BEAM-BEAM DYNAMICS

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Abstract
This lecture discusses the dynamics of the beam-beam interaction in storage rings. The force which a bunch exerts on particles in another bunch, circulating in the opposite direction is derived from a beam-beam potential, and used to compute the incoherent beam-beam kick, the beam-beam tune shift and the beam-beam limit. The same potential is also used to compute the coherent beam-beam kick, tuneshift and limit, describing the action on the whole bunch circulating in the opposite direction. The variation of the beam-beam tune shift with the betatron amplitude, and the resulting tune spreads in the beams and footprints are also discussed. The lecture concludes with a discussion of the nonlinear resonances driven by the beam-beam effect.

1 INTRODUCTION

Beam-beam effects are one of the most important phenomena which limit the luminosity in a storage ring which for two equal beams is given by:

\[
L = \frac{N^2 f k}{4\pi \sigma_x \sigma_y}
\]  

(1)

Here, \(N\) is the bunch population, \(f\) is the revolution frequency, and \(k\) is the number of bunches in one beam. The co-ordinate axes are labelled \(x\) for the radially outwards direction, \(y\) for the direction perpendicular to the median plane, and \(s\) along the equilibrium orbit; \(z\) may be either \(x\) or \(y\). The horizontal and vertical rms beam radii at the collision points are called \(\sigma_x\) and \(\sigma_y\), respectively. The bunch length will be called \(\sigma_s\). For good luminosity \(L\) we want large values of \(N\), \(f\) and \(k\), and small values of \(\sigma_x\) and \(\sigma_y\). Small values of \(\sigma_x\) and \(\sigma_y\) are achieved in low-\(\beta\) insertions. Their discussion is outside the scope of this lecture. It will become clear later on that beam-beam dynamics causes an upper limit on \(N/\sigma_x \sigma_y\), and therefore needs to be studied.

1.1 Charge distributions

The most general charge distribution which will be considered is a Gaussian distribution in three dimensions (3D), given by:

\[
\rho(x, y, s) = \frac{Ne}{(2\pi)^{3/2}\sigma_x \sigma_y \sigma_s} \exp \left( -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{s^2}{2\sigma_s^2} \right)
\]  

(2)

We will often do calculations for cases in which the bunch length \(\sigma_s\) is much larger than the transverse radii \(\sigma_x\) and \(\sigma_y\). The appropriate charge distribution in this case is a Gaussian distribution in two dimensions (2D) with line number density \(n\):

\[
\rho(x, y) = \frac{ne}{2\pi \sigma_x \sigma_y} G(x, y; \sigma_x, \sigma_y) = \frac{ne}{2\pi \sigma_x \sigma_y} \exp \left( -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} \right)
\]  

(3)
1.2 Simplifying assumptions

In most of this paper, I will make the following simplifying assumptions:

1. The beam-beam collisions are head-on, i.e. the two colliding bunches follow the same trajectory in opposite direction, without offset and without crossing angle.

2. The bunches are long with \( \sigma_z \gg \sigma_z \). Hence, I can do the electrodynamics in the two transverse dimensions only.

3. The orbit functions \( \beta_x \) and \( \beta_y \) are constant through beam-beam collisions, or at least \( \beta_x, \beta_y \gg \sigma_z \). Hence, I can neglect their variation along the beam-beam collisions, which was called the “hourglass effect” [1], although the consequences, a reduction of the luminosity [2], and an increase in the beam-beam tune shift [3], were published much earlier.

4. When the mathematics gets too heavy and tables of integrals would be needed, I do not treat bunches with arbitrary \( \sigma_x \) and \( \sigma_y \), but treat round beams with \( \sigma_x = \sigma_y = \sigma \) instead.

5. I often assume collision between bunches moving at relativistic speeds with \( \gamma \gg 1 \), and assume \( \beta \approx 1 \). Here \( \beta = v/c \) is the bunch velocity in units of the speed of light \( c \).

6. I consider collisions between particles of opposite charge, and hence attractive beam-beam forces. For collisions of particles of the same charge and repulsive beam-beam forces, the signs of many equations must be changed.

2 ELECTRO-MAGNETIC FIELD OF A MOVING BUNCH

In the case of a Gaussian charge distribution in two dimensions, given by Eq. (3), the beam-beam fields can be derived from a beam-beam potential \( U(x, y; \sigma_x, \sigma_y) \) which has the following integral representation for arbitrary \( \sigma_x \) and \( \sigma_y \):

\[
U(x, y; \sigma_x, \sigma_y) = \frac{n\epsilon}{4\pi\epsilon_0} \int_0^\infty \exp\left(-\frac{x^2}{2\sigma_x^2 + t} - \frac{y^2}{2\sigma_y^2 + t}\right) \frac{dt}{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}
\]  

(4)

This expression was derived by Kellogg [4], Houssais [5] at the University of Rennes, Kheifets [6], and Takayama [7]. The expression for the potential \( U \) in three dimensions is very similar. As usual, the electric field \( \vec{E} \) is obtained by taking the gradient of \( U \), i.e. \( \vec{E} = -\nabla U \). The components of \( \vec{E} \) were derived in closed form by Augustin [8] and Bassetti and Erskine [9]. Talman [10] gives a derivation of the potential \( U \) and of the components of \( \vec{E} \) which are for \( \sigma_x > \sigma_y \):

\[
E_x - iE_y = -\frac{in\epsilon}{2\epsilon_0\sqrt{2\pi(\sigma_x^2 - \sigma_y^2)}} \left[ w\left(\frac{x + iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right) - \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right) w\left(\frac{x\sigma_x + iy\sigma_y}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right)\right]
\]  

