Beam Breakup in Bunch Trains

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Beam Breakup in Bunch Trains

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Abstract

Stability of bunch train is theoretically studied employing a simple macroparticle approach. Of interest here is a single-pass effect originating from the wake fields covering only single train of bunches. Equation of the macroparticle motion is derived taking into account the closed-orbit distortion generated by the wakes. It is found that betatron motion of a preceding bunch can give rise to non-exponential blowups in the amplitudes of the trailing-bunch oscillations. Several possibilities are examined to suppress the instability mechanism. Effect of radiation damping is also included in the present analysis.

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1 Introduction

Transverse mode coupling instability imposes rigorous restrictions on bunch current in LEP[1], limiting the achievable luminosity. To overcome this limitation, some possible schemes have been proposed and studied so far. Replacing a single bunch by a bunch train is one of the solutions which we might consider to increase the LEP luminosity[2].

Since spacing between neighboring bunches in a train should be taken relatively tight, e.g. about 74 m in LEP, there arises a possibility that an instability mechanism which has not been so serious in large circular machines plays a dominant role; namely single-pass instability due to the wake fields extending over only single train may severely take place, leading to the non-exponential-type beam blowup similar to BBU effect in linear accelerators[3].

In this report, we try to explore, based on a macroparticle model, stability of a bunch train circulating in a storage ring of the average radius R. The bunch train considered here consists of N_b bunches traveling at the speed of light c. The position of the n-th bunch centroid is located at the distance z_n (z_1=0) behind the first-bunch centroid. The distance between the m-th and n-th bunch is, accordingly, indicated by d_{nm}=|z_n-z_m|. We here concentrate our attention upon effect of the middle-range wake fields mentioned above, localized at RF cavities. Thus, the wakes originating from other sources, as well as long-range wakes, are neglected. Moreover, the cavity wakes are assumed to be damped away before arrival of the next train, since the single-pass effect is of interest. The RF cavities are sitting at s=s_1, s_2, ..., s_{N_c} where N_c is total cavity number, and s is distance along the design particle orbit.

2 Closed Orbit Distortions

Before proceeding to stability analysis, we first determine the closed orbits associated with individual bunches. Because of existence of the cavity wakes as well as of field imperfections and misalignments of the magnetic components installed along the ring circumference, the design closed orbit is distorted. In particular, the distorted closed orbit is bunch-dependent since different bunches see different wakes. Under the assumption we have made, the leading bunch is not kicked by the wake fields and, therefore, the closed-orbit distortion, Δx_1, can be given by the periodic solution to the equation

\[ \frac{d^2 \Delta x_1}{ds^2} + K(s)\Delta x_1 = F(s), \]  

(2.1)
where $K(s)$ corresponds to quadrupole focusing strength, and $F(s)$ is the periodic function, i.e. $F(s) = F(s + 2\pi R)$, characterizing the field errors and misalignments along the orbit. The periodic solution is readily found to be

$$\Delta x_i(s) = \sqrt{\beta(s)} \sum_{m=-\infty}^{\infty} \frac{v^2 f_m}{\nu^2 - m^2} e^{im\theta}, \quad (2.2)$$

where $\nu$ and $\beta(s)$ are, respectively, transverse tune and betatron function of the ring, the variable $\theta$ is related to the longitudinal coordinate $s$ as

$$\theta(s) = \frac{1}{\nu} \int_0^s ds', \quad (2.3)$$

and the Fourier coefficients are represented by

$$f_m = \frac{1}{2\pi \nu} \int_0^{2\pi R} \sqrt{\beta(s)} F(s) e^{-im\theta} ds. \quad (2.4)$$

Even if all RF cavities have been perfectly constructed and precisely aligned, the first bunch will still leave wakes behind it as far as there exists $F(s)$ causing a closed-orbit distortion. The second bunch is then kicked by the first-bunch wakes, leaving additional wake fields, and the wakes seen by the third bunch are now sum of the wakes left by the preceding two bunches, and so on. Thus the closed-orbit distortion of the $n$-th bunch, $\Delta x_n$, is derived from

$$\frac{d^2 \Delta x_n}{ds^2} + K(s) \Delta x_n = F(s) + \sum_{i=1}^{N} F^{(i)}_n(s) \delta_p(s - s_i), \quad (2.5)$$

where $\delta_p(s)$ denotes the periodic $\delta$-function with the periodicity $2\pi R$, and $F^{(i)}_n(s = s_i)$ is concerned with the kick force at the $i$-th cavity felt by the $n$-th bunch, given by

$$F^{(i)}_n(s) = \sum_{m=1}^{n-1} W^{(i)}_{nm} [\Delta x_m(s) + \Delta^{(i)}]. \quad (2.6)$$

Here, $\Delta^{(i)}$ stands for misalignment of the $i$-th cavity, and the coefficient $W^{(i)}_{nm}$ can be introduced, for a specific deflecting mode, as

$$W^{(i)}_{nm} = \frac{eQ_m W}{E_0} \sin \left( \omega_i \frac{d_{nm}}{c} \right), \quad (2.7)$$
where \( \omega_i \) is the frequency of the mode at the \( i \)-th cavity, \( W \) corresponds to the wake strength written in the unit of \( \text{V/C/m} \), and total charge of the \( n \)-th bunch and energy of the design particle have been denoted by \( Q_n \) and \( E_0 \) respectively. From Eq.(2.5) together with Eq.(2.2), we obtain the closed-orbit distortion of the \( n \)-th bunch (\( n \geq 2 \))

\[
\Delta x_n(s) = \Delta x_i + \sqrt{\beta(s)} \sum_{m=-\infty}^{\infty} \frac{v^2}{v^2 - m^2} \sum_{i=1}^{N_\xi} \sqrt{\beta_i} F_n^{(i)}(s_i) e^{im(\theta - \theta_i)} ,
\]

where \( \beta_i \equiv \beta(s=s_i) \), and \( \theta_i \), defined in the region \( 0 \leq \theta_i < 2\pi \), indicates \( \theta \)-coordinate of the \( i \)-th cavity location.

