Beam stacking at high values of $T = \sin \varphi$

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BEAM STACKING AT HIGH VALUES OF $P = \sin \varphi$  

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SUMMARY

The process of accumulating a beam in a storage ring by means of radio-frequency stacking at values of $\Gamma = \sin \phi$ between 0.5 and 0.84 is discussed. Results of statistical studies on a computer at these values of $\Gamma$ are presented. They indicate that such stacking processes can be done with efficiencies well above 50% for stacks consisting of 100 or more pulses.

RESUME

On discute le procédé d'accumulation dans un anneau de stockage HF pour des valeurs de $\Gamma = \sin \phi$ comprises entre 0,5 et 0,84. On présente les résultats d'études statistiques sur une calculatrice pour ces valeurs de $\Gamma$. Ils indiquent qu'un tel procédé d'accumulation peut se faire avec des rendements nettement supérieurs à 50% pour des stockages de 100 impulsions ou davantage.
BEAM STACKING AT HIGH VALUES OF $\Gamma = \sin \theta$

1. INTRODUCTION

Some years ago a statistical study of the beam stacking process for $\Gamma = 0.3$ and $\Gamma = 0.5$ was undertaken by Swenson\(^1\). Such low values of $\Gamma$ means, in practice, that the r.f. buckets must move very slowly through the already stacked particles, and this may lead to excessive stacking times for the highest stacked currents in the ISR. For these reasons Swenson's work has been extended to higher values of $\Gamma$, including both stacking at the top and at the bottom.

Inspection of Swenson's curves for the stacking efficiency $\eta$ versus the number of cycles with $\Gamma = 0.3$ and $\Gamma = 0.5$ shows that the number of stacking cycles necessary to obtain reasonable values of $\eta$ increases rapidly with increasing $\Gamma$. It is therefore important to perform a considerably higher number of stacking cycles than Swenson. In order to do this the existing computer program\(^2\) had to be modified.

2. ALGORITHM FOR A STACKING CYCLE

For convenience and as a definition of the terms to be used later we will give a short description of the method employed in these studies although it is not much different from the one used by Swenson\(^1\).

A stacked beam is generally represented by a distribution $V(E, \varphi)$ in phase and energy. What we want to know is the effect on the beam of a "stacking cycle" of the following form. Particles are captured in an r.f. bucket at an injection energy $E_i$ far below the energy $E_s$ of the stack. By frequency modulation, this bucket, uniformly occupied by particles, is moved up in energy, and the particles are released from it at a predetermined energy
by abruptly switching off the r.f. voltage. To describe this 
operation computationally we must know

i) the effect of the moving bucket on the distribution function

(V(E,φ): let \( \tilde{V}(E,φ) \) be the modified distribution;

ii) the particle distribution \( V_s(E,φ) \) added to the stack by the bucket.

For the final distribution \( V_{f,N} \) after the \( N^{th} \) cycle we may write

\[
V_{f,N} = \tilde{V}_{i,N} + V_s
\]

(1)

where \( V_{i,N} \) is the distribution before the \( N^{th} \) cycle.

By letting

\[
V_{i,N+1} = V_{f,N}
\]

(2)

the operation can be carried through any number of cycles.

The computational work involved can be greatly reduced by making three assumptions:

1) We assume that the azimuthal distribution of particles is uniform at the beginning of the stacking cycle, hence \( V \) is a function of \( E \) alone, \( V(E) \).

2) We divide the beam stacking region into 100 channels labelled 1 to 100 from the highest energy downwards, and assume that all energy spectra are adequately represented by histograms constructed on these channels.

3) We assume that there is no interaction between the particles in a stacked beam. Thus all possible stack shapes can be investigated by linearly superimposing the effects of the moving bucket on samples of particles placed in each channel.

