INTRA-BUNCH MOTION

E. Métral

Abstract

Impedance-driven coherent beam instabilities are usually studied analytically with the linearised Vlasov equation, ending up with an eigenvalue system to solve. The eigenvalues describe the mode-frequency shifts, leading in particular to the Transverse Mode-Coupling Instability (TMCI) intensity threshold in the absence of chromaticity. This can be directly compared to measurements in particular for the lowest modes and in the absence of tune spread. Another important observable is the intra-bunch motion, which can be also accessed analytically thanks to the eigenvectors. The different regimes, below-at-above TMCI, are described and represented using a simple analytical model, which helps to really understand what happens at each step.
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Impedance-driven coherent beam instabilities are usually studied analytically with the linearised Vlasov equation, ending up with an eigenvalue system to solve. The eigenvalues describe the mode-frequency shifts, leading in particular to the Transverse Mode-Coupling Instability (TMCI) intensity threshold in the absence of chromaticity. This can be directly compared to measurements in particular for the lowest modes and in the absence of tune spread. Another important observable is the intra-bunch motion, which can be also accessed analytically thanks to the eigenvectors. The different regimes, below-at-above TMCI, are described and represented using a simple analytical model, which helps to really understand what happens at each step.

INTRODUCTION
The intra-bunch motion for independent longitudinal or transverse modes (i.e. at “low intensity”) has been explained analytically already several decades ago in Ref. [1], and it has been observed and confirmed in many machines and simulation codes. These intra-bunch signals correspond to standing-wave patterns with the number of nodes linked to the radial mode number and with, in particular, a left/right (head/tail) symmetry. Therefore, how can we understand theoretically Fig. 1 (and in particular the left/right asymmetry), which has been obtained through single-bunch PyHEADTAIL [2] simulations with the CERN SPS transverse impedance model [3]?

Figure 1: Simulated (with PyHEADTAIL) transverse single-bunch instability with the SPS transverse impedance model and a chromaticity $Q' = 3$. The amplitude of the instability is plotted over time with respect to the centre of the bucket over 2000 consecutive turns. Courtesy of M. Beck [3].

Figure 2: Several pictures from Ref. [5] vs. bunch intensity where it can be clearly observed how both modes 0 and -1 evolve with intensity for zero chromaticity: (a) low-intensity (well below TMCI); (b) just below TMCI; (c) above TMCI; (d) after mode-decoupling. Courtesy of D. Amorim [5].

Recently, the intra-bunch motion in the presence of two modes has been obtained with the DELPHI Vlasov solver [4], revealing different regimes, well below the TMCI...
threshold, close to the TMCI threshold and above the TMCI threshold [5]. As it can be observed on Fig. 2 where several snapshots are depicted, the general case considering the interaction between two (or several) modes is more involved than the low-intensity case as it now depends on the bunch intensity.

Can we understand the observed asymmetries of Fig. 2 (and Fig. 1), the fixed points, the shifts towards the head or the tail, the travelling wave? The purpose of this paper is to explain the different regimes with a simple analytical approach.

**GENERAL APPROACH WITH THE GALACTIC VLASOV SOLVER**

In Ref. [6], the GALACTIC Vlasov solver has been presented and used to describe the destabilising effect of resistive transverse dampers. In this general approach, a decomposition on the low-intensity eigenvectors is used and the problem to solve is described by these two equations (e.g. in the horizontal x-plane)

$$\sigma_x(l) = \sum_{i,j} a_{ij} \sigma_{x,y}(l), \quad \frac{\omega}{\omega_c} a_{ij} = H^x a_{ij},$$  

(1)

where the $\sigma_{x,y}$ are the low-intensity eigenvectors (solutions of the low-intensity eigenvalue problem with constant inductive impedance), $l$ is an integer, $\omega_c$ the (complex) angular betatron frequency, $\omega$ the angular synchrotron frequency and $H^x$ is the matrix to be diagonalised. The eigenvalues give the mode-frequency shifts (with both real and imaginary parts) and the eigenvectors give the coefficients $a_{ij}$ to be used in the equation on the left-hand side to be able to plot the intra-bunch signal (following Ref. [1]).

