ON THE CORRECTION OF LARGE RANDOM SKEW QUADRUPOLAR ERRORS DURING THE RAMP IN LHC

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The complex design of the super-conducting dipole magnets in LHC will cause unwanted multipoles of which the skew quadrupolar is one of the most prominent. Due to the two-in-one design of the magnets it will have a systematic and also a large random (i.e. varying from magnet to magnet) component. During the ramp these multipole components will be time dependent and, due to the finite inter-strand resistance, the magnitude could be considerably larger than the values at injection energy. In this note we will devise methods to detect and correct the coupling and test their success in short-term (1000 turn) dynamic aperture studies. This will result in bounds on the acceptable random skew quadrupolar errors which may be present in the dipoles. We then combine the detection and correction methods into a feedback system that will prove to be able to control coupling during the ramp. Simulation results of the feedback system assuming realistic BPM resolutions are reported.

KEY WORD: Skew quadrupolar

1 INTRODUCTION

The super-conducting dipole magnets of LHC\(^1\) will contain unwanted multipoles which are due to the peculiarities of the two-in-one design and the manufacturing process. The multipoles, of which the skew quadrupole is the most prominent, already have a sizeable magnitude in a static configuration\(^2\) in which the magnets are excited by a constant current. Moreover, during the ramp, the finite inter-strand resistance will allow flux loops to appear which in turn will cause increased and time-dependent multipoles. The inter-strand resistance can be increased to avoid the flux loops to appear, but this will make the magnet more susceptible to quenches, because it impedes the distribution of excess current in a single strand among its neighbors. Furthermore, it is very difficult to guarantee a uniform thickness of the inter-strand resistance layers such that the multipolar contents of the dipoles will vary from magnet to magnet and also within a given magnet.

In order to assess the effect of a largely increased skew quadrupole which has a "systematic" component, identical in all dipoles, and a "random" component that varies

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155
from magnet to magnet on beam dynamics we investigate methods to control the effect of skew quadrupoles as well as means to experimentally diagnose coupling. We augment this by studying the dynamic aperture as a function of the skew quadrupole magnitude.

This report is organized as follows: In the next section we develop a Hamiltonian theory of coupling that allows us to calculate resonance strengths of up to third order in the skew quadrupole gradients followed by a section about constructing knobs of different correction skew quadrupoles that correct single resonance parameters without affecting others and find rules for grouping skew quadrupoles in order to avoid generating higher order coupling. The next section is devoted to a procedure that can be used to experimentally diagnose coupling resonance parameters non-invasively. The subsequent section reports results of tracking runs in which different seeds of skew quadrupole components are corrected in different ways. The next section discusses the simulation of a feedback system which proves to be capable of correcting temporally varying coupling dynamically and is followed by the conclusions.

Before turning to the main sections of this report we will give a few definitions. The multipole errors in LHC dipole magnets are usually given as expansion coefficients $a_n$ and $b_n$ of the following equation

$$B_y + iB_x = B_0 \sum_{n=1}^{\infty} (b_n + ia_n) \left( \frac{x + iy}{r} \right)^{n-1}$$

(1)

where $B_0$ is the vertical nominal deflecting field of the dipole, $B_x$ and $B_y$ are the horizontal and vertical fields, respectively. $r$ is a reference radius of 1 cm at which the multipole coefficients are measured (usually with a rotating coil). $b_n$ are the normal multipole coefficients and $a_n$ the skew coefficients. We are mainly interested in skew quadrupole coefficients $a_2$. The $a$ and $b$ coefficients can be related to the integrated normalized gradients $K_nL$ that are used in optics codes such as MAD³ by the equation

$$\frac{K_{n-1}L}{(n-1)!} = b_n \frac{\Delta \phi}{r^{n-1}}$$

(2)

and a similar equation for $a_n$. $\Delta \phi$ is the bending angle per dipole, which is about 4.868 mrad for the 13.145 m long dipoles in LHC.

A convenient estimator of the magnitude of coupling in a circular accelerator is the minimum achievable tune separation $\Delta Q$. It can be calculated from the full-turn transfer matrix $R$, which may be written in $2 \times 2$-block form as

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(3)

For the minimum tune separation (or width of the difference resonance) we then obtain Ref. 4

$$\Delta Q \approx \frac{\sqrt{\det |C + B|}}{\pi (\sin(2\pi Q_1) - \sin(2\pi Q_2))}$$

(4)
where $\tilde{B}$ denotes the symplectic conjugate of $B$ and $Q_1$ and $Q_2$ are the coupled eigen tunes. In some of the figures below, $\Delta Q$ is also referred to as $\text{CW}$, for coupling width.

In LHC $\Delta Q$ is on the order of 0.5 for the “nominal” systematic skew gradients in the dipoles. Coupling of this magnitude certainly requires correction. Following the experience made in HERA and correcting the coupling by moving the vertical orbit in the chromaticity correction sextupoles requires orbit changes of about 6 mm, which is too much to apply routinely. Thus, in order to correct this strong coupling, skew quadrupoles are required.

In the following section we will develop a method that quantifies the strength of coupling perturbatively. Equation 4 gives the strength of the difference resonance non-perturbatively and can be used to estimate the remainder of perturbative calculations of coupling. The latter are developed in the next section.

2 HAMILTONIAN THEORY TO THIRD ORDER

The influence of perturbing magnetic fields in an accelerator is most compactly described in a Hamiltonian framework. Here we are mainly interested in the influence of skew quadrupolar elements for which the (thin-lens) Hamiltonian is given by

$$H_s(x, y) = \frac{1}{f} xy$$

where $f$ is the focal length of the skew quadrupole. The transfer map due to this Hamiltonian is given by

$$e^{-H_s} = \sum_{n=0}^{\infty} \frac{(-H_s)^n}{n!}$$

where the colon denotes a “Poisson Bracket about to happen”. The Poisson Bracket of two functions $f$ and $g$ which depend on $(x, x', y, y')$ is given by

$$: f : g = [f, g] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial x'} - \frac{\partial f}{\partial x'} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y'} - \frac{\partial f}{\partial y'} \frac{\partial g}{\partial y}.$$  

Applying $H_s$ e.g. to a function $x'$ we obtain the normal form of the kick in $x'$ due to a skew quadrupole

$$e^{-H_s} x' = x' - [H_s, x'] + \frac{1}{2} [H_s, [H_s, x']] - \cdots = x' - \frac{1}{f} y$$

where the infinite series truncates.

The Hamiltonian of a beam line is then given by a sequence of individual Hamiltonians. Note that a transfer matrix $R_i$ may also be written as $\exp(-H_i :)$ where $H_i$ is a symmetric
quadratic polynomial in \((x, x', y, y')\). A very convenient representation of a beam line can be obtained from the following observation\(^a\)

\[
\mathcal{M} = e^{-H_1} e^{-H_2} e^{-H_3} e^{-H_4} e^{-H_5} = e^{-H_1} e^{-H_2} e^{-H_3} \\
\times e^{H_2} e^{H_3} e^{-H_2} e^{-H_3} \\
\times e^{H_3} e^{-H_3} e^{-H_3}.
\]  

(9)

where we inserted unit maps in the form of \(\exp(\cdot H)\). The first line is simply the linear transfer map through the entire section and we can now utilize the similarity relation for Lie-transformations\(^5\)

\[
e^{-\tilde{H}_s} = e^{-H_1} e^{-H_2} (x, x', y, y') e^{H_1} \\
\quad \quad \quad = e^{-H_s} \exp(\cdot H) x, \exp(\cdot H) x', \exp(\cdot H) y, \exp(\cdot H) y');
\]

(10)

which means that a similarity transformation of a Hamiltonian \(H_s\) can be evaluated by merely re-expressing it in the transformed variables. The second and third lines in Equation 10 are thus only the Hamiltonians \(H_s\) and \(H_t\), respectively, expressed in variables at the end of the beam line and we will write

\[
\mathcal{M} = e^{-L} e^{-\tilde{H}_s} e^{-\tilde{H}_t};
\]

(11)

where we defined \(e^{-L} = e^{-H_1} e^{-H_2} e^{-H_3}\) as the Hamiltonian that generates the linear transport through the beam line. We thus effectively pushed all perturbing Hamiltonians to the end of the beam line and \(\exp(\cdot \tilde{H}_s)\) is the kick that the skew quadrupole would do if it were situated there.