All the information for doing stacking experiments on the computer can thus be obtained by putting a sample of particles
into each of the 100 channels and computing their energy
distribution after one stacking cycle in terms of a "histogram"
constructed on the same channels. We will use the term
"histogram" only for the energy spectrum after one stacking
cycle of samples of particles initially within a single channel.
There will be a total of 100 of these histograms with serial
numbers equal to the number of the channel in which the particles
are put at the beginning. We consider the histograms as columns
in a matrix \( A \). The effect of the moving bucket is thus described
by a multiplication of the matrix \( A \) with a column vector representing
the energy spectrum of the particles:

\[
\vec{V}_{i,N} = A \vec{V}_{i,N}.
\]  

(3)

Obviously, the effect of the particles brought up in
the last bucket is just an addition of two column vectors.
Finally, we have

\[
\vec{V}_{f,N} = A \vec{V}_{i,N} + \vec{V}_5.
\]  

(4)

The amount of computer time for computing the matrix \( A \)
is then reduced to acceptable limits by the fourth assumption
that the histograms obtained for a sample of particles initially
within one channel after one stacking cycle, depend only on the
difference of the energy of that channel and the energy where
the bucket is released (given by the frequency of the r.f.).
Thus we can replace samples of particles in each of 100 channels
and a bucket stopping at one and the same energy, by a sample of
particles in one channel only and a bucket stopping at 100
different energies, which takes about half a minute on an IBM 7090
for one particle.
Furthermore, particles starting on two different trajectories whose Hamiltonians in scaled co-ordinates differ by $2\pi T$ will also behave approximately in the same way. It is therefore sufficient to distribute the particles in the space between two separatrices since they just differ in the Hamiltonian by $2\pi T$.

In order to have a simple recipe for choosing the initial conditions of the particles, one wants to distribute them at equidistant energies or phases.

It can be shown that a uniform distribution in phase between 0 and $2\pi$ yields a rather peaked distribution in the Hamiltonian, where half the particles are quite near to one particular value and only the rest is distributed smoothly over the remainder. This effect is more pronounced for high values of $T$ and makes this distribution unsuitable for our purpose.

However, a uniform distribution in energy also yields an almost uniform distribution in the Hamiltonian. Therefore, all particles were put at the same phase and at equidistant intervals in energy. The range of energies covered was always about one channel, which is of the same order of magnitude as the distance in energy between two adjacent separatrices.

The particle energies used to construct histogram No. 100 show very clearly a dependence on their initial energy, or aerial number, as is shown for $T = 0.7$ in Fig. 1. The missing particle No. 70015 got stuck near the unstable fixed point when the bucket was moving upwards in energy, and hence has an energy above the range of the energy scale (0.702). Since the number of separatrices crossed within the range of initial energies is only two, one cannot expect that the particles chosen will constitute a fair representation of the behaviour of a large number of particles. This shows up in the average energy of all
particles after the passage of the bucket, which may be compared to the result of analytical theory stating that the change in energy due to deceleration by phase displacement should be equal to the bucket area divided by $2\pi^2$.

The number of separatrices crossed within the range of initial energies decreases rapidly with increasing $\Gamma$. It must therefore be expected that the correct selection of initial conditions becomes more and more difficult with increasing $\Gamma$.

We therefore decided to eliminate from the subsequent analysis those particles which have become stuck near to the bucket, and to make the average change in energy equal to the analytical value by also suppressing a certain number of low-energy particles if this was necessary. The effect of these operations is roughly equal to that of only taking particles between two adjacent separatrices.

4. RESULTS

The stacking experiments to be described were performed with the number of particles given in the table below:

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>Number of particles</th>
<th>$\lambda$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>50</td>
<td>3</td>
<td>19</td>
</tr>
<tr>
<td>0.6</td>
<td>50</td>
<td>2</td>
<td>19</td>
</tr>
<tr>
<td>0.7</td>
<td>44$^\ast$</td>
<td>2</td>
<td>19</td>
</tr>
<tr>
<td>0.8</td>
<td>100</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>0.84</td>
<td>43$^\ast$</td>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>

$^\ast$) In these cases, the particles were eliminated from the histograms.
Here \( \lambda \) is the number of channels equivalent to the area of a bucket and \( K \) is the starting channel for computing the stacking efficiencies.

For all values of \( \Gamma \), particles were stacked both "at the top" and "at the bottom".

When stacking "at the top", the bucket with the last injected pulse passes through the whole stack which is thus accelerated by phase displacement. The particles of the last pulse are always deposited at the same energy by switching off the r.f. voltage. This energy corresponds to channel number 20, in our case.

When stacking "at the bottom", the bucket with the last injected pulse is released as soon as it enters the stack. This can be simulated in the programme by shifting the stack \( \lambda \) channels upwards before the matrix multiplication is performed, and by depositing the particles of the last bucket in the same channel all the time.

