ADDENDUM TO THE
1993 EUROPEAN SCHOOL OF HIGH-ENERGY PHYSICS

Zakopane, Poland
12–25 September 1993

PROCEEDINGS
Eds. N. Ellis, M. B. Gavela

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QCD and Deep Inelastic Scattering

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Contents

Introduction, general references and acknowledgements 1
Deep inelastic scattering – a first look 5
Deep inelastic scattering – structure functions $F_i(x, Q^2)$ 9
Naive quark model – for $ep \rightarrow eX$
  – for $\nu N \rightarrow \mu X$
  – sum rules 14
Colour and local SU(3)$_c$ gauge theory (QCD) 17
Coupling constant renormalisation – RGE and the $\beta$ function 29
The QCD-improved parton model
  – mass factorisation and Altarelli-Parisi evolution 37
Global analysis to determine the parton densities 60+
A commentary on small $x$
  – references for small $x$ 61
Use of moments to show Altarelli-Parisi equation sums
  leading $\log Q^2$ 64
Small $x$ and the double leading logarithm approximation 67
BFKL or Lipatov equation 72
Parton shadowing and the GLR equation 76
Applications of the BFKL equation 83
Conclusions
Introduction

The use of energetic electrons (and other leptons) to probe the structure of hadronic matter has had a long and extremely successful history. We begin with a quick pictorial review and then introduce the Quark Parton Model with its prediction of scaling for deep inelastic $ep \rightarrow eX$ and $\nu N \rightarrow \mu X$ scattering. Here by “scaling” we mean that the structure functions $F_i(x, Q^2)$ depend only on the Bjorken variable $x$, and not on $Q^2$. A study of the QCD-improved parton model is preceded by an introductory discussion of QCD, renormalisation and the running of the coupling constant $\alpha_s$. We indicate how the running of $\alpha_s$, governed by the universal beta function, can, in principle, sum all the logarithmic ultraviolet divergences associated with the quark-gluon vertices. In an analogous way we indicate how universal running parton densities, governed by Altarelli-Parisi evolution equations, can, in principle, sum all the logarithmic initial state collinear singularities. The analogy is summarised at the bottom of transparency 30.

We show how well the parton densities are determined by current experiments, using the MRS analysis as an example. We then turn to the perturbative QCD expectations in the small $x$ region – a regime which has recently become experimentally accessible with the advent of HERA, the electron-proton collider at DESY, Hamburg.

The viewgraph transparencies used in the lectures are reproduced here, together with a few typewritten commentaries to help the reader. Sometimes you will see a reference “HM” to the book “Quarks and Leptons” by F. Halzen and A.D. Martin (Wiley 1984) where further discussion can be found. This should not be taken to imply that this is the best reference but merely that it is the one that I had most easily to hand. More specialized reviews on QCD and Deep Inelastic Scattering can be found in

R.D. Field, Applications of Perturbative QCD, Addison Wesley (1989)
Z. Kunszt, Perturbative QCD, Proc. of Boulder Theoretical Advanced Study
Institute, 1990, eds. M. Cvetić and P. Langacker, World Scientific, p. 229


*Handbook of QCD*, computer-retrievable text prepared by the CTEQ collaboration, 1993, G. Sterman et al.

The parton distributions are determined by high precision deep inelastic scattering and related data. For deep inelastic muon scattering, $\mu p \rightarrow \mu X$ and $\mu D \rightarrow \mu X$, the latest data are from the BCDMS collaboration [Phys. Lett. B223 (1989) 485] and NMC [Phys. Lett. B295 (1992) 159]. For neutrino scattering, $\nu N \rightarrow \mu X$, the relevant data are from the CCFR collaboration (to be published). Examples of these data are shown on transparencies 39, 41b, 44-47. More details of the MRS parton analysis described in these lectures can be found in


Between viewgraphs 60 and 61 you will find an extensive commentary on the lectures about small $x$ physics. With the advent of HERA, small $x$ physics has become a “hot topic” and many developments can be expected. A list of some of the relevant small $x$ papers is given at the end of the commentary.
Acknowledgements

I would like to thank many people for enjoyable discussions on the subject of these lectures including Adrian Askew, Dmitri Bardin, Yuri Dokshitzer, Nick Ellis, Victor Fadin, Nigel Glover, Jan Kalinowski, Valery Khoze, Witold Krasny, Jan Kwieciński, Genya Levin, Dick Roberts, James Stirling, Peter Sutton and Bryan Webber. The excellent atmosphere created at the Zakopane School by Michal Turala, Egil Lillestol, Susannah Tracy and all their helpers from Krakow, combined with the enthusiasm of the students, made giving these lectures a very enjoyable duty.
Deep inelastic scattering — a first look

Electron-nucleus scattering:

\[ q = (z, \vec{q}) \quad \text{where} \quad z = \frac{E - E'}{E} \]

\[ p_i = (M_N, \vec{0}) \]

\[ p_f = (M_N + z, \vec{q}) \]

\[ p_f^2 = M_N^2 = (M_N + z)^2 - \vec{q}^2 \]

\[ M_N^2 = M_N^2 + 2M_N z + z^2 - \vec{q}^2 \]

\[ q_i^2 = -Q^2 \]

Scaling variable for el. scatt:

\[ x_N = \frac{Q^2}{2M_N z} = 1 \]

\[ (\frac{Q^2}{2p_i q_i}) \quad \text{"Bjorken" x} \]

The lower picture on the next page shows that with increasing \( Q^2 \) the wavelength of the photon probe becomes small enough to resolve individual protons. We also see that the elastic peak is drastically suppressed with increasing \( Q^2 \), since the chance of the \( N \) spectator nucleons lining up with the outgoing struck proton \( \sim (Q^2)^{-N} \). The action now moves to elastic and incoherent scattering from individual protons, with a peak at \( x_N \approx M_p / M_N \approx 1/N \) (where \( N \) is the number of nucleons in the nucleus) smeared out by the Fermi momentum of the struck proton in the nucleus (see D.H. Perkins, *Introduction to High Energy Physics*, Addison-Wesley, 1982 and F.E. Close, *Quarks and Partons*, Academic Press, 1979).
$Q^2R^2 \ll 1$

pure elastic nuclear scatt.

\[ z' = \frac{Q^2}{2M_n} \]

$Q^2R^2 \sim 1$

excite nuclear states

$Q^2R^2 \gg 1$

resolve protons

\[ z' = \frac{Q^2}{2M_p} \]
As we increase $Q^2$ even further ($Q^2 R_p^2 \gg 1$) the photon resolves the constituent valence quarks of the proton (or neutron) and the whole picture repeats itself a layer down, see the above sketch. Now the plot is drawn in terms of the proton Bjorken scaling variable.

If this historical development were to be repeated yet again, then a further increase of $Q^2$ would (after a period of scaling) eventually allow the photon to resolve the ("preon") constituents of the quarks and scaling would set in again based on a peak at $x \simeq (1/3)(1/N_{preons})$. To date, there is not the slightest evidence for any composite preonic nature of quarks. Instead, with increasing $Q^2$, a different kind of scaling violation occurs, originating from QCD, see the next page. The photon resolves "sea" (as well as "valence") quarks. There now appear to be more partonic constituents to share the energy and momentum of the proton, so more are found at small $x$ and less at large $x$, as indicated by the dot-dashed curve.
QCD scale violations

scattering off "sea" quark

$F_2(x, Q^2)$

scaling
Deep Inelastic Lepton Scatt.

\[ e \rightarrow e^\prime X \]

\[ Q^2 \equiv -q^2 \gg M^2 \quad \text{"deep"} \]

\[ W^2 = (p+q)^2 \gg M^2 \quad \text{"inelastic"} \]

(assume \( Q^2 \ll M^2 \) so \( \gamma \) exchange dominates)

Spin-averaged matrix element squared:

\[
| M \|^2 = \frac{e^4}{Q^4} L_{\mu \nu} W^{\mu \nu} (4\pi M) \]

**LEPTONIC**

\[ L_{\mu \nu} = \frac{1}{2} \sum_{\text{spin}} \left[ \bar{u}(k') \gamma_\mu u(k) \right] \left[ \bar{u}(k') \gamma_\nu u(k) \right]^* \]

\[ = 2 \left( k_\mu k'_\nu + k'_\mu k_\nu - k_\nu k'_\mu - q_\mu q_\nu - q_\mu q_\nu - q'^\mu q'^\nu \right) \quad \text{(HM 6.26)} \]

**HADRONIC**

\[
W^{\mu \nu} = -W_1 g^{\mu \nu} + \frac{W_2}{M^2} p^\mu q^\nu + \frac{W_3}{M^2} q^\mu q^\nu + \frac{W_4}{M^2} (p^\mu q^\nu + q^\mu p^\nu) \quad \text{(HM 8.24)}
\]

Current conserv. \( q_\mu W^{\mu \nu} = q_\nu W^{\mu \nu} = 0 \)

\[ W^{\mu \nu} = (W_1 (-g^{\mu \nu} + \frac{q^{\mu} q^{\nu}}{q^2}) + \frac{W_2}{M^2} (p^\mu - \frac{p \cdot q}{q^2} q^\mu)(p^\nu - \frac{p \cdot q}{q^2} q^\nu) \quad \text{(HM 8.27)} \]

\[ F_1 = M W_1 \]

\[ F_2 = \frac{p \cdot q}{M} W_2 \]
Hadronic vertex: 2 indep. (Lorentz inv.) variables

\[ q^2 = q \cdot q, \quad p \cdot q \quad (p^2 = M^2 \text{ fixed}) \]

or \[ Q^2 = -q^2, \quad x = \frac{Q^2}{2p \cdot q} \quad \text{"Bjorken x"} \]

\( x = \frac{p \cdot q}{M} \)

For (unpolarised) scatt., vertex is described by two structure functions: \( F_1(x, Q^2), F_2(x, Q^2) \)

Using \( l \mu \nu, W^{\mu \nu} \) (and some algebra) (HM 9.23)

\[
\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{xQ^4} \left\{ y^2xF_1 + (1-y)F_2 \right\}
\]

\( N_\text{B} \gg M \)

where \( y = \frac{2p \cdot q}{s} = \frac{Q^2}{xs} \)

\( s = (k+p)^2 \approx 2k \cdot p \)

To measure both \( F_1 \) and \( F_2 \) as functions of \( x, Q^2 \) need expts. at diff. energies \( \sqrt{s} \)

\[ F_2^{\text{FP}} \quad \text{(fixed target)} \]

\[ 0.45 \quad \log Q^2 \]

\[ 0.4 \]

\[ 0.35 \]

\[ 0.25 \]

\[ 0.2 \]

\[ x = 0.07 \]

\[ x = 0.18 \]

\[ x = 0.35 \]

\[ Q^2 = 30 \text{ GeV}^2 \]

\[ F_2^{\text{FP}} \]

\[ 1 \]

\[ 0.5 \]

\[ 0.1 \]

\[ 10^{-4} \]

\[ 10^{-3} \]

\[ 10^{-2} \]

\[ 10^1 \]

\[ x \]

HERA 1

Fixed target
Lines of constant $E_e'$ and $\theta_e$

Good measurement of $x, Q^2$ from $\theta_e, E_e'$ in region where $\theta_e$ and $E_e'$ rapidly change ($x \approx 10^{-3}$)
Lines of constant $E_j$ and $\theta_j$

Information about $x$, $Q^2$ from $\theta_j$ when $x \leq 10^{-2}$
Interpretation?

Naive quark parton model

Basic idea: $p^*p$ interaction at large $Q^2$ can be expressed as sum of incoherent scatt. from point-like quark constituents. Over short time-scale $1/\sqrt{Q^2}$ photon sees state of non-interacting quarks. Final hadronization process occurs long after.

$$\frac{d^2\sigma}{dx dq^2} = \sum_q \int_0^1 dq \ f_q(x) \ \frac{d^2\hat{s}_{eq}}{dx dq^2},$$

where $f_q$ is probability of finding quark in proton with fraction $x$ of its momentum.

$$m_q^2 = (\not{p} + q)^2 = 2p \cdot q - Q^2 = \frac{Q^2}{x} \bar{x} - Q^2,$$

$$\therefore \bar{x} = (1 + \frac{m_q^2}{Q^2}) \approx \bar{x} \text{ for } Q^2 \gg m_q^2$$

fractional mom. $\approx$ Bjorken $x$.