The method to push elements to the end of the beam line is easily extended for many perturbing elements and we are then left with an expression such as

\[
\mathcal{M} = e^{-L} e^{-\tilde{H}_{s1}} \ldots e^{-\tilde{H}_{sn}}
\]

(12)

if \(n\) perturbing elements are present. We are now faced with the task of concatenating the Hamiltonians, which is easily performed by the Campbell-Baker-Haussdorff (CBH) formula\(^5\) which reads

\[
e^{-\tilde{H}_s} e^{-\tilde{H}_t} = e^{-(\tilde{H}_s - \tilde{H}_t + 1/2[\tilde{H}_s, \tilde{H}_t] - 1/12[\tilde{H}_s - \tilde{H}_t, [\tilde{H}_s, \tilde{H}_t]])}.
\]

(13)

In this way the combined effect of many perturbing elements can be calculated easily element by element up to fifth order in monomials in \(x, x', y, y'\). In the end we are left

\(^a\)Note that, contrary to matrix equations, Hamiltonian or Lie-type equations are read from left to right.\(^5\) Thus \(-H_1\) is applied first, \(-H_2\) second, and \(-H_3\) last.
with the linear unperturbed transfer matrix \( R = e^{-L} \) and a map \( e^{-\tilde{H}_0} \) that represents the cumulative effect of all perturbing elements pushed to the end of the beam line.\(^6,\)\(^8\) Note that this representation is still manifestly symplectic and that the only approximation made in deriving the one-turn-map in the representation \( \exp(-L) \exp(-\tilde{H}_0) \) is that we use three terms in the CBH formula, only.

A further simplification can be obtained by transforming the one-turn-map into normalized phase space by transforming the variables by the matrix

\[
\begin{pmatrix}
\tilde{x} \\
\tilde{x'}
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{\beta_x} & 0 \\
\alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x}
\end{pmatrix} \begin{pmatrix}
x \\
x'
\end{pmatrix}
\]

(14)

and a similar expression for the variables \( y \) and \( y' \). In these variables the uncoupled part is simply given by a rotation matrix and \( -\tilde{H}_0 \) is transformed to \( -\tilde{H} \) which depends on the variables in normalized phase space \((\tilde{x}, \tilde{x'}, \tilde{y}, \tilde{y'})\). Then \( -\tilde{H} \) is a quadratic form in those variables. Being in normalized phase space it is easy to transform the polynomial to action angle variables using the following transformations

\[
\begin{align*}
\tilde{x} &= \sqrt{2J_x} \cos(\psi_x), & \quad \tilde{x}' &= \sqrt{2J_x} \sin(\psi_x), \\
\tilde{y} &= \sqrt{2J_y} \cos(\psi_y), & \quad \tilde{y}' &= \sqrt{2J_y} \sin(\psi_y).
\end{align*}
\]

(15)

\( -\tilde{H} \) then depends on the actions \( J_x, J_y \) and the phases \( \psi_x, \psi_y \). It is shown in the next section that the coefficients of \( \sin(\psi_x \pm \psi_y) \) and \( \cos(\psi_x \pm \psi_y) \) yield the driving terms of the coupling resonances, up to a factor \( 2\pi \sqrt{J_x J_y} \). Forthwith we call them resonance parameters. Since the Hamiltonian \( \tilde{H}_0 \) is calculated to third order in CBH the resonance parameters are correct to the same order.

We have coded the presented method up to 5th Hamiltonian order (decapoles) and third order in the CBH formula (combined effect of 3 sextupoles). The main ingredient of the code\(^9\) are subroutines to calculate transfer matrices, the Poisson Bracket, and one to perform a linear change of variables in a polynomial. Given an optics file the code first calculates the transfer matrices of the unperturbed lattice, then steps through the beam line. If it encounters a perturbing element it sets up the Hamiltonian, changes its variables to that at the end of the beam line and then concatenates with the Hamiltonian that constitutes the effect of previously encountered elements.

Applied to a beam line that contains only skew quadrupoles using the third order CBH formula implies that the interaction of three skew quadrupoles is calculated properly. Considering a skew quadrupole Hamiltonian pushed to the end of the beam line (using an uncoupled transfer matrix \( R \)) we observe that it contains only the following terms

\[
xy, xy', x'y, x'y'.
\]

(16)

In the concatenation using CBH we have to calculate the Poisson Bracket of terms appearing in Equation 16 and find that in the first commutator only terms such as

\[
x^2, xx', x'^2, y^2, yy', y'^2
\]

(17)
appear, which are terms in the Hamiltonian generating beta beat. The third order part in CBH requires the calculation of the Poisson Bracket of these expressions with those of Equation 16, resulting again in terms such as those in Equation 16. This line of argument can be pursued further with the result that odd numbered orders contribute to coupling terms in the Hamiltonian (which can be directly fixed with correction skew quadrupoles) and that even numbered orders contribute to beta beat terms (which can not be fixed by skew quadrupoles in a first order correction). The best way to avoid problems with higher orders is to use correction schemes with small correction skew quadrupole excitations that avoid the generation of higher order terms altogether.

In the next section we investigate means to optimally place correction skew quadrupoles in a beam line such as to avoid their excessive excitation.

3 COUPLING RESONANCE CONTROL KNOBS

It is very informative to calculate the Hamiltonian of a single skew quadrupole pushed to the end of the beam line in terms of beta functions and phase advances. To simplify things further we add a map to the end of the beam line that maps into normalized phase space. The entire transfer matrix from the position of the skew quadrupole to normalized phase space at the end of the beam line is then given by

\begin{equation}
R_x = \begin{pmatrix}
\cos(\bar{\phi}_x) & \sin(\bar{\phi}_x) \\
-\sin(\bar{\phi}_x) & \cos(\bar{\phi}_x)
\end{pmatrix}
\begin{pmatrix}
1/\sqrt{\beta_x} & 0 \\
\alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x}
\end{pmatrix}
\end{equation}

where we define \( \bar{\phi}_x = 2\pi Q_x - \phi_x \). \( \alpha \) and \( \beta \) are the usual twiss parameters, \( Q \) is the tune and \( \phi \) is the betatron phase of the uncoupled machine. The transfer matrix for the motion in the vertical plane can be obtained by substituting \( y \) for \( x \).