4.1 Histograms

Figure 2a shows some typical histograms for \( \Gamma = 0.8 \), which should be compared to those obtained by Swenson \(^1\) for \( \Gamma = 0.3 \) reproduced here as Fig. 2b. "The shaded portion of each histogram shows the energy spectrum after one cycle of the stacking process for a sample of particles placed initially in the open rectangle. The stacking process consists of a moving r.f. bucket, which starts far to the right of the figure, and moves to the left until it arrives at channel 20, where it is abruptly turned off". \(^1\)

The samples of particles placed in channels 2 and 10 always remain above the bucket, those in channels 18 and 22 are in the vicinity of the energy where the bucket is dropped, and those in channels 80 and 100 are passed by the bucket which is turned off far above them in energy.
Bearing in mind that the area of the bucket corresponds to one channel at $\Gamma = 0.8$, and to five channels at $\Gamma = 0.3$, the considerably higher ratio of energy spread to average displacement in all the histograms is immediately noticeable.

4.2 Average acceleration

Figures 3 and 4 show for $\Gamma = 0.5$ and $\Gamma = 0.6$, respectively, where no particles were eliminated, the average energy $\bar{E}$ of the sample of particles initially in channel No. 20 as a function of the position of the bucket indicated by the channel numbers along the abscissa. According to the analytical theory of synchrotron phase space, one expects the energy to be reduced by three channel widths for $\Gamma = 0.5$, and two channel widths for $\Gamma = 0.6$ when the bucket passes the channel No. 20. The channel limits drawn in show that the average accelerations by phase displacement are indeed quite near to the expected ones. This agrees with the result of Swenson for $\Gamma = 0.3$.

4.3 Build-up of the stack

Figure 5 shows a set of 12 energy spectra demonstrating the accumulation of a stacked beam resulting from 120 repetitions of the beam-stacking process with full buckets at $\Gamma = 0.8$. The dashed rectangle represents a stacked beam of unit density in a rectangular region of phase space having the same area as the energy spectrum after 120 cycles. To produce Fig. 5, the average of the contents of five neighbouring channels in the computer output was taken and a smooth curve was drawn through the points thus obtained.

An example of how a stacked beam is accumulated by stacking at the bottom at $\Gamma = 0.8$ is shown in Fig. 6. The slightly more rectangular shape than that of the previous one is to be expected, on the basis of the argument that a stack must be considerably less disturbed by stacking at the bottom since the bucket never actually passes it.
4.4 Average stacking efficiencies versus number of cycles \( N \)

We define the average stacking efficiency \( \bar{\eta}(N) \) as the ratio of the sum of the contents of channels \( K \) to \( K + N_{\Lambda} - 1 \) and \( N \) times the sum of the contents of a bucket. That is to say that we take a slice of the stack of the width of an ideal stacking process and compare its area to the area of the ideal rectangle. \( K \) defines the position of the slice in the stack and is chosen in such a way that the slice of the stack taken contains the highest number of particles. The values of \( K \) used are given in Table 1.

Figures 7 to 11 show these stacking efficiencies \( \bar{\eta}(N) \) for all the computed values of \( \Gamma \), both for stacking at the top and at the bottom.

The efficiencies for stacking at the top and at the bottom do not differ appreciably, although there is a tendency for the bottom efficiency to be higher than the other one at low numbers of cycles, as one would expect. For the higher numbers of cycles both stacking efficiencies tend towards common values at \( \Gamma = 0.5, 0.6, 0.7, \) and \( 0.8 \), although there are differences in their behaviour which do not depend on \( \Gamma \) in an obvious way. At \( \Gamma = 0.84 \), the bottom stacking efficiency is lower than the other one above 25 cycles. This result, which cannot be explained by physical reasoning, may be taken as an indication of the difficulties and errors involved in such a computer experiment at high values of \( \Gamma \).

The stacks build up more and more slowly with increasing \( \Gamma \), but nevertheless reach reasonable efficiencies even for the highest value of \( \Gamma \) considered.
5. Discussion of the results

From the various stacking efficiency curves one gets the impression that they all have more or less the same shape and that only their horizontal scales are different. This led Hereward\(^3\) to the following conjecture.

The energy variance \(\sigma^2\) of a set of particles which have been well passed by the bucket is known and printed out by the programme.

The mean energy change \(\delta\) of these particles is also known, even from analytical theory.