$\bullet$ $ep \rightarrow eX$

$$\text{eq. scatt: } \frac{d\hat{\sigma}_{eq}}{d\hat{S}} = \frac{2\pi \alpha^2 e_q^2}{\hat{S}^2} \left( \frac{\hat{S}^2 + \hat{u}^2}{\hat{E}^2} \right).$$

$$\hat{S} = (xp + k)^2 \approx 2xp \cdot k \approx xS$$

$$\hat{E} = -Q^2 = -xyS$$

$$\hat{u} = -\hat{S} - \hat{E} = -x(1-y)S$$

$$\frac{d\hat{\sigma}_{eq}}{dx dQ^2} = \frac{2\pi \alpha^2 e_q^2}{Q^4} \left\{ 1 + (1-y)^2 \right\} \delta(x-y).$$
Quark parton model

\[ \frac{d^2 \sigma}{dx dq^2} = \frac{4\pi \alpha^2}{x Q^4} \sum_q \int dy f_q(y) e_q^2 \frac{1}{2} \left\{ 1 + (1-y)^2 \right\} \delta(x-y) \]

to be compared with general formula

\[ \frac{d^2 \sigma}{dx dq^2} = \frac{4\pi \alpha^2}{x Q^4} \left\{ y^2 x F_1 + (1-y) F_2 \right\} \]

\[ = \frac{4\pi \alpha^2}{x Q^4} \left\{ \frac{1}{2} y^2 (2xF_1 - F_2) + \frac{1}{2} \left\{ 1 + (1-y)^2 \right\} F_2 \right\} \]

\[ F_2 = 2xF_1 = \frac{\Sigma}{\frac{1}{2}} \sum_q \int dy f_q(y) x e_q^2 \delta(x-y) = \frac{\Sigma}{\frac{1}{2}} e_q^2 x f_q(x) \]

Callan-Gross relation: spin½ quarks

(Spin½ quarks: \( F_1 = 0 \))

Insight into \( y \) dependence

\[ \frac{d\hat{\sigma}}{d\hat{\kappa}} = \frac{2\pi \alpha^2 e_q^2}{\hat{\kappa}^2} \left( \frac{\hat{S}^2 + \hat{\mu}^2}{\hat{S}^2} \right) \]

\[ \hat{S} = \frac{x S}{x} \]

\[ \hat{\kappa} = -\frac{xy S}{x} \]

\[ \hat{\mu} = -x(1-y) S \]

\[ \frac{1}{y} \frac{(1-y)^2}{z} \]

\[ e, q, \text{ same helicity} \]

\[ e, q, \text{ opp. helicity} \]

\[ \text{eq. scatt in c.m. frame} \]

\[ e \rightarrow \frac{\vec{e}}{\vec{\kappa}} \rightarrow \frac{\vec{e}}{q} \rightarrow \frac{\vec{q}}{q} \]

\[ \hat{S} = \frac{4k^2}{(1-\cos\hat{\theta})} \]

\[ \hat{\kappa} = -2k^2 (1-\cos\hat{\theta}) \]

\[ \hat{\mu} = -2k^2 (1+\cos\hat{\theta}) \]

\[ \therefore \hat{y} = \frac{1}{2} (1-\cos\hat{\theta}) \]

\[ \text{eq. scatt in forward scatt} \]

\[ y = 0 \text{ forward scatt} \]

\[ y = 1 \text{ backward scatt} \]

At HE

\( (\text{fermion helicity conserved at gauge boson vertex}) \)

\[ (6.37) \]
Flavour sum rules

proton = uud + q\bar{q} pairs

"valence" quarks  "sea" quarks
(carry q. nos)

When probed at scale Q all flavours \( m_q \leq Q \) are active

Notation:
\[ f_u(x) \equiv u(x) = u_v + u_{\text{sea}} \]
\[ f_{\bar{u}}(x) \equiv \bar{u}(x) = u_{\text{sea}} \]

\[ \int_0^1 (u - \bar{u}) \, dx = \int_0^1 u_v \, dx = 2 \]
\[ \int_0^1 (d - \bar{d}) \, dx = \int_0^1 d_v \, dx = 1 \]

QPM:
\[ F_2 = \sum_q \frac{e_q^2}{4\pi} x f_q(x) \]

\[ F_2^{\text{ep}} = x \left( \frac{4}{9} u + \frac{1}{9} d + \frac{1}{9} s +... + \frac{4}{9} \bar{u} + \frac{1}{9} \bar{d} + \frac{1}{9} \bar{s} +... \right) \]

\[ F_2^{\text{en}} = x \left( \frac{4}{9} d + \frac{1}{9} u + \frac{1}{9} s +... + \frac{4}{9} \bar{d} + \frac{1}{9} \bar{u} + \frac{1}{9} \bar{s} +... \right) \]

\((u \leftrightarrow d)\)

Gottfried sum rule

\[ \int_0^1 \frac{dx}{x} (F_2^{\text{ep}} - F_2^{\text{en}}) = \frac{1}{3} \int dx \, (u - d + \bar{u} - \bar{d}) \]
\[ = \frac{1}{3} \int dx \left\{ (u - \bar{u}) - (d - \bar{d}) \right\} \]
\[ = \frac{1}{3} \left( \int dx \, (u - \bar{u}) \right) \]
\[ = \frac{1}{3} \quad \text{if} \quad \bar{u} = \bar{d} \]

Expt: NMC \[ \int_{0.004}^{0.8} = 0.236 \pm 0.008 \pm 0.014 \]
\[ \Rightarrow \int_0^1 = 0.258 \pm 0.017 \]
\[ 2N \rightarrow \mu X \]

\[ (\nu \bar{\nu}) \quad L_{\mu \nu} \approx G_F^2 \frac{(\nu \bar{\nu})}{M^2} W^{\mu \nu} \]

Parity violation - extra contributions

\[ L_{\mu \nu} = L_{\mu \nu} \pm 2i \epsilon^{\mu \nu \lambda \sigma} k^\lambda k^\sigma \]

\[ W^{\mu \nu} = \ldots - \frac{i}{M} \epsilon^{\mu \nu \lambda \sigma} p_2 q_\sigma W_3 \]

\[ \left( Q^2 < M_W^2 \right) \]

\[ F_3 = \frac{f_3^2}{M^2} W_3 \]

\[ \frac{d^2 \sigma^{\nu \bar{\nu}}}{dx \, dQ^2} = \frac{G_F^2}{\pi \alpha} \left( \frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left\{ y^2 x F_2^{\nu} + (1-y)^2 F_2^{\nu} + 2x \left[ 1 + (1-y)^2 \right] F_2 \right\} \]

\[ = \frac{G_F^2}{\pi \alpha} \left\{ \frac{y^2}{2} (2x F_2^{\nu} - F_2^{\nu}) + \frac{1}{2} \left[ 1 + (1-y)^2 \right] F_2^{\nu} \pm \frac{1}{2} \left[ 1 - (1-y)^2 \right] x F_2^{\nu} \right\} \]

Quark parton model: (assume \( \theta_c = 0 \))

\[ \nu \rightarrow \bar{d} \]

\[ \frac{d \sigma}{dQ^2} (\nu d \rightarrow \mu^- u) = \frac{G_F^2}{\pi} \frac{1}{x} = \frac{d \sigma}{dQ^2} (\bar{\nu} \bar{d} \rightarrow \mu^+ \bar{u}) \]

\[ \bar{d} \rightarrow u \]

\[ \frac{d \sigma}{dQ^2} (\bar{d} u \rightarrow \mu^+ d) = \frac{G_F^2}{\pi} (1-y)^2 = \frac{d \sigma}{dQ^2} (\nu u \rightarrow \mu^+ \bar{d}) \]

as for \( ep \rightarrow eX \)

\[ 2x F_2^{\nu} = F_2^{\nu} \]

\[ F_2^{\nu} p = 2x (d + \bar{u} + s + c) \]

\[ F_2^{\nu}  \bar{p} = 2x (u + \bar{d} + c + \bar{s}) \]

\[ x F_3^{\nu} p = 2x (d - \bar{u} + s - \bar{c}) \]

\[ x F_3^{\nu}  \bar{p} = 2x (u - \bar{d} + c - \bar{s}) \]

For \( \nu n (\bar{\nu} n) \): \( u \leftrightarrow d \) and \( \bar{u} \leftrightarrow \bar{d} \)
For an isoscalar target $N$ assume $s = \bar{s}$.

- $F_2^{\nu N} = \frac{1}{2} (F_2^{\nu p} + F_2^{\nu n}) = x (u + \bar{u} + d + \bar{d} + 2s + 2c)$
- $xF_3^{\nu N} = \frac{1}{2} (xF_3^{\nu p} + xF_3^{\nu n}) = x (u - \bar{u} + d - \bar{d} + 2s - 2c)$
- $F_2^{eN} = \frac{1}{2} (F_2^{ep} + F_2^{en}) = \frac{5}{18} x \left( u + \bar{u} + d + \bar{d} + \frac{4}{5} s + \frac{16}{5} c \right)$

\[ \frac{F_2^{eN}}{F_2^{\nu N}} \simeq \frac{5}{18} \] difference det. $s(x)$?

- $F_2^{ep} - F_2^{en} = \frac{1}{3} x (u + \bar{u} - d - \bar{d})$

4 structure fn. measurements $F_2^{u p}$, $F_2^{u d}$, $F_2^{\nu N}$, $xF_3^{\nu N}$ determine 4 parton combos. If $c(x)$ is neglected $\rightarrow u + \bar{u}$, $d + \bar{d}$, $\bar{u} + \bar{d}$, $s$ (NB: they do not determine five $u, d, \bar{u}, \bar{d}, s$).

Sum rules

1. Adler sum rule
   \[ \int_0^1 \frac{dx}{x} (F_2^{d p} - F_2^{u p}) = 2 \int_0^1 \{(u - \bar{u}) - (d - \bar{d})\} dx = 2 \]
   (Exact: follows from current conserv.)

2. Gross–Llewellyn Smith sum rule
   \[ \int_0^1 \frac{1}{2} (F_3^{u p} + F_3^{d p}) dx = \int (d - \bar{u} + u - \bar{d}) dx = 3 \left( 1 - \frac{x s}{x \bar{s}} + \ldots \right) \]

3. Bjorken sum rule
   \[ \int_0^1 dx (F_1^{u p} - F_1^{d p}) = 1 \left( 1 - \frac{2 \alpha_s}{3 \frac{1}{x} \bar{s} + \ldots} \right) \]

[QPM: $2xF_1 = F_2$]
A bit of history....

Momentum sum rule

\[
\sum_{q} \int_{0}^{1} x (q(x) + \bar{q}(x)) \, dx = 0.5 \quad \text{(not 1)}
\]

First (indirect) evidence for gluon

\[
\int_{0}^{1} x g(x) \, dx \simeq 0.5
\]


Evidence for SU(3) of colour

- Originally need for hadron spectroscopy (1964 on)

\[
\Delta^{++}(J^P = \frac{3}{2}^-) = u^\uparrow u^\uparrow u^\uparrow
\]

\[
\begin{align*}
\text{fermion} & : \Psi_{\text{space}} \Psi_{\text{spin}} \Psi_{\text{flavour}} \\
\text{Antisym} & : \Psi_{\text{sym}} \times \Psi_{\text{colour singlet}}
\end{align*}
\]

- Now "see" colour everywhere e.g. T decay

\[\begin{array}{c}
\text{SU(3)\_c gauge thry} \\
\text{Explain confinement?}
\end{array}\]

branching ratio \( \sim 20\% \quad 20\% \quad 60\% \)
First we indicate how QED emerges from imposing local $U(1)_Q$ gauge invariance on the Lagrangian density of a free electron. ($Q$ is the charge operator.)

Then we extend the ideas and impose non-Abelian $SU(3)_c$ gauge invariance on the Lagrangian describing a triplet of coloured quarks. (The crucial self-coupling of the gluons arises from the non-Abelian nature of $SU(3)$; in particular from the structure constants $f_{abc}$.)

The Feynman rules can be readily deduced from the form of the final $\mathcal{L}$ (HM p 313).

Two questions come to mind. If local gauge invariance requires that the gauge bosons are massless then

(i) why do strong interactions, which are mediated by massless gluons, have short range?

(ii) how can weak interactions be mediated by massive gauge bosons?

These questions are addressed pictorially on the following page entitled “Range of interactions”. The gauge symmetry of weak interactions is “broken” or “hidden” in the ground state, such that the vacuum is full of Higgs bosons which impede the progress of the $W$ bosons (and other massive particles).
Range of Interactions.

QED

\[ e \longrightarrow e \]

long-range

Weak

\[ W^+ \ell^{-} \rightarrow W^+ \ell^{-} \]

Short-range

QCD

\[ \text{gluon} \]

\[ q \rightarrow q \]

"m_q = 0" so long-range? Yes!

No! As no colour charges exist at long range. All colour confined in colour singlet (colour neutral hadrons)

\[ \text{Short-range} \]

(compare the van der Waals force between electrically neutral molecules)
Coupling constant renormalisation

Consider a dimensionless QCD observable $R$

Naive prediction for energies $Q \gg m$:

\[ R \to \text{const. independent of } Q! \]

\[ \uparrow \quad \text{no scale in } \mathcal{L}_{\text{QCD}} \]

Not true in renormalizable field theory, like QCD

So where does second mass scale enter?

When calculate $R = \sum_n c_n \alpha_s^n$ encounter

Feynman diagrams with loops which diverge logarithmically

Need to renormalise (reparametrize) the theory.

\[ \Rightarrow \text{introduces renormalisation scale } \mu \]

\[ \Rightarrow R \left( \log \frac{Q^2}{\mu^2} \right) \]
Q.E.D:

\[ e^2 = \left( e_0 + e_3 \ldots \right)^2 \]

\[ \sum_{\text{phys}} e^2 = \sum_{\text{bare}} e_0^2 + \sum_{\text{bare charged screened}} e_0^2 + \ldots \]

Physical or effective charge

\[ \alpha(Q^2) = \alpha_0 \left\{ 1 + \frac{\alpha_0}{3\pi} \log \frac{Q^2}{M^2} + \frac{1}{2} \left( \frac{\alpha_0}{3\pi} \log \frac{Q^2}{M^2} \right)^2 + \ldots \right\} \]

Large \( Q^2 \) leading log sum

\( \alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \log \frac{Q^2}{M^2}} \)

(Subtract)

\[ \alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \log \frac{Q^2}{\mu^2}} \]

Infinities removed at price of ren. scale \( \mu \).

