamiltonian we need a sufficiently large number of frequencies to observe. In the simulation used in this note we use two horizontal (CHA.QL7.R1, CHA.QL11.R1) and two vertical (CVA.QL6.R1, CVA.QL8.R1) orbit correctors, all oscillating at different frequencies. The frequencies were chosen such that the smallest difference between two mixed frequencies is maximum. Fixing one frequency then determines the other three. We choose the following four $f_1 = 0.0073722 f_0, f_2 = 0.0131550 f_0, f_3 = 0.0161366 f_0$, and $f_4 = 0.0175514 f_0$ where $f_0$ is the revolution frequency which is 11 245 kHz in LEP or LHC. We then choose two horizontal (PU.QL7.R1, PU.QL9.R1) and two vertical (PU.QL6.R1, PU.QL8.R1) BPM which are used to detect the response of the beam. The BPM need to record the position of the beam on 32 768 consecutive turns which may require special hardware. For simplicity the wobbling correctors, the used BPM and the reference point need to lie in a section of beam line that contains no non-linear elements.

The four wobbling frequencies $f_1, \ldots, f_4$ can mix to the following 17 different frequencies: $0, 2f_1, f_1 + f_2, f_1 - f_2, f_1 + f_3, f_1 - f_3, f_1 + f_4, f_1 - f_4, 2f_2, f_2 + f_3, f_2 - f_3, f_2 + f_4, f_2 - f_4, 2f_3, f_3 + f_4, f_3 - f_4, 2f_4$. Observing all frequencies at all BPM we obtain 68 different observables of which we use all, except the constant term for a total of 64 observables, which turn out to be sufficient to determine the 20 terms in the hamiltonian.

The order of magnitude of the signals can be easily assessed by the following analysis. Assume that the reference point is in normalized phase space and that the BPM and orbit correctors are also situated at the same location. Then the normalized kick effected to the
beam is $\sqrt{\beta} \epsilon$, where $\epsilon$ is the kick and the mixing signal is proportional to the square of that, with the coefficient $\hbar$ of the Hamiltonian being part of the proportionality constant. The observed BPM signal, transformed into normalized phase space, is given by $z/\sqrt{\beta}$ such that we obtain the approximate relation

$$\frac{z}{\sqrt{\beta}} \approx \hbar \beta^3 \epsilon^2$$

yielding the simple relation $z \approx \hbar \beta^3 \epsilon^2$. Inserting typical values for LEP $\beta = 100 \text{ m}$, $\hbar = 10/\sqrt{\text{m}}$ and $\epsilon = 10 \mu\text{rad}$ we obtain $z \approx 1 \mu\text{m}$. We must remember that this is the amplitude of a harmonic oscillation with a known frequency and can thus be measured with high accuracy.

In the simulations reported in this paper we use a detuned LEP optics at high energy (K21P46) which contains about 500 sextupoles. The coefficients of the Hamiltonian for this beam line are shown in fig. 1. The correctors and BPM were chosen from the RF section of IR 1 such that the phase advances between respective elements are around 90 degrees and the corresponding beta functions are maximum. The correctors are then excited with an amplitude of 10 $\mu$rad which generate orbit deviations of $\pm 1 \text{mm}$ in the arcs.

Figure 2 shows the Fast-Fourier-Transform (FFT) of position of the beam at the horizontal position monitor PU.QL7.R1. We clearly observe the exciting frequencies as large peaks with amplitudes on the order of 1 mm, but also other frequencies which come from the mixing due to sextupoles. In a separate run we turned off all sextupoles and the secondary peaks vanished as well. We also observe that the large peaks swamp the
secondary signals which are two to four orders of magnitude weaker than the primary ones. The width of the primary peaks is a consequence of the finite number of turns that the beam position is sampled. We know, however, the exact frequency of the primary signals and can thus construct a notch-filter to remove them from the raw data. We proceed as follows: First we generate time series (sine-like and cosine-like) of the same length as the raw data, namely 32768 turns, which contain only the primary frequency and Fourier-Transform them. The resulting spectra are then fitted to the data points near the primary peaks (normally 9 frequency bins for the sine- and cosine-like transform) which yield the amplitudes of both phases. The primary frequencies with the proper amplitudes are then removed from the raw data and the resulting time series is Fourier-Transformed which leads to the spectrum shown in fig. 3. There the primary frequencies are removed and the secondary frequencies with sub-micron amplitudes are clearly visible, consistent with the above estimate.

The small amplitudes are not prohibitive for an experimental detection, because the signal is sampled over many turns and the random BPM noise is reduced by a factor $\sqrt{N}$. 10 $\mu$m thus result in 6 $10^{-8}$ m resolution on the spectral peaks which is marginal in the light of fig. 3. But improving the BPM resolution or increasing the number of sampled turns will improve the situation.

An interesting point to note is that the number of peaks in fig. 3 is larger than that of the secondary frequencies. Most of the observed peaks can be attributed to the mixing of two frequencies, but there are a few that are left unaccounted, implying that octupolar effects are also visible in the spectrum which is consistent with the effect that the hamiltonian coefficients of the sextupoles in octupolar order are about three orders of magnitude larger than those of sextupolar order which compensates the extra factor.

Figure 2: The spectrum of the beam response at position monitor PU.QL7.R1.
The hamiltonian terms are essentially resonance driving terms evaluated at emittances of 1 mrad, a single coefficient $h_i$. For example, the change in $z'$ corresponding to $h_i$ being non-zero non-linearity at the reference point is introduced which produces a kick corresponding to $s_i$. We therefore mix the raw data after filtering as described in the previous section with all combinations of $1/2 \cos(2\pi(f_i \pm f_j)t)$ for $i, j = 1, \ldots, 4$ for each of the four BPM resulting in $4 \times 17 = 68$ spectral coefficients $s$ which are assembled into a vector $\vec{s}$.