3 Non-Exponential Instability

It is almost obvious that, for the \( n \)-th macroparticle bunch, the betatron motion about the design orbit is governed by exactly the same equation as Eq.(2.5). Basic equation for bunch-train study is thus of the form

\[
\frac{d^2 X_n}{ds^2} + K(s) X_n = \sum_{i=1}^{N_x} \sum_{m=1}^{n-1} W_m^{(i)} X_m \delta_p(s-s_i) ,
\]

where \( X_n \) is the transverse displacement of the \( n \)-th bunch centroid measured from its own closed orbit given by Eq.(2.8). Scaling the coordinate as \( Y_n = X_n / \sqrt{\beta(s)} \) and changing the independent variable to \( \theta \) defined in Eq.(2.3), Eq.(3.1) is transformed to yield

\[
\frac{d^2 Y_n}{d\theta^2} + v^2 Y_n = \sum_{i=1}^{N_x} \sum_{m=1}^{n-1} W_m^{(i)} Y_m \delta_p(\theta - \theta_i) .
\]

It is straightforward to somewhat generalize this equation employing two-macroparticle approach which enables us to include effect of synchrotron motion as outlined in Appendix A.

Since the right hand side of Eq.(3.2) depends only on the preceding bunch motions, we can easily solve it resulting in the general solution for \( Y_n(\theta) \)

\[
Y_n(\theta) = a_n \cos(\nu \theta) + b_n \sin(\nu \theta) \\
+ \sum_{i=1}^{N_x} \sum_{m=1}^{n-1} W_m^{(i)} \int_0^\theta Y_m(\theta') \sin[\nu(\theta - \theta')] \delta_p(\theta - \theta_i) d\theta' .
\]
where \( n \geq 2 \), and the initial conditions have been introduced according to

\[
a_n = Y_n(\theta = 0), \quad \text{and} \quad b_n = \frac{1}{v} \left( \frac{dY_n}{d\theta} \right)_{\theta=0}.
\]

For the first bunch, the third term in the right side of Eq.(3.3) disappears, and the solution simply becomes

\[
Y_1(\theta) = a_1 \cos(v\theta) + b_1 \sin(v\theta).
\]  \((3.4)\)

Eq.(3.4) contains enough information to determine the second-bunch motion from Eq.(3.3) leading to the betatron amplitude after \( N \) turns

\[
Y_2(2\pi N) = a_2 \cos(N\mu) + b_2 \sin(N\mu) + \frac{NS_{21}a_1}{2} \sin(N\mu)
\]

\[
+ \frac{a_1}{2} \sum_{i=1}^{N} \beta_i W_{2i}^{(i)} \sin(\mu - 2v\theta_i),
\]  \((3.5)\)

where \( \mu = 2\pi v \), \( S_{nm} = \sum_{i=1}^{N} \beta_i W_{nm}^{(i)} \), and we have assumed \( b_1 = 0 \) for simplicity. The third term in the right hand side of Eq.(3.5) expresses a growing oscillation proportional to the number of turns \( N \).

Solution for the third bunch can be derived, from \( Y_1 \) and \( Y_2 \), in an analogous way. Obviously, contribution from the first bunch results in the same form of the terms involved in Eq.(3.5) yielding a linearly growing amplitude, while the wake parameter \( W_{2i}^{(i)} \) must be replaced by \( W_{2i}^{(i)} \). The stable part in the second-bunch solution also gives rise to the growing terms linearly dependent on the turn number \( N \). But, on the other hand, the unstable part, namely the third term in Eq.(3.5), generates not only \( N \)-dependent amplitude but also the \( N^2 \)-dependent one evaluated, after \( N \) turns, as

\[
\frac{N^2 S_{32} S_{21} a_1}{8}.
\]  \((3.6)\)

It is an easy matter to see that this term leads to a \( N^3 \)-dependent amplitude in the solution for the fourth bunch. Thus we reach the conclusion that growing oscillation of the \( n \)-th bunch has the amplitude involving all powers of \( N \) up to the order of \( N^{n-1} \), provided that the leading first bunch executes betatron motion about the closed orbit[4]. More correctly, betatron motion of the \( m \)-th bunch produces the divergent terms proportional to
settings for the fundamental mode could widen the spread of the HOM frequencies. A simple approximate
formula for the n-th bunch amplitude has been presented in Ref.[5].

It is worth while to note that the coefficients of the unstable terms may be very small since the wake function \( W_{nm}^{(i)} \) oscillates rapidly as clear from Eq.(2.7). It can even be accidentally equal to zero depending on the mode frequency and bunch spacing. Practically, the wake function can not be identical in all cavities because of the fabrication tolerances, temperature, tuner settings to compensate for frequency errors of the accelerating mode, and so on. Then the deflecting-mode frequency \( \omega_i \) is no longer unique but should be expressed as \( \omega_i = \omega_0 + \Delta \omega_i \) where \( \Delta \omega_i \) denotes the frequency deviation at the i-th cavity from the average \( \omega_0 \) over all cavities. It is thus necessary to perform a statistical estimate for the most probable strength of the wake kicks.

If a large number of cavities are set on the ring, we can approximately replace the sum over the cavity number i by an integral with some normalized weight function \( \bar{f}(\omega) \), writing

\[
\sum_{i=1}^{N_x} \beta_i W_{nm}^{(i)} \rightarrow \frac{eQ_m W_{\bar{\beta}}}{E_0} \int_{-\Delta \omega/2}^{\Delta \omega/2} d\omega \cdot \bar{f}(\omega) \sin \left[ (\omega_0 + \omega) \frac{d_{nm}}{c} \right]. \tag{3.7}
\]

where \( \Delta \omega \) is maximum deviation of the deflecting-mode frequency, and we have assumed for simplicity that values of the betatron function at the cavity positions have little difference from their average \( \bar{\beta} \). Assuming the distribution of the frequency errors to be Gaussian, we can put

\[
\bar{f}(\omega) = \frac{N_e}{\sqrt{2\pi \sigma}} \exp \left( -\frac{\omega^2}{2\sigma^2} \right), \tag{3.8}
\]

where \( \sigma \) is a constant which should be smaller than \( \Delta \omega \).

To make a rough estimate, we substitute Eq.(3.8) into Eq.(3.7) expanding the range of integration to infinity, obtaining

\[
S_{nm} = \frac{N_e e Q_m W_{\bar{\beta}}}{E_0} \sin \left( \frac{\omega_0 d_{nm}}{c} \right) \exp \left[ -\frac{1}{2} \left( \frac{\sigma d_{nm}}{c} \right)^2 \right]. \tag{3.9}
\]

We now have the extra factor which may remarkably reduce the integrated effect of the wake kicks depending quite sensitively on the product \( \sigma d_{nm} \). In the LEP cavities, an estimated frequency spread of the higher-order-modes (HOMs) due to the fabrication tolerances is, unfortunately, even less than 0.1 percent of \( \omega_0 \), but hopefully different tuner settings for the fundamental mode could widen the spread of the HOM frequencies.
Existence of many HOMs requires us to sum up the contributions from all modes for final evaluation of the kick amount received by each bunch in one turn. The factor \( \sin(\omega_0 \delta_{nm}) \) can then be either positive or negative depending on the mode frequency \( \omega_0 \), and we have an additional reduction factor in the kick strength accordingly. The effect of this factor may be more significant than the one considered above.