According to Equation 10, we now have to express the Hamiltonian at betatronic phase \( \phi_x, \phi_y \) with coordinates \( (x, x', y, y') \) in terms of the normalized phase space coordinates at the end of the beam line \( (x_0, x_0', y_0, y_0') \) at \( (\phi_x/y = 2\pi Q_x/y) \). Of course we have

\begin{equation}
\begin{pmatrix}
x \\
x'
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \\
-\sin(\bar{\phi}_x) & \cos(\bar{\phi}_x)
\end{pmatrix}
\begin{pmatrix}
R_x^{-1} & 0 \\
0 & R_y^{-1}
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_0'
\end{pmatrix}
\end{equation}

Explicitly written we get

\begin{align*}
x &= \sqrt{\beta_x} \cos(\bar{\phi}_x)x_0 - \sqrt{\beta_x} \sin(\bar{\phi}_x)x_0' \\
y &= \sqrt{\beta_y} \cos(\bar{\phi}_y)y_0 - \sqrt{\beta_y} \sin(\bar{\phi}_y)y_0'
\end{align*}
and inserting into Equation 5 we obtain for the Hamiltonian pushed to the end of the line

\[ \tilde{H}(x_0, x_b, y_0, y_b) = \frac{1}{f} \sqrt{\beta_x \beta_y} \cos(\phi_x) \cos(\phi_y) x_0 y_0 \]

\[ - \frac{1}{f} \sqrt{\beta_x \beta_y} \cos(\phi_x) \sin(\phi_y) x_0 y'_b \]

\[ - \frac{1}{f} \sqrt{\beta_x \beta_y} \sin(\phi_x) \cos(\phi_y) x'_0 y_0 \]

\[ + \frac{1}{f} \sqrt{\beta_x \beta_y} \sin(\phi_x) \sin(\phi_y) x'_0 y'_0 . \]

\( \tilde{H} \) is given in terms of variables in normalized phase space and therefore we can express those in terms of action angle variables \( J \) and \( \psi \)

\[ x_0 = \sqrt{2J_x} \cos(\psi_x), \quad x'_b = \sqrt{2J_x} \sin(\psi_x), \]

\[ y_0 = \sqrt{2J_y} \cos(\psi_y), \quad y'_0 = \sqrt{2J_y} \sin(\psi_y) \]

with the result

\[ \tilde{H}(J_x, J_y, \psi_x, \psi_y) = \frac{1}{f} \sqrt{\beta_x \beta_y} \frac{1}{2} \sqrt{2J_x 2J_y} \]

\[ \{ \cos(\psi_x + \psi_y) \cos(\tilde{\phi}_x + \tilde{\phi}_y) - \sin(\psi_x + \psi_y) \sin(\tilde{\phi}_x + \tilde{\phi}_y) \]

\[ + \cos(\psi_x - \psi_y) \cos(\tilde{\phi}_x - \tilde{\phi}_y) - \sin(\psi_x - \psi_y) \sin(\tilde{\phi}_x - \tilde{\phi}_y) \} . \]

Comparing with Ref. 2, Equation 2,3 we see that the terms are (apart from a factor \( 2\pi \sqrt{J_x J_y} \) and being complex conjugate) similar to the driving terms of the zeroth harmonic of the complex coupling coefficients \( c_+^0 \) and \( c_-^0 \). Recovering only the zeroth harmonic in the Hamiltonian framework is not surprising, because we pushed all elements to the end of the beam line such that their localization properties are only kept modulo phase advance \( 2\pi \).

We therefore define the contribution of a given skew quadrupole to the strengths of the resonances, which we will call \textit{resonance parameters} collectively, by

\[ \delta \sigma_c = \frac{1}{2\pi} \frac{1}{f} \sqrt{\beta_x \beta_y} \cos(\phi_x + \phi_y) \]

\[ \delta \sigma_s = - \frac{1}{2\pi} \frac{1}{f} \sqrt{\beta_x \beta_y} \sin(\phi_x + \phi_y) \]

\[ \delta \Delta_c = \frac{1}{2\pi} \frac{1}{f} \sqrt{\beta_x \beta_y} \cos(\phi_x - \phi_y) \]

\[ \delta \Delta_s = - \frac{1}{2\pi} \frac{1}{f} \sqrt{\beta_x \beta_y} \sin(\phi_x - \phi_y) . \]
Now that we know how much a given skew quadrupole affects the four resonance parameters, we will analyze what configuration of four skew quadrupoles will affect only a single parameter, but not the other three. In other words, we construct orthogonal knobs for the resonance parameters. Two knobs of this sort were already found in Ref. 2. They consist of two skew quadrupoles separated by 90 degree both in horizontal and vertical betatron phase. Powered in parallel they affect the cosine-like phase of the difference resonance and powered anti-parallel they affect the cosine-like phase of the sum resonance. Here we supply the other two knobs.

First we express the resonance parameters in terms of the skew quadrupole excitations and obtain the following matrix-equation

\[
\begin{pmatrix}
\sigma_c \\
\sigma_s \\
\Delta_c \\
\Delta_s \\
\end{pmatrix} =
\begin{pmatrix}
\cos(\phi_1^x + \phi_1^y) & \ldots & \cos(\phi_4^x + \phi_4^y) \\
-\sin(\phi_1^x + \phi_1^y) & \ldots & -\sin(\phi_4^x + \phi_4^y) \\
\cos(\phi_1^x - \phi_1^y) & \ldots & \cos(\phi_4^x - \phi_4^y) \\
-\sin(\phi_1^x - \phi_1^y) & \ldots & -\sin(\phi_4^x - \phi_4^y) \\
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
k_3 \\
k_4 \\
\end{pmatrix}
\] (25)

where we define \( k_i = \sqrt{\beta_x \beta_y} / f_i \). All we now have to do is to find phases that make many elements of the matrix to zero and a few others unity. For convenience, but without loss of generality we chose the reference point just after the quadrupole labeled 4. If we choose the placement according to the following table

<table>
<thead>
<tr>
<th>skew quadrupole</th>
<th>( \phi_x )</th>
<th>( \phi_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>270</td>
<td>180</td>
</tr>
<tr>
<td>2</td>
<td>135</td>
<td>135</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

we see that the previous equation acquires the following form

\[
\begin{pmatrix}
\sigma_c \\
\sigma_s \\
\Delta_c \\
\Delta_s \\
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & -1 & 1 \\
-1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
k_3 \\
k_4 \\
\end{pmatrix}
\] (26)

and the knob coefficients for the orthogonal knobs are the columns of the inverse of the matrix in Equation 26 which can be easily calculated and yield
Note that the previous exercise merely serves as a guideline as to where the skew quadrupoles should be placed. In particular the relative placement of skew quadrupole 1 with respect to the others is not important, however, the demand for a skew quad pair with a $\phi_x - \phi_y$ near 90 degree is important. Ignoring this simple rule leads to knobs in which the correcting skew quadrupoles fight each other which in turn implies large excitations of the skew quadrupoles that will cause higher order coupling and beta beat.

For the analysis of LHC we placed the skew quadrupoles according to the following rules in all eight insertions:

1. **QSK4** is placed next to the first regular quadrupole in each insertion (Q10).

2. **QSK5** is placed one 90 degree cell away from QSK4 into the adjacent arc. QSK4 and QSK5 constitute the pair that was proposed in Ref. 2.

3. **QSK3** is placed next to the second regular quadrupole in the dispersion suppressor (Q9), which is about 45 degree in horizontal and vertical betatron phase away from QSK4.

4. **QSK2** is placed inside the insertion where phase difference $\phi_x - \phi_y$ as measured from QSK5 is close to 90 degrees, which in most insertions is next to the outermost quadrupole of the outer triplet (Q6).

Using knobs constructed according to these rules we are able to control the coupling without introducing higher order coupling. We have to note, however, that the knobs and the excitations of the skew quadrupoles are calculated for a given optics. If, e.g. during the $\beta^*$—squeeze the optics in the insertions changes, the knob coefficients change and the skew quadrupoles must be included in the squeeze-program.

Having found knobs which can correct the coupling in the next section we turn to methods which can be used to diagnose it.