**APPROXIMATED MODEL FOR THE LOW-INTENSITY HEAD-TAIL MODES**

At low intensity, the same results as in Ref. [1] are obtained with the GALACTIC Vlasov solver, as can be seen in Fig. 3 (left), where the modes 0 and -1 can be treated independently and do not depend on intensity (except that they are only valid at low intensity). As can be seen in Fig. 3 (right), and as discussed by Sacherer in Ref. [7], the intra-bunch signals can be approximated by

$$S_0(t, n) = \cos\left(\frac{\pi l}{\tau_b}\right) \cos(2\pi n Q),$$

(2)

$$S_{-1}(t, n) = \sin\left(\frac{\pi l}{\tau_b}\right) \cos(2\pi n Q),$$

(3)

where $t$ is the time, $n$ the turn number, $\tau_b$ the full (4-sigma) bunch length and $Q$ the transverse tune. These approximated sinusoidal modes will be used below in the simple analytical model.

![Figure 3](image-url)

**SIMPLE MODEL CONSIDERING MODES 0 AND -1 TOGETHER**

Considering the simplified model used in the past to study the destabilising effect of the LHC transverse damper for $Q' = 0$ (deduced from studies with the GALACTIC Vlasov solver) but without damper [6], the following matrix needs to be diagonalised (with $x$ a normalised parameter proportional to the bunch intensity [6])

$$\begin{pmatrix}
-1 & -0.23jx \\
-0.55jx & -0.92x
\end{pmatrix},$$

(4)

where $j$ is the imaginary unit. The related eigenvalues are depicted in Fig. 4 (left), while the eigenvectors are represented in Fig. 4 (right).

![Figure 4](image-url)

Below TMCI, the Intra-Bunch Motion (IBM) is given by

$$IBM(t, n) \propto a_0 S_0(t, n) - a_{-1} S_{-1}(t, n),$$

(5)

with $a_0$ and $a_{-1}$ reals, which are given by the blue curve in Fig. 4 (right) and which therefore depend on the bunch intensity. At very low bunch intensity, the signals of Fig. 3 are recovered and the two modes are independent and not perturbed by the other one. As the bunch intensity increases, mode 0 is more and more perturbed by mode -1 and vice versa.

At the TMCI threshold, the IBM is given by

$$IBM(t, n) \propto a [S_0(t, n) - S_{-1}(t, n)],$$

(6)
with \( a_0 = a_{-1} = a \) real. The signal is zero at both bunch extremities but there is also a fixed point inside the bunch when
\[
\cos\left(\frac{\pi t}{\tau_B}\right) - \sin\left(\frac{2\pi t}{\tau_B}\right) = 0 ,
\]
i.e. when \( t = \tau_B / 6 \): the signal is asymmetric and shifted towards the tail.

Above TMCI, the IBM is given by
\[
IBM(t, n) \propto (a + j b) S_0(t, n) - (a - j b) S_{-1}(t, n)
\]
\[
= \sqrt{a^2 \left[ \cos\left(\frac{\pi t}{\tau_B}\right) - \sin\left(\frac{2\pi t}{\tau_B}\right) \right]^2 + b^2 \left[ \cos\left(\frac{\pi t}{\tau_B}\right) + \sin\left(\frac{2\pi t}{\tau_B}\right) \right]^2} \cos[2\pi n Q + \varphi(t)] ,
\]
with \( a \) and \( b \) reals (deduced from Fig. 4 (right)) and
\[
\varphi(t) = \arctan\left( \frac{b \left[ \cos\left(\frac{\pi t}{\tau_B}\right) + \sin\left(\frac{2\pi t}{\tau_B}\right) \right]}{a \left[ \cos\left(\frac{\pi t}{\tau_B}\right) - \sin\left(\frac{2\pi t}{\tau_B}\right) \right]} \right) .
\]

Due to the latter term (which is coming from the fact that the eigenvectors from Fig. 4 (right) have now both a real and an imaginary part), a travelling wave along the bunch is created: the coupling of two standing waves is a travelling wave, which is another way to see that the bunch is in the TMCI regime.

Once the modes decouple (for \( x = 4.8 \), see Fig. 4), the signal is the symmetric of the one when the two modes couple (for \( x = 0.61 \)). The signal is zero at both bunch extremities but there is also a fixed point inside the bunch when
\[
\cos\left(\frac{\pi t}{\tau_B}\right) + \sin\left(\frac{2\pi t}{\tau_B}\right) = 0 ,
\]
i.e. when \( t = -\tau_B / 6 \): the signal is asymmetric and shifted towards the tail.