4 Effect of Tune Shift

The fundamental mechanism of the described instability is clearly a resonance driven by the forcing terms with the frequency identical to free-oscillation tune. The mathematical situation can be imitated by the simple equation

\[
\frac{d^2 x}{d\theta^2} + \nu^2 x = f(\theta),
\]

where \( f(\theta) \) is a periodic driving function with the frequency \( \nu \). For the second bunch, \( f(\theta) = \cos(\nu \theta) \) representing the stable betatron motion of the first bunch. The general solution to Eq.(4.1) then contains the unstable part given by \( \theta \sin(\nu \theta) \) where the amplitude grows linearly with respect to \( \theta \). This growing term turns to one of the driving forces for the third bunch, so \( f(\theta) \) is now sum of the stable oscillation terms plus \( \theta \sin(\nu \theta) \). Consequently, the third-bunch solution involves not only the linearly growing part but also quadratically growing one like \( \theta^2 \cos(\nu \theta) \). Thus the most dangerous term in the n-th bunch solution has the amplitude proportional to \( \theta^{n-1} \).

Noting that the frequency of the driving force is essential to the instability mechanism, it is realized that the unstable terms can be eliminated by somehow modifying the frequency of \( f(\theta) \) from the resonant value \( \nu \). For this purpose, we here investigate two possibilities introducing two types of additional forces into Eq.(3.1).

4.1 Amplitude-Dependent Tune Shift

We first consider the effect of octupole components, which can be a source of tune shift. An approximate equation of the n-th bunch motion including the octupole nonlinearity can be written as

\[
\frac{d^2 Y_n}{d\theta^2} + \nu^2 Y_n + A(\nu \beta_{\text{oct}})^2 Y_n \frac{d^2}{d\theta^2} (\theta - \theta_{\text{oct}}) = \nu \sum_{i=1}^{N_B} \beta_i \sum_{m=1}^{n-1} W_{nm}^{(i)} Y_m \delta_p (\theta - \theta_i),
\]

where \( A \) is the constant associated with field strength of the octupole magnet sitting at the position \( \theta = \theta_{\text{oct}} \), \( \beta_{\text{oct}} = \beta(\theta = \theta_{\text{oct}}) \), and we have dropped some small terms assuming a weak
octupole field. The closed-orbit distortions incorporated here are the same as those defined in Eq.(2.8) since effect of the weak nonlinearity on \( \Delta x_n \) is negligible. To perform a simple theoretical exploration, we take one-turn average of the kick forces in Eq.(4.2) to get

\[
\frac{d^2 Y_n}{d\theta^2} + v^2 Y_n + \alpha Y_n^3 = D \sum_{m=1}^{n-1} W_{nm} Y_m, \tag{4.3}
\]

where \( W_{nm} \equiv W^{(1)}_{nm} \), \( D=v\beta_1/2\pi \), the parameter \( \alpha \) is understood to be the first-order quantity defined by \( \alpha=\frac{A(v\beta_{oct})^2}{2\pi} \), and only single cavity has been taken into account.

Let us look at the second-bunch motion, putting \( n=2 \) in Eq.(4.3). As long as betatron amplitude of the first bunch is not too large, the centroid motion is nearly a simple harmonic oscillation around the closed orbit. Thus \( Y_1 \) can still be expressed as Eq.(3.4), and Eq.(4.3) results in

\[
\frac{d^2 Y_2}{d\theta^2} + v^2 Y_2 + \alpha Y_2^3 = D a_2Y_2 \cos(v\theta), \tag{4.4}
\]

where we have put \( b_1=0 \). Since \( \alpha \) is a first-order parameter, Eq.(4.4) is roughly the linear equation analogous to Eq.(4.1) while the oscillation amplitude stays small. The lowest-order solution is therefore given by

\[
Y_2 = a_2 \cos(v\theta) + b_2 \sin(v\theta) + \frac{D W_{21}}{2v} a_1 \theta \sin(v\theta), \tag{4.5}
\]

corresponding to the unstable motion in Eq.(3.5).

Although the oscillation described by the above solution grows linearly with respect to \( \theta \), we now have the nonlinear component which becomes more and more dominant with the increasing amplitude. This implies that, when going to the first-order solution, we observe a beam behavior essentially different from that of the linear system. Specifically, it is well-known that betatron frequency of nonlinear system is no longer constant but depends on the oscillation amplitude. Therefore, recalling the argument in the beginning of this section, it is expected that the octupole nonlinearity might suppress the divergent oscillation generating a tune shift.

To proceed further, let us assume the solution to Eq.(4.4) to be

\[
Y_2(\theta) = \rho(\theta) \cos \chi(\theta), \tag{4.6}
\]

introducing the two unknown functions \( \rho(\theta) \) and \( \chi(\theta) \) related to each other through the condition
\[ \dot{\rho}\cos\chi - (\dot{\chi} - \nu)\rho\sin\chi = 0, \quad (4.7) \]

where the dot stands for \( \frac{d}{d\theta} \). Here, \( \rho \) is a slowly varying function, and the derivative \( \dot{\rho} \) is of the first order accordingly. \( \dot{\chi} \) is generally of the zero-th order, but it is reasonable to claim that the difference \( \dot{\chi} - \nu \) should be a first-order quantity slowly varying in \( \theta \) since the zero-th order solution to Eq. (4.4) has the frequency \( \nu \).

Employing the phase-amplitude averaging method based on Eqs. (4.6) and (4.7), we eventually reach the averaged version of Eq. (4.4)

\[
\begin{align*}
\dot{\rho} &= -\frac{D_{a_1}W_{21}}{2\nu}\sin\Psi, \\
\dot{\Psi} &= \frac{3\alpha}{8\nu}\rho^2 - \frac{D_{a_1}W_{21}}{2\nu}\frac{\cos\Psi}{\rho},
\end{align*}
\quad (4.8) \quad (4.9)
\]

where \( \Psi(\theta) = \chi(\theta) - \nu\theta \). Moreover, it can be shown that the system governed by Eqs. (4.8) and (4.9) has a constant of motion

\[ \rho\cos\Psi - \Gamma \cdot \rho^4 = C_0 (= \text{const.}), \quad (4.10) \]

where \( \Gamma = 3\alpha/16D_{a_1}W_{21} \), and the value of \( C_0 \) is determined, for example, with initial conditions. The maximum amplitude \( \rho_{\text{max}} \) of the second bunch can be derived from Eq. (4.10), putting either \( \cos\Psi = -1 \) or \( +1 \) depending on the sign of \( \Gamma \). For a small initial offset, \( \rho_{\text{max}} \) is approximately equal to \( \Gamma^{1/2} \).