## 4 NON-INVASIVE COUPLING RESONANCE DIAGNOSTIC

In this section we investigate the feasibility to determine the resonance parameters $\sigma_c$, $\sigma_s$, $\Delta_c$, and $\Delta_s$ experimentally without perturbing the beam. It is shown in Appendix A that the closed-orbit response coefficients $C_{ij}^{\xi} = \partial Y B P M_i / \partial X C O R_j$ between orbit correctors labeled $j$ and BPM labeled $i$ are linearly dependent on the resonance parameters, if there are no coupling elements between the correctors, BPM, and the position where the resonance parameters are evaluated. Clearly, having four resonance parameters requires to use at least four response coefficients which can be obtained from using two horizontal
orbit correctors and two vertical BPM. We may then write the relation among the resonance parameters and the response coefficients in the form of a matrix equation

\[
\begin{pmatrix}
\sigma_c \\
\sigma_s \\
\Delta_c \\
\Delta_s
\end{pmatrix} = \begin{pmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & m_{22} & m_{23} & m_{24} \\
m_{31} & m_{32} & m_{33} & m_{34} \\
m_{41} & m_{42} & m_{43} & m_{44}
\end{pmatrix} \begin{pmatrix}
C_{11} \\
C_{12} \\
C_{13} \\
C_{14}
\end{pmatrix}.
\]

The matrix \( M = (m_{ij}) \) can be determined in a simple fit, in which four skew quadrupoles are weakly excited and for each configuration the resonance parameters and the response coefficients are calculated. The matrix elements are then found by solving linear sets of equations.

The optimum place for the correctors and BPM can be found from the condition that the rows of the matrix \( M \) are orthogonal, because that would minimize accidental cross talk due to errors in the \( C \)'s. Playing with various configurations we find that this condition is fulfilled if the two correctors are 90 degrees betatron phase apart and sit at positions with equal beta functions and that the BPM should also be 90 degree apart from each other and have equal beta functions. These requirements are, however, not very strict, but, if they are fulfilled the system is less error-prone.

The response coefficients can be measured in a static configuration by exciting a corrector and measuring the change in orbit position at the corresponding BPMs. This, however, requires large excitations in order to reach reasonable accuracies and would perturb the beam significantly. A better way is to excite the orbit correctors with low-frequency low-amplitude sinusoidal excitations on the order of 50 to 100 Hz and record the BPM signals at the same frequencies. The frequencies should be chosen such that they are not too low, because that mandates long sampling times, and lower than the revolution frequency of LHC, which is \( f_0 = 11245 \) Hz. The excitation-signal is then mixed with the BPM signals. The DC component of the residual signal is proportional to the response coefficient \( C \). The mixing can either be done by inexpensive analog circuitry, because the involved signals have low frequency or the corrector excitation and the BPM signals can be digitized first and mixed digitally. This would require hardware similar to that of the existing 1000-turn monitor on LEP.

In order to test the required accuracy of the BPM and the excitation strength of the orbit correctors we write a small code that determines the matrix \( M \) for the version 2 LHC injection optics without errors. As reference point where the resonance parameters are

<table>
<thead>
<tr>
<th>element</th>
<th>phase after IP1</th>
<th>beta function</th>
<th>position</th>
</tr>
</thead>
<tbody>
<tr>
<td>XCOR_1</td>
<td>( \phi_x = 89.9 )</td>
<td>( \beta_x = 232.7 ) m</td>
<td>after first separator</td>
</tr>
<tr>
<td>XCOR_2</td>
<td>( \phi_x = 190.0 )</td>
<td>( \beta_x = 300.0 ) m</td>
<td>before Q5</td>
</tr>
<tr>
<td>YBPM_1</td>
<td>( \phi_y = 113.0 )</td>
<td>( \beta_y = 156.8 ) m</td>
<td>before second separator</td>
</tr>
<tr>
<td>YBPM_2</td>
<td>( \phi_y = 183.6 )</td>
<td>( \beta_y = 117.8 ) m</td>
<td>before Q5</td>
</tr>
</tbody>
</table>
TABLE 2: The relative resonance excitations as a function of the BPM resolution for an excitation of $\Delta_e = 0.005$.

<table>
<thead>
<tr>
<th>BPM error</th>
<th>rel. $\sigma_c$</th>
<th>rel. $\sigma_s$</th>
<th>rel. $\Delta_c$</th>
<th>rel. $\Delta_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ideal, abs.</td>
<td>0.000</td>
<td>0.000</td>
<td>0.005</td>
<td>0.000</td>
</tr>
<tr>
<td>ideal, rel.</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0 $\mu$m</td>
<td>0.052</td>
<td>-0.032</td>
<td>0.998</td>
<td>0.027</td>
</tr>
<tr>
<td>10 $\mu$m</td>
<td>-0.008</td>
<td>-0.121</td>
<td>0.990</td>
<td>0.064</td>
</tr>
<tr>
<td>20 $\mu$m</td>
<td>-0.058</td>
<td>-0.196</td>
<td>0.974</td>
<td>0.096</td>
</tr>
<tr>
<td>30 $\mu$m</td>
<td>-0.100</td>
<td>-0.257</td>
<td>0.954</td>
<td>0.121</td>
</tr>
<tr>
<td>40 $\mu$m</td>
<td>-0.135</td>
<td>-0.307</td>
<td>0.931</td>
<td>0.142</td>
</tr>
<tr>
<td>50 $\mu$m</td>
<td>-0.164</td>
<td>-0.348</td>
<td>0.909</td>
<td>0.159</td>
</tr>
</tbody>
</table>

calculated we choose IP1 and the chosen BPM and orbit correctors are displayed. The same matrix $M$ is then used for all other optics with perturbation or without.

The response coefficients are determined in a small tracking code that consists of transfer matrices between IP1, the orbit correctors, and the BPM. Every turn the beam receives a sinusoidally modulated kick with amplitude $\varepsilon_j$ and frequency $f_j$ at corrector XCOR$_j$ and the positions at YBPM$_i$ are recorded. We arbitrarily choose $f_1/f_0 = 0.008467$ and $f_2/f_0 = 0.005830$. The four response coefficients $C_{ij}$ are calculated from multiplying the excitation of the corrector $j$ with the position of BPM $i$ and summing over all turns. $^b$ Obviously the resulting estimates for the $C_{ij}$ do not have an absolute meaning, only their relative magnitudes and signs carry meaning and thus only the relative magnitude of the resonance parameters (normalized by $\sqrt{\sigma_c^2 + \sigma_s^2 + \Delta_c^2 + \Delta_s^2}$) can be deduced. In the code only the normalized resonance parameters are printed and compared with the equally normalized resonance parameters from the third order calculation described in the previous section.

Testing various orbit corrector excitation magnitudes and numbers of turns we find that excitations of 2 $\mu$rad (4 $\mu$rad peak to peak) and sampling over 1024 turns is sufficient. In order to determine the required BPM resolution we use the same simulation code as above but add random numbers with a given rms, say 20 $\mu$m, truncated at three standard deviations, to the signal as reported from the BPM. We then use an uncoupled LHC optics and excite the resonance knob that excites the cosine phase of the difference resonance by 0.005 (but leaves the other phase of the difference resonance and both phases of the sum resonance untouched). Using this optic in the tracking code we vary the BPM resolution between 10 and 50 $\mu$m and report the relative strength of the excitations in Table 2.

We see that with 20 $\mu$m BPM resolution we can achieve about a 5-fold rejection of the “wrong” resonance parameter. Repeating the same analysis with a deliberately excited resonance of 0.001 we obtain the following table.