(5)

Here \( w(z) \) is the complex error function [11]. It is sometimes convenient to leave the electric field components in their integral form, as will be done for calculating the coherent
beam-beam kicks. The components of the magnetic field $\vec{B}$ in the laboratory frame are most easily obtained by observing that the electro-magnetic field of a bunch is an electrostatic field in its rest frame, and that $\vec{B}$ vanishes there. Hence, the magnetic field $\vec{B} = \beta \times \vec{E}/c$ of a bunch moving with speed $\beta c$ in the negative $s$-direction of the laboratory frame follows from the electric field $\vec{E}$. In Cartesian coordinates, the relations between the components of the electric field $\vec{E}$ and magnetic field $\vec{B}$ are:

$$B_x = \beta E_y/c \quad B_y = -\beta E_x/c$$  \hspace{1cm} (6)

In round beams with $\sigma_x = \sigma_y = \sigma$, we use cylindrical coordinates and find that the electric field $E_r$ and magnetic field $B_\phi$ have only a radial and an azimuthal component, respectively:

$$E_r = -\frac{n\epsilon}{4\pi\epsilon_0} \frac{\partial}{\partial r} \int_0^\infty \frac{\exp\left(-\frac{r^2}{2\sigma^2+t}\right)}{2\sigma^2+t} \, dt$$  \hspace{1cm} B_\phi = \frac{n\epsilon\beta c\mu_0}{4\pi} \frac{\partial}{\partial r} \int_0^\infty \frac{\exp\left(-\frac{r^2}{2\sigma^2+t}\right)}{2\sigma^2+t} \, dt$$  \hspace{1cm} (7)

It can be verified without tables of integrals and derivatives, differentiating first and then integrating, using $1/(2\sigma^2 + t)$ as integration variable, that these expressions, combined with the Lorentz force in Eq. (8), yield Eq. (10).

3 INCOHERENT BEAM-BEAM EFFECTS

In this section, we first use the electro-magnetic field of a moving bunch to calculate the incoherent beam-beam force which a bunch exerts on the particles in another bunch, moving in the opposite direction, during the collision. We then integrate this force over the collision, and obtain the incoherent beam-beam kick. Finally, we compute the incoherent linear beam-beam tune shift and the incoherent beam-beam limit.

3.1 Incoherent beam-beam force

The incoherent beam-beam force $\vec{F}$ is that force which a bunch exerts on individual particles of the bunch travelling in the opposite direction. It follows from the Lorentz force acting on a particle with charge $-\epsilon$:

$$\vec{F} = -\epsilon \left( \vec{E} + \vec{\sigma} \times \vec{B} \right)$$  \hspace{1cm} (8)

In the case of round beams with $\sigma_x = \sigma_y = \sigma$, and of long bunches with $r \ll \sigma_y$, the electro-magnetic field has only two non-vanishing components, $E_r$ and $B_\phi$. The most direct and transparent way of finding $E_r$ and $B_\phi$ is using Gauss's and Ampere's laws, respectively, in cylindrical coordinates, instead of working with Eqs. (6) and (7). The result is:

$$2\pi r E_r = \left(1/\epsilon_0\right) \int_0^r 2\pi r'\rho(r')dr' \quad 2\pi r B_\phi = \mu_0 \int_0^r 2\pi r' \beta c \rho(r')dr'$$  \hspace{1cm} (9)

The dependences of $E_r$ and $B_\phi$ on the radius $r$ are the same. Hence, the Lorentz force has only an $r$ component $F_r$. For a round beam with a Gaussian distribution in two dimensions, given by Eq. (3), the integral can be expressed in closed form:

$$F_r(r) = \frac{n\epsilon^2 (1 + \beta^2)}{2\pi \epsilon_0 r} \left[ 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right) \right]$$  \hspace{1cm} (10)
The sign of $\beta^2$ corresponds to the case of the particles and the bunch moving in opposite directions. When particle and bunch travel in the same direction, the sign of $\beta^2$ is negative. The sign of the incoherent beam-beam force $F_\tau(r)$ depends on the signs of the charges in the two bunches. If the two beams contain particles of opposite charge, the force is attractive, and the sign is negative, as shown in Eq. (10). In Cartesian co-ordinates with $r = \sqrt{x^2 + y^2}$, the forces become:

$$F_x(x, y) = F_\tau(r)x/r \quad F_y(x, y) = F_\tau(r)y/r$$

(11)

The incoherent beam-beam force $F_\tau(r)$ is shown in Figure 1. For $r \ll \sigma$, the force $F_\tau(r)$ increases linearly, and is given by:

$$F_\tau(r) = -\frac{ne^2(1 + \beta^2)r}{4\pi\epsilon_0\sigma^2} \quad \text{for } r \ll \sigma$$

(12)

The linear beam-beam force is similar to that in a quadrupole, but in contrast to a quadrupole it has the same sign in both $x$ and $y$ directions. For $r \gg \sigma$, the force drops like $1/r$:

$$F_\tau(r) = -\frac{ne^2(1 + \beta^2)}{2\pi\epsilon_0r} \quad \text{for } r \gg \sigma$$

(13)

For elliptical Gaussian beams with $\sigma_x > \sigma_y$, it follows from Eqs. (5), (6) and (8) that the beam-beam force can be written as:

$$F_x - iF_y = -e(1 + \beta^2)(E_x - iE_y)$$

(14)

As Eq. (6) this result applies to arbitrary charge distributions in a bunch.