Q.C.D:

\[ -\frac{1}{3\pi} \to \]

\[ b_0 = \frac{33 - 2n_f}{12\pi} \]

\[ \alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + b_0 \alpha_s(\mu^2) \log \frac{Q^2}{\mu^2}} \]

(HM p. 169)
\[ e(e) = e(\mu) \]

Ward identities \( \rightarrow \) Slavnov-Taylor identities of QED

\[ \alpha_s(q\bar{q},g)_\text{with loops} = \alpha_s(ggg)_\text{with loops} \]

preserved by renormalisation (gauge theory)

Confinement of colour (hadrons - col. singlets)

non-pert. techniques:
- lattice QCD
- chiral \( \chi \)
- Regge theory

unique to non-Abelian gauge theory

QCD is a “dream” theory: it has the possibility of explaining the confinement of colour (via lattice QCD) and yet allows the calculation of “hard” hadronic processes via a perturbation expansion in \( \alpha_s \).
First formal approach via Renorm. Eq. — then example

Go back to our observable \( R \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \), but \( \mu \) is arbitrary

\[
\frac{\mu^2}{Q^2} \frac{dR}{d\mu^2} = \left( \mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right) R = 0
\]

\[
t = \log \frac{Q^2}{\mu^2}
\]

\[
\therefore \frac{\mu^2}{Q^2} \frac{\partial}{\partial \mu^2} = \frac{\partial}{\partial \log \mu^2} = -\frac{\partial}{\partial t}
\]

Claim soln of RGE is \( R \left( \frac{Q^2}{\mu^2}, \alpha_s \right) = R \left( 1, \bar{\alpha}_s(Q^2) \right) \)

where function \( \bar{\alpha}_s(Q^2) \) is introduced by

\[
t = \int_{\bar{\alpha}_s(Q^2)}^{\alpha_s(\mu^2)} \frac{d\alpha}{\beta(\alpha)} \quad \text{such that} \quad \bar{\alpha}_s(Q^2, \mu^2) = \alpha_s
\]

"Proof"

\[
\frac{\partial}{\partial t} \quad 1 = \frac{1}{\beta(\bar{\alpha}_s(Q^2))} \frac{\partial \bar{\alpha}_s(Q^2)}{\partial t} \quad \Rightarrow \quad \frac{\partial \bar{\alpha}_s(Q^2)}{\partial t} = \beta(\bar{\alpha}_s(Q^2)) \quad \text{(1)}
\]

\[
\frac{\partial}{\partial \alpha_s} \quad 0 = \frac{1}{\beta(\bar{\alpha}_s(Q^2))} \frac{\partial \bar{\alpha}_s(Q^2)}{\partial \alpha_s} - \frac{1}{\beta(\alpha_s)} \frac{\partial \alpha_s}{\partial \alpha_s} \quad \Rightarrow \quad \frac{\partial \bar{\alpha}_s(Q^2)}{\partial \alpha_s} = \frac{\beta(\bar{\alpha}_s(Q^2))}{\beta(\alpha_s)} \quad \text{(2)}
\]

Indeed any function \( f(\bar{\alpha}_s(Q^2)) \) will satisfy RGE

\[
\left( -\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) f(\bar{\alpha}_s) = \left( -\frac{\partial \bar{\alpha}_s}{\partial t} + \beta(\bar{\alpha}_s) \frac{\partial}{\partial \bar{\alpha}_s} \right) f(\bar{\alpha}_s) = 0
\]

using (1)

using (2)

In particular \( R(1, \bar{\alpha}_s(Q^2)) \) is a solution,

with \( \bar{\alpha}_s(Q^2, \mu^2) = \alpha_s \) and \( \frac{\partial \alpha_s}{\partial t} = \beta(\alpha_s) \).
\text{RGE} \Rightarrow R\left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right) = R\left(1, \alpha_s(Q^2)\right)

RGE implies entire Q^2 dep. of R comes from running \alpha_s(Q^2).

That is if we know
\[ R(1, \alpha_s) = c_0 + c_1 \alpha_s + c_2 \alpha_s^2 + \ldots \]

then Q^2 dep. is
\[ R(1, \alpha_s(Q^2)) = c_0 + c_1 \alpha_s(Q^2) + c_2 \alpha_s(Q^2)^2 + \ldots \]

but we also need to know Q^2 dep. of \alpha_s
\[ \beta(\alpha_s) = -b_0 \alpha_s^2 - b_1 \alpha_s^3 + \ldots \]

The beta function
\[ \beta(\alpha_s) = \frac{\partial \alpha_s}{\partial \log \frac{Q^2}{\mu^2}} \]

Recap: 1-loop \( \beta \text{ fn} \) (sums leading \( \log \frac{Q^2}{\mu^2} \))

\[ \frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(\mu^2)} + b_0 \log \frac{Q^2}{\mu^2} \]

\[ \frac{b}{\partial \log \frac{Q^2}{\mu^2}} = b_0 \]

\[ \beta(\alpha_s) \]

\( \therefore \beta(\alpha_s) = -b_0 \alpha_s^2 \)

2-loop \( \beta \text{ fn} \) (sums next-to-leading \( \log s \))

\[ \frac{1}{\alpha_s(Q^2)} = -b_0 \alpha_s^2 - b_1 \alpha_s^3 + \ldots \]

\[ b_1 = \frac{153 - 19 n_f}{24\pi^2} \]

\[ \beta(\alpha_s) = -b_0 \alpha_s^2 - b_1 \alpha_s^3 \]
Example \[ R = \frac{\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})}{\sigma_0} \]

Lowest order\[ \begin{array}{c}
\text{virtual} \\
\text{interference} \\
\text{real} \\
e^+e^- \rightarrow q\bar{q}g
\end{array}\]

\[ \sigma(q\bar{q}g) = \frac{2\alpha_s}{3\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \]

Collinear and soft singularities at \( x_i \rightarrow 1 \)

Not problem at issue. Cancel in inclusive quantities between real and virtual contributions.

Method 1: Gluon mass regulator \( m_g \)

\[ \sigma_i(q\bar{q}g) = \sigma_0 \frac{2\alpha_s}{3\pi} \left( -\log^2 \frac{\Lambda^2}{m_h^2} + 3\log \frac{\Lambda^2}{m_q^2} - \frac{7}{2} + \frac{\pi^2}{3} \right) \]

\[ \sigma_i(q\bar{q}) = \sigma_0 \frac{2\alpha_s}{3\pi} \left( \log^2 \frac{\Lambda^2}{m_h^2} - 3\log \frac{\Lambda^2}{m_q^2} + 5 - \frac{\pi^2}{3} \right) \]

\[ \sigma_i = \sigma_0 \frac{\alpha_s}{\pi} \]
Method 2: Dimensional regularisation

Use \( n = 4 + 2\varepsilon \) space-time dimensions

singularities appear as \( \frac{1}{\varepsilon} \) poles

(method has the advantage that it can be used, with \( \varepsilon < 0 \), for UV divergences)

\[
\sigma_i(q\bar{q}) = \sigma_0 \frac{2\alpha_s}{3\pi} \text{He}(\varepsilon) \left[ \frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \frac{19}{2} + O(\varepsilon) \right]
\]

\[
\sigma_i(q\bar{q}) = \sigma_0 \frac{2\alpha_s}{3\pi} \text{He}(\varepsilon) \left[ -\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - 8 + O(\varepsilon) \right]
\]

\[
\sigma_i = \sigma_0 \frac{\alpha_s}{\pi} \quad \text{where} \quad \text{He}(\varepsilon) = 1 + O(\varepsilon)
\]

To \( O(\alpha_s) \):

\[
R = 1 + c_1 \alpha_s \quad \text{where} \quad c_1 = \frac{1}{\pi}
\]

At \( O(\alpha_s^2) \) UV divergences associated with \( \alpha_s \) occur

\[
R \left( \frac{Q^2}{\mu^2}, \alpha_s \right) = 1 + c_1 \alpha_s - \left( c_1 b_0 \log \frac{Q^2}{\mu^2} - c_2 \right) \alpha_s^2
\]

\[
\alpha_s(Q^2) = \frac{\alpha_s}{1 + \alpha_s b_0 \log \frac{Q^2}{\mu^2}} = \alpha_s - b_0 \log \frac{Q^2}{\mu^2} \alpha_s
\]

\[
R \left( \frac{Q^2}{\mu^2}, \alpha_s \right) = 1 + c_1 \frac{\alpha_s}{1 + \alpha_s b_0 \log \frac{Q^2}{\mu^2}} + c_2 \frac{\alpha_s}{1 + \alpha_s b_0 \log \frac{Q^2}{\mu^2}} \alpha_s^2
\]

Alternately check \( R \left( \frac{Q^2}{\mu^2}, \alpha_s \right) \) is indep. of \( \mu^2 \)

\[
\frac{dR}{d\log \mu^2} = c_1 (-b_0 \frac{\alpha_s}{1 + \alpha_s b_0 \log \frac{Q^2}{\mu^2}}) + c_2 \frac{b_0 \alpha_s}{1 + \alpha_s b_0 \log \frac{Q^2}{\mu^2}} + O(\alpha_s^3)
\]
At $O(\alpha_s^3)$

Need

$$\beta(\alpha_s) = -b_0 \alpha_s^2 - b_1 \alpha_s^3$$

$$R(1, \alpha_s(\mu^2)) = 1 + c_1 \alpha_s(\mu^2) + c_2 \alpha_s(\mu^2)^2 + c_3 \alpha_s(\mu^2)^3$$

$$c_3 = -\frac{12 \times 805}{7\pi^3}$$

$$R(\frac{Q^2}{\mu^2}, \alpha_s) = 1 + c_1 \alpha_s - \left(c_1 b_0 \log \frac{Q^2}{\mu^2} - c_2\right) \alpha_s^2$$

$$+ \left[c_1 b_0^2 \log^2 \frac{Q^2}{\mu^2} - (c_1 b_1 + 2 c_2 b_0) \log \frac{Q^2}{\mu^2} + c_3\right] \alpha_s^3$$

3 loops!

We see the coeffs. of $\alpha_s^n$ depend on $\mu$—but change exactly compensated by change in $\alpha_s(\mu^2)$ so that $R(\frac{Q^2}{\mu^2})$ is indep. of choice of renormalisation scale $\mu$.

Breaks down when series is truncated.

An observable calc. to $O(\alpha_s^n)$ will be changed at $O(\alpha_s^{n+1})$.
Scale dependence at $Q = 100 \text{GeV}$

Scale dependence of the corrections to $\sigma(e^+e^- \rightarrow \text{hadrons})$

$\Lambda^{(b)} = 230 \text{ MeV}, Q=100 \text{ GeV}$

$\mu = Q$

renormalisation scale $\mu$ (GeV)
Further notes on $\alpha_s$

1. $\alpha_s(Q^2)$ at 2-loops

$$\frac{\partial \alpha_s}{\partial t} = -b_0 \alpha_s^2 - b_1 \alpha_s^3$$

$$\frac{1}{\alpha_s(\mu^2)} - \frac{1}{\alpha_s(\mu^2)} + \frac{b_1}{b_0} \log \left( \frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \frac{b_0 + b_1 \alpha_s(\mu^2)}{b_0 + b_1 \alpha_s(Q^2)} \right) = b_0 \log \frac{Q^2}{\mu^2}$$

2. Renormalization scheme dependence

- base

$\alpha_s$ depends on scheme

$\alpha_s^A = Z^A \alpha_s^0$

$\alpha_s^B = Z^B \alpha_s^0$

- infinite parts same

e.g. MS scheme uses dim. reg. and Minimally Subtracts $\frac{1}{\varepsilon}$ poles (and replaces $\alpha_s^0$ by $\alpha_s(\mu^2)$).

*** MS subtracts (naturally occurring comb.) $\frac{1}{\varepsilon} + \log 4\pi - \gamma_E$

favoured by QCD ers

$\alpha_s^B = \alpha_s^A (1 + c \alpha_s^A)$

$\beta = -b_0 \alpha_s^2 - b_1 \alpha_s^3 - b_2 \alpha_s^4 - b_3 \alpha_s^5 - ....$

- scheme dependent

Check: $\alpha' = \alpha(1 + c \alpha)$

then $\frac{\partial \alpha'}{\partial t} = -b_0 \alpha'^2 - b_1 \alpha'^3$

$\rightarrow \frac{\partial \alpha}{\partial t} + 2\alpha(-b_0 \alpha^2) = -b_0 \alpha^2 (1 + 2c \alpha ..) - b_1 \alpha^3 ..$

Scheme is specified by $m, b_2, b_3, ...$

Observable $R = \alpha_s^N (c_0 + c_1 \alpha_s + ... + c_n \alpha_s^n)$ then $\frac{\partial R}{\partial (RS)} = O(\alpha_s^{N+n+1})$
3. Flavour thresholds

$b_0, b_1, \ldots$ depend on no. of active flavours

\[ Q^2 \leq m_c^2 \quad \eta_3 = 3 \]

\[ m_c^2 \leq Q^2 \leq m_b^2 \quad \eta_4 = 4 \]

\[ m_b^2 \leq Q^2 \leq m_t^2 \quad \eta_5 = 5 \]

need prescription for continuity of $\alpha_s$ at thresholds

4. Fundamental QCD parameter. QCD coupling defined by (exptd) value of $\alpha_s(M_Z^2) |_{\eta_5 = 5} \equiv \frac{\alpha_s}{M_Z^2}$

5. Demise of $\Lambda(QCD)$ (Can LEP kill it and replace by $\alpha_s(M_Z^2)$?)

$\Lambda$ defined as the point where $\alpha_s \to \infty$

\[
\log \frac{Q^2}{\Lambda^2} \equiv -\int_{\alpha_s(Q^2)}^\infty \frac{d\alpha}{\beta(\alpha)}
\]

At one-loop

\[
\frac{1}{b_0} \int_{\alpha_s(Q^2)}^\infty \frac{d\alpha}{\alpha^2} = \frac{1}{b_0 \alpha_s(Q^2)}
\]

\[
\alpha_s(Q^2) = \frac{1}{b_0 \log \frac{Q^2}{\Lambda^2}}
\]

6. Exptd det. of $\alpha_s(M_Z^2)$
\[ R_t = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau \bar{\nu}_\mu)} \]

\[ \alpha_s(Q) \]

\[ \alpha_s(M_Z^2) \]

\[ \Lambda^{(6)} = 50 - 450 \text{ MeV} \]
Bethke

$$\alpha_s(M_Z^2) = 0.118 \pm 0.007$$

In addition to the DIS determination, note that the ratios $R_i$ of inclusive hadron/lepton rates have less theoretical uncertainty than event shapes.
DIS: a third look. We left QPM at

\[
\frac{F_2}{x} = \frac{1}{q} \int_0^1 dy \, f_1(y) \, e^2 \, \delta(y-x)
\]

\[
= \frac{1}{q} \int_0^1 dy \, f_q(y) \, e^2 \, \delta(\frac{y-x}{y})
\]

Notation \( y \rightarrow y \).

(Note \( y = Q^2/\alpha_s \)).

The QCD-improved parton model

- \( O(\alpha_s) \) corrections:

\[
\left| \begin{array}{ccc}
\rightarrow & \rightarrow & \rightarrow \\
\text{real} & + & \text{virtual}
\end{array} \right|^2
\]

- Singularities: \( \nu \) divergences \( \rightarrow \) running \( \alpha_s \)

Final state "mass or collinear" sing. \( \rightarrow \) cancel in inclusive proc.