$^b$ The corrector excitations on successive turns serve as the filter coefficients of a FIR filter, with which the BPM signal is analyzed.
TABLE 3: The relative resonance excitations as a function of the BPM resolution for an excitation of $\Delta_c=0.001$.

<table>
<thead>
<tr>
<th>BPM error</th>
<th>rel. $\sigma_c$</th>
<th>rel. $\sigma_s$</th>
<th>rel. $\Delta_c$</th>
<th>rel. $\Delta_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ideal, abs.</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>ideal, rel.</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0 $\mu$m</td>
<td>0.052</td>
<td>-0.032</td>
<td>0.998</td>
<td>0.027</td>
</tr>
<tr>
<td>10 $\mu$m</td>
<td>-0.164</td>
<td>-0.348</td>
<td>0.909</td>
<td>0.159</td>
</tr>
<tr>
<td>20 $\mu$m</td>
<td>-0.254</td>
<td>-0.473</td>
<td>0.817</td>
<td>0.211</td>
</tr>
<tr>
<td>30 $\mu$m</td>
<td>-0.299</td>
<td>-0.533</td>
<td>0.756</td>
<td>0.235</td>
</tr>
<tr>
<td>40 $\mu$m</td>
<td>-0.324</td>
<td>-0.567</td>
<td>0.715</td>
<td>0.249</td>
</tr>
<tr>
<td>50 $\mu$m</td>
<td>-0.340</td>
<td>-0.588</td>
<td>0.687</td>
<td>0.258</td>
</tr>
</tbody>
</table>

Here we only have a two-fold rejection of the “wrong” resonance parameter with 20 $\mu$m BPM resolution. Finally we report the analysis of a realistically perturbed LHC optics with corrected systematic dipole skew quadrupole excitation and added random skew quadrupoles which are not corrected. The resonance parameters for this optics are $\sigma_c = 0.025$, $\sigma_s = 0.001$, $\Delta_c = -0.004$, and $\Delta_s = 0.022$. We see that the larger resonance parameters are found irrespective of the BPM resolution, but that the weaker excited resonance parameters are only found with more accurate BPM. The very weakly excited $\sigma_s$ phase of 0.001 is not discernible, but the order of magnitude of the $\Delta_c$ phase is.

The wobbling of horizontal correctors will not generate emittance growth, because the beam sees only the 60 Hz signal, sampled at the revolution frequency of 11 245 Hz which will generate 60 Hz sidebands of the revolution frequency. Emittance growth, on the other hand, is caused by frequencies within the tune spread of the beam, which is around 3 kHz for LHC.$^c$

TABLE 4: The relative resonance excitations as a function of the BPM resolution for a realistically perturbed optics.

<table>
<thead>
<tr>
<th>BPM error</th>
<th>rel. $\sigma_c$</th>
<th>rel. $\sigma_s$</th>
<th>rel. $\Delta_c$</th>
<th>rel. $\Delta_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ideal, abs.</td>
<td>0.025</td>
<td>0.001</td>
<td>-0.004</td>
<td>0.022</td>
</tr>
<tr>
<td>ideal, rel.</td>
<td>0.750</td>
<td>0.040</td>
<td>-0.114</td>
<td>0.651</td>
</tr>
<tr>
<td>0 $\mu$m</td>
<td>0.754</td>
<td>0.031</td>
<td>-0.099</td>
<td>0.648</td>
</tr>
<tr>
<td>10 $\mu$m</td>
<td>0.748</td>
<td>0.016</td>
<td>-0.087</td>
<td>0.658</td>
</tr>
<tr>
<td>20 $\mu$m</td>
<td>0.741</td>
<td>-0.001</td>
<td>-0.075</td>
<td>0.667</td>
</tr>
<tr>
<td>30 $\mu$m</td>
<td>0.734</td>
<td>-0.016</td>
<td>-0.062</td>
<td>0.676</td>
</tr>
<tr>
<td>40 $\mu$m</td>
<td>0.726</td>
<td>-0.032</td>
<td>-0.050</td>
<td>0.685</td>
</tr>
<tr>
<td>50 $\mu$m</td>
<td>0.718</td>
<td>-0.048</td>
<td>-0.038</td>
<td>0.693</td>
</tr>
</tbody>
</table>

$^c$The author is grateful to L. Vos for clarifying discussions regarding this point.
We conclude that the presented method will be useful to diagnose resonance parameters on the order of 0.005 with wobbling horizontal correctors on the order of 2 μrad and using BPM with a resolution of about 20 μm if all elements are positioned where the beta functions are on the order of 150 m or larger. Moreover, we conclude that coupling correction algorithms which minimize resonance parameters are operationally possible.

5 TRACKING STUDIES FOR LHC

In this section we analyze the effect of the random skew quadrupolar component of the dipoles and different correction algorithms on the 4-dimensional dynamic aperture which corresponds to tracking with $\Delta p/p = 0$. Since we are dealing with transverse imperfections and are mainly interested in relative improvements this should be sufficient. Extending this study to include longitudinal phase space is desirable, but requires considerable extra effort and computer capacity. For the present study we prepare a LHC, version 2 lattice with injection optics, include chromaticity sextupoles, and add random sextupole components to the dipoles of integrated strength with mean zero and $b_{3, rms} = 0.882 \times 10^{-4}$. The dynamic aperture as well as the resonance parameters are calculated at IP1, where both design beta functions are 8 m. The dynamic apertures of both configurations are shown in the following table.

<table>
<thead>
<tr>
<th>horiz. dyn. ap.</th>
<th>vert. dyn. ap.</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>21.9 mm</td>
<td>13.7 mm</td>
<td>chromaticity sextupoles only</td>
</tr>
<tr>
<td>9.6 mm</td>
<td>6.9 mm</td>
<td>random sextupoles added to dipoles</td>
</tr>
</tbody>
</table>

The dynamic aperture with the random sextupoles drops by about a factor of two and we also observe that the vertical dynamic aperture is about 35% smaller than the horizontal.

To the such prepared optics we then add the systematic skew quadrupole gradient and the random component with different strengths to the dipoles as given in the following equation

$$a_2 = 0.770 \times 10^{-4} \pm n \times 1.227 \times 10^{-4}.$$  

We use $n$ between one and five and prepare 10 different seeds for the random part of $a_2$ which is scaled up, such that we obtain 10 different machines with the same systematic $a_2$ but the random part is scaled by $n$ for a total of 50 different optics.

In order to estimate the severity of the effect we calculate the dynamic aperture for the 10 seeds with $n = 3$ and find that their average is on the order of 2 mm, but in 8 out of those seeds the dynamic aperture is less than 1 mm. Clearly, some correction method is needed.

The different optics are corrected by minimizing the resonance parameters calculated to third order with various skew quadrupole control knobs we describe below:

1. SYSM.ARC: This knob is constructed by using four consecutive skew quadrupoles next to an IP and zeroes the first order resonance coefficients of the difference resonance $\Delta_c$ and $\Delta_s$ due to the systematic skew in the dipoles for the adjacent arc.
2. **SYSM1P**: This knob is the superposition of 16 **SYSM_ARC** knobs and zeroes the difference resonance for the entire LHC to first order.

3. **COSP, S1NP, COSM, S1NM**: These knobs use four consecutive skew quadrupoles in an arc and affect $\sigma_c$, $\sigma_s$, $\Delta_c$, and $\Delta_s$, only. There are 16 knobs each.

4. **COSP, S1NP, COSM, S1NM**: These knobs are superpositions of the 16 **COSP, S1NP, COSM, S1NM** knobs and affect only $\sigma_c$, $\sigma_s$, $\Delta_c$, and $\Delta_s$, respectively.