After \(Na\) stacking cycles, the scatter in energy has increased as \(\sqrt{N\delta^2}\) and the displacement as \(Na\delta\). If, for different values of \(\Gamma\), these quantities have the same ratio, that is at a certain value of

\[
\frac{N\delta}{\sqrt{N\delta^2}} = \sqrt{N} \frac{\delta}{\sigma},
\]

it is reasonable to expect the final energy spectrum to have the same ratio of scatter to ideal width, the same shape on an adjusted horizontal scale, and therefore the same stacking efficiency.

The result of plotting the stacking efficiency for all values of \(\Gamma\), including Swenson's \(\Gamma = 0.3\) results on this horizontal scale, is shown in Fig. 12.

One can indeed draw a common smooth curve through all the points. Unfortunately the \(\sigma^2\) is available only when most of the computer time has already been spent. Thus there is not much point in stopping here and not going one step further and actually computing the shape of the stacking efficiency curve.
6. CONCLUSIONS

Two stacking techniques have been considered for the intersecting storage rings for the CPS; stacking at the top and stacking at the bottom \(^6\).

Since, when stacking at the top, the acceleration of the injected pulse across almost the full aperture has to be done with a bucket of the size of the bunch, one has to use \(\Gamma \approx 0.84\) in order to finish a stacking cycle within the repetition time of the CPS.

When stacking at the bottom, on the other hand, most of the actual acceleration of the injected pulse happens outside the stack and can be done with a large bucket. Only when the bucket gets near to the stack has it to be reduced in size so as to fit tightly around the bunch. As a consequence, small values of \(\Gamma\) are sufficient.

Swenson's results already indicate that stacking at the bottom can be done with satisfactory efficiency for almost any number of pulses as far as the last phase of the stacking process is concerned, when the small bucket moves into the edge of the stack.

The results presented in this report now show that stacking at the top at values of \(\Gamma\) as high as \(\Gamma = 0.84\) is also still an efficient technique for large stacks. Thus there is considerable freedom in the choice of the stacking technique from the point of view of efficiency for large stacks.

For small stacks one should bear in mind that it is only important to obtain a filling time which is a certain (small) fraction of the beam lifetime. It is therefore possible to stack only every \(N^{th}\) pulse at the top at low \(\Gamma\) which is also efficient without getting an excessive filling time.
ACKNOWLEDGEMENTS

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REFERENCES

1) D.A. Swenson, Int. Conf. on High-Energy Accelerators, Brookhaven (1961), p. 17.

2) The original programmes were written by J.S. Horsby and are described in the following internal CERN documents:
   a) A Programme for Accelerator Studies (Programme 02004/A), Ref. 932/p, 20.3.1961.
   b) Addendum to A Programme for Accelerator Studies (Programme 02004/A), Ref. 4083/p, 5.6.1962.
   c) Revised Input Facilities for Programme 02004/A, DB/CO, Ref. 3633/p, 2.4.1962.
   d) Ancillary Programme 1 for Processing the Results of a Programme for Accelerator Studies (Programme 02004/B) Ref. 933/p, 30.5.1961.

3) H.G. Harcourt, private communication.


6) AR/Int. SG/64-9.
Fig. 1 Final energy $E$ of the particles vs. initial energy given by serial number for $\Gamma = 0.7$

The bucket is at an energy $E_B = 0.702$
Fig. 2a  Histograms for $\Gamma = 0.8$
Fig. 2b
Histograms for \( \Gamma = 0.3 \) obtained by Swenson
Fig. 3  Average energy $\overline{E}$ of a sample of particles initially in channel 20 vs. position of bucket given by channel number for $\Gamma = 0.5$
Fig. 4  Average energy $\bar{E}$ of a sample of particles initially in channel 20 vs. position of the bucket given by channel number for $\Gamma = 0.6$
Fig. 5

Accumulation of stacked beam by stacking full buckets at the top at $\Gamma = 0.8$
Fig. 6
Accumulation of stacked beam by stacking full buckets at the bottom at $\Gamma = 0.8$
Fig. 7  Average efficiency for stacking at top and bottom  $\Gamma = 0.5$
Fig. 8  Average efficiency for stacking at top and bottom $\Gamma = 0.6$
Fig. 9 Average efficiency for stacking at top and bottom $\Gamma = 0.7$
Fig. 10: Average efficiency for stacking at top and bottom $\Gamma = 0.8$.
Fig. 11 Average efficiency for stacking at top and bottom $\Gamma = 0.84$