The Burkhardt-Cottingham Sum Rule in Perturbative QCD

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Abstract

We have computed in perturbative QCD the structure function $g_2(x,Q^2)$ of polarized lepton production at order $\alpha_s$ on a quark target of mass $m$. Upon taking the first moment we find that it vanishes exactly at all orders in $m^2/Q^2$ as demanded by the validity of the Burkhardt-Cottingham sum rule, contrary to a recent claim.

In a recent paper [1] it was claimed that the Burkhardt-Cottingham (BC) sum rule [2] for the structure function $g_2$ of polarized deep inelastic scattering:

$$\int_0^1 \frac{dx}{Q^2} g_2(x,Q^2) = 0$$

(1)
is violated at order $\alpha_s$ for a target quark in QCD and an explicit calculation was presented with a non vanishing result. This conclusion is somewhat puzzling because it is well known [3] that in the light cone expansion there is no operator corresponding to a finite value of the BC moment. Thus finding a non vanishing value at order $\alpha_s$ in perturbative QCD would require some very peculiar mechanism to take place. Also in presence of a non vanishing result it would be intriguing to understand how to go from a quark to a nucleon target. In fact, in absence of an operator, the usual place (i.e. the nucleon matrix element) where the structure of the nucleon is hidden would be missing in this case. This issue is made particularly important by the perspective of a near future measurement of $g_2$ in experiments at CERN, SLAC and HERA. Since the validity of the BC sum rule is one of the planned experimental checks it is essential to know whether the right hand side of eq.(1) is expected to be of order $\alpha_s$, which at moderate $Q^2$ is not that small. In view of the conceptual and practical interest of this issue we have made an independent calculation of $g_2$ at order $\alpha_s$. Our result is at variance with that of ref.[1] and indeed we find that the BC sum rule is satisfied. In this note we present our calculation and we show that the BC sum rule at order $\alpha_s$ is valid exactly at all orders in $m^2/Q^2$ and not just in a linear approximation in that ratio. Finally we discuss some further features of our calculation.

In order to define $g_2$ one starts from the Fourier transform of the forward matrix element between polarized nucleons of the product of two electromagnetic currents which is given by the sum of a symmetric (S) and an antisymmetric (A) tensor [3,4]:

$$W_{\mu\nu} = \frac{-i}{4\pi} \int d^4x \epsilon^{\mu\nu} (p,s) J_{\mu}(x) J_{\nu}(0) \cdot (p,s) = W^S_{\mu\nu} + iW^A_{\mu\nu}$$

(2)
The symmetric part is independent of the polarization and contains the unpolarized structure functions. The antisymmetric part, linear in the polarization vector $\epsilon_\mu$, can be written in terms of two structure functions $g_1$ and $g_2$:

$$iW^A_{\mu\nu} = \frac{ie}{2m} \epsilon^\alpha \epsilon^\beta \delta_{\mu\alpha} \epsilon_{\nu\beta} q^\lambda \left[ \frac{q^2}{(pq)^2} g_1(x,Q^2) + \frac{(pq) x^\lambda - (qs) p^\lambda}{(pq)^2} g_2(x,Q^2) \right]$$

(3)

where $p$ is the target (of mass $m$) four-momentum, $q$ the virtual photon four-momentum, and $x$ is the Bjorken variable $x = Q^2/(2pq)$. With the present normalisation, at lowest order, $g_1 = 1/2 \epsilon_\mu^\nu \delta(1-x)$ for a target quark of charge $q_1$. It is simple to show [4] that for longitudinal polarization what matters is the combination $[g_1 - (2m\nu/Q)^2 g_2]$ so that the $g_2$
contribution is negligible at large $Q^2$. Instead, for transverse polarization, $iW_{\mu\nu}$ is proportional to $2mQ/Q_1 [g_1 + g_2]$, so that $g_1$ and $g_2$ enter with equal coefficients but the whole contribution is of order $1/Q$. Note that, in spite of the factor of m, $W^\mu_{\lambda\nu}$ does not vanish for m=0, because the longitudinal component of $s^\mu$ is proportional to 1/m.