Initial state "mass or collinear" sing. \( \rightarrow \) pose a problem

Calculation gives

\[
\frac{F_2}{x} = \frac{1}{q} \int_0^1 dy \, f_q(y) \, e^2 \left[ \delta\left(1 - \frac{x}{y}\right) + \frac{\alpha_s}{2\pi} \left\{ P(y) \log \frac{Q^2}{\mu^2} + R(y) \right\} \right]
\]

\[
\left( x = \frac{Q^2}{m^2} \right)
\]

Origin of sing:

Small \( k_T^2 \) limit

of Feyn. diagram

(\( \alpha_s = 10.30 \))

\[
\frac{1}{\delta_0} \frac{d\sigma}{d\vec{k}_T^2} = e^2 \frac{1}{\pi} \frac{\alpha_s}{2\pi} \frac{\alpha_s}{d\vec{k}_T^2} P(x)
\]

\[
P = \frac{4}{3} \frac{1+x^2}{1-x}
\]

\[
\frac{\delta_0(y^2 q \rightarrow q q)}{\delta_0} = e^2 \int \frac{d\vec{k}_T^2}{\delta_0} \frac{\alpha_s}{2\pi} \frac{\alpha_s}{d\vec{k}_T^2} P(x) \log \frac{Q^2}{\mu^2}
\]

Helicity fac \( e^2 \left| \frac{k_T^2}{k_T^4} \right| \)

\( \mu = \) artificial regulator
\[ \frac{F_2}{x} = \sum_q \int_x^1 \frac{dy}{y} f_q(y) e_y^2 \left[ \delta \left( 1 - \frac{x}{y} \right) + \frac{\alpha_s}{\pi} \left\{ P \left( \frac{\Delta}{y} \right) \log \frac{Q^2}{M^2} + R_i \left( \frac{\Delta}{y} \right) \right\} \right] \]

Logs again! Looks bad! Don't panic.

Factorize sing. into bare densities \( f_q^{bare}(y) \rightarrow f_q^{ten}(y, Q^2) \)

This renormalisation is called mass factorization

\[ \frac{F_2}{x} = \sum_q \int_x^1 \frac{dy}{y} f_q(y, M^2) e_y^2 \left[ \delta \left( 1 - \frac{x}{y} \right) + \frac{\alpha_s}{\pi} \left\{ P \left( \frac{\Delta}{y} \right) \log \frac{Q^2}{M^2} + R_i \left( \frac{\Delta}{y} \right) \right\} \right] \]

Observable so finite \( \Rightarrow \) finite

\( f_q^{ten}(y, M^2) \) has absorbed all infrared sensitivity. Specific to hadron but universal. Process dependent.

where renormalized densities are

\[ f_q(M^2) = f_q^{ten}(x) + \frac{\alpha_s}{\pi} \int_x^1 \frac{dy}{y} f_q(y) \left\{ P \left( \frac{\Delta}{y} \right) \log \frac{M^2}{x^2} + R_i \left( \frac{\Delta}{y} \right) \right\} \]

Finite, but no absolute pred. But \( M^2 \) dep. known from QCD

\[ \frac{\partial f_q(M^2)}{\partial \log M^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} f_q(y, M^2) P \left( \frac{\Delta}{y} \right) \]

Altarelli-Parisi eq.

\[ \mathcal{A} \]

\[ \frac{\partial \alpha_s(M^2)}{\partial \log M^2} = \frac{\alpha_s}{\pi} \]

Analogy:

- \( M = \) mass factorization scale
- \( \text{expt} \rightarrow f_q(x, Q^2) \)
- "Running of \( f_q(x, Q^2) \) given by QCD

- \( \mu = \) renorm. scale
- \( \text{expt} \rightarrow \alpha_s(Q^2) \)
- "Running \( \alpha_s(Q^2) \) given by QCD
Further discussion

- Sep. of sing./non-sing. parts at scale $M$ is not unique. Provided coll. sing. absorbed by $f_q(xM^2)$, can add any finite term. Must specify scheme: $\overline{\text{MS}}$ scheme favoured

[An alternative DIS scheme factors all QCD contributions to $F_2$ into $f_q(x, M^2)$, so

$$\frac{F_2}{x} = \sum_q \frac{e_q^2}{\beta_0} f_q(x, \Lambda^2)$$

but then all other struct. funs. and QCD contributions to other processes, and to $\alpha_s$, have to be compensated for the $F_2$ contrib.]

- $P_{qg}(z) = \frac{4}{3} \frac{1+z^2}{1-z} \rightarrow z$

$z=1$ sing. (soft infrared gluon) cancelled by virtual contrib. \(\rightarrow\)

$$\left| \begin{array}{c}
\frac{z}{2} \\
\frac{z}{3} \\
\frac{z}{4} \\
\frac{z}{5}
\end{array} \right|$$

After cancell. of singularity there remains a residual $8(1-z)$ type contribution from virtual diag.

$$P_{qg}(z) = \frac{4}{3} \frac{1+z^2}{(1-z)^2} + 2 \delta(1-z) \quad \text{(HM page 222)}$$

+ prescription ensures cancell.? \(\rightarrow\) $\int_0^1 dz \frac{f(x)}{(1-z)^2} = \int_0^1 dz \frac{f(x) - f(1)}{1-z}$

Easy to find from \(\int_0^1 P_{qg}(z) dz = 0\) \((q - \bar{q} \text{ is conserved})\)
At $O(xs)$ need to include $q x^* \rightarrow q \bar{q}$

\[ \mathcal{S}_q(x, Q^2) \equiv q(x, Q^2) \]

\[ \frac{dq(x, Q^2)}{d \log Q^2} = \frac{\alpha_s(Q^2)}{2 \pi} \int_x^1 \frac{dy}{y} \left[ P_{qq}(\frac{x}{y}) q(y, Q^2) + P_{qg}(\frac{x}{y}) g(y, Q^2) \right] \]

**GLAP eqs**

Gribov-Lipatov
Dokshitzer
Altarelli-Parisi

Similarly

\[ \frac{dq(x, Q^2)}{d \log Q^2} = \frac{\alpha_s(Q^2)}{2 \pi} \int_x^1 \frac{dy}{y} \left[ P_{gg}(\frac{x}{y}) g(y, Q^2) + \sum_i P_{qi}(\frac{x}{y}) q(y, Q^2) \right] \]

\[ \frac{dy}{y} \left. \right|_{y = \text{max}} \]

\[ P_{qg} \approx q \]

**Notation:**

\[ \frac{dq}{d \log Q^2} = \frac{\alpha_s}{2 \pi} \left( P_{qq} \otimes q + P_{qg} \otimes q \right) \]
Flavour singlet  \( q_s = \sum_i (q_i + \bar{q}_i) \)

\[ e.g. \text{non-singlet} \quad u, \bar{u} = u - \bar{u} \quad \text{gluon cancels} \]

\[
\frac{d\alpha_s^{NS}}{\log Q^2} = \frac{\alpha_s}{2\pi} \quad P_{qq} \otimes q_s^{NS}
\]

\[
\left[ \begin{array}{l}
\frac{d\alpha_s}{\log Q^2} = \frac{\alpha_s}{2\pi} \left( P_{qq} \otimes q_s + 2f P_{qg} \otimes g \right) \\
\frac{d\alpha_s}{\log Q^2} = \frac{\alpha_s}{2\pi} \left( P_{gg} \otimes q_s + P_{qg} \otimes g \right)
\end{array} \right.
\]

Can solve for evolution in \( Q^2 \)

- by step-by-step integration

- or by inverting moments (see later)
Determination of parton distributions \( f_i(x, Q^2) \)

From p.30

\[
\frac{F_2}{x} = \sum_i \int_0^1 \frac{dy}{y} f_i(y, Q^2) \left( x \right)^2 \epsilon_f^2 \left[ \delta \left( 1 - \frac{x}{y} \right) + \frac{\alpha_s}{2 \pi} \left( \frac{P_i}{y} \right) \log \frac{Q^2}{m_i^2} + \mathcal{R}_i \left( \frac{x}{y} \right) \right] + O(\alpha_s^2)
\]

Choose \( M = Q \)

\[
\epsilon_f^2 \left[ \delta \left( 1 - \frac{x}{y} \right) + \alpha_s \mathcal{B}_1 \left( \frac{x}{y} \right) + \alpha_s^2 \mathcal{B}_2 \left( \frac{x}{y} \right) \right].
\]

calc. from pQCD. No dangerous log \( Q^2/m^2 \) terms (recall they were swept into \( f_i(y, Q^2) \))

Very non-trivial to calc. \( B_2, \ldots \). Need to go to Leiden!

In general \( V = g, W \)

\[
F_a(x, Q^2) = \sum_{i=1}^{3} \int_0^1 \frac{dy}{y} f_i(y, Q^2) \left( x \right)^2 \epsilon_f^2 \left[ \delta \left( 1 - \frac{x}{y} \right) + \frac{\alpha_s}{2 \pi} \left( \frac{C_i}{y} \right) \log \frac{Q^2}{m_i^2} + \mathcal{R}_i \left( \frac{x}{y} \right) \right] + O(\alpha_s^2)
\]

calculable as \( \sum \alpha_s^n (B_n) C_i \)

but process dependent

pQCD only gives evolution in \( Q^2 \)

Need to input \( f_i(x, Q_0^2) \)

but UNIVERSAL.

\[
\frac{\partial f_i(x, Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2 \pi} \sum_i \int_0^1 \frac{dy}{y} f_i(y, Q^2) P_i(x, y)
\]

\[
P_{ij} = P_{ij}^{(0)} + \alpha_s P_{ij}^{(1)} + \ldots
\]

Sums LL: \( \alpha_s^n \log \frac{Q^2}{m^2} \)

Sums NLL: \( \alpha_s^n \log \frac{Q^2}{m^2} \)
<table>
<thead>
<tr>
<th>DIS subprocesses</th>
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<tbody>
<tr>
<td>$\alpha_s^0$: $V + q(\bar{q}) \rightarrow q(\bar{q})$</td>
</tr>
<tr>
<td>$\alpha_s^1$: $V + q(\bar{q}) \rightarrow q(\bar{q})$ (one loop correction)</td>
</tr>
<tr>
<td>$V + q(\bar{q}) \rightarrow q(\bar{q}) + g$</td>
</tr>
<tr>
<td>$V + g \rightarrow q + \bar{q}$</td>
</tr>
<tr>
<td>$\alpha_s^2$: $V + q(\bar{q}) \rightarrow q(\bar{q})$ (two loop correction)</td>
</tr>
<tr>
<td>$V + q(\bar{q}) \rightarrow q(\bar{q}) + g$ (one loop correction)</td>
</tr>
<tr>
<td>$V + q(\bar{q}) \rightarrow q(\bar{q}) + g + g$</td>
</tr>
<tr>
<td>$V + q(\bar{q}) \rightarrow q(\bar{q}) + q(\bar{q}) + \bar{q}(q)$</td>
</tr>
<tr>
<td>$V + g \rightarrow q + \bar{q}$ (one loop correction)</td>
</tr>
<tr>
<td>$V + g \rightarrow q + \bar{q} + g$</td>
</tr>
</tbody>
</table>

Table 1. List of deep inelastic lepton-parton subprocesses up to $\mathcal{O}(\alpha_s^2)$. 
Diagrams which contribute to the DIS subprocesses listed in the table.
Summary:

\[ F_2^{ep} = x \left( \frac{4}{9} u + \frac{1}{3} d \ldots \right) \]
\[ F_2^{en} = x \left( u + \bar{u} + \ldots \right) \]
\[ x F_3^{en} = x \left( u - \bar{u} + \ldots \right) \]

QCD-improved Parton Model

Singularity / loops everywhere

- UV divergences \( \rightarrow \) "swept" into running \( \alpha_s(Q^2) \)
  controlled by \( \beta(\alpha_s) \)
  \[ = -b_0 \alpha_s - b_1 \alpha_s^2 \ldots \]

- Final state sing. \( \rightarrow \) cancel between real and virtual diagrams in an inclusive process.

- Initial state sing. \( \rightarrow \) "swept" into running \( f_i(x, Q^2) \)
  controlled by \( A \) (eq. with \( P(\frac{N}{9}) \)

\[ P(0) + P(1) \alpha_s + \ldots \]

"Higher twist" neglected

\[ O\left( \frac{M^2}{Q^2} \right) \]

(axial gauge)

(see 61-63)
Global analysis to determine $f_i(x, Q^2)$

Parametrize $x$ dependence of $f_i(x, Q^2)$.

Evolve in $Q^2$ and fit to all available data (in pQCD region): $F_2^{np/\mu n}$, $F_i^{\perp N}$, Drell-Yan, prompt $\gamma$...

Why so much fuss? Why bother?

- $f_i$ are universal: enter every "hard" process involving hadrons
  
  e.g. Nick Ellis wants them to estimate
  
  $pp \rightarrow b\bar{b}X$ at LHC (for $\xi^B$)
  
  $gg \rightarrow b\bar{b}$, $gg \rightarrow b\bar{b}g$

- \[ (p_{\beta_1} + p_{\beta_2})^2 = 4m_b^2 + \text{bit} \]
  
  \[ x_1 x_2 s = 4m_b^2 + \frac{2m_b^2}{\Lambda^2} \approx \frac{15}{14000} \approx 10^{-3} \]

  Nick wants $g(x \approx 10^{-3}, Q^2 \approx 100 \text{GeV}^2)$

- Correlate wide range of processes: stringent and very broad test of pQCD. Even $\propto \xi^B(M_Z^2)$

- HERA enters new regime ($x \lesssim 10^{-3}$)
  
  Novel effects. Not appropriate to sum log $Q^2$.
  
  Need to sum log ($x$). Shadowing?
**PROTON STRUCTURE FN. ANALYSES**

**History of next-to-leading order analyses**

<table>
<thead>
<tr>
<th></th>
<th>( \mu \text{-DIS} )</th>
<th>( \nu \text{-DIS} )</th>
<th>Prompt ( \gamma )</th>
<th>D-Yan</th>
<th>( W, Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRS ’88</td>
<td>EMC + ..</td>
<td>CDHSW</td>
<td>AFS((+J/\psi))</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>DFLM ’88</td>
<td>(EMC + ..)</td>
<td>CHARM + ..</td>
<td>-</td>
<td>E288 + ..</td>
<td>-</td>
</tr>
<tr>
<td>ABFOW ’89</td>
<td>BCDMS</td>
<td>-</td>
<td>WA70</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HMRS ’90</td>
<td>EMC</td>
<td>CDHSW</td>
<td>WA70</td>
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<td>-</td>
</tr>
<tr>
<td>BCDMS</td>
<td>NMC(n/p)</td>
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<td>E605</td>
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</tr>
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<tr>
<td>MRS (Apr ’92) (sets D₀,D₋)</td>
<td>BCDMS</td>
<td>NMC(p,n) Canadiens-( C)</td>
<td>CDHSW</td>
<td>WA70</td>
<td>E605</td>
</tr>
<tr>
<td>MRS (Nov ’92) (sets D₀,D₋)</td>
<td>BCDMS</td>
<td>NMC(p,n) Canadiens-( C)</td>
<td>CCFR</td>
<td>WA70</td>
<td>E605</td>
</tr>
<tr>
<td>CTEQ ’93</td>
<td>BCDMS</td>
<td>NMC(p,n) Canadiens-( C)</td>
<td>CCFR</td>
<td>WA70</td>
<td>E605</td>
</tr>
</tbody>
</table>

Table 1: NLO determinations of parton distributions [6,7,9,10,11] together with the data used in the various analyses. Data marked † were used in preliminary form.