In order to establish relations between the 20 hamiltonian coefficients $h$ and the spectral coefficients $s$ we perform 20 different tracking runs in which only a single artificial non-linearity at the reference point is introduced which produces a kick corresponding to a single coefficient $h_i$. For example, the change in $z'$ corresponding to $h_i$ being non-zero.

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1. The hamiltonian terms are essentially resonance driving terms evaluated at emittances of 1 mrad, which accounts for their large magnitude.
and all other coefficients being zero is given by
\[ \Delta x' = [-H, x'] = -3h_1x^2 \] (5)

where \([f, g]\) denotes the Poisson-Bracket between \(f\) and \(g\). In this way a basis in the space of spectral coefficients is constructed. Subjecting the resulting BPM time series to the analysis described in the previous paragraph we obtain the spectral coefficients for \(h_1\), which we label \(\tilde{\iota}_1\). We then repeat the same procedure for each of the 20 \(h\) and assemble the \(\tilde{\iota}_i\), with \(i = 1, \ldots, 20\) into a matrix \(\mathbf{T} = (\tilde{\iota}_1, \ldots, \tilde{\iota}_{20})\) where the \(\tilde{\iota}\) are the columns of the matrix \(\mathbf{T}\). The problem of finding the hamiltonian coefficients \(h\) that generate an observed pattern of spectral coefficients we simply have to solve the following linear equations \(s_i = \sum_j T_{ij}h_j\) where \(j\) runs from 1 to 20 and \(i\) from 1 to 68. We can suppress certain measurements by left-multiplying by a weight-matrix \(\Lambda_i = \text{diag}(1/\sigma_1, \ldots, 1/\sigma_{68})\) which is done in the simulation to ignore all observations with frequency zero, i.e. the dc-component. The previous over-determined linear equation can be easily solved in the least-squares sense, yielding
\[ h_j = \left( (\mathbf{T}^\top \mathbf{T})^{-1} \mathbf{T}^\top \right)_{ji} s_i \] (6)

where errors are given by the diagonal elements of the covariance matrix \((\mathbf{T}^\top \mathbf{T})^{-1}\). Performing this procedure for the sample LEP file we can reconstruct the hamiltonian coefficients as shown in fig. 4.

Comparing fig. 4 with fig. 1, we see a close resemblance. All strongly excited aberrations are present with the correct magnitudes. We also see some aberrations which should not be present because there are only upright sextupoles in the lattice. These rather small unwanted aberrations are a consequence of the experimental setup and may serve as an estimate of the accuracy of the method which we find in the 5 % range in fig. 4 where the BPM resolution is negligible. Redoing the scan with finite BPM resolution errors cause the unwanted aberrations to become more and more prominent. Simulations indicate that \(1 \mu m\) BPM resolution is sufficient but \(10 \mu m\) is marginal such that we advocate a resolution of \(5 \mu m\) or better.

## 5 Hardware

The hardware requirements for this method are moderate. We need four accurate power supplies that provide a sinusoidal signal in the range between 40 and 300 Hz that can drive an orbit corrector such as to kick the beam up to \(10 \mu rad\). The power supply is required to provide a computer-readable digitized signal of its output current on a turn-by-turn basis, i.e. at about 10 kHz. The four BPM need to provide the beam's position on a turn-by-turn basis with an accuracy of better than about \(5 \mu m\). The hardware need to be capable of storing the power supply data and the BPM data over a period of 32 768 turns. Of course the number of recorded turns and BPM accuracy may be traded, but this needs to be checked carefully in the future. Given a large file that contains eight columns of data, the four BPM positions and the four kicks for each of he 32 768 turns the remainder of the algorithm will reside in a computer and will not require any special hardware.
Moreover, currently the filtering requires minutes of CPU time, because 32,000 turns need to be sampled which corresponds to about 3 s in real time. The feedback would be necessarily slow, about once every few seconds, because 32,000 turns need to be sampled which corresponds to about 3 s in real time in LEP or LHC. Currently the filtering requires minutes of CPU time, and is hard to extend to diagnose higher multipoles as well.

In the present state the algorithm used for the reconstruction is unsatisfactory, because it needs to run many simulations in order to construct the matrix $T$. So far attempts to calculate $T$ from “first principles” are only partly successful. Much more work along these lines is required. Nevertheless, even in the present state, the method should be capable of diagnosing sextupolar aberrations in LEP and later in LHC.

In the future this diagnostic method may become part of a feedback system in LHC that counteracts slowly varying sextupolar components which may arise from persistent currents in the LHC dipoles. The second component of such a feedback system are knobs consisting of different sextupoles that affect individual aberrations. These knobs are very easy to construct using the hamiltonian formalism described in ref. 1 and are similar to those discussed in ref. 4. The feedback would be necessary slow, about once every few seconds, because 32,000 turns need to be sampled which corresponds to about 3 s in real time in LEP or LHC. Moreover, currently the filtering requires minutes of CPU time.

6 Conclusions and Outlook

We presented a method to determine sextupolar aberration coefficients in LEP with a wobbling diagnostic method and tested that method in simulations. We showed that non-linear elements such as sextupoles act as frequency mixers for primary orbit oscillations effected to the beam by orbit correctors. Upon careful removal of the primary signals with a notch-filter the mixed signals clearly show up in the FFT of the BPM signals. Furthermore we showed that it is possible to reconstruct the coefficients with an accuracy of 5%, if no BPM errors are present. Observations corroborate that the method can be extended to diagnose higher multipoles as well.

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but can certainly be sped up considerably. The reconstruction of the hamiltonian takes less than 2 seconds and can also be sped up. However, careful tuning of the software is required.

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References