It is also possible to demonstrate that oscillations of all bunches in a train can be bounded by the nonlinear force. According to an analytic estimate by Gluckstern, peak amplitude of the \( n \)-th bunch can also be characterized by the parameter \( \Gamma \) and is roughly proportional to \( \Gamma^{(n-1)/(2n-1)} \). Hence the maximum growth is saturated at the value proportional to \( \Gamma^{1/2} \) as number of bunches in a train increases.

The amplitude-dependent tune shift treated in this section could be a possibility to limit the growth in betatron amplitude if a sufficiently large \( |\Gamma| \) can be provided. However, in the LEP case, there currently exist only eight octupole magnets whose field strength is too weak to anticipate a strong suppression of the non-exponential beam blowup. On the other hand, a large number of sextupoles, which also yield a tune shift, are installed on LEP. But, unfortunately, the effect of sextupole nonlinearity on betatron tune should be much weaker than that of octupole. Therefore, the nonlinear components on LEP will not help much to prevent the amplitude from reaching the critical size.
4.2 Current-Dependent Tune Shift

It is well-known that image charges and currents induced on beam environments can be a source of tune shift. Since amount of the shift is related to beam intensity, it is possible to develop bunch-dependent tunes by intentionally setting different currents for different bunches. Leaving only linear part of the image forces and making use of the same transformation as applied to Eq.(3.1), we find the equation

$$\frac{d^2 Y_n}{d\theta^2} + K_n(\theta) Y_n = \nu \sum_{i=1}^{N_b} \sum_{m=1}^{n-1} W_{nm}^i \delta_{ij} (\theta - \theta_i).$$

(4.11)

where the closed orbits have been redefined incorporating the image forces, and $K_n(\theta)$ is the periodic function including the image effect on the $n$-th bunch. If we smooth this periodic linear force writing an approximate tune of the $n$-th bunch as $v_n$, Eq.(4.11) becomes

$$\frac{d^2 Y_n}{d\theta^2} + v_n^2 Y_n = \nu \sum_{i=1}^{N_b} \sum_{m=1}^{n-1} W_{nm}^i \delta_{ij} (\theta - \theta_i).$$

(4.12)

The tune $v_n$ is decomposed into the zero-current tune $v$ and the bunch-dependent shift $\Delta v_n$ due to the image fields; namely $v_n = v + \Delta v_n$. $\Delta v_n$ can be represented as $\Delta v_n = (dv/dI)_n I_n$ where $I_n$ is current of the $n$-th bunch, and an explicit analytic form of the detuning factor $(dv/dI)$ is given, for example, by the Laslett's formula[6]. Practically, value of $(dv/dI)$ should be determined through experimental observations. In LEP, past experiments show that $(dv/dI) = 0.129 \text{ [mA}^{-1}]$ vertically and $(dv/dI) = 0.064 \text{ [mA}^{-1}]$ horizontally[7].

Needless to say, Eq.(4.12) yields the solution similar to Eq.(3.3). However, the first two terms in the right hand side of Eq.(3.3) now have the frequency $v_n$ and the factor $\sin[v(\theta-\theta')]$ in the third term must be modified to $\sin[v_n(\theta-\theta')]$. The first-bunch solution is again a harmonic oscillation with the frequency $v_1$. The second-bunch amplitude after $N$ turns can then be derived to be

$$Y_2(2\pi N) = a_2 \cos(2\pi N v_2) + \frac{vb_z}{v_2} \sin(2\pi N v_2)$$

$$+ \frac{v a_1}{2v_2} \sum_{i=1}^{N_b} \beta_i W_{21}^i \frac{\sin(N\mu_+)}{\sin\mu_+} \sin\left(1 - \frac{\theta_i}{\pi}\mu_+ - N\mu_-\right) - \frac{\sin(N\mu_-)}{\sin\mu_-} \sin\left(1 - \frac{\theta_i}{\pi}\mu_- - N\mu_+\right),$$

(4.13)

where we have put $b_1 = 0$ and $\mu_\pm = \pi(v_1 \pm v_2)$. It has been confirmed that the growing term no longer exists because $v_1 \neq v_2$. Eq.(4.13) is substantially a superposition of two stable
oscillations with the frequencies $v_1$ and $v_2$. In actual cases, these two frequencies might probably be chosen as close to each other as possible, and the resulting motion is beating.

To evaluate the third-bunch solution, Eq.(4.13) as well as the first-bunch solution are substituted into Eq.(4.12). The forcing terms now involve two modes having the frequencies $v_1$ and $v_2$, but the free-oscillation tune of the third-bunch is $v_3$ different from both of them. Therefore the third-bunch motion is also stable and beating. Because of linearity of the problem, it is obvious that the $n$-th bunch solution $Y_n(\theta)$ is generally composed of the $n$ stable modes oscillating at the frequencies $v_m$ $(m=1,2,\cdots,n)$. If each frequency is different from the others, the endless growth of the betatron amplitude can be totally suppressed. Thus the use of current-dependent tune shift is a simple and effective way to avoid the beam breakup in bunch trains. An analogous preferable situation can also be established by means of RF focusing element as discussed in Appendix B.

It is important to notice that the oscillation amplitude in Eq.(4.13) can be considerably large owing to the factor $1/\sin u$, when the difference of the bunch currents is too small. When $v_1=v_2$, the last term in the right hand side plays a dominant role, and the peak beating amplitude is of the order of

$$\frac{a_1v_1}{2v_2} \frac{S_{21}}{\sin u}$$

(4.14)

For the third and the later bunches, it may not be straightforward to obtain such a compact formula as Eq.(4.14) for quick evaluation of the maximum amplitudes. It is, however, possible to conclude a rough criteria when current difference between any adjacent bunches are sufficiently small and approximately the same; namely $\Delta v=v_n-v_{n-1}\ll 1$ regardless of the bunch number $n$. In this case, the maximum beating amplitude of the $n$-th bunch can be estimated from the formula

$$\text{max}(Y_n) = \frac{1}{a_1} \left( \frac{1}{2\pi \Delta v} \right)^{n-1} \prod_{m=1}^{n-1} S_{m+1,m} = h_n.$$  (4.15)

The tune shift $\Delta v$ must be set up such that the parameter $h_n$ takes an acceptable number depending on various conditions; e.g. minimum aperture size, expected value of $a_1$, and so on. If $h_n=1$ is adopted for all bunches, the beating could be completely eliminated but this choice might be too conservative. Note that, with strong wake fields and/or a very small $\Delta v$, $h_n$ usually becomes larger for a later bunch.