**Note only last 3 had new NMC, CCFR data available**
To illustrate the dangers of extrapolation.
Naive estimates of $f_i(x)$

**$x \to 1$ behaviour**

\[ f_i(x) \sim (1-x)^{2n_s-1} \]  

(when $n_s = \#\text{spectators}$)

\[ xqV \sim x^{\frac{1}{2}} \]

\[ xq_s \sim x^{-\frac{1}{2}} \]

\[ xq \sim x^0 \]

\[ xq_s \sim x^0 \]

Naive Regge pomeron

\[ xq, xq_s \sim x^0 \]

\[ \left( 1-x \right)^3 \]

\[ \left( 1-x \right)^5 \]

\[ \left( 1-x \right)^7 \]

\[ x \]

\[ \sum \beta_i \beta_j \sim x^0 \]

\[ \sum \beta_i \beta_j \sim x^{-\alpha_R} \]

**$x = \frac{Q^2}{2p \cdot q} \to 0$ behaviour**

fixed $Q^2$ so $p \cdot q \to \infty$  

so $s_{\gamma p} \to \infty$  

(Regge limit)

\[ G_t \sim \beta_p \alpha_{\gamma p}^{-1} + \beta_R \alpha_{\gamma R}^{-1} \]

flavourless valence  

\[ g, \omega, \rho_1, \rho_2 \]

\[ \alpha_{\gamma p}(0) \approx 1 \]

\[ \alpha_{\gamma R}(0) \approx \frac{1}{2} \]

\[ \left( 1-x \right)^3 \]

\[ \left( 1-x \right)^5 \]

\[ \left( 1-x \right)^7 \]

\[ x \]
Parametrization of input distributions $f_i^b(x, Q^2 = 4 Q^2 Y^2)$

MRS: $x f_i^b(x, Q^2) = A_i x^{\alpha_i}(1-x)^{\beta_i} (1 + Y_i x^{1/2 + \delta_i x})$

- $i = u_v, d_v, g, \frac{1}{2}(\bar{u} + \bar{d}) \approx 2\bar{s}$
- at most $\delta_i \approx 0$ for gluon

- Naive estimates give some idea of $\alpha_i, \beta_i$
- In fact for quon and sea distributions, MRS set $\alpha_i = 0$ conventional Pomeron; $D_0$ set of partons
- or $\alpha_i = -\frac{1}{2}$ "Lipatov" type behaviour; $D_-$ set of partons

- 3 of the 4 $A_i$'s determined by sum rules
  $$\sum_i x f_i(x, Q^2) dx = 1$$
  $$\int (u - \bar{u}) dx = 2, \quad \int (d - \bar{d}) dx = 1$$

- Evidence that $\bar{d} > \bar{u}$ (on average) from Gottfried s-rule.
- $\bar{u} \neq \bar{d}$ can arise from breaking of $g - a_2$ Regge exchange degeneracy

- $\frac{1}{2}(\bar{u} + \bar{d}) \approx 2\bar{s}$ from CCFR dimuon data $\pi N \rightarrow \mu^{+}\mu^{-}X$

- $x (\bar{d} - \bar{u}) = A_D x^{\frac{1}{2}} (1-x)^{Y_{sea}}$

- $x \alpha(t)$: $s$-channel $t$-channel exchange
\[
\mathbf{d - \bar{u}}
\]

\[
F_2^{\mu\nu} - F_2^{\mu\nu} = \frac{1}{3} x (u + \bar{u} + d + \bar{d}) + \ldots
\]
\[
\frac{1}{2} (F_2^{\mu\nu} + F_2^{\nu\mu}) = \frac{5}{18} x (u + \bar{u} + d + \bar{d} + \frac{4}{5} s) + \ldots
\]
\[
F_2^{2N} = F_2^{\bar{2}N} = x (u + \bar{u} + d + \bar{d} + 2s) + \ldots
\]
\[
\frac{1}{2} x (F_3^{2N} + F_3^{\bar{2}N}) = x (u - \bar{u} + d - \bar{d}) + \ldots
\]

Note: \( \mathbf{d - \bar{u}} \) not constrained \([\text{set } \equiv 0 \text{ until '92 analyses}]\)

First hint \( \mathbf{d} \neq \bar{u} \) came from Gottfried S.R. test by NMC

\[
I(GSR) \equiv \int_0^1 \frac{dx}{x} (F_2^{\mu\nu} - F_2^{\mu\nu}) = \frac{1}{3} \int_0^1 (u - d) dx + \frac{2}{3} \int_0^1 (\bar{u} - \bar{d}) dx
\]

\[
\text{NMC}
\]
\[
\int_{0.004}^{0.8} \frac{dx}{x} (F_2^{\mu\nu} - F_2^{\mu\nu}) = 0.236 
\]
\[
\pm 0.016
\]

\[
\Rightarrow I(GSR) = 0.258 \pm 0.017
\]

If corr. for deuteron screening

\[
I(GSR) \approx 0.23
\]

\[
\text{Caution: small } x \text{ behaviour is v. important}
\]

eg. MRS(\( S'_0 \)) with \( \bar{u} = \bar{d} \) is good fit. \( I(GSR) \approx 0.333 \)

(reason: v. different \( x^\alpha \) behaviour of \( u, d \))

MRS(\( D'_0 \)) \( \bar{d} - \bar{u} = A_0 x^\alpha (1-x)^\beta \) sea

\[
\Rightarrow I(GSR) = 0.256
\]

Conclude: much flexibility in \( \bar{d} - \bar{u} \) and \( I(GSR) \)
$F_2^p(x) - F_2^{\mu n}(x)$ at $Q^2 = 7 \text{ GeV}^2$

NMC data corrected for shadowing

- MRS D0'
- MRS S0'
- CTEQ2M

$S_0': \bar{u} = \bar{d}$ \hspace{1cm} $I(\text{GSR}) = 0.333$

$D_0': \bar{u} \neq \bar{d}$ \hspace{1cm} $I(\text{GSR}) = 0.256$
This CCFR determination was not available for MRS or CTEQ1

strange sea

CCFR dimuon
NLO analysis

Q^2 = 4 GeV^2

also find
\frac{s(x)}{\bar{u}(x) + \bar{d}(x)}
is indep. of x
• **Data**

<table>
<thead>
<tr>
<th>LO subprocess</th>
<th>Parton determination</th>
</tr>
</thead>
</table>

- **DIS (μN→μX)**
  - BCDMS, NMC
  - $F_2^{μP}$, $F_2^{μN}$

- **DIS (υN→μX)**
  - CCFR (CDHSHW)
  - $F_2^{υN}$, $x F_3^{υN}$

- **pp → γX**
  - WA70
  - $q q → γ q$
  - $g(x \sim 0.4)$

- **pN → μμX**
  - E605
  - $q q → γ q$
  - $\bar{q} = ... (1-x)\gamma^*$

- **pP → WX**
  - ZX
  - UA2, CDF, D0
  - $u, d$ at $x_1 x_2 s \approx M_W^2$
  - i.e. $x \approx \frac{M_W}{\sqrt{s}} \approx 0.13 / 0.05$
  - CERN, FNAL

- **υN → μμX**
  - CCFR
  - $u s → μ^- c \rightarrow μ^+$
  - $s \sim \frac{1}{2} \bar{u}$ or $\frac{1}{2} \bar{d}$

- Also **pP → dijet (forward)**
  - CDF, D0
  - $g q, g q$
  - $g(x \sim 0.01)$

  **pP → bb**
  - HERA
  - $F_2^{ep}$

**Note:** Must include nuclear corrections, deuteron screening.

Relative normalisation:
- CCFR (0.95)
$F_2^{2JN}$ (CCFR)

Must make heavy target corrections.
Allow for relative normalization.
constrains g at \( x = \frac{2p_T}{M} = \frac{p_T}{1.4 \text{ GeV}} \) for WA70

\[
\begin{align*}
\frac{\mathcal{E} d^3\sigma}{d^3p|_{y=0}} & (\text{pb GeV}^{-2}) \\
10^2 & \quad 10^1 & \quad 1 \quad 10^{-1} & \quad 10^{-2} \\
\text{WA70 } pp \rightarrow \gamma X & \\
\text{D}_{0} & \quad \text{gg} \rightarrow \gamma g & \text{dominant}
\end{align*}
\]
$W^\pm$ Asymmetry

Valence partons

\[\begin{array}{c}
\text{u} \\
\text{d}
\end{array}\]

\[0 \quad x \quad 1\]

\[\begin{array}{c}
\text{d} \\
\text{u}
\end{array}\]

\[u \rightarrow W^+ \rightarrow d\]

\[d \rightarrow W^- \rightarrow \bar{u}\]

\[p \rightarrow \quad \rightarrow \bar{p}\]

\[u \text{ more forward than } d\]

\[W^+ \quad \quad \quad \quad W^-\]

CDF (uncorr.)

CTEQ1M

MRS(D0)

CDF measure

\[A_e(y_e) = \frac{\sigma(e^+) - \sigma(e^-)}{\sigma(e^+) + \sigma(e^-)}\]

So have to fold in $W^\pm \rightarrow e^\pm \nu$

CDF: when corrected errors $\times \frac{1}{2}$ for '92-'93 data

errors $\times \frac{1}{2}$ for present run!
$Q^2 = 20 \text{ GeV}^2$

MRS ($D_0'$): $s = \frac{1}{4} (\bar{u} + \bar{d})$ at $Q^2 = 4$
\[ g(x_{10}^2) \]

MRS (D')

\[ Q^2 = 20 \text{ GeV}^2 \]

"Lipatov"
Conclusions on parton determinations so far

Wide range of DIS and related data well described by NLO perturbative QCD with very economical parametrization

\( \sim 15 \) parameters or less (and some constrained \( x^6(1-x)^6 \))

- including QCD coupling

\[ \alpha_s(M_Z^2) = 0.113 \pm 0.005 \]

Partons well determined for \( x \geq 0.02 \) (e.g. \( D_0 \approx D_- \))

(but need data to pin down \( u, d, \) quon).

For smaller \( x \): much freedom (\( D_0 \) v. different to \( D_- \))

\( F_2^{op} \) at HERA, only determines comb. of sea.

\[
\frac{\partial^2 f_n(x, Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \ldots
\]

We urgently need determination of \( g(x \sim 10^{-3}) \) for Nick

HERA \rightarrow \frac{\partial F_2}{\partial \log Q^2}, F_2, J/\psi, Q\bar{Q} \ldots

\( pp \) collider \rightarrow same side dijets, …
Determination of gluon at $x \sim 10^{-3}$

**Longitudinal St. fn. $F_L$**

\[ \sigma(x^* p) \sim \varepsilon^\mu \varepsilon^{\nu^*} W_{\mu^*} \]

\[ (-g^\mu + q^\mu q^\nu) W_1 + \frac{q_2^2}{M^2} (...) \]

For transverse photons (helicity $\lambda = \pm 1$)

\[ \varepsilon^\mu \Rightarrow (0, \vec{\varepsilon}, 0) \Rightarrow \sigma_T \sim W_1 \]

For longitudinal photons (helicity $\lambda = 0$)

\[ \varepsilon^\mu = \frac{1}{\sqrt{q^2}} (\frac{1}{2} + q^2; \vec{0}, \vec{q}) \Rightarrow \sigma_L \sim (1 + \frac{q_2^2}{M^2}) W_2 - W_1 \]

since $\varepsilon_\mu q^\mu = 0$ and

\[ q^\mu = (q; \vec{0}, \sqrt{q^2 + q^2}) \]

\[ \infty : F_2 - 2x F_1 \equiv F_L \]

in high $Q^2$, fixed $x$

**Checks, as in QPM**

\[ x \rightarrow \frac{\alpha_s(Q^2)}{\pi} \]

\[ \Rightarrow \varepsilon_\mu \varepsilon^{\nu^*} \]

$\lambda = 0$ forbidden

\[ \therefore \sigma_L = 0 \]

\[ \therefore F_2 = 2x F_1 \]

But at $O(\alpha_s)$ contributes from

\[ \int_{m^2}^{m^2} + \int \]

find

\[ F_L = \frac{\alpha_s(Q^2)}{\pi} \left[ \frac{4}{3} \int_{0}^{1} dy \left( \frac{\alpha_s}{\pi} \right)^2 F_2(y, Q^2) + 2 \sum_{q} \int_{0}^{1} \frac{dy}{y} (\frac{\alpha}{\pi})^2 (1 - \frac{y}{x}) n_q(y, Q^2) \right] + O(\alpha_s^3) \]

64
Sensitivity of $F_L$ and $\frac{\partial F_L}{\partial \log Q^2}$ to the gluon at small $x$.

\[ Q^2 = 20 \text{GeV}^2 \]

\[ \frac{\partial F_L}{\partial \log Q^2} = O(\alpha_s^2) \text{ included.} \]

\[ dF_2 \quad \frac{\partial F_2}{\partial \log Q^2} \]

\[ xg(x) \]

\[ 10^{-4} \quad 10^{-3} \quad 10^{-2} \quad 10^{-1} \]

\[ x \]

\[ \frac{\partial F_2}{\partial \log Q^2} = \frac{\alpha_s(\alpha_s)}{2\pi} \left\{ \sum_{x} dy \left( \frac{y}{y} \right) R \left( \frac{x}{y} \right) F_2 \left( y, Q^2 \right) + 2 \sum_{y} \left\{ \sum_{x} dy \left( \frac{y}{y} \right) R \left( \frac{x}{y} \right) yg \left( y, Q^2 \right) \right\} \right\} + O(\alpha_s^2) \]
Same side jets at large $y$ come from $g_{\text{small } x} q_{\text{val } \text{large } x}$

$x_1 x_2 \approx \frac{(2 p_T)^2}{s}$

$\approx \frac{(60)^2}{1800}$

$x_1 x_2 \approx 10^{-3}$

0.2 val.