![Figure 1: Comparison of the incoherent (full line) and coherent (dashed line) beam-beam kicks for round Gaussian beams. The abscissa is drawn in units of $\sigma$; the ordinate is arbitrary.](image)
3.2 Incoherent beam-beam kick

The beam-beam force due to a bunch moving in the negative $s$-direction at speed $v$ can be obtained from the two-dimensional force in Eq. (10) by multiplying it with the longitudinal density distribution, centered at $s = -vt$:

$$F_r(r, s, t) = \frac{-N\varepsilon^2 (1 + \beta^2)}{(2\pi)^{3/2}\varepsilon_0 r\sigma_z} \left[ 1 - \exp \left( -\frac{r^2}{2\sigma_z^2} \right) \right] \exp \left[ -\frac{(s + vt)^2}{2\sigma_z^2} \right]$$  \hspace{1cm} (15)

This approximation is justified when the bunch length $\sigma_z$ is much larger than the transverse beam radii $\sigma_x$ and $\sigma_y$. The beam-beam kick $\Delta r'$ is obtained from the beam-beam force by integration over the collision, remembering that the test particle is at $s = vt$:

$$mc\beta\gamma \Delta r' = \int_{-\infty}^{+\infty} F_r(r, s, t) \, dt$$  \hspace{1cm} (16)

Introducing the classical particle radius $r_0 = e^2/4\pi\varepsilon_0 mc^2$, assuming $\beta \approx 1$, and performing the integration over $t$, we find:

$$\Delta r' = -\frac{2N r_0}{\gamma r} \left[ 1 - \exp \left( -\frac{r^2}{2\sigma_z^2} \right) \right]$$  \hspace{1cm} (17)

For $r \ll \sigma$, the beam-beam kick approaches the following limit, where $\delta$ is the inverse focal length of the quadrupole representing the beam-beam kick:

$$\Delta r' = -\frac{N r_0 r}{\gamma \sigma_z^2} = -r\delta \quad \text{for} \quad r \ll \sigma$$  \hspace{1cm} (18)

3.3 Incoherent linear beam-beam tune shift

The following calculation applies to both the horizontal and the vertical plane. Therefore, I use the general subscript $z$. Using Eq. (18) for $z \ll \sigma$, the linear map $M$ in the $z$-plane through an arc of the storage ring with phase advance $2\pi Q_z$ and amplitude function $\beta_z$ at the collision point and the beam-beam collision becomes:

$$M = \begin{pmatrix} \cos 2\pi Q_z & \beta_z \sin 2\pi Q_z \\ -\beta_z^{-1} \sin 2\pi Q_z & \cos 2\pi Q_z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\delta & 1 \end{pmatrix}$$  \hspace{1cm} (19)

The trace of the map $M$ yields the perturbed tune $(Q_z + \xi_z)$:

$$\text{Tr}(M) = 2\cos 2\pi (Q_z + \xi_z) = 2\cos 2\pi Q_z - \beta_z \delta \sin 2\pi Q_z$$  \hspace{1cm} (20)

Expanding the trigonometric functions and assuming $\beta \approx 1$ as in Eq. (18) yields for the linear beam-beam tune shift $\xi_z \ll 1$:

$$\xi_z = \beta_z \delta / 4\pi = \frac{N r_0 \beta_z}{4\pi \gamma \sigma_z^2}$$  \hspace{1cm} (21)

It is easy enough to compute the linear beam-beam tune shifts for Gaussian beams with an elliptical cross section and rms radii $\sigma_x$ and $\sigma_y$, by expanding the functions appearing in Eq. (5) into Taylor series, keeping only the lowest order terms in the real and imaginary
parts, respectively, and repeating the calculations leading to Eq. (21). For \( \beta \approx 1 \) the result for the beam-beam tune shift in the \( z \)-plane, \( \xi_z \), is:

\[
\xi_z = \frac{N r_0 \beta_z}{2 \pi \gamma \sigma_z (\sigma_z + \sigma_y)}
\]

Inserting the vertical beam-beam tune shift \( \xi_y \) into the luminosity formula Eq. (1), and assuming \( \sigma_y \ll \sigma_z \), yields the following expression which is often used in the discussion of storage ring performance:

\[
L \approx \frac{N f k \xi_y \gamma}{2 r_0 \beta_y}
\]

(23)

The product \( N f k \) in the numerator is the circulating current in one beam, often limited by effects other than beam-beam collisions, i.e. collective effects and/or RF power. Beam-beam effects enter into Eq. (23) by the factor \( \xi_y \) in the numerator. Therefore, all storage rings operate close to the beam-beam limit. The energy in terms of \( \gamma \) also appears in the numerator. It is also advantageous to have a small value of \( \beta_y \) at the collision point, since it appears in the denominator. Therefore, the beam-beam collisions take place in low-\( \beta \) insertions in all modern storage rings.