We now show simulation results obtained from a simple tracking code where only single kick is applied to a macroparticle in every turn. The kick strength is evaluated on the basis of the LEP parameters; namely $E_0=20$ GeV, $R=4.2429$ km, $N_c=128$, $\bar{\beta}=40$ m,
\(v=90.27\) in the horizontal plane, and \(v=76.24\) in the vertical plane. According to a result from a mesh-code analysis for the LEP 5-cell Cu cavity, the peak transverse wake generated by a bunch of about 2-cm long is around 7 V/pC/m in the range of a few tens of meters[8]. Thus, in the following numerical examples, we assume \(W\) to be 5 V/pC/m within the whole range covering a single train while later bunches might feel somewhat less wakes because of slight resistive damping. This value contains the contributions from all HOMs, so the sinusoidal factor in Eq.(3.9) is now unnecessary. We further drop the reduction factor to perform a pessimistic estimate. The value of \(S_{nm}\) is then about 0.0342 for the bunch current of 0.3 mA.

Fig.1 shows the severe growths of betatron amplitudes originating from the non-exponential instability. The ordinate of the figures represents the ratio of betatron displacement to the initial value taken to be common in all bunches. A variation in bunch currents is then introduced, resulting in Fig.2. The parameter \(h_n\) corresponding to this example is about three, enough to kill the large beating. In fact, we observe well-bounded motions in all four bunches.

\[
\text{Tune } v=90.27; \text{ Detuning Factor } \frac{dv}{dl}=-0.06 \text{ [mA}^{-1}] \text{; No Damping}
\]

\[
\text{Bunch Currents [mA]} : I_1=I_2=I_3=I_4=0.3
\]

Fig.1
5 Radiation Damping

Effect of radiation damping is now incorporated. Although natural damping force is thought to be quite weak at the injection energy of LEP, it does have some preferable effect for limiting the non-exponential beam blowup. We here consider betatron motion about the closed orbit given in Eq.(2.8) while $\Delta x_n$ can be redefined including the damping force. Starting equation is then written, adding a frictional term to Eq.(3.1), as

$$\frac{d^2 X_n}{ds^2} + 2\Lambda \frac{d(X_n + \Delta x_n)}{ds} + K(s)X_n = \sum_{i=1}^{N_s} \sum_{m=1}^{n-1} W^{(i)_{nm}} X_m \delta_p(s - s_i), \quad (5.1)$$

where $\Lambda$ is damping constant. Eq.(5.1) can be transformed to give

$$\frac{d^2 Y_n}{d\theta^2} + 2\lambda \frac{dY_n}{d\theta} + v^2 Y_n = v \sum_{i=1}^{N_s} \beta_i \sum_{m=1}^{n-1} W^{(i)_{nm}} Y_m \delta_p(\theta - \theta_i) - 2\lambda G_n(\theta), \quad (5.2)$$
where we have assumed $(\Lambda/\nu)|d\beta/d\vartheta|<1$, and the damping force has been averaged introducing the parameter $\lambda=\Lambda R$. $G_n(\theta)$ is the 2\pi-periodic function related to the closed-orbit distortion $\Delta x_n$ as

$$G_n(\theta) = \frac{\sqrt{\beta}}{R} \left( \frac{d\Delta y_n}{d\theta} + \frac{1}{2\beta} \frac{d\beta}{d\theta} \Delta y_n \right) = \sum_{m=-\infty}^{\infty} g_{nm} e^{im\theta}, \quad (5.3)$$

where $\Delta y_n = \Delta x_n / \sqrt{\beta(s)}$, and $g_{nm}$ is complex Fourier coefficient. Noting $v \gg \lambda$, the general solution to Eq.(5.2) can be obtained, to a good approximation, as

$$Y_n(\theta) = e^{-\lambda\theta}[a_n \cos(v\theta) + b_n \sin(v\theta)]$$
$$+ e^{-\lambda\theta} \sum_{i=1}^{N_k} \hat{W}_{n,1}(\theta_i) \int_{\theta_i}^{\theta} d\theta' Y_m(\theta') e^{\lambda\theta'} \sin[v(\theta - \theta')] \delta_p(\theta' - \theta)$$
$$- \frac{2\lambda}{v} e^{-\lambda\theta} \int_{0}^{\theta} d\theta' G_n(\theta') e^{\lambda\theta'} \sin[v(\theta - \theta')] \quad (5.4)$$

Natural damping time at the LEP injection energy is about 0.5 second corresponding to 6000 turns. The parameter $\lambda$ is then estimated to be $\lambda=2.7 \times 10^{-5}$. Although this number seems too small, a finite betatron amplitude, the origin of the non-exponential instability, is almost completely damped only in a few seconds. It is, therefore, an easy matter to make the instability mechanism ineffective unless injection procedure of a later bunch affects the orbits of the preceding bunches. All we should do in this case is simply to wait for some time until the bunch newly added to a train naturally comes down to its own closed orbit. Then, we only need to consider the last term in Eq.(5.4) for $Y_1(\theta)$ substituted back into the second term to determine $Y_2(\theta)$. The approximate solution for the first bunch after some seconds is thus expressed by

$$Y_1(\theta) = 2\lambda \sum_{m=-\infty}^{\infty} \frac{g_{1m} e^{im\theta}}{(j\lambda - m)^2 - v^2}, \quad (5.5)$$

where the terms oscillating at the frequency $v$ have been damped away. Eq.(5.4) together with Eq.(5.5) leads to the second-bunch solution in the N-th turn

$$Y_2(\theta) = \sum_{i=1}^{N_k} \hat{W}_{2i}(\theta_i) Y_1(\theta_i) e^{-\lambda(\theta - \theta_i)} e^{2\pi\lambda} \sin[v(\theta - \theta_i) + \mu] - \sin[v(\theta - \theta_i)]$$
$$\frac{e^{2\pi\lambda} - \cos\mu}{(e^{2\pi\lambda} - \cos\mu)^2 + \sin^2\mu} + \sum_{i=1}^{n_k} \hat{W}_{2i}(\theta_i) Y_1(\theta_i) e^{-\lambda(\theta - \theta_i)} \sin[v(\theta - \theta_i)]$$
$$+ 2\lambda \sum_{m=-\infty}^{\infty} \frac{g_{2m} e^{im\theta}}{(j\lambda - m)^2 - v^2}. \quad (5.6)$$
where \( n_c \) is less than or equal to \( N_c \), \( \vartheta \) varies in the region 
\( \theta_{n_c} \leq \vartheta < \theta_{n_c+1} \) with the definition 
\( \theta_0=0 \) and \( \theta_{N_c+1}=2\pi \), the second term vanishes when \( n_c \leq 0 \), and we have 
dropped some small damping terms in the derivation process of this equation under the 
assumption \( N>1 \). We see that all terms in Eq.(5.6) execute stable oscillations with rather 
small amplitudes proportional to \( \lambda \).