5. $10^{-3}$ gluon

CDF dijet data (also D0)

27 GeV $< E_T < 60$ GeV

$|y_1| = |y_2| = y$

CDF: prelim. data

MRS use "Ellis, Kunszt, Soper" to estimate $O(x^3)$ effects

$R_1$ compares $\sigma($dijet$)$ at diff dijet c.m. kinematics - scale dep

Better to use $R_2 = \frac{\sigma(\text{same side})}{\sigma(\text{central})} = \frac{\text{same c.m. kin.}}{\text{}}$

(Are expected systematic error?)
Now to small $x \to$  

We wish to understand why pQCD predicts rise as $x \to 0$.  

à la Jan Kureckinski
$F_2$ from H1

$Q^2 = 8.5 \text{ GeV}^2$

$Q^2 = 15 \text{ GeV}^2$

$Q^2 = 30 \text{ GeV}^2$

$Q^2 = 60 \text{ GeV}^2$

- MRS D^m
- MRS D^v
- CTEQ1MS
- GRV
- DOLA
$$F_2 \text{ (Hera)}$$

$Q^2 = 120$

$Q^2 = 240$

$Q^2 = 30$

$Q^2 = 60$

$F_2 \text{ from ZEUS}$
A commentary on small $x$

Altarelli-Parisi evolution and LL($Q^2$) summation

To prepare for the study of the small $x$ region it is helpful to return to the (leading order) Altarelli-Parisi or GLAP equations and to show how they effectively resum the leading log$Q^2$ (LL($Q^2$)) terms. For simplicity we take fixed $\alpha_s$, and consider a non-singlet quark distribution (e.g. $u - \bar{u}$), since it decouples from the evolution of the gluon distribution. We show that Altarelli-Parisi equation factorizes in moment space and is directly amenable to analytic solution.

Dokshitzer [1] proved that, in an axial gauge, the LL($Q^2$) sum comes from the sum of ladder diagrams; the diagram with $n$ gluon rungs corresponding to the $(\alpha_s \log Q^2)^n$ contribution. (An axial gauge is one in which the gluon has only two physical polarization states - in such a gauge we do not need unphysical “ghost” contributions which in general would be required to cancel the scalar polarization component of the gluon). Here we give a heuristic discussion of how the Altarelli-Parisi equation is equivalent to the LL($Q^2$) sum of ladder diagrams. We find that the LL($Q^2$) contributions come from the region where the transverse momenta are strongly ordered along the ladder of Fig. 1

$$Q^2 \gg k_{nT}^2 \gg \ldots \gg k_{1T}^2.$$  

Further discussion of the origin of the ladder structure can be found, for example, in Chapter 1 of ref. [2].

Fig. 1

Fig. 2
DLL approximation

When we go to small $x$ then we encounter new logarithmic effects associated with $\log(1/x)$ contributions. These need to be resummed. We consider the double leading logarithmic (DLL) approximation of the Altarelli-Parisi equation for the gluon (which is the dominant parton at small $x$). This involves the summation of the DLL contributions $\alpha_s \log Q^2 \log(1/x)$. We find

$$xg(x, Q^2) \sim \exp \left( 2 \left[ \frac{3 \alpha_s}{\pi} \log Q^2 \log \left( \frac{1}{x} \right) \right]^{\frac{1}{2}} \right),$$

up to slowly varying logarithmic factors. This behaviour is identified with the sum of ladder diagrams where both the transverse and the (fractional) longitudinal gluon momenta are strongly-ordered along the ladder of Fig. 2

$$Q^2 \gg k_{nT} \gg \ldots \gg k_{1T},$$

$$x \ll x_{n-1} \ll \ldots \ll x_1.$$  

Aside: The above DLL form applies for evolution from a non-singular starting distribution such as $xg(x, Q_0^2) \sim \text{constant}$, as $x \to 0$. An example is the GRV partons [3] which evolve from a very low scale, $Q_0^2 = 0.3$ GeV$^2$, and develop a DLL form in the HERA regime. However for a singular form such as $xg(x, Q_0^2) \sim x^{-\lambda}$ with $\lambda = \frac{1}{2}$, as in MRS($D^\perp$) [4], the singular behaviour is stable to evolution and overrides the DLL form – then in moment space $\tilde{g}(n, Q_0^2) \sim 1/(n - \lambda - 1)$ has the leading singularity in the $n$ plane, see p. 66.

BFKL equation and LL($1/x$) summation

If we are interested in small $x$ but moderate $Q^2$, then the BFKL or Lipatov equation [5] is appropriate. We must sum the leading $\alpha_s \log(1/x)$, i.e. LL($1/x$), terms but keep the full $Q^2$ dependence, not just the LL($Q^2$) terms. The strong-ordering of the gluon transverse momenta, associated with LL($Q^2$), is no longer appropriate, and we have to integrate over the full $k_T$ phase space. As in the other limits, we can picture the LL($1/x$) behaviour as a sum of “ladder” diagrams, but now the QCD calculation is more involved. The resulting structure does indeed look like a summation of ladder diagrams but actually they are only an effective representation for a whole set of Feynman diagrams, most of which are of non-ladder form, that were originally summed by BFKL [5]. The virtual contributions can be shown to collectively amount to the reggeization of the $t$-channel (i.e. “vertical”) gluon exchanges in the ladder. Thus somewhat miraculously both the real and virtual contributions reduce to a ladder form with exactly the correct colour structure. Further details can be found in, for example, ref. [6].
We summarize the situation in Fig. 3.

Figure 3: The three different limits of $g(x, Q^2)$ in the log$Q^2$, log$(1/x)$ plane: (i) the LL($Q^2$) region where $\alpha_s \log Q^2 \sim 1$ but $\alpha_s \log (1/x)$ is small, reached by Altarelli-Parisi or GLAP evolution from input distributions $f_i(x, Q_0^2)$, (ii) the DLL($Q^2, 1/x$) region where $\alpha_s \log Q^2 \log (1/x) \sim 1$ but $\alpha_s \log Q^2$ and $\alpha_s \log (1/x)$ are both small, and (iii) the LL($1/x$) limit where $\alpha_s \log (1/x) \sim 1$ but $\alpha_s \log Q^2$ is small, reached by evolution from $g(x_0, Q^2)$ using the Lipatov or BFKL equation. We also indicate the regions giving the dominant contributions in the integrations over the phase space of the emitted gluons; these are strongly-ordered in the longitudinal and/or transverse momenta of the gluons along the chain.

As mentioned earlier the next-to-leading order, NLL($Q^2$), contribution to the Altarelli-Parisi or GLAP evolution is known. The same is not true for BFKL, which is a LL($1/x$) equation. Much theoretical effort is being devoted to obtaining the NLL($1/x$) terms.

Another relevant development, which is not covered in the lectures, is the unified equation derived by Marchesini et al. [7]. This equation reduces to the BFKL equation at small $x$ and to the Altarelli-Parisi equation at large $x$. The virtual corrections at small $x$ are included by means of a so-called
“non-Sudakov” form factor. The implications of this unified equation are under study.

Solutions of the BFKL or Lipatov equation

In addition to attempting to amplify the various evolutions shown in Fig. 3, the lectures also explore the solutions of the BFKL equation. For fixed $\alpha_s$, there is an analytic expression for the leading behaviour of the (unintegrated) gluon distribution, $f$, at small $x$. The solution has the form

$$f(x, k_T^2) \sim x^{-\lambda}(k_T^2)^{1/2} \exp \left( \frac{-\log^2(k_T^2/k_f^2)}{\log(1/x)} \right)$$

with $\lambda = (3\alpha_s/\pi)4\log2 \approx 0.5$, which displays (i) the advertised $x^{-\lambda}$ singular small $x$ behaviour and (ii) a Gaussian structure in $\log k_T^2$ that broadens (diffuses) with decreasing $x$. This diffusion in $k_T$ arises because of the relaxation of the strong-ordering which leads to a “random walk” in $k_T$ as we proceed along the gluon ladder. Further details can be found, for example, in refs. [8,9].

For running $\alpha_s$, numerical solutions of the BFKL equation have been obtained [10]. These were used to predict [11] the behaviour of $F_2(x, Q^2)$ at small $x$. A singular $x^{-\lambda}$ form with $\lambda \approx 0.5$ was found. This dramatic growth with decreasing $x$ has recently been observed by the H1 [12] and ZEUS [13] experiments at HERA. It is a surprise that this perturbative QCD behaviour appears to shine through possible corrective effects so precociously and clearly.

Parton shadowing and the GLR equation

The increase $f \sim x^{-\lambda}$ or $2g \sim x^{-\lambda}$, as $x$ decreases, cannot go on indefinitely. If the density of gluons within the proton becomes too large they can no longer be treated as free partons. The growth, as $x \to 0$, must eventually be suppressed by gluon recombination. By considering QCD Feynman diagrams which become important at small $x$, these $2g \to g$ recombination effects were estimated some time ago by GLR [6], and a little later by Mueller and Qiu [14]. To a first approximation the normal evolution is corrected by including a quadratic term in which the gluon ladder branches into two ladders which then couple to the proton. The resulting “GLR equation” has the symbolic form

$$\Delta g = P \otimes g - V \otimes g^2$$

where the extra term with the minus sign reflects the suppression due to recombination or shadowing. Recently it has been established [15] that there
are interactions between gluons in the recombining ladders, which put the validity of the GLR equation into doubt. However it appears reasonable to use an “enhanced” form [16] of the $V \otimes g^2$ term to estimate the onset of shadowing, see transparency 73a. Numerical studies [8,11] indicate that shadowing effects will be small in the HERA regime, unless gluons are concentrated in small “hot-spots” within the proton, see transparency 78.

**DIS + jet events**

The idea that these events are, in principle, an ideal way to identify the $x^{-\lambda}$ BFKL behaviour is due to Mueller [17], see transparency 81. Numerical QCD estimates can be found in refs. [18,19].

**The Pomeron and diffractive scattering**

More than 30 years ago Pomeranchuk predicted that hadronic total cross sections would approach a constant asymptotic limit. The Regge trajectory whose exchange ensures this behaviour become known as the Pomeron, with intercept $\alpha(t = 0) = 1$ so that $\sigma_{\text{tot}} \sim s^{\alpha - 1} \sim \text{constant}$. Even today the observed slowly rising (high energy) total cross sections and forward elastic scattering shapes are remarkably well described [20] by an effective trajectory $\alpha(t) = 1.08 + \alpha' t$ with $\alpha' = 0.25 \text{ GeV}^{-2}$ representing, not just single, but multiple Pomeron exchange. This “soft” or “non-perturbative” Pomeron, which accounts for the behaviour of the $Q^2 = 0$ photon-proton total cross section $\sigma_{\text{tot}}(\gamma p) \sim s^{0.08}$, should be contrasted with the so-called “BFKL” or “perturbative QCD” Pomeron, which for $x \lesssim 10^{-3}$ would imply the $Q^2 \approx 10 \text{ GeV}^2$ virtual-photon-proton cross-section behaves as

\[ \sigma_{\text{tot}}(\gamma^* p) \sim xq \sim x^{-\lambda} \sim s^{-1} \sim s^{0.5}, \]

see transparency 40. Of course at sufficiently high $s$, these power law behaviours must be suppressed by unitarity/shadowing effects.

More recently interest in Pomeron physics has been revived by studies of “diffractive events” which contain a rapidity gap in the final state, where hadrons produced in the collision only populate part of the detector away from the outgoing proton direction. A supplementary condition for the presence of (“soft”) Pomeron exchange is that there should be a slow variation of the cross section as a function of the width of the rapidity gap. Ingelman and Schlein [21] anticipated that there would be “hard” diffractive events in which the final state contains jets as well as a rapidity gap, and proposed a model in which the Pomeron is made up of quarks and gluons which take part
in the hard scattering. Subsequently the UA8 experiment at the CERN $p\bar{p}$ collider observed such events. Now the ZEUS and H1 experiments at HERA are observing diffractive events in deep inelastic scattering at a rate of about 5-10%. It remains to be seen whether the Pomeron constituent models can explain the rate and shape of such events. Clearly HERA will be a probe to help study the fundamental nature of the Pomeron.

Finally, further general discussions of small $x$ physics can be found in refs. [6,22].