3.4 Incoherent beam-beam limit

The limit of stability of the linear betatron oscillations is reached when \( \text{Tr}(M) = \pm 2 \) in Eq. (20). Solving it for \( \dot{\xi}_z \) leads to the following expression for \( \dot{\xi}_z > 0 \):

\[
\dot{\xi}_z = \begin{cases} 
(1/2\pi) \cot \pi Q_z & \text{for } 0 < Q_z \text{ mod } 1 < 0.5 \\
-(1/2\pi) \tan \pi Q_z & \text{for } 0.5 < Q_z \text{ mod } 1 < 1
\end{cases}
\]

(24)

The linear beam-beam limit \( \dot{\xi}_z \), shown in Figure 2, is periodic in \( Q_z \) with period one half. For most fractional tunes this limit is much higher than the observed limit \( \xi \approx 0.03 \). This suggests that the observed beam-beam limit \( \xi \) is not caused by the linear beam-beam limit \( \dot{\xi}_z \).

4 COHERENT BEAM-BEAM EFFECTS

Coherent beam-beam effects arise from the forces which an exciting bunch exerts on a whole test bunch during a beam-beam collision. Based on the material already presented, we will first calculate the coherent beam-beam kicks, then linearize them to obtain the linear coherent tune shift, and finally compute the coherent beam-beam limit.

4.1 Coherent beam-beam kicks

Coherent beam-beam kicks are the kicks which a whole test bunch receives by colliding with a whole bunch travelling in the opposite direction. They are obtained from the incoherent beam-beam kicks by integration over the charge distribution in the test bunch. The symmetry of the beam-beam force implies that the coherent beam-beam kick vanishes for head-on collisions. Let us treat the two beams on an equal footing, with the first beam acting on the second beam and vice-versa, and introduce offsets \( \bar{x}_\pm \) and \( \bar{y}_\pm \) of the two beams. In order to evaluate the coherent horizontal beam-beam kick \( \Delta \bar{x}_\pm \) for Gaussian beams with elliptical cross section, we follow [18] and start from the integral
representation of the beam-beam potential in Eq. (4), integrated over the beam-beam collision, using Eqs. (3), (6), (14), and (16):

\[-\frac{2\pi \sigma_x \gamma \Delta \bar{x}}{Nr_0} = \int_{-\infty}^{+\infty} dxdyG(x + \bar{x}, y + \bar{y}; \sigma_x, \sigma_y) \frac{\partial}{\partial x} \int_0^\infty \exp \left( -\frac{(x+\bar{x})^2}{2\sigma_x^2 + t} - \frac{(y+\bar{y})^2}{2\sigma_y^2 + t} \right) \frac{dt}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \]

(25)

The equation for the coherent vertical beam-beam kick \( \Delta \bar{y} \) is very similar. An integration by parts of the right hand side moves the differential operator from the beam-beam potential to the Gaussian density distribution:

\[-\int_{-\infty}^{+\infty} dxdy \frac{\partial}{\partial x} G(x + \bar{x}, y + \bar{y}, \sigma_x, \sigma_y) \int_0^\infty \exp \left( -\frac{(x+\bar{x})^2}{2\sigma_x^2 + t} - \frac{(y+\bar{y})^2}{2\sigma_y^2 + t} \right) \frac{dt}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \]

(26)

Because of the symmetry of \( G \) in \( x \) and \( \bar{x} \) the differentiation with respect to \( x \) may be replaced by a differentiation with respect to \( \bar{x} \):

\[= \int_{-\infty}^{+\infty} dxdy \frac{\partial}{\partial \bar{x}} G(x + \bar{x}, y + \bar{y}, \sigma_x, \sigma_y) \int_0^\infty \exp \left( -\frac{(x+\bar{x})^2}{2\sigma_x^2 + t} - \frac{(y+\bar{y})^2}{2\sigma_y^2 + t} \right) \frac{dt}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \]

(27)

Interchanging integrals and differentiation moves the differentiation outside the integrals:

\[= \frac{\partial}{\partial \bar{x}} \int_{-\infty}^{+\infty} dxdyG(x + \bar{x}, y + \bar{y}, \sigma_x, \sigma_y) \int_0^\infty \exp \left( -\frac{(x+\bar{x})^2}{2\sigma_x^2 + t} - \frac{(y+\bar{y})^2}{2\sigma_y^2 + t} \right) \frac{dt}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \]

(28)
The integrals in $x$ and $y$ have the same form, and can be done analytically. The remaining integral in $t$ is very similar to the beam-beam potential in Eq. (4):

$$U_c = \frac{n\epsilon}{4\pi\epsilon_0} \int_0^\infty \exp \left( \frac{-(\vec{x}_+ - \vec{x}_-)^2}{4\sigma_x^2 + t} - \frac{-(\vec{y}_+ - \vec{y}_-)^2}{4\sigma_y^2 + t} \right) \frac{dt}{\sqrt{(4\sigma_x^2 + t)(4\sigma_y^2 + t)}}$$

(29)

The coherent beam-beam potential $U_c$ contains the differences of the offsets $\vec{x}_\pm$ and $\vec{y}_\pm$ instead of $x$ and $y$, and a factor 4 instead of 2 in front of the $\sigma$’s. In the case of round Gaussian beams this yields the following expression for the coherent beam-beam kick $\Delta \vec{r}'$ with $\vec{r} = \sqrt{(\vec{x}_+ - \vec{x}_-)^2 + (\vec{y}_+ - \vec{y}_-)^2}$:

$$\Delta \vec{r}' = -\frac{2N\gamma\sigma_0}{\gamma \vec{r}} \left[ 1 - \exp \left( -\frac{\vec{r}^2}{4\gamma^2} \right) \right]$$

(30)

Figure 1 shows a comparison of the incoherent and the coherent beam-beam kicks. The coherent kick has half the slope of incoherent kick for $\vec{r} \ll \sigma$. The two kicks are equal for $\vec{r} \gg \sigma$.