Note that in the above knob calculation we assume that all skew quadrupoles can be controlled independently and each has its own power supply.\(^d\)

We then use a minimization algorithm we call **COUPFIX** to decouple the machine. This minimization algorithm applies given knobs (typically **COSP**, etc) to zero the resonance parameters which are calculated to third order and re-calculates those resonance parameters. The procedure iterates until the resonance parameters are smaller than $10^{-5}$.

In the first attempt we correct the 50 optics by first applying the **SYSM1P** knob to cancel the first order resonance parameters of the difference resonance and the use the **COUPFIX** algorithm with the **COSP, S1NP, COSM, S1NM** knobs to cancel all four

---

\(^d\) Hooking up all QSkn in series, which requires only four power supplies we find that the required skew quadrupole excitations are very large and lead to the generation of higher order coupling. We therefore abandon these knobs.
resonance parameters to third order. Note that no extra care was taken to correct the
tune which after correction returned to its design value of 68.28/68.31 to within 0.01,
with few exceptions. The dynamic apertures of all 50 optics are shown in Figure 1.
The asterisks show the dynamic apertures of the 50 seeds and the solid line shows the
average over the ten different seeds. The vertical error bars depict the spread over the
same ten seeds. We see that there is a small drop of less than 1 mm in both horizontal
and vertical dynamic aperture when the skew quadrupole component is added, compared
to the uncoupled optics, shown on Figure 1 as the point at zero. Furthermore we see that
two or three times the random skew can be corrected without further reducing the dynamic
aperture or increasing the spread between the seeds unduly. However, if the random skew is
increased to four or five times its normal rms we experience a further reduction in dynamic
aperture and an increase in spread between the seeds, which is more prominent in the
vertical plane.

In order to show the quality of the correction we display the beta function and the
dispersion for the uncoupled optics in Figure 2 and a strongly perturbed seed with five times
the nominal random $a_2$. Note that the tunes and the widths of the difference resonance are
displayed above the graphs and that the tunes between both graphs agree to within 0.02.
The resonance widths $\Delta Q$ reported is $1.2 \times 10^{-3}$, which is a residual due to fourth and higher
order, because the third order is corrected to better than $10^{-5}$. The horizontal beta beat at the
QF and the vertical beta beat at the QD quadrupoles in the arcs is about 20 m rms with about
100 m peak-to-peak variation for this seed. An important point to note is the drastically
perturbed vertical dispersion, which is of the same magnitude as the horizontal dispersion
and is mainly caused by the correction skew quadrupoles because they are placed in the
dispersion suppressors. This may necessitate moving the skew quadrupoles to a dispersion­
free section or adding extra ones to cancel the vertical dispersion. In the present study the
effect on the dispersion is not visible, because the code is 4-dimensional only. The coupling
angle $\eta$ meanders around 20 degree.

In a second attempt we only correct the resonance parameters using the COUPFIX
algorithm to first order (e.g. only a single iteration of COUPFIX) and display the resulting
dynamic aperture as dashed lines and crosses in Figure 1. There is only a small degradation
visible compared to the solid line from the third order correction. We conclude that a first
order correction is likely to be sufficient, provided the right knobs are used.

In a third round we investigate the effect of localizing the correction in a single IR, instead
of using a spread out correction scheme as the one described in the previous paragraph.
Now we use only correction skew quadrupoles in (arbitrarily chosen) IR4 and construct
the $\text{COSP}$, etc knobs from them and correct in the same fashion as before. The resulting
dynamic aperture is shown in Figure 3 which shows that one or two times the nominal
random skew can be corrected, but larger excitations will reduce the dynamic aperture
significantly. The reason for this behavior is the abundant generation of higher orders due
to the required stronger excitation of the correcting skew quadrupoles. In the following
table we show the average magnitude of the residual difference resonance as calculated by
Equation 4 averaged over the 10 seeds as a function of the random skew.

---

$^e$ The author is grateful to A. Verdier for pointing this out.
FIGURE 2: The beta function and the dispersion of the uncoupled LHC, version 2 injection optics (above) in arc 2 and a seed with five times the nominal random $\alpha_2$. The upper graph shows the horizontal and vertical beta functions as solid and dashed lines and the lower one shows the respective dispersions. The middle graph shows the coupling angle.\textsuperscript{11}
FIGURE 3: The horizontal and vertical dynamic aperture as a function of the random skew excitation after correction with the SYSMIP knob and minimizing the resonance parameters to third order with the C0UFPFX algorithm using knobs in IR4 only. The average of the third order global C0UFPFX correction depicted by asterisks in Figure 1 is shown as dashed line.

<table>
<thead>
<tr>
<th>random $a_2$ of dipoles in units of $1.227 \times 10^{-4}$</th>
<th>rms resonance widths global correction</th>
<th>rms resonance widths correction in IR4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5.1 \times 10^{-6}$</td>
<td>$7.5 \times 10^{-6}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.7 \times 10^{-5}$</td>
<td>$6.3 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>$5.9 \times 10^{-5}$</td>
<td>$4.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.7 \times 10^{-4}$</td>
<td>$7.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>5</td>
<td>$4.2 \times 10^{-4}$</td>
<td>$2.4 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Clearly the average residual difference resonance is much stronger if the correction is only done in IR4 as opposed to a correction spread out over the entire machine. We conclude that a local correction is not advantageous, because the required correction skew quadrupole strengths are larger and give rise to higher order coupling, which then diminishes the dynamic aperture.

We conclude that random skew excitations of the dipoles in LHC can be reasonably well corrected up to three times the nominal random $a_2$ if all power correction skew quadrupoles have independent power supplies. It is mandatory to have small skew quadrupolar correction strengths which implies the use of many distributed skew quadrupoles in order to avoid higher order coupling. The latter requirement mandates an intelligent placement of the
correction skew quadrupoles as discussed in Section 3. Note, however, that for practical purposes other correction schemes which require less independent power supplies, should be investigated.

6 RAMP SIMULATION WITH COUPLING FEEDBACK

We are now in a position to simulate the behavior of LHC during the ramp and the effect of a feedback system constructed from the control knobs described in Section 3 and the diagnostic system, described in Section 4. For the sake of simplicity we assume that the normalized gradients (K-values) of all magnets are constant during the ramp, except the skew quadrupole components in the dipoles, which will follow a temporal profile as shown in Figure 4 which is given by the equation

\[ P(t) = \frac{t^2}{t^2 + T_1^2} e^{-t/T_2} \]  \hspace{1cm} (29)

where \( T_1 \) is a variable to control the rise time at the beginning of the ramp and \( T_2 \) governs the decay of the \( a_2 \)-perturbation through the ramp. In the simulation each one of the 1280 full-length dipoles is assigned the magnitude of the perturbing \( a_2 \) and individual time constants \( T_1 \) and \( T_2 \) which are typically chosen to be 150 s and 300 s, respectively.\(^\text{f}\) In the code \( P(t) \)

\(^{f}\) One may argue that introducing a spread in \( T_1 \) and \( T_2 \) among the different magnets alleviates the severity of the perturbation, because different magnets reach their peak \( a_2 \) at different times and thus smear out the peak. This effect, however, is found to be very small and we use constant \( T_{1/2} \) in all simulations.
is normalized such that its peak value is unity. The simulation then performs the following things typically once per second for 500 seconds:

- Update the skew quadrupolar errors in the dipoles using the time profile shown in Figure 4.
- Calculate the eigen-tunes, resonance width $\Delta Q$ as shown in Equation 4, and the 200-turn dynamic aperture.
- Estimate the response coefficients $C_{ij}$ from tracking the linear lattice for 1000 turns using two horizontal orbit correctors which are wobbled by about 60 and 80 Hz and two vertical BPM with 20 $\mu$m random error.
- Using the $C_{ij}$ to calculate the resonance parameters $\sigma_c$, $\sigma_s$, $\Delta_c$, and $\Delta_s$.
- Apply the COSP, SINP, COSM, SINM knobs to cancel the resonance parameters. In the simulation we only apply 70% of the correction in order to simulate a conservative feedback gain.
- After the correction is applied the tunes, $\Delta Q$, the dynamic apertures, and the resonance parameters are re-calculated for diagnostic purposes.