Several pictures of the different regimes are shown in Fig. 5.

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**Figure 5:** Several pictures of the different regimes: (a) \( x = 0.1 \); (b) \( x = 0.5 \); (c) \( x = 0.61 \); (d) \( x = 1.0 \); (e) \( x = 4.0 \); (f) \( x = 4.8 \); (g) \( x = 5.9 \).
SIMPLE MODEL CONSIDERING TWO HIGHER-ORDER MODES TOGETHER

A similar approach can be used to study the mode-coupling between two higher-order modes, such as e.g. between modes -2 and -3 and pictures like in Fig. 6 can be obtained, revealing a huge amplification from head to tail with zero growth rate (as recently discussed by A. Burov in the context of convective instabilities with space charge [8]). It is worth mentioning that in this case the three fixed points of the intra-bunch signal can also be computed and are solutions of

\[
\cos \left( \frac{3\pi t}{\tau_B} \right) + \sin \left( \frac{4\pi t}{\tau_B} \right) = 0 .
\]

Figure 6: Example of picture obtained in the presence of mode-coupling between modes -2 and -3.

Another example is shown in Fig. 7 for the case of mode-coupling between modes -9 and -10.

Figure 7: Example of picture obtained in the presence of mode-coupling between modes -9 and -10.

EFFECT OF A RESISTIVE TRANSVERSE DAMPER

Considering the simplified model used in the past to study the destabilising effect of the LHC transverse damper for \( Q^* = 0 \) (deduced from studies with the GALACTIC Vlasov solver) [6], the following matrix needs to be diagonalised

\[
\begin{pmatrix}
-1 & -0.23j x \\
-0.5j x & -0.92 x + 0.48j
\end{pmatrix},
\]

where the term “+0.48 j” is the contribution from the resistive damper with a damping time of 100 turns. The corresponding plots of the eigenvalues and eigenvectors vs. intensity are revealed in Fig. 8, comparing the cases without and with a resistive transverse damper. A similar approach can be adopted and similar results are obtained [9].

![Figure 8: Eigenvalues (top) and eigenvectors (bottom) vs. intensity for the cases (left) without damper (called ADT, corresponding to Eq. (4), see Fig. 4) and (right) with damper (corresponding to Eq. (12)).](image)

EFFECT OF CHROMATICITY

A similar approach can be adopted with non-zero chromaticity and similar results are obtained [10]. The corresponding plots of the eigenvalues and eigenvectors vs. intensity are revealed in Fig. 9.

\[
Q' = 5, \text{ no ADT} \quad Q' = -5, \text{ no ADT}
\]

![Figure 9: Eigenvalues (top) and eigenvectors (bottom) vs. intensity for the cases (left) with \( Q' = 5 \) and (right) with \( Q' = -5 \), both without transverse damper (ADT).](image)

CONCLUSION

The intra-bunch motion, and its main features below, at and above the TMCI intensity threshold, can be explained with a simple analytical model, revealing clearly what happens when the bunch intensity is increased. The pictures of intra-bunch signals obtained with the simple analytical model are very similar to the ones obtained with the DELPHI Vlasov solver (see Fig. 2) in most cases but a difference seems to be observed in the TMCI regime, with the traveling wave propagating along the bunch, which remains to be fully understood.

It was interesting to observe that in some cases a huge amplification factor can be observed from Head to Tail...
with 0 growth rate, as recently discussed by A. Burov in the context of convective instabilities with space charge [8].

This simplified model should help us to better understand some observations in different particle accelerators and in simulations, as e.g. in Fig. 1. Furthermore, can something like this explain some past measurements in CERN PS & PSB (in the presence of strong space charge), as depicted in Fig. 10? Can something like this explain some (parts of) simulations in the presence of electron cloud, as depicted in Fig. 11? This will be investigated in detail in the future.

Figure 10: Example of pictures measured in the CERN PS and PSB in the presence of strong space charge. Courtesy of E. Koukovini Platia for the PSB measurements.

Figure 11: Example of simulated instability in the presence of electron cloud. Courtesy of L. Sabato.

REFERENCES