4.2 Linear coherent beam-beam tune shift

Using the same arguments as in the case of the incoherent beam-beam kicks, we find for the coherent beam-beam kick for $\vec{r} \ll \sigma$ for round Gaussian beams with $\beta \approx 1$:

$$\Delta \vec{r}' = -\frac{N\gamma\sigma_0}{2\gamma \vec{r}} \Delta \quad \text{for} \quad \vec{r} \ll \sigma$$

(31)

Here, $\Delta = \delta/2$ is the inverse focal length of the quadrupole representing the linear coherent beam-beam kick. The linear coherent beam-beam tune shift $\Xi_z \ll 1$ can be calculated for $\vec{x}_+ - \vec{x}_- \ll \sigma$ and $\vec{y}_+ - \vec{y}_- \ll \sigma$ and becomes just one half of the linear incoherent tune shift $\xi_z$ shown in Eq. (21):

$$\Xi_z = \frac{N\gamma\sigma_0}{8\pi\gamma \sigma^2} \quad \text{for} \quad \vec{x}_+ - \vec{x}_- \ll \sigma \text{ and } \vec{y}_+ - \vec{y}_- \ll \sigma$$

(32)

This result can be generalized for elliptical Gaussian beams:

$$\Xi_z = \frac{N\gamma\sigma_0}{4\pi\gamma \sigma_x(\sigma_x + \sigma_y)} \quad \text{for} \quad \vec{x}_+ - \vec{x}_- \ll \sigma_x \text{ and } \vec{y}_+ - \vec{y}_- \ll \sigma_y$$

(33)

Not surprisingly, this is one half of the incoherent beam-beam tune shift in Eq. (22). If the two beams have different rms radii at the interaction points, all $\sigma$’s appearing in Eq. (33) must be replaced by $\sqrt{(\sigma_x^2 + \sigma_y^2)/2}$ [18].

4.3 Coherent beam-beam limit

I work in the approximation $z \ll \sigma$ and describe the bunch motion in two beams and one plane by matrices. The rotation matrix $R$ operates on the transpose of the vector $\vec{z} = (z_+, z'_+, z_-, z'_-)$, and transports bunches between collision points. In the case of just
one bunch in each of the two beams, the order of $R$ is four:

$$
R = \begin{pmatrix}
  C_+ & \beta_+ S_+ & 0 & 0 \\
-\frac{S_+/\beta_+}{C_+} & C_- & 0 & 0 \\
0 & 0 & C_- & \beta_- S_- \\
0 & 0 & -\frac{S_-/\beta_-}{C_-} & C_-
\end{pmatrix}
$$

(34)

Here $C_\pm = \cos \mu_\pm$ and $S_\pm = \sin \mu_\pm$, $\mu_\pm$ is the phase advance between the collision points, and the index $\pm$ marks the beam. The $\beta_\pm$ are taken at the interaction point(s), the $\alpha_\pm$ there are assumed to vanish. The motion of each bunch is described by a $2 \times 2$ block matrix. The block matrices for all bunches are arranged along the main diagonal. The kick matrix $K$ describes the beam-beam kicks and in the case of equal bunches is given by:

$$
K = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-\Delta & 1 & +\Delta & 0 \\
0 & 0 & 1 & 0 \\
+\Delta & 0 & -\Delta & 1
\end{pmatrix}
$$

(35)

The kick matrix $K$ has ones along the main diagonal. For each pair of bunches there are four kick terms $\Delta$ which link the changes of the slopes to the positions of the two bunches which are colliding at a collision point. The generalization to $k > 1$ bunches is obvious. Then the kick matrix $K$ contains four kick terms for all pairs of bunches which collide at the same time. The stability of the one-turn map consisting of $2k$ $R$-matrices alternating with $2k$ $K$-matrices of order $4k$ can be analysed by computing its eigenvalues. Closed analytical solutions for the coherent beam-beam limit $\Xi$ are known in symmetrical cases [19]. When all $2k$ interaction points and all $k$ bunches in the two beams have the same parameters, and if all $2k$ phase advances between the interaction points are the same, the coherent beam-beam limit $\Xi$ is:

$$
\Xi = \frac{\cos(\pi Q/k) - \cos((Q + 1)\pi/k)}{2\pi \sin(\pi Q/k)}
$$

(36)

Here, $\{\ldots\}$ denotes the integral part. Figure 3 shows the coherent beam-beam limit $\Xi$ for $k = 4$ bunches in each beam, colliding in eight collision points. $\Xi$ is periodic every four units of tune $Q$, and approaches zero when the tune $Q$ approaches an integer from below. With increasing $k$, $\Xi$ decreases. For $k = 4$, it is already much smaller than the incoherent limit $\xi$, as becomes obvious by comparing Figures 2 and 3.

5 NONLINEAR BEAM-BEAM EFFECTS

The nonlinear variation of the beam-beam force with $r$ in a round Gaussian beam, shown in Eq. (10), causes a tune shift $\Delta Q(a)$ with amplitude $a$ and a tune spread in the beams, and drives nonlinear resonances. The usual method for treating these effects is Hamiltonian perturbation theory. Figure 4 shows $\Delta Q(a)/\xi$ as a function of $a/\sigma$ for round Gaussian beams.