A simulation run without dynamic aperture calculation takes a few minutes on a fast workstation and about a day with dynamic aperture calculations.
The seed used for the simulation is one of those presented in Figure 1 as having three times the nominal random skew quadrupole component. The tunes, resonance width $\Delta Q$, and the dynamic apertures of this seed before and after static correction used for Figure 1 are shown in the following table.

<table>
<thead>
<tr>
<th>quantity</th>
<th>uncorrected</th>
<th>corrected</th>
<th>uncoupled</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_x$</td>
<td>0.22815</td>
<td>0.27960</td>
<td>0.27999</td>
<td></td>
</tr>
<tr>
<td>$Q_y$</td>
<td>0.36696</td>
<td>0.31502</td>
<td>0.30996</td>
<td></td>
</tr>
<tr>
<td>$\Delta Q$</td>
<td>0.14397</td>
<td>0.00001</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>Dyn Ap X</td>
<td>–</td>
<td>0.00871</td>
<td>0.00961</td>
<td>m</td>
</tr>
<tr>
<td>Dyn Ap Y</td>
<td>–</td>
<td>0.00478</td>
<td>0.00694</td>
<td>m</td>
</tr>
</tbody>
</table>

We see that this seed is indeed perturbed strongly by the added random skew quadrupoles, e.g. the tunes are moved by 0.05 and the width of the difference resonance $\Delta Q$ is on the order of 0.140. After the correction reported in Section 5 the tunes are within 0.005 of the nominal tunes of 0.280 and 0.310 and $\Delta Q$ is virtually zero. The horizontal dynamic aperture is close to that of the uncoupled optics, but the vertical is reduced by 30% from the uncoupled value. We now show that the feedback system is capable of achieving the same quality of correction as the static correction.

Figure 5 shows the results of a simulation run over 500 seconds. The bottom graph shows the rms excitation of the skew quadrupole component in 1280 dipoles in arbitrary units. Note that the curve starts at a non-zero value, which is due to the systematic component. The top two graphs show the tunes, of which the horizontal is kept to within 0.001 of the nominal tune. The vertical tune varies by 0.005. Note that the worse tunes agree quite well with the corrected tunes of the static correction shown in above table. The third graph shows $\Delta Q$ which is also corrected quite well to better than 0.005 in agreement with the estimated accuracy reported in Section 4. The fourth graph, labeled DA X displays the 200-turn horizontal dynamic aperture which varies only weakly around 9 mm of $\pm 0.5$ mm. The fifth graph labeled DA Y shows the vertical dynamic aperture which does vary through the ramp in correlation with the rms excitation of the skew quadrupole excitation. Note, however, that the minimum corresponds to about 5 mm which is in good agreement with the achievable dynamic aperture in the static correction reported in the above table.

We conclude that a feedback system such as the one described in this section is capable to correct the coupling in real time to a level that guarantees the dynamic apertures to maintain reasonable levels and be comparable to static correction schemes.

7 CONCLUSION AND OUTLOOK

We developed a method to systematically calculate the effect of skew quadrupoles on the sum and difference coupling resonance parameters perturbatively up to third order in a Hamiltonian framework. Four resonance coefficients, which appeared quite natural in the
Hamiltonian analysis closely resemble the zeroth harmonics of the conventional coupling parameters. Using these concepts we were able to design knobs, i.e. linear combinations of skew quadrupoles excitations that affect one particular resonance parameter but not the others. As an added benefit we found rules for the placement of the skew quadrupoles that avoid the generation of higher order contributions to the resonance parameters. An advantageous way of placing the skew quadrupoles is to have 90 degree horizontal and vertical phase advance between two of the skew quadrupoles, 45 degree in both planes to the next skew quadrupole and a phase split of $\phi_x - \phi_y = 90$ degree to the fourth skew quadrupole. Moreover we presented a method to measure the resonance parameters non-invasively. In LHC, using a sinusoidal excitation with an amplitude of $2 \mu$rad of two horizontal orbit corrector at two different frequencies between 50 and 100 Hz and assuming BPM resolutions of 20 $\mu$m of two monitors used by this method we could discern relative resonance parameter magnitudes on the order of $10^{-3}$ if data over 1000 turns were utilized. The high resolution is a consequence of the data analysis which mixes the exciting signal for the correctors with the signal from the BPM and thus allows a very sensitive discrimination of the beam’s oscillation at that frequency. We then utilized the devised correction methods to correct a LHC, version 2 injection optics that was perturbed by systematic skew errors of the dipoles and random skew errors with an rms of one to five times the nominal errors with 10 seeds each. We found that the vertical dynamic aperture is typically 35 % smaller than the horizontal. Moreover, placing correction skew quadrupoles according to the above rules and all 64 of them powered independently (eight per insertions) we could maintain equal dynamic apertures with one, two, or three times the nominal random skew error in the dipoles for all seeds. An increased spread and a slight reduction among the 10 seeds was visible for larger random errors. It turned out that correction schemes that use skew quadrupoles distributed all over the ring cause less degradation of the dynamic aperture than those using only a few localized correction skew quadrupoles.

The knobs constructed in Section 3 can be combined with the diagnostic method of Section 4 to form a feedback system that corrects the temporally varying coupling during the ramp dynamically. A simulation code was written to analyze the performance of such a feedback system in the presence of BPM errors with the skew quadrupolar contents of the dipoles varying in a realistic way. It turned out that the feedback system is capable to correct the coupling in real time to a level that guarantees the dynamic apertures to maintain reasonable levels and be comparable to static correction schemes.

The non-invasive resonance parameter detection can be generalized to analyze other non-linear resonances because the non-linear elements in a circular accelerator act as a frequency mixer for the exciting low-frequency orbit corrector wobbling frequency. Thus the BPM signal mixed with a signal of twice the exciting frequency contains the information about resonance parameters of third order resonances. Possibly this can be generalized to even higher order. A careful analysis of achievable accuracies is mandatory and will be undertaken soon.