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Also articles in
Moments:  \( A-P \) equation sums leading \( \log Q^2 \)

For simplicity, consider \( q \) non-singlet, and fixed \( \alpha_s \)

\[
A-P \text{ eq.: } \frac{dq(x,Q^2)}{d\log Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} P_{qq}(\frac{y}{x}) q(y,Q^2)
\]

\[
= \frac{\alpha_s}{2\pi} \int_0^1 dy \int_0^1 dz \delta(x-yz) P(x) q(y,Q^2)
\]

**Take moments — then eq. factorizes**

\[
\int_0^1 dx \frac{dx}{x} \frac{dq(x,Q^2)}{d\log Q^2} = \frac{\alpha_s}{2\pi} \int_0^1 dy \int_0^1 dz \int_0^1 \frac{dx}{x} \frac{dx}{\alpha_s} z^n \delta(x-yz) P(z) q(y,Q^2)
\]

\[
= \frac{\alpha_s}{2\pi} \int_0^1 dz \frac{z^n}{z} P(z) \int_0^1 \frac{dy}{y} \frac{y^n}{\alpha_s} q(y,Q^2)
\]

\[
\frac{dM_n(Q^2)}{d\log Q^2} = \frac{\alpha_s}{2\pi} A_n M_m(Q^2)
\]

\[
\therefore \quad M_n(Q^2) = c_n \exp \left( \frac{\gamma_n(\alpha_s)}{\alpha_s} \log Q^2 \right) = c_n (Q^2)^{\gamma_n}
\]

\[
\gamma_n(\alpha_s) = \frac{\alpha_s A_n}{2\pi}
\]

\[
\therefore \quad M_n(Q^2) = M_n(Q_o^2) \left[ \frac{Q^2}{Q_o^2} \right]^{\gamma_n}
\]

Aside: Can recover \( q(x,Q^2) \) by inverse (Mellin) transform

\[
q(x,Q^2) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} M(n,Q^2) x^{-n} \, dn
\]

Choose \( c \) to right of singularities in \( M(n,Q^2) \)
Another aside: incorporate running $\alpha_s$

Method 1:

\[ \frac{dM_n}{d\log Q^2} = \frac{\alpha_s}{2\pi} A_n M_n = \frac{1}{2\pi b_0 \log \frac{Q^2}{\Lambda^2}} A_n M_n \]

Integrate →

\[ \log M_n = \frac{A_n}{2\pi b_0} \log \left( \log \frac{Q^2}{\Lambda^2} \right) + \log c_n \]

\[ M_n = c_n \left[ \log \frac{Q^2}{\Lambda^2} \right]^{A_n/2\pi b_0} \]

\[ M_n(Q^2) = M_n(Q_0^2) \left[ \frac{\alpha(Q^2)}{\alpha(Q_0^2)} \right]^{A_n/2\pi b_0} \]

Method 2:

\[ M_n = c_n \exp \left( y_n(\alpha_s) \log Q^2 \right) \]

\[ = c_n \exp \int Q^2 \frac{\gamma_n(\alpha_s)}{Q^2} d\frac{Q^2}{Q} \]

\[ = c_n \exp \int \frac{\nu_n(Q^2)}{\beta(\alpha_s)} \frac{\gamma_n(\alpha_s)}{\beta(\alpha_s)} d\alpha_s \]

\[ y_n = \frac{A_n}{2\pi} \alpha_s + \ldots \]

\[ \beta = -b_0 \alpha_s^2 + \ldots \]

Hints at RGE for $M_n$:

\[ \left[ -\frac{\partial}{\partial \alpha_s} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \gamma_n(\alpha_s) \right] M_n(\frac{Q^2}{\mu^2}, \alpha_s) = 0 \]

Solution is

\[ M_n(\frac{Q^2}{\mu^2}, \alpha_s) = M_n(1, \alpha_s(Q^2)) \exp \left\{ \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{\nu_n(\alpha)}{\beta(\alpha)} d\alpha \right\} \]

now sums all logs.
Altarelli-Parisi in ladder form (fixed $\Lambda$ for simplicity)

$$M_n(Q^2) = c_n \exp \left( \gamma_n \log Q^2 \right) = c_n \sum \frac{1}{x_1^{n+1}} (\gamma_n \log Q^2)^n$$

Claim: $n^{th}$ term $\leftrightarrow$ ladder with $n$ runs

Must use axial gauge in which gluon has only two (physical) states of polarizability.

Start with 1-rung

Recall: $L_L(Q^2)$

$$\frac{1}{x} \frac{d}{dy} q(x) \frac{Q^2}{k_T^2}$$

$$q(x,q^2) = \int_0^1 dx \int dy \frac{\delta(x-y) q(y)}{y} \frac{\alpha_s}{\pi} \frac{Q^2}{k_T^2} P(z) \log Q^2$$

Take moments:

$$M_n(Q^2) = \sum \frac{d}{dy} q(x) \int_n^{Q^2} \frac{Q^2}{k_T^2}$$

$$\frac{\alpha_s A_n}{\pi} \frac{Q^2}{k_T^2}$$

checks.

2-rung

transv: $L_L(Q^2)$ come from $Q^2 \gg k_T^2 \gg k_T^2$

$$\int Q^2 \frac{d}{dy} q(x) \int k_T^2 \frac{d}{dy} q(y) = \int d\log k_T^2 \log k_T^2 = \frac{1}{2} (\log Q^2)^2$$

Take moments:

$$A_n \int_0^1 \frac{d}{dy} q(x) \int x^n F(x)$$

moment of 1-rung $= c_n A_n$

$$\therefore M_n(Q^2) = \frac{1}{2} c_n \left( \frac{\alpha_s}{\pi} A_n \log Q^2 \right)^2 = c_n \frac{1}{2} (\gamma_n \log Q^2)^2$$

checks.
Small $x$

gluon dominates

Approx.:  
- Keep only $g$
- $P_{gg} = 6 \left( \frac{1-x}{x} + \frac{1}{1-x} + x(1-x) \right) \approx \frac{6}{x}$

Find $\alpha_s \log^{-\frac{1}{2}}$ terms.
Must resum

Double Leading Log. Approx (Fixed $\alpha_s$)

A-P eq.
\[
\frac{d g(x, Q^2)}{d \log Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} P_{gg} \left( \frac{x}{y} \right) g(y, Q^2)
\]
\[
\frac{d x g(x, Q^2)}{d \log Q^2} = \frac{3\alpha_s}{\pi} \int_x^1 \frac{dy}{y} y g(y, Q^2)
\]

We will find the solution is
\[
x g(x, Q^2) \sim \exp \left\{ 2 \left[ \frac{3\alpha_s}{\pi} \log^{-\frac{1}{2}} \log Q^2 \right] \right\}
\]

grows faster than any power of $\log (\frac{1}{x})$ as $x \to 0$.  

if $x g(x, Q^2) \sim a_0$
**DLL from summing ladders**

Let $k^2_i = k^2_n$

Fix $\alpha_s$

1 rung

$$Q^2 \to x, k^2_i$$

$$x g(x, Q^2) = \int \frac{dQ^2}{Q^2} \frac{\alpha_s(k^2_i)}{2\pi} \int \frac{dy}{y} \left( \frac{y}{x} \right)^{\frac{1}{2}} g(y)$$

$$x g(x, Q^2) = \frac{3\alpha_s}{\pi} \int \frac{dQ^2}{Q^2} \int \frac{dy}{y} y g(y)$$

**N rungs**

Transverse: LL($Q^2$) from $Q^2 \gg k^2_i \gg \ldots$

$$(\frac{3\alpha_s}{\pi}) n \int \frac{dQ^2}{Q^2} \int \frac{d^2 k_n}{k^2_n} \ldots \int \frac{d^2 k_{n-1}}{k^2_{n-1}} \ldots \int \frac{d^2 k_1}{k^2_1}$$

$$= \frac{1}{n!} \left( \frac{3\alpha_s}{\pi} \log \frac{Q^2}{Q_0^2} \right)^n \sim \frac{1}{n!} \text{ from nested } \int s$$

Longitudinal: LL($1/x$) from $x << x_{n-1} \ll \ldots \ll y$

$$\int \frac{dx_{n-1}}{x_{n-1}} \ldots \int \frac{dx_1}{x_1} \int \frac{dy}{y} g(y) = \frac{1}{n!} \left( \log \frac{1}{x} \right)^n C_0$$

Sum:

$$x g(x, Q^2) = \sum \left( \frac{1}{n!} \right)^2 \left( \frac{3\alpha_s}{\pi} \log \frac{Q^2}{Q_0^2} \log \frac{1}{x} \right)^n C_0$$

Use:

$$\sum \left( \frac{1}{n!} \right)^2 \left( \frac{y}{x} \right)^n = I_0(\alpha) \sim \frac{e^\alpha}{\sqrt{2\pi} n}$$

$$x g(x, Q^2) \sim C_0 \exp \left( 2 \left[ \frac{3\alpha_s}{\pi} \log \frac{Q^2}{x} \log \frac{Q_0^2}{Q_0^2} \right]^\frac{1}{2} \right)$$

Aside: If $y g(y, Q^2) \sim y^{-\frac{1}{2}}$ then $x g(x, Q^2) \sim x^{-\frac{1}{2}}$
Alternatively, DLL from moments

To solve AP

eq. in DLLA

\[ \frac{d \tilde{g}(x, Q^2)}{d \log Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \tilde{g}(y, Q^2) \tilde{g}(y, Q^2) \]

\[ \frac{d \tilde{g}(n, Q^2)}{d \log Q^2} = \frac{\alpha_s}{2\pi} A_n \tilde{g}(n, Q^2) \]

\[ A_n = \int_0^1 \frac{dz}{z} z^n P_Q(z) \int_0^1 \frac{dy}{y} y^n g(y, Q^2) \]

\[ \approx 6 \int_0^1 dz z^{n-2} \]

\[ t = \log \frac{Q^2}{Q_0^2} \]

\[ \approx \frac{6}{n-1} \]

\[ \tilde{g}(n, Q^2) = \tilde{g}(n, Q_0^2) \exp \left( \frac{3\alpha_s t}{n-1} \right) \]

\[ x g(x, Q^2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dn}{2\pi i} \frac{1}{x^{n-1}} \tilde{g}(n, Q^2) \]

\[ = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp \left\{ (n-1) \log \frac{1}{x} + \frac{3\alpha_s t}{n-1} \right\} \tilde{g}(n, Q_0^2) \]

\[ \equiv H(n) \]

\[ \text{For large } \log \frac{1}{x} \text{ and } t \text{ estimate by expanding about saddle point of } H: \]

\[ \frac{dH}{dn} = \log \left( \frac{1}{x} \right) - \frac{3\alpha_s t}{(n-1)^2} = 0 \text{ when } (n-1)^2 = \frac{3\alpha_s t}{\pi \log (1/x)} \]

\[ H \approx H(n_0) + K(n-n_0)^2 \]

\[ = H(n_0) - K \nu^2 \]

\[ \text{where } n-n_0 = i\nu \]

\[ x g(x, Q^2) \sim \exp \left\{ \int_x^1 \log \frac{1}{x} + \frac{3\alpha_s t}{n-1} \right\} \]

\[ \sim \exp \left\{ 2 \left[ \left( \frac{3\alpha_s}{\pi} \log \frac{1}{x} \log \frac{Q^2}{Q_0^2} \right)^{\frac{1}{2}} \right] \right\} \text{ as before} \]
LL($\frac{1}{x}$) sum. Lipatov (or BFKL) kernel

Small $x$, but moderate $Q^2$

DLL approx. (of APEq.) only included LL($\frac{1}{x}$) approx. by LL($Q^2$).
For finite $Q^2$ must sum LL($\frac{1}{x}$) but keep full $Q^2$ dep.,
not just LL($Q^2$).

Clearly, must relax strong ordering of $k_T$'s which
gave LL($Q^2$) $(\ln^2(a^2)/(n!))$ behaviour — and
integrate over full $k_T$ phase space.

Need to unfold (last) $k_T^2$ integration and work
with unintegrated gluon distribution $f(x,k_T^2)$

$$xg(x,Q^2) = \int_{Q^2}^{\infty} \frac{dk_T^2}{k_T^2} f(x,k_T^2)$$

Recursion relation: n-ming contrib. $f_n$ in terms of $f_{n-1}$

$$f_n(x,k^2) = \int_{x}^{1} \frac{dx'}{x'} \int dk_T^2 K(k,k') f_{n-1}(x',k_T^2)$$

Lipatov kernel:

$$K(k,k') = \frac{3\alpha_s}{\pi} k^2 \left\{ \frac{1}{k^2 |k^2-k'^2|} - \beta(k^2) \delta(k^2-k'^2) \right\}$$

where

$$\beta(k^2) = \int \frac{dk_T^2}{k_T^2} \left\{ \frac{1}{|k^2-k'^2|} - \frac{1}{(4k_T^4+k^4)^{1/2}} \right\}$$

Apparent $k^2=k_T^2$ singularity cancels: real $\leftrightarrow$ virtual
Aside: BFKL kernel is clever

• real emissions

\[
\begin{align*}
\text{all sum to} & \quad \frac{k^2}{-k_T^2} \quad \text{with vertex factor} \\
\text{give effective diagram} & \quad \frac{3\alpha_s}{\pi} \frac{k_T^2 k_T'^2}{q_T^2} \\
& \quad \text{and correct colour structure!}
\end{align*}
\]

• virtual emissions (same order)

\[
\text{overall effect of virtual contributions is simply to modify above real emission diagram by Reggeizing the gluon exchanged } \left( \frac{s}{k_T^2} \rightarrow \frac{\alpha_s(k_T^2)}{-k_T^2} \right) \text{ Again colour structure is correct.}
\]

Conclude: Must regard as effective ladder diagram incorporating many different contributions
Aside: DLL reproduced

\[ k^2 \gg k'^2 \quad \text{so} \quad K(k,k') = \frac{3\nu s}{\pi} \frac{1}{k'^2} \]

recursion relation becomes

\[ f_n \sim \frac{3\nu s}{\pi} \frac{1}{n!} \log \left( \frac{1}{x} \right) n \log \left( \frac{k'^2}{\mu^2} \right) f_{n-1} \]

so

\[ f = \sum f_n \sim \exp \left( 2 \left[ \frac{3\nu s}{\pi} \log \left( \frac{1}{x} \right) \log \left( \frac{k'^2}{\mu^2} \right) \right]^{1/2} \right) \]

LL(1/x) from recursion relation

Not so simple \( K(k,k') \)

We will find

\[ f(x,k^2) \sim k(x^2) x^{-\lambda} \]

where \( \lambda \) is max. evalve of \( K \)

Insight from toy model

Assume \( K \) factorises

\[ K(k,k') = \mu(k) \nu(k') \]

(Not true)

\[ f_n(x,k^2) = \frac{\mu(k)}{x} \int_0^1 \frac{dx'}{x'} \int dk'^2 \frac{\nu(k') f_{n-1}(x',k'^2)}{tm(x') \mu(k') tn_{n-1}(x')} \]

\[ t_m(x) = \lambda \int_0^1 \frac{dx'}{x'} t_{n-1}(x') \quad \text{where} \quad \lambda = \int dk'^2 \nu(k') \mu(k') \]

\[ t_m(x) \sim \frac{2^m}{n!} \ln^n \left( \frac{1}{x} \right) \]

\[ f = \sum f_n = \mu(k) e^{\lambda \ln(1/x)} = \mu(k) x^{-\lambda} \]

\( \lambda \) is an evalve of \( K \)

\[ K \otimes u = \int dk'^2 \mu(k') \nu(k') u(k') = \lambda u \]

Max \( \lambda \) dominates small \( x \)
Lipatov (or BFKL) equation

Recursion:

\[ f_n(x, k^2) = \int_1^x \frac{d\alpha'}{\alpha'} \int d\alpha'' K(\alpha', \alpha'') f(\alpha'', k^2) \]

\[ \sum_{n=0}^{\infty} f_n = A \otimes f_{n-1} \]

\[ f = \sum_{n=0}^{\infty} f_n \]

Integral form

\[ f(x, k^2) = f_0(x_0, k^2) + \int_1^x \frac{d\alpha'}{\alpha'} \int d\alpha'' K(\alpha', \alpha'') f(\alpha'', k^2) \]

Differential form

\[ -x \frac{df}{dx} = \int d\alpha'' K(\alpha', \alpha'') f(\alpha', k^2) \]

Lipatov or BFKL eq

\[ \frac{\partial f}{\partial \log(\lambda x)} = K \otimes f = \lambda f \]

\[ f \sim e^{\lambda \log(\lambda x)} \sim x^{-\lambda} \]

For fixed \( \alpha_s \) can solve analytically (see next page)

\[ f(x, k^2) \sim \left( \frac{x}{x_0} \right)^{-\lambda} \frac{\exp \left( -\frac{\log^2(k^2, k_0^2)}{2 \pi^2 \log(\alpha_s, x_0)} + A \right)}{2 \pi^2 \log(\alpha_s, x_0) + A} \]

\[ \lambda = \frac{3\alpha_s}{\pi} 4 \log 2 \approx 0.5 \]

\[ \lambda'' = \frac{3\alpha_s}{\pi} 28 f(3) \approx \ln x_0 \]

Need \( k_0^2 \) cut-off.