5.1 Hamiltonian formalism

Following [20] I start from the differential equation for betatron oscillations in the $z$ coordinate with a periodic linear driving term $K(s)$ and a periodic beam-beam kick,
Figure 3: Coherent beam-beam limit $\Xi$ for $k = 4$ bunches colliding in eight collision points as a function of the tune $Q$. The abscissa is $Q$. The ordinate $\Xi$ is periodic every four units of $Q$.

Figure 4: Tune shift versus amplitude for round Gaussian beams. The abscissa is the amplitude $a$ in units of $\sigma$. The ordinate is the ratio $\Delta Q(a)/\Delta Q(0)$. 
obtained from Eqs. (17) and (21), and restrict the derivation to one dimension and round Gaussian beams:
\[
\frac{d^2 z}{ds^2} + K(s) z = -\frac{8\pi \sigma^2 \xi}{\beta_z} \left[ \frac{1 - \exp(-z^2/2\sigma^2)}{z} \right] \delta_p(s)
\]  
(37)

Here \(\delta_p(s)\) is the periodic \(\delta\)-function with period \(2\pi R\), where \(R\) is the average radius, assuming just one beam-beam collision per turn. I now apply a Courant-Snyder transformation \([21]\) to normalized betatron phase space \((\eta, \eta')\) with \(\eta = z/\sqrt{\beta_z}\) and \(\theta = \int ds/(Q\beta_z)\):
\[
\frac{d^2 \eta}{d\theta^2} + Q^2 \eta = -\frac{8\pi Q \xi \sigma^2}{\beta_z \eta} \left[ 1 - \exp\left( -\frac{\beta_z \eta^2}{2\sigma^2} \right) \right] \delta_p(\theta)
\]  
(38)

The new independent variable \(\theta\) and \(\delta_p(\theta)\) have period \(2\pi\), \(\beta_z\) is taken at the collision point, and \(Q\) is the tune. The solutions of the homogeneous equation are circles, and the constants \(\epsilon\) and \(\phi\) are given by the initial conditions:
\[
\eta(\theta) = \sqrt{\epsilon} \cos(Q\theta + \varphi) = \sqrt{\epsilon} \cos \phi
\]  
(39)

To solve the inhomogeneous Eq. (38), we assume that \(\epsilon\) and \(\phi\) are slowly varying functions of \(\theta\), use the method of variation of constants, and obtain the following set of equations for \(\eta\) and \(\eta'\), where in the second equation we already neglect terms in \(\epsilon'\) and \(\phi'\), where the prime \('\) denotes differentiation with respect to \(\theta\):
\[
\eta = \sqrt{\epsilon} \cos \phi \qquad \eta' = -Q\sqrt{\epsilon} \sin \phi
\]  
(40)

Changing to action-angle variables with
\[
\epsilon = \eta^2 + (\eta'/Q)^2 \quad \phi = -\arctan(\eta'/Q\eta)
\]  
(41)

and differentiating, we obtain first-order differential equations for \(\epsilon\) and \(\phi\):
\[
\frac{d\epsilon}{d\theta} = 2\eta \eta' + \frac{2\eta'' \eta'}{Q^2} = \frac{16\pi \xi \sigma^2}{Q \eta \beta_z} \left[ 1 - \exp\left( -\frac{\beta_z \eta^2}{2\sigma^2} \right) \right] \delta_p(\theta)
\]  
(42)
\[
\frac{d\phi}{d\theta} = -\frac{1}{1 + [\eta'/Q(\eta)^2]} \left( \frac{\eta''}{Q} - \frac{\eta'^2}{Q \eta^2} \right) = Q + \frac{8\pi Q \xi \sigma^2}{\epsilon \beta_z} \left[ 1 - \exp\left( -\frac{\beta_z \eta^2}{2\sigma^2} \right) \right] \delta_p(\theta)
\]  
(43)

The final step in first order perturbation theory is substituting the unperturbed solution into the right-hand sides, and replacing the periodic \(\delta\)-function by its Fourier expansion
\(\delta_p(\theta) = (1/2\pi) \sum_{m=-\infty}^{\infty} \exp(-im\theta)\) which yields:
\[
\frac{d\epsilon}{d\theta} = 8\xi \sigma^2 \sin \phi \sqrt{\epsilon/\beta_z} \left[ \frac{1 - \exp(-\beta_z \eta^2/2\sigma^2)}{\eta \sqrt{\beta_z}} \right] \sum_{m=-\infty}^{\infty} \exp(-im\theta)
\]  
(44)
\[
\frac{d\phi}{d\theta} = \left( Q + \frac{4\xi \sigma^2 \cos \phi}{\sqrt{\epsilon \beta_z}} \right) \left[ \frac{1 - \exp(-\beta_z \eta^2/2\sigma^2)}{\eta \sqrt{\beta_z}} \right] \sum_{m=-\infty}^{\infty} \exp(-im\theta)
\]  
(45)
5.2 Tune shift with amplitude

The tune shift with amplitude follows from Eq. (45) by only taking the constant term in the Fourier expansion of the $\delta$-function, and averaging the remainder over all phases $\phi$ [20]. Observing that $\cos^2\phi = (1 + \cos 2\phi)/2$, and using the integral representation of the modified Bessel function $I_0$ of order 0 [12], we find:

$$\frac{d\phi}{d\theta} = Q + \frac{\xi}{\alpha \pi} \int_0^{2\pi} \left[ 1 - \exp(-\alpha \cos^2 \phi) \right] d\phi = Q + \frac{(2\xi/\alpha)}{1 - \exp(-\alpha/2)I_0(\alpha/2)}$$

(46)