ACKNOWLEDGEMENTS

I am grateful to J.-P. Koutchouk for suggesting this problem and many interesting discussions.
REFERENCES


APPENDIX A: RESONANCE PARAMETERS AND RESPONSE COEFFICIENTS

In this appendix we derive the relation between resonance parameters $\sigma_c$, $\sigma_s$, $\Delta_c$, $\Delta_s$ and the orbit response coefficients $C^{ij}$ between a corrector labeled $j$ and a BPM labeled $i$. Therefore we have to make a connection between the Hamiltonian framework, presented in Section 2 and transfer matrices. We do so by considering the transfer matrix of a single skew quadrupole which we write as

$$ S = 1 - \frac{1}{f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 1 - \frac{1}{f} \begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix} $$ \hspace{1cm} (30)

and project its effect to the end of the beam line. We define $G$ as the upper right two by two block of the matrix in the previous equation. The full transfer matrix of the skew quadrupole and the rest of the machine is given by

$$ \begin{pmatrix} R_x & 0 \\ 0 & R_y \end{pmatrix} S = \begin{pmatrix} 0 & R_x GR_y^{-1} \\ R_y GR_x^{-1} & 0 \end{pmatrix} \begin{pmatrix} R_x & 0 \\ 0 & R_y \end{pmatrix} \hspace{1cm} (31) $$
Using $R_x$ and $R_y$ from Equation 18 we can evaluate the expressions in the previous equation and get

$$R_x G R_y^{-1} = \sqrt{\beta_x \beta_y} \begin{pmatrix} \sin(\tilde{\phi}_x) \cos(\tilde{\phi}_y) & -\sin(\tilde{\phi}_x) \sin(\tilde{\phi}_y) \\ \cos(\tilde{\phi}_x) \cos(\tilde{\phi}_y) & -\cos(\tilde{\phi}_x) \sin(\tilde{\phi}_y) \end{pmatrix}$$

$$= \frac{1}{2} \sqrt{\beta_x \beta_y} \begin{pmatrix} \sin(\tilde{\phi}_x - \tilde{\phi}_y) + \sin(\tilde{\phi}_x + \tilde{\phi}_y) \\ \cos(\tilde{\phi}_x - \tilde{\phi}_y) + \cos(\tilde{\phi}_x + \tilde{\phi}_y) \end{pmatrix}$$

$$- \cos(\tilde{\phi}_x - \tilde{\phi}_y) + \cos(\tilde{\phi}_x + \tilde{\phi}_y)$$

$$\sin(\tilde{\phi}_x - \tilde{\phi}_y) - \sin(\tilde{\phi}_x + \tilde{\phi}_y)$$

Now we can compare coefficients with Equation 24 and rewrite Equation 32 in terms of resonance parameters with the result

$$-\frac{1}{f} R_x G R_y^{-1} = \pi \begin{pmatrix} \delta \Delta_s + \delta \sigma_s & \delta \Delta_c - \delta \sigma_c \\ -\delta \Delta_c - \delta \sigma_c & \delta \Delta_s - \delta \sigma_s \end{pmatrix}$$

$$-\frac{1}{f} R_y G R_x^{-1} = \pi \begin{pmatrix} -\delta \Delta_s + \sigma_s & \delta \Delta_c - \delta \sigma_c \\ -\delta \Delta_c - \sigma_c & -\delta \Delta_s - \delta \sigma_s \end{pmatrix}$$

The effect of many skew quadrupoles can then be determined from the product of the terms coming from the individual skew quadrupoles each of which looks like

$$\hat{G} = 1 + \pi \begin{pmatrix} 0 & 0 & \delta \Delta_s + \delta \sigma_s & \delta \Delta_c - \delta \sigma_c \\ 0 & 0 & -\delta \Delta_c - \delta \sigma_c & \delta \Delta_s - \delta \sigma_s \\ -\delta \Delta_s + \sigma_s & \delta \Delta_c - \delta \sigma_c & 0 & 0 \\ -\delta \Delta_c - \sigma_c & -\delta \Delta_s - \delta \sigma_s & 0 & 0 \end{pmatrix}$$

To first order this amounts to simply adding the off (block-) diagonal matrix elements due to the individual skew quadrupoles. Note that the matrix in Equation 34 is only symplectic to first order in the resonance parameters. It is easy to see that symplecticity is broken in second order, unless $\Delta_c^2 + \Delta_s^2 = \sigma_c^2 + \sigma_s^2$.

The presented analysis shows that the off (block-) diagonal matrix elements of the transfer matrix through a coupled beam line are proportional to the resonance parameters. We choose the reference point where the resonance parameters are calculated in normalized phase space. This restriction can easily be remedied by mapping the transfer matrix back into real space. Then a general coupled full turn matrix can be written as

$$R = \begin{pmatrix} A_x^{-1} & 0 \\ 0 & A_y^{-1} \end{pmatrix} \hat{G} \begin{pmatrix} A_x & 0 \\ 0 & A_y \end{pmatrix} R_0 = \left[ 1 + \begin{pmatrix} 0 & \hat{G}_1 \\ \hat{G}_2 & 0 \end{pmatrix} \right] R_0 = [1 + \hat{G}] R_0$$
where $A_x$ is the $2 \times 2$ matrix containing the beta functions in Equation 18 and $R_0$ is the uncoupled one-turn map and $\tilde{G}_{1/2}$ are non-zero in the off-diagonal blocks, which are proportional to the resonance parameters.

Now we want to calculate the response coefficient for a beam line in which a corrector and BPM are situated at the beginning of the beam line with no coupling elements placed in-between. The response coefficient matrix $C$ can be written as

$$C = R_B (1 - R)^{-1} R_c^{-1}$$  \hspace{1cm} (36)

where $R_c$, $R_B$ are the transfer matrices from the start of the beam line to the corrector and BPM, respectively. Inserting $R$ from Equation 35 and some algebraic manipulations we get

$$C = R_B (1 - R_0 - \tilde{G} R_0)^{-1} R_c^{-1} = R_B \left[ (1 - R_0)(1 - (1 - R_0)^{-1} \tilde{G} R_0) \right]^{-1} R_c^{-1}.$$  \hspace{1cm} (37)

In the limit of small coupling the matrix elements of $\tilde{G}$ are small and we can rewrite the previous equation as

$$C = R_B \left[ 1 + (1 - R_0)^{-1} \tilde{G} R_0 \right] (1 - R_0)^{-1} R_c^{-1}$$  \hspace{1cm} (38)

$$= R_B (1 - R_0)^{-1} R_c^{-1} + R_B (1 - R_0)^{-1} \tilde{G} R_0 (1 - R_0)^{-1} R_c^{-1}.$$

The first term in the square brackets yields the response coefficients without coupling and the second term the off (block-) diagonal response coefficients due to coupling, which, in the approximation of weak coupling, are linear in the resonance parameters.

In the main body of this report, however, we did not use this method to calculate the matrix in Equation 27, but excited four independent skew quadrupoles, calculated the response coefficients and solved for the matrix coefficients.

**APPENDIX B: TRACKING AND DYNAMIC APERTURE**

The software used to determine the dynamic aperture consists of a conversion program that reads a MAD SURVEY file and converts it to a format very similar to the one used in TRANSPORT with (at least) one line per element. Each line contains a code of the element, the length, the strength, a second parameter, the energy and a name-string. Such a file (typically about 10000 lines for LHC) is read into memory and all transfer matrices from the beginning of the beam line to any element in the beam line are calculated. This allows very fast subsequent evaluation of transfer matrices and response coefficients. Assigning errors to elements is usually done once and the optics is saved as another ASCII file that can be used at a later time, or modified in any other way. In this way it is guaranteed that the same seeds are used at different times. Once a given file contains all required errors and corrections (which are just the strengths of the correction elements, after all) it is read again and the transfer matrices are calculated. Then a lumping process is started which detects
non-linear elements, stores their positions, and stores the linear transfer matrix and the
centroid shift due to misalignments. For LHC we typically have 1500 lumps. The tracking
then simply consists of applying the non-linear kick and multiplication with the lumped
transfer matrix. This speeds up tracking considerably.

We set up two FORTRAN functions that have the $x-$ or $y-$coordinate as input variable
and returns the number of survived turns as negative number or, if all required turns are
survived, that number as positive number. These functions are fed to a bisection algorithm
to determine the point where the function changes sign which is interpreted as the dynamic
aperture. In this way we evaluate the horizontal and vertical dynamic aperture separately.
For the horizontal the $x-$coordinate is varied while $x', y,$ and $y'$ are zero at the beginning
of the tracking while for the vertical dynamic aperture we find vary the $y-$coordinate and
keep $x, x'$, and $y'$ at zero.