Notice that Eq.(5.6) is totally independent of the turn number \( N \), which suggests 
that the solution is periodic in a single turn. It is not so evident from Eq.(5.6) but the orbit 
described by the first two terms in Eq.(5.6) is actually closed and identical in every turn 
and, therefore, can be expanded into Fourier series in the same way as Eq.(5.5). Since the 
last term in the right hand side also has \( 2\pi \)-periodicity, \( Y_2 \) can be rewritten in the form of 
Fourier expansion like \( Y_1 \) in Eq.(5.5) apart from the coefficients. As a result, substitution 
of Eqs.(5.5) and (5.6) into Eq.(5.4) yields the third-bunch solution analogous to Eq.(5.6). 
Taking advantage of linearity of the problem, it is possible to write down the general 
solution for the \( n \)-th bunch as

\[
Y_n(\vartheta) = \sum_{i=1}^{N_c} \sum_{m=1}^{n_c-1} W_{nm}^{(i)} Y_m(\theta_i) e^{-\lambda(\vartheta-\theta_i)} e^{2\pi i \nu (\vartheta-\theta_i)} 
\]

\[
+ \sum_{i=1}^{n_c} \sum_{m=1}^{n_c} W_{nm}^{(i)} Y_m(\theta_i) e^{-\lambda(\vartheta-\theta_i)} \sin[\nu(\vartheta-\theta_i)] + 2\lambda \sum_{m=-\infty}^{\infty} \frac{g_{mn} e^{jm\vartheta}}{(j\lambda - m)^2 - \nu^2}.
\]

Thus all bunches, if they survive the instability, eventually drive themselves onto their own 
closed orbits in Eq.(5.7) slightly different from \( \Delta x_n \) in Eq.(2.8).

Let us now take into account the effect of the first term in the right hand side of 
Eq.(5.4), which has so far been neglected assuming perfect injection processes and no 
noise. In reality, when a required distance between neighboring bunches in a train is rather 
short, it should be difficult, due to residual fields of kicker magnet, to inject a bunch 
without causing any influence on the preceding bunches existing already. These bunches 
might receive successive weak kicks by the fields during accumulation of the additional 
bunch. The damped betatron terms are then re-generated. They are clearly of substantial 
importance because the oscillation frequency coincides with the resonant value leading to 
the non-exponential instability. Although these oscillatory components are eventually 
damped away again and all bunches get back to the closed orbits derived above, the 
betatron amplitudes initially grow reaching some maxima. If this peak amplitude of a 
bunch exceeds radius of a beam pipe, the bunch could be lost or accumulating the current 
up to the same level as those of the preceding bunches might be hopeless.

To evaluate the peak amplitude of the second bunch, we simply adopt the function 
\( a_1 e^{\lambda \vartheta} \cos(\nu \vartheta) \) as the first-bunch solution, neglecting the stable small oscillation with \( 2\pi- \)

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periodicity. Then of concern is only the second term in the right hand side of Eq.(5.4), which yields the amplitude after $N$ turns

$$Y_2(2\pi N) = \frac{NS_2|a_1|e^{-2\pi \lambda N}}{2} \sin(N\mu) + \frac{a_1e^{-2\pi \lambda N}}{2} \frac{\sin(N\mu)}{\sin \mu} \sum_{i=1}^{N_i} \beta_i W_2^{(i)} \sin(\mu - 2\nu \theta_i), \quad (5.8)$$

identical to the last two terms in Eq.(3.5) except for the damping factor $e^{-2\pi \lambda N}$. Specifically, the first term in the right side makes a dominant contribution to the maximum amplitude. The timing when $Y_2$ comes to its peak is clearly $\theta=1/\lambda$, and the corresponding amplitude is evaluated from

$$\frac{S_2|a_1|}{4\pi e \lambda} \quad (5.9)$$

It is straightforward to show that the peak amplitude of the $n$-th bunch takes place at around $\theta=(n-1)/\lambda$, and the value is roughly of the order of

$$a_1 \left( \frac{n-1}{2(n+2)^{1/2}} \right)^n \frac{n-1}{\prod_{m=1}^{n-1} S_{m+1,m}} \quad (5.10)$$

---

**Fig.3**

_Tune $\nu=90.27$; Detuning Factor $dv/dl=-0.06 \text{ [mA]^{-1}}$; Damping Time 3000 [turns]_

_Bunch Currents [mA]: $I_1=I_2=I_3=I_4=0.3$_
when \( n \) is a modest number. Note that this is on the assumption that the leading first bunch has a finite betatron amplitude \( a_1 \) initially. Needless to say, bunch stability is much more improved when \( a_1=0 \). Provided that the peak amplitude of the last bunch in a train is well below minimum aperture size, no bunches will be lost, at least, due to the non-exponential instability.

Fig.3 demonstrates the effect of radiation damping. The assumed damping time is 3000 turns corresponding to the LEP case with all damping wigglers switched on at 20 GeV. Other parameters are identical to those adopted in Fig.1. We recognize that the peak amplitude actually occurs at \((n-1)\times6000\) turns for the \( n \)-th bunch, which agrees with the expectation drawn above. Further, the magnitudes of the maximum amplitudes are also in accord with Eq.(5.10).