Here $\alpha = \epsilon \beta_z/(2\sigma^2) = (a/\sigma)^2/2$ where $a$ is the betatron amplitude. The extra term on the right hand side is the nonlinear tune shift $\Delta Q(a)$ with amplitude, which is shown in Figure 4:

$$\Delta Q(a)/\xi = \left(\frac{2}{\alpha}\right) \left[ 1 - \exp(-\alpha/2)I_0(\alpha/2) \right]$$

(47)

The tune shift $\Delta Q(a)$ is approximately equal to $\xi$ for $a \ll \sigma$. However, for $a \gg \sigma$, the tune shift $\Delta Q(a)$ is much smaller than $\xi$. This behaviour of $\Delta Q(a)$ is completely different from that due to multipole components of the magnetic guiding field in a storage ring. For components with eight or more poles, the absolute value of the tune shift is a monotonically increasing function of the amplitude in first order. The tune shifts with amplitude of elliptical Gaussian beams are functions of the horizontal and vertical amplitudes, $a$ and $b$, and of the ratio $\sigma_y/\sigma_x$ [22, 23]. The tune shifts with amplitude cause a tune spread within the beam. For the Gaussian density distribution in Eq. (3), the distributions in the amplitudes $a$ and $b$ are Rayleigh distributions given by:

$$R(a) = \left(\frac{a}{\sigma_x^2}\right) \exp(-a^2/2\sigma_x^2) \quad R(b) = \left(\frac{b}{\sigma_y^2}\right) \exp(-b^2/2\sigma_y^2)$$

(48)
The results of a Monte Carlo calculation [22] of the distribution functions of the horizontal and vertical beam-beam tune shifts for a beam with $\sigma_y/\sigma_x = 0.1$, based on $10^6$ random pairs of amplitudes $(a, b)$, are shown in Figure 5. The tune spreads are almost as large as the linear beam-beam tune shifts $\xi_x$ and $\xi_y$, respectively. The horizontal distribution has a peak close to $\xi_x$ and a long tail towards smaller tune shifts. The vertical distribution is more symmetrical.

![LHC footprint for head-on collisions](image)

Figure 6: Footprint of head-on beam-beam collisions of round Gaussian beams in the LHC. Abscissa and ordinate are the amplitude-dependent beam-beam tune shifts $\Delta Q_x$ and $\Delta Q_y$, respectively, in units of the linear beam-beam tune shift $\xi$. The lines mark constant amplitudes of horizontal and vertical betatron oscillations.

Another way of representing the effects of the beam-beam tune shift is known as footprints, and shown in Figure 6. Here one plots the tunes $Q_x$ and $Q_y$ in the $(Q_x, Q_y)$-plane with the amplitudes of the betatron oscillations as a parameter. In the case of equal emittances, equal $\beta$-functions at the interaction points in the horizontal and vertical plane, and head-on collisions, the footprint is symmetrical with respect to the diagonal line $Q_x = Q_y$. Particles with vanishing betatron amplitudes have the highest tune shifts $\Delta Q_x \approx \Delta Q_y \rightarrow \xi$, particles with large amplitudes the smallest ones. The tune shifts are concentrated in the neighbourhood of the diagonal line $\Delta Q_x \approx \Delta Q_y$. 
5.3 Beam-beam resonances

In order to compute the excitation of nonlinear resonances driven by the beam-beam collisions, the slowly varying terms in the equations of motion, Eqs. (44) and (45), must be identified and extracted. We obtain closed expressions for the resonance excitation by following [20], and replacing the function in the square bracket by its Fourier transform \( \tilde{g}(\omega) \):

\[
1 - \exp(-\beta_z \eta^2 / 2\sigma^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) \exp(i\omega \sqrt{\beta_z} \cos \phi) d\omega
\]

Using the identities for Bessel functions \( J_n \) of order \( n \) [13]

\[
J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x) \quad J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x)
\]

and the following relation which follows from [24] after a little algebra

\[
\exp(i\omega \sqrt{\beta_z} \cos \phi) = \sum_{k=-\infty}^{\infty} J_k(\omega \sqrt{\beta_z}) \exp[i(k\phi + \pi/2)]
\]

and integrating once by parts, the equations for \( \epsilon \) and \( \phi \) take the form:

\[
\frac{d\epsilon}{d\theta} = -\frac{4\xi \sigma^2}{\pi \beta_z} \sum_{n,m=-\infty}^{\infty} n^2 \exp[i(n\phi - m\theta)] \int_{-\infty}^{\infty} \tilde{g}(\omega) J_n(\omega \sqrt{\beta_z}) d\omega / \omega
\]

\[
\frac{d\phi}{d\theta} = Q + \frac{2i\xi \sigma^2}{\beta_z \epsilon} \sum_{n,m=-\infty}^{\infty} i^n \exp[i(n\phi - m\theta)] \int_{-\infty}^{\infty} \tilde{g}'(\omega) J_n(\omega \sqrt{\beta_z}) d\omega
\]