Width of Gaussian

Diffusion into dangerous IR region.
Solution of BFKL equation for fixed $\alpha_s$ (compare p.66)

$$\frac{\partial f(x,k^2)}{\partial \log (1/x)} = \frac{3\alpha_s}{\pi} k^2 \int_0^\infty \frac{dk'^2}{k'^2} \left[ \frac{f(x,k'^2) - f(x,k^2)}{|k^2 - k'^2|} + \frac{f(x,k^2)}{(4k'^4 + k^4)^{1/2}} \right]$$

- **Step 1**
  - Take moments
  - $\tilde{f}(x,\omega) \equiv \int_0^\infty \omega^{-1} f(x,k^2) \, dk^2$
  - Factorizes
  - Then: $\frac{\partial \tilde{f}(x,\omega)}{\partial \log (1/x)} = \tilde{K}(\omega) \tilde{f}(x,\omega)$

- **Step 2**
  - Perform integral for coefficients

- **Step 3**
  - Solve

- **Step 4**
  - Inverse transform

- **Step 5**
  - Saddle pt integral

\[ \tilde{f}(x,\omega) = \tilde{f}(x_0,\omega) e^{\tilde{K}(\omega) \log \frac{x}{x_0}} = \tilde{f}(x_0,\omega) \left( \frac{x}{x_0} \right)^{-\tilde{K}(\omega)} \]

\[ \tilde{K}(\frac{1}{2} + i\omega) = \lambda - \frac{1}{2} \lambda'' \omega^2 \]

\[ \lambda = \frac{3\alpha_s}{\pi} \ln 2 \]

\[ \lambda'' = \frac{3\alpha_s}{\pi} 2 \theta \Gamma(3) \]

\[ \tilde{f}(x_0,\frac{1}{2}) \left[ 1 + \left( \omega - \frac{1}{2} \right) \frac{1}{2} \frac{d^2}{d\omega^2} - \ldots \right] e^{-\log \frac{k^2}{\omega} - \frac{1}{2} \frac{d^2}{d\omega^2} - \ldots} \equiv A \]

\[ \int_{-\infty}^{\infty} e^{-a\omega + ib}\, d\omega = \int_{-\infty}^{\infty} dv \, e^{-a(v + \frac{1}{2} \frac{b^2}{2a})^2} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \]

\[ f(x,k^2) = \left( \frac{k_0}{x} \right)^{-\lambda} \tilde{f}(x_0,\frac{1}{2}) \left( \frac{k}{k_0} \right)^{1/2} \exp \left( \frac{-\log \frac{k^2}{k_0^2}}{2\pi \lambda'' \log \left( \frac{k^2}{k_0^2} \right) + A} \right) \]

87
KMS

Detailed numerical solution of BFKL eq. in small $x$ region confirms $x^{-\lambda}$ form of analytic (leading behaviour) estimate. Explicitly, solve

$$\frac{df(x, k^2)}{dx} = \int_{k_0^2}^{k^2} dk' K(k^2, k'^2) f(x, k'^2)$$

use running $\alpha_s$

Solve from starting distribution $f(x_0=0.01, k_0^2)$ determined from MRS(D) gluon

infrared cut-off needed to prevent diffusion into infrared region of large $\alpha_s$

Find $f(x, k^2) \to C x^{-\lambda}$ very soon as $x$ decreases

$\lambda \sim 0.4-0.5$ weakly dep. on $k_0^2$ more sensitive to choice of $k_0^2$

$$x g(x, Q^2) = \int_{k_0^2}^{Q^2} \frac{dk_T^2}{k_T^2} f(x, k_T^2) \sim x^{-\lambda}$$

P.S. See solution when we have discussed and included shadowing.
Parton saturation / shadowing

$q$ growth as $x \to 0$ must eventually be tamed by $q$ recombination

\[
\begin{align*}
&x = x_0 \\
&x \ll x_0
\end{align*}
\]

\[
\frac{xp}{Q} \gg 1 \\
\frac{1}{xp} \ll \frac{1}{Q}
\]

long $\ll$ trans. size

Estimate of when shadowing is important:

\[ g(x,Q^2) = \text{no. density of gluons with transverse size } \sim 1/Q \]

\[ N = \text{no. of gluons/unit of rap. } \sim x g(x,Q^2) \]

$qg$ cross section \[ \hat{\sigma} \sim \frac{\alpha_s(Q^2)}{Q^2} \]

Crucial parameter:

\[ W' \sim \frac{N \hat{\sigma}}{\pi R^2} \sim \frac{\alpha_s(Q^2)}{\pi R^2 Q^2} xg(x,Q^2) \]

If $W' \ll 1$ can neglect shadowing
Shadowing leads to additional (non-linear) terms in evolution equation

\[ \frac{\partial (xg)}{\partial \log Q^2} = P_{gg} \otimes g + P_{gq} \otimes q - \frac{81 \alpha_s^2}{16 R^2 Q^2} \int \frac{dx'}{x} \left[ x' g \right]^2 \]

LO: quadratic in $g$

Similar addition to Lipatov eq.

Gribov, Levin, Ryarkin, Mueller, Qiu, Collins, Kwiecinski

Quantitatively, $R$ is the crucial parameter - says how the gluons are concentrated in proton:

\[ R \approx 5 \text{ GeV}^{-1} \quad \rightarrow \quad R \approx 2 \text{ GeV}^{-1} \]

gluons uniformly spread across proton  

Above eq. (LO shift) is only valid if additional term is small:

\[ W = \frac{C_{\text{quadratic}}}{C_{\text{linear}}} \leq O(\alpha_s) \]
**GLR equation**

- **Normal evolution**
- **Additional GLR shadowing term**: $2g + g$ recomb.
- **Called "FAN" diagram**
- **Iteration via eq., produces whole series of fan diagrams**

**Simplifying assumptions**:

(i) Form of vertex which combines $2 \rightarrow 1$ ladder

(ii) Coupling of $n$ ladders to $p \sim (\text{single ladder})^n$

(iii) No interaction between the gluon ladders which recombine. True only to $O\left(\frac{1}{N_c^2}\right) \sim 0.1$

GLR eq. probably only valid in weak shadowing limit and then effect of 10% coupling:

$\Rightarrow$ quadratic term $\times 1.7$  
Bartels, Ryskin

This is not so bad since

(i) shows onset of shadowing

(ii) $\frac{1}{R^2}$ dep. is much larger uncertainty.
pQCD applicable below dotted line for sufficiently large $Q^2$. 

Non pert. Regge region

BFKL eq.

AP eq.

$\log Q^2$

$W = 0(\xi)$

$M = g^2\, \text{threshold}$

$M > 0(\xi)$

Critical line
- numerical sol. of Lip. eq. with shadowing included i.e. GLR eq. Kwiecinski M Sutton

\[ \text{xg} \]

\[ \begin{align*}
\text{Q}^2 = 4 \text{ GeV}^2 & \quad \text{Q}^2 = 20 \text{ GeV}^2 \\
\text{unshad.} & \quad \text{unshad.} \\
R = 5 & \quad R = 5 \\
R = 2 \\
(\text{hot spot}) & \quad R = 2
\end{align*} \]

\[ \begin{align*}
\text{C}_{\text{quad}} / C_{\text{lin.}} & \quad \text{C}_{\text{quad}} / C_{\text{lin.}} \\
R = 5 & \quad R = 5 \\
R = 2 & \quad R = 2 \\
\text{HERA} & \quad \text{HERA}
\end{align*} \]

- \( x^{-\lambda} \) apparent in unshad. gluon \( \rightarrow \) we see \( \lambda \leq 0.5 \)
- Shadowing of gluon small in HERA regime (unless hot-spots), but recall \( F_2 \) measures ~ sea ; since \( g \rightarrow (q\bar{q}) \) sea shadowing should be less in \( F_2 \).
QCD prediction of $F_2$ at small $x$

\[ F_2 = f \otimes F^{(0)} + F_2 \text{non-Lipatov} \]

\[ F_2(x, Q^2) = \int_0^1 \frac{dz}{z} \int \frac{dk_T^2}{k_T^4} f \left( \frac{x}{z}, k_T^2 \right) F_2^{(0)} \left( x', k_T^2, Q^2 \right) \]

\[ f / k_T^2 \sim C x^{-\lambda} \quad \text{feeds through} \quad \frac{F_2(x, Q^2)}{Q^{2z}} \sim C' x^{-\lambda} \]

Known as $k_T$ factorization formula

Sensitive to infrared region

\[ \text{(unintegrated) gluon obtained from numerically solving BFKL equation.)} \]
Prediction, not extrapolation

$F_2^{ep}(x, Q^2 = 30 \text{GeV}^2)$

HERA data

$\alpha_S = 1 \text{GeV}^2$

$\alpha_S = 2$

$R = 5 \text{ (conv)}$

$R = 2 \text{ GeV}^{-1}$

(pQCD) = No shad.

With shad. (AKMS)
"Shape" $\lambda$ is much less sensitive to infrared than normalisation $C$

$$F_2 = C x^{-\lambda} + F_2^{\text{non-lip.}}$$

\begin{itemize}
  \item $k_a^2 = 1 \text{GeV}^2$
  \item $k_a^2 = 2$
  \item $k_a^2 = 1$ shad. \( R = 5 \text{GeV}^{-1} \) "conv."
  \item $R = 2 \text{GeV}^{-1} \) "hot spot"
\end{itemize}

From HERA data:

- $Q^2 = 15 \text{GeV}^2$
- $Q^2 = 30 \text{GeV}^2$
Identification of $x^{-\lambda}$ behaviour

Measurements at HERA of $F_2 \to$ sea at small $x$
$F_2 \to$ gluon

but interpretation needs $f_i(x, Q^2) \to$ "non-part" input
BFKL (or Lip.) eq. $\to$ predicts behaviour at very small $x$
(and so far only at leading log ($1/x$)).

Steep behaviour observed could be due to
(i) BFKL pQCD $x^{-\lambda}$
(ii) Steep non-part. input $x$ distribution
(this in itself would be remarkable)

Also
(iii) GRV partons: steep behaviour arises from
     long evolution length with A-P eqs. $\to$ in fact
     they evolve from valence-like input at very
     low scale $Q_0^2 \approx 0.3 \text{GeV}^2$
     Essentially they have DLL
     $xg \sim \exp \left[ 2 \left( \frac{3 \alpha_s}{\pi} \log \frac{1}{x} \log Q^2 \right)^2 \right]$ mimics $x^{-0.4}$
     at $Q^2 \sim 20 \text{GeV}^2$
(or rather running $\alpha_s$ form)

Idea was to originally use only val. quark input,
but now find need val. gluon & sea quarks! ?

* make model unattractive

Note: Other ways to identify $x^{-\lambda}$, like
     $e^+ e^-$, J/$\psi$, prompt $\phi$ prod. suffer from
     same reliance on non-part. input
     but...
DIS events containing measured jet are special \((x, Q^2)\) and \((x_j, k_T^2)\).

\[ q \rightarrow x_j, k_T^2 \quad (\text{JET}) \]

Choose \(k_T^2 \approx Q^2\) so neutralize evol. in \(Q^2\) focus on \(x\) dep.

\[ x_j \gg x \quad (\frac{x}{x_j})^{-\lambda} \]

Choose as large as poss. \(x_j \sim 0.1\)

\[ q + \frac{4}{6}(q + \bar{q}) \quad \text{known} \]

Measure differential st. fn.

\[ x_j \; k_T^2 \frac{\partial F_2}{\partial x_j \partial k_T^2} \sim \alpha_s(k_T^2) \left[ \sum_{a} x_j \hat{f}_a(x_j, k_T^2) \right] \left( \frac{x}{x_j} \right)^{-\lambda} \]
Diffusion in $k_T^2$ as proceed along chain shown below as width of gaussian:

(Bartel's 'cigar')
CONCLUDE  small $x$ physics is in its infancy.

- The dramatic growth of $F_2$ seen at HERA is very suggestive of BFKL pQCD effects (as predicted by AKMS) — but alternative interpretations based on conventional AP are still possible.

- Need to open up final state to make less inclusive measurements than $F_2$, e.g. DIS+jet, $E_T$ flow... (encounter jet recognition, hadronization.)

- **Exp:** HERA is only just starting to probe small $x$.
  Much more information to come.

  Concerning DIS: $F_2$, $F_L$, +jets
  - deep inelastic diffraction
  - heavy quark, $J/\Psi$ production
  - photon structure function...

- **Theory:** BFKL eq. just sums leading $\log(1/x)$ terms.
  Need NLO $\log(1/x) \Rightarrow$ Fadin, Lipatov, Bartels...
  Need solutions of unified eq. which reduce to

$$\begin{cases} 
\text{AP eq. at large } x & \Rightarrow \text{Marchalini, Webber} \\
\text{BFKL eq. at small } x & \Rightarrow \text{KMS}
\end{cases}$$

- GLR eq. for shadowing is based on approx.;
  validity in question, but probably OK to estimate onset of shadowing
  $\Rightarrow$ Bartels, Ryskin, Leri...