6 Discussions

It follows from the present results that the radiation damping in LEP works rather effectively, even at the injection energy, in limiting the linear-growth regime of bunches with a modest intensity. However, the maximum amplitudes of the third and fourth bunch shown in Fig.3 are apparently beyond a permissible range. This may be a possible explanation to the recent experimental observation on LEP, where accumulation of the third

\[
\begin{align*}
\text{Tune } v &= 90.27 ; \text{ Detuning Factor } \frac{dv}{dl} = -0.06 \text{ [mA}^{-1}] ; \text{ Damping Time } 3000 \text{ [turns]} \\
Bunch \text{ Currents [mA]} : &I_1 = 0.45, I_2 = 0.45, I_3 = 0.35, I_4 = 0.30
\end{align*}
\]
and fourth-bunch to the first-bunch intensity, i.e. 0.4~0.45 mA, failed while the second-bunch accumulation to the level was no problem[9].

To approximately simulate the situation of the experiment, let us increase the second-bunch current in Fig.2 to 0.45 mA keeping other parameters unchanged. Fig.4 illustrates the result with the damping time being set at 3000 turns again. While the maximum amplitude in Fig.2 was, even without the damping force, only twice greater than the initial value, we now observe large growths in all trailing bunches. This drastic change strongly suggests avoiding use of equal currents for the leading two bunches. In fact, it is interesting to note that the peak amplitudes of all three trailing bunches occur at 3000 turns indicating excitation of the linear-growth mechanism, though we observe no such behavior in Fig.2.

Let us next test the case where four bunches in a train are very closely populated but any of the two have slightly different intensities. As expected, the non-exponential instability is no longer dominant under this setup. Instead, the beating motion now becomes responsible for large amplitudes as seen in Fig.5.

Using the vertical parameters of LEP, Fig.5 is altered to Fig.6. Comparison of these two figures leads us to the conclusion that horizontal beam blowup in LEP should be much more severe than vertical growth because of the detuning factor almost half of the vertical value. The LEP beam pipe geometry is, however, an ellipse whose horizontal semi-axis, $=70\text{mm}$, is twice larger than the vertical size, $=35\text{mm}$. Therefore, the vertical growth may be as dangerous as the horizontal one.

Let us employ Eq.(4.15) to evaluate a desirable tune difference for realizing modest betatron amplitudes. Maintaining the total current of a train in Figs.5 and 6, i.e. 1.3 mA and requiring the condition $\Delta v=0.0012$ corresponding to the current difference of 0.02 mA in the horizontal plane. We thus try the set of bunch currents $I_1=0.355\text{ mA}$, $I_2=0.335\text{ mA}$, $I_3=0.315\text{ mA}$, and $I_4=0.295\text{ mA}$, resulting in Fig.7. While the first-bunch intensity is even higher than the example in Fig.5, the stability of the bunch train has been remarkably improved. Further, the maximum amplitudes are in reasonable agreement with the prediction from Eq.(4.15); i.e. $h_2=5.4$, $h_3=13.6$, and $h_4=21.6$.

It should be noticed that, except for the nonlinear situation discussed in the section 4.1, the possible largest amplitude always has linear dependence on $a_1$ supposed to express maximum initial offset of the first bunch measured from its closed orbit. Therefore, if an error at injection or some other undesirable factor doubles the value of $a_1$, it immediately results in twice larger amplitudes in all following bunches. Inversely speaking, if the size of $a_1$ is minimized by performing better injections of the trailing bunches, it would considerably improve stability of all bunches. In fact, leading two bunches become completely free from the non-exponential instability when $a_1=0$ and, furthermore, the third-bunch amplitude only suffers the linear-growth regime which is much less dangerous than the quadratic one. Thus it is quite crucial to avoid accidental kicks to preceding bunches.
Fig. 5

Tune \( \nu = 90.27 \); Detuning Factor \( \frac{dv}{dl} = -0.06 \) [mA\(^{-1}\)]; Damping Time 3000 [turns]

Bunch Currents [mA]: \( I_1 = 0.345 \), \( I_2 = 0.335 \), \( I_3 = 0.315 \), \( I_4 = 0.305 \)

Fig. 6

Tune \( \nu = 76.24 \); Detuning Factor \( \frac{dv}{dl} = -0.13 \) [mA\(^{-1}\)]; Damping Time 3000 [turns]

Bunch Currents [mA]: \( I_1 = 0.345 \), \( I_2 = 0.335 \), \( I_3 = 0.315 \), \( I_4 = 0.305 \)
As readily understood from some present formulae, e.g. Eq.(5.10), maximum amplitude is generally concerned with the product of $S_{m+1,m}$. It is because the most severe term in the $n$-th bunch solution originates only from the most severe term in the $(n-1)$-th bunch solution. This fact implies that, for the bunch train filled with equally-spaced bunches, magnitude of the wake function at the distance $\Delta d = d_{m+1,m}$ is of particular importance. Minimization of the sum of $S_{m+1,m}$ over all HOMs optimizing $\Delta d$ is thus essential to achieving better stability of a bunch train.

7 Summary

It has been shown that a finite offset of a bunch from its own closed orbit can be an origin of non-exponential breakup of the trailing bunches in the bunch train. Provided that the residual kicker-field accompanied with beam injection causes no disturbance to motions of already-existing bunches, and that thermal noises of the devices installed along the ring are negligible, the described instability is not essentially troublesome. It may, however, be possible in actual situations that one or more preceding bunches in a train receive slight kicks during accumulation sequence of the later bunch, starting to execute betatron
oscillations about the individual closed orbits. Then the bunches would suffer rapid increase in the amplitudes, and may eventually be intercepted by beam pipe.

In order to cure this instability, the following procedures might be helpful:

(a). improvement of injection-kicker performance,

Clearly, this is quite effective since peak betatron amplitude is proportional to the offsets of preceding bunches which should mostly be generated in the injection process.

(b). use of different currents for different bunches in a train,

This scheme is probably easiest to employ. In LEP, the threshold current due to transverse mode coupling is about 0.6 mA per bunch. In order to achieve a small $h_n$ for all bunches keeping the intensities as close to this threshold as possible, a desirable tune difference should be around 0.0036 corresponding to $h_n=3-4$ and to the current difference of 0.06 mA in the horizontal plane. Thus, as far as the pessimistic estimate for the wake fields is concerned, about 10-percent variation in bunch currents might be recommended under high-intensity operations in LEP.

(c). enhancement of radiation damping,

It is definitely better to excite all damping wigglers enhancing radiation, since maximum amplitude of the n-th bunch is roughly proportional to $\lambda^{1-n}$.

(d). optimization of bunch spacing,

This would also work remarkably. For instance, if the wake factor $S_{m+1,m}$ is halved by optimizing bunch spacing, then the peak amplitude of the fourth bunch becomes 1/8.

(e). use of superconducting cavities,

If most of the present RF cavities in LEP are replaced by superconducting ones, we will not have to worry about the non-exponential beam blowup.