These equations are still an exact representation of the original differential equation and the beam-beam kicks. Let us define a resonance by choosing two relatively prime integers \( p \) and \( q \) such that the phase \( \chi = p\phi - q\theta \) can be considered as varying more slowly than other phases, and keep in the double sum only terms of the form \( \exp[i\ell(p\phi - q\theta)] \), where \( \ell \) is any integer – positive, negative, or zero. Then the equations for \( \epsilon \) and \( \chi \) become approximately:

\[
\frac{d\epsilon}{d\theta} = -\frac{4p\xi \sigma^2}{\pi \beta_z} \sum_{\ell=-\infty}^{\infty} \ell \exp[i\ell(\chi + p\pi/2)] \int_{-\infty}^{\infty} \tilde{g}(\omega) J_{\ell p}(\omega \sqrt{\beta_z}) d\omega / \omega
\]

\[
\frac{d\chi}{d\theta} = (pQ - q) + \frac{2i\xi \sigma^2}{\beta_z \epsilon} \sum_{\ell=-\infty}^{\infty} \exp[i\ell(\chi + p\pi/2)] \int_{-\infty}^{\infty} \tilde{g}'(\omega) J_{\ell p}(\omega \sqrt{\beta_z}) d\omega
\]

The sums in the equations for \( \epsilon \) and \( \chi \) can be simplified, using \( J_{-2n}(x) = J_{2n}(x) \):

\[
\frac{d\epsilon}{d\theta} = -\frac{8i\xi \sigma^2}{\beta_z} \sum_{\ell=1}^{\infty} \ell \sin[\ell(\chi + p\pi/2)] \int_{-\infty}^{\infty} \tilde{g}(\omega) J_{\ell p}(\omega \sqrt{\beta_z}) d\omega / \omega
\]

\[
\frac{d\chi}{d\theta} = (pQ - q) + \frac{4i\xi \sigma^2}{\beta_z \epsilon} \left[ \frac{1}{2} \int_{-\infty}^{\infty} \tilde{g}'(\omega) J_0(\omega \sqrt{\beta_z}) d\omega \\
+ \sum_{i=1}^{\infty} \cos[\ell(\chi + p\pi/2)] \int_{-\infty}^{\infty} \tilde{g}'(\omega) J_{\ell p}(\omega \sqrt{\beta_z}) d\omega \right]
\]
We now limit the sums over \( \ell \) to the term with \( \ell = 1 \). This is a good approximation [20] for resonances of high order \( p \). According to Eq. (56), the fixed points occur at \( \chi + p\pi/2 = 0 \) and \( \pi \). Their amplitudes are obtained by setting Eq. (57) equal to zero. The result is:

\[
-\frac{pQ - q}{\xi} = \frac{2ip\alpha^2}{\pi \beta \varepsilon} \left[ \int_{-\infty}^{\infty} \tilde{g}'(\omega) J_0(\omega \sqrt{\varepsilon \beta}) d\omega \pm 2 \int_{-\infty}^{\infty} \tilde{g}'(\omega) J_p(\omega \sqrt{\varepsilon \beta}) d\omega \right]
\]

(58)

\[
= (2p/\alpha) \left[ 1 - \exp(-\alpha/2) I_0(\alpha/2) \pm 2 \exp(-\alpha/2) I_{p/2}(\alpha/2) \right]
\]

(59)

Here, the abbreviation \( \alpha = \beta \varepsilon / 2\sigma^2 \) in Eq. (47) has been used again, as well as the explicit equation [20] \( \tilde{g}'(\omega) = -2i\delta(\omega) + i\sigma \sqrt{2\pi} \exp(-\omega^2\sigma^2/2) \) and the identity

\[
(1/\sqrt{2\pi}) \int_{-\infty}^{\infty} du \exp(-u^2/2) J_p(u\sqrt{2\alpha}) = \exp(-\alpha/2) I_{p/2}(\alpha/2)
\]

(60)

Since \( \tilde{g}(\omega) \) is an even function of \( \omega \), the integral vanishes when \( p \) is odd, i.e. resonances of

![Figure 7: Resonance width for round Gaussian beams versus amplitude. The abscissa is the amplitude in units of \( \sigma \). The ordinate is \( \log_{10}(W_p/\xi) \), i.e. the resonance width in units of \( \xi \). The orders of the resonances are \( p = 4, 6, 8, 10, 12 \), starting at the top.](image)

odd order are not driven by head-on beam-beam collisions. For even \( p \) it can be expressed [14] in terms of the confluent hypergeometric function \( M(\frac{p+1}{2}, p + 1, -\alpha) \). Applying a Kummer transformation [15] changes the sign of \( \alpha \). The resulting expression in \( M \) can be written in terms of modified Bessel functions [16]. The accumulated expression of \( \Gamma \) functions, powers of 2, etc., is reduced to unity by applying the duplication formula [17] to \( \Gamma(2\alpha) \). The first two terms in the square bracket of Eq. (59) are the amplitude dependent beam-beam tune shift which we already encountered in Eq. (47). The term following the \( \pm \) sign yields the range of the tune \( Q \) for which the amplitudes of the fixed points are
smaller than that of the particle with scaled amplitude $\alpha$, i.e. for which the particle is in the resonance. Solving Eq. (59) for this range of $Q$, we obtain the width of the resonances $W_p(\alpha)$ of order $p$ driven by the beam-beam collisions:

$$W_p(\alpha) = \xi(4/\alpha) \exp(-\alpha/2)I_{p/2}(\alpha/2)$$  

(61)

Figure 7 shows $\log_{10}(W_p/\xi)$ of the resonance width as a function of the betatron amplitude $a$ in units of the rms beam radius $\sigma$, i.e. of $a/\sigma = \sqrt{2}\alpha$. The resonance width decreases with the order $p$ of the resonances. The maximum width moves to higher $a/\sigma$ with increasing $p$. Resonances of odd order are not excited because of the symmetry of the beam-beam force for head-on collisions.

References


[17] ibid., 256.