It is straightforward to compute $iW_{\mu\nu}$ for a free quark, no matter if massive or massless. In either case it is found that $g_2$ vanishes (while $g_1$ is non-vanishing and finite even for m = 0). The quarks must be interacting and/or virtual in order to contribute to $g_2$. Since the free quark case is so peculiar it is important to ground the discussion of $g_2$ on the general light cone expansion method. In the light-cone expansion of $iW_{\mu\nu}$, two classes of operators occur [3,4]. For the simplest case of non-singlet channels, one class is represented by the operator of twist 2 given by:

$$\hat{q}_{\lambda i} \gamma^\mu p_{i} [D^\mu_{i_1} \ldots D^\mu_{i_k}] \lambda_{q}, \quad k \geq 0$$

(4)

where $\lambda_i$ is a flavour matrix. To the second class belong operators that need at least two indices (we call them twist-3 operators), given by:

$$\hat{q}_{\mu i} \gamma^\mu p_{i} [D^\mu_{i_1} \ldots D^\mu_{i_k}] \lambda_{q}, \quad k > 0$$

$$\hat{q}_{\mu i} \gamma^\mu p_{i} [D^\mu_{i_1} \ldots D^\mu_{i_k}] \lambda_{q}$$

(5)

(first antisymmetrise in $\mu$ and $\lambda$ and then symmetrise in all $\mu$:s). The last operator is proportional to the quark mass matrix $m$. The moments of $g_1$ and $g_2$ are given by (only even moments are accessible to the light-cone method):

$$\int_0^{1} dx x^k g_1(x, Q^2) = a_k, \quad k = 0, 2, 4, ...$$

$$\int_0^{1} dx x^k g_2(x, Q^2) = \frac{k}{k+1} (d_k - a_k), \quad k = 2, 4, ...$$

(6)

where $a_k$ and $d_k$ are the matrix elements of the $k$th operators of twist 2 and 3 respectively.

We see that $g_1$ is only determined by the twist-2 operator sequence, while the second class of operators only enters in the expression of $g_2$.

The first important issue about $g_2$ is whether or not the operators of twist 3 are important. Note that if the twist-3 operators are negligible, i.e. all $a_k = 0$, then $g_2$ is completely determined in terms of $g_1$ by the so-called Wandzura–Wilczek relation (WW) [3,5]:

$$g_2 = g_2^{WW} = -g_1 + \int_0^{1} dx x g_1(x, Q^2)$$

(7)

while in general

$$g_2 = g_2^{WW} + \hat{g}_2$$

(8)

with $\hat{g}_2$ being the contribution from the $d_k$ sequence. Also note that for a free quark the $WW$ sum rule is violated. In fact for a free quark $g_2^{WW}$ is exactly cancelled by $\hat{g}_2$ because we have said that $g_2$ is zero for a free-quark target. In this case we can show that the contribution to $g_2$ is entirely given by the mass operator in eq. (5) which plays a crucial role on determining $g_2$ for a quark (also at non trivial order in $\alpha_0$) [6].

Going back to eq. (6) we see that the light-cone approach does not associate an operator to the first moment of $g_2$. Because of this fact and adding some specific assumptions on the Regge behaviour at small $x$ of the forward photon-hadron amplitude one can derive the BC sum rule, i.e. the vanishing of the first moment of $g_2$. However, the presence of the factor of $k$ in the r.h.s. of eq. (6) suggests the validity of the (BC) sum rule, eq.(1). In a partonic approach or, more generally, from a field theory point of view, based on, say, Feynman diagrams to all orders, the primary quantities are not moments but x-dependent structure functions. If we imagine one well-defined x density which generates all moments, the simplest thing is to continue the dependence of moments on $n$ to all values of $n$ where the resulting dependence is non-singular. Then one way for the BC sum rule to hold would be that $d_k - 1/k$ for $k > 0$ (we know that $a_k$ is non-singular because the first moment of $g_1$ is finite). Alternatively one can think of contributions which, at least for $Q^2 \rightarrow \infty$, become proportional to $\ln(x)$; this would make the first moment discontinuous with respect to the extrapolation in $n$. However, in this case, the corresponding violation of the BC sum rule would not show up from the data at finite x for sufficiently large $Q^2$.