(f). introduction of strong nonlinear components and/or RF focusing elements, and

The effect of the amplitude-dependent tune shift might be currently negligible in LEP.

(g). application of feedback.

Feedback system works if the kick supplied by the system is applicable only to a single bunch of interest. Note that each bunch in a train has different closed orbit.

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References
Appendix A: Two-Macroparticle Model

Let us introduce effect of synchrotron oscillation, dividing a single bunch into two macroparticles whose transverse displacements measured from the closed orbit $\Delta x_n$ in Eq.(2.8) are denoted by $u_n$ and $v_n$ for the n-th pair. It is assumed in the following that bunch length is short enough to claim that the inter-bunch wakes seen by a macroparticle pair are roughly the same.

In our simple model, only the trailing piece of a macroparticle pair is kicked by the constant single-bunch wake left by the leading partner when it passes through an RF cavity. Further, the inter-bunch wake force generated by a preceding bunch is proportional to the center-of-mass position of the bunch. Considering the bunches having a common intensity, $u_n$ and $v_n$ satisfy the coupled equations

\[
\frac{d^2 u_n}{ds^2} + K(s)u_n = \sum_{i=1}^{N_c} \sum_{m=1}^{n-1} \frac{W_{nm}^{(i)}}{2} (u_m + v_m)\delta_p(s - s_i) + \sum_{i=1}^{N_c} \xi(s)W_{ni}^{(i)}(v_n + \Delta x_n)\delta_p(s - s_i),
\]

(A.1)

\[
\frac{d^2 v_n}{ds^2} + K(s)v_n = \sum_{i=1}^{N_c} \sum_{m=1}^{n-1} \frac{W_{nm}^{(i)}}{2} (v_m + u_m)\delta_p(s - s_i) + \sum_{i=1}^{N_c} \xi(s + \frac{\pi R}{v_s})W_{ni}^{(i)}(u_n + \Delta x_n)\delta_p(s - s_i),
\]

(A.2)

where $W_{nm}^{(i)}$ represents single-bunch wake parameter, $v_s$ is synchrotron tune, and the cavity misalignment $\delta^{(i)}$ has been neglected. $\xi(s)$ is the periodic step function with the periodicity $2\pi R/v_s$, defined by

\[
\xi(s) = \begin{cases} 
0, & \text{for } \left( n - \frac{1}{4} \right) \frac{2\pi R}{v_s} \leq s \leq \left( n + \frac{1}{4} \right) \frac{2\pi R}{v_s} \quad (n = \text{integer}) \\
1, & \text{for other regions,}
\end{cases}
\]

where the origin of the longitudinal coordinate has been chosen so that $\xi(s)$ becomes an even function. This function can be expanded into Fourier series as $\xi(s) = \frac{[1+2\xi(s)]}{2}$ with

$$\xi(s) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos \left[ (2n-1)v_s \frac{s}{R} \right].$$

From Eqs.(A.1) and (A.2) together with Eqs.(2.5), we find

$$\frac{d^2 Y^+_n}{d\theta^2} + K_+ (\theta) Y^+_n = \bar{F}(\theta) + \nu \sum_{i=1}^{N_\xi} \beta_i \sum_{m=1}^{n-1} W^{(i)}_{nm} Y^+_m \delta_p (\theta - \theta_i) - \nu \sum_{i=1}^{N_\xi} \beta_i \xi(\theta) W^{(i)}_{nm} Y^+_m \delta_p (\theta - \theta_i),$$

$$\frac{d^2 Y^-_n}{d\theta^2} + K_- (\theta) Y^-_n = \nu \sum_{i=1}^{N_\xi} \beta_i \xi(\theta) W^{(i)}_{nm} Y^+_m \delta_p (\theta - \theta_i),$$

where $Y^+_n = (u_n + v_n + 2\Delta x_n) / \sqrt{\beta(s)}$, $Y^-_n = (u_n - v_n) / \sqrt{\beta(s)}$, we have again used the function $\theta(s)$ in Eq.(2.3) as independent variable, and

$$\bar{F}(\theta) = 2v^2 [\beta(\theta)]^{3/2} F(\theta), \quad K_\pm (\theta) = v^2 \left[ 1 \mp \frac{1}{2\nu} \sum_{i=1}^{N_\xi} \beta_i \delta^{(i)} \delta_p (\theta - \theta_i) \right].$$

**Appendix B: Effect of RF Focusing**

In order to make the tune $\nu$ bunch-dependent, we here briefly explore application of a time-dependent linear kick to a bunch train, modifying Eq.(3.1) to

$$\frac{d^2 X_n}{ds^2} + K(s) X_n = \sum_{i=1}^{N_\xi} \sum_{m=1}^{n-1} W^{(i)}_{nm} X_m \delta_p (s - s_i) + \kappa(t) X_n \delta_p (s),$$

where the new kick has been set at the position $s=0$, $\kappa(t)$ is the periodic function representing a time-variation of the kick strength, and the closed-orbit distortions have been redefined including the term added. This additional effect can be provided, for example, by installing an RF focusing device on the ring. The function $\kappa(t)$ is then of the form $\kappa(t) = q \cos(\Omega t)$, where $q$ is a constant related to the RF voltage, and $\Omega$ denotes the angular RF frequency which should be an integer multiple of angular revolution frequency of the design particle. Magnitude of the kick strength depends on the timing when a bunch
traverses the focusing element. The resulting tune shift can thus be made bunch-dependent by making a proper choice of the RF frequency and initial phase.

Eq.(B.1) turns out to be the same form as Eq.(4.11) together with the coefficient of the linear force term changed to

\[ K_n(\theta) = v^2 - v\beta_0\kappa_n\delta_p(\theta). \]  

(B.2)

where \( \beta_0 = \beta(s=0) \), and \( \kappa_n \) corresponds to the RF kick strength experienced by the n-th bunch. The second term yields a tune shift, and an approximate relation between the original tune \( v \) and the shifted tune \( v_n \) can be given by

\[ v_n - v \approx \frac{\kappa_n\beta_0}{4\pi}. \]  

(B.3)

The smoothed version of Eq.(B.1) is identical to Eq.(4.12) except that we now need to employ Eq.(B.3) instead of the Laslett tune shift. The n-th bunch solution is made stable anyway if \( v_m \neq v_n \) for \( m \neq n \). Although the RF kick described here is a simple source to eliminate the resonant growth of betatron amplitude, it may be difficult to get a sufficiently large tune shift by means of an RFQ element when the beam energy is high.