The real and virtual diagrams that contribute to $g_2$ at order $\alpha_0$ are shown in fig.(1). From the tensor $W_{\mu\nu}$ defined in eq.(2) one projects $g_2$ out by multiplication with the tensor

$$L_{\mu\nu} = \delta_{\mu\nu} [q^2 + (p \cdot q)]$$

(9)

where p, s and q are the initial quark four-momentum, its polarisation four-vector and the virtual photon four-momentum, respectively. One obtains:

$$L_{\mu\nu} W^\mu = -2m(\alpha_0) [1 + m^2(pq)_{2} [Q^2 - (q^2)^2] g_2$$

(10)

One great simplifying element in this calculation is that real and virtual contributions to $g_2$ are separately finite for $m > 0$, without need to introduce a cutoff for infrared soft gluon divergences. In fact it is well known that soft gluon real emission gives a factorised contribution proportional to the lowest order rate. However we have already mentioned that $g_2$ is zero at lowest order for a free quark with mass m. Then the real emission term is infrared finite and consequently the virtual contribution is also finite. Similarly there are no ultraviolet divergences, because the $q^0$ part of the vertex gives a zero contribution to $g_2$ again due to the vanishing of the lowest order result. Collinear singularities are
regulated by the quark mass. In conclusion no regularisation has to be introduced beyond keeping the mass \( m \) finite.

The calculation of the real part is done by computer algebraic manipulation of the Dirac traces plus reducing the phase space integrals to a few basic ones which could be computed exactly in analytic form. The final result is:

\[
\zeta_2\text{-real} = \frac{C_F a_s}{8\pi} \frac{\xi^2}{4(\xi^2 + 1)^2} \left[ c_1 + \frac{2c_2}{\sqrt{4\xi^2 + 1}} \log \frac{2\xi + 1 + \sqrt{4\xi^2 + 1}}{2\xi + 1 - \sqrt{4\xi^2 + 1}} \right]
\]

(11)

with \( r = m^2/Q^2 \) and

\[
c_1 = -\frac{1}{(\xi + 1)^2} \left[ 52r^3 \xi^3 - 72r^2 \xi^2 + 78r \xi^2 + 32r \xi^2 + 86r^2 + 6r^2 + 12r^2 + 2 + 2r^2 + 51r - 25 \xi + 3r + 13 \right]
\]

(12)

\[
c_2 = \xi \left[ 16 r^2 + 4 r^2 + 15r + r + 2 \right]
\]

(13)

Note that both \( c_1 \) and \( c_2 \) are proportional to \( \xi \) and vanish at \( \xi = 0 \), so that the BC moment is convergent.

As for the virtual contribution we first observe that self energy diagrams can be disregarded because they give a \( \gamma_5 \) term of no effect on \( g_2 \). The vertex correction can be cast into the form:

\[
V^\mu = e_q \bar{u} (p') F_1(Q^2) \gamma^\mu + \frac{a_s F_2(Q^2)}{4 \pi} \left[ q \gamma^\mu \gamma^\nu \gamma^\rho \right] u(p)
\]

(14)

with \( p' = p + q \). For the same reason as above, only \( F_2 \) contributes to \( g_2 \). The function \( F_2 \) was computed in QED in ref.\( [8] \) where is given by

\[
\left( F_2(x) \right)_{\text{QED}} = \frac{\log \theta}{1 - \theta^2} ; \quad \theta = -\frac{1 + \sqrt{1 + 4r}}{1 + \sqrt{1 + 4r}}
\]

(15)

Note the limits \( F_2(0) = 1/2 \) and \( F_2(Q^2 = 0) = -m^2/Q^2 \log(m^2/Q^2) + c(m^2/Q^2) \). One can check that the signs are correct by writing the vertex at \( Q^2 = 0 \). By using \( F_1(0) = 1 \), one obtains

\[
V^\mu = \bar{u} (p') \left[ \frac{(p + p')^\mu}{2m} + (1 + \frac{a_s}{2\pi} \log \frac{1 + \gamma^\nu}{1 - \gamma^\nu} \right] u(p)
\]

(16)

which is the well known correction to the electron anomalous magnetic moment. Multiplying the QED result for \( F_2 \) by \( C_F = 3 \) and performing a simple algebra one arrives at the result:

\[
\zeta_2\text{-virtual} = -\frac{C_F a_s}{8\pi} \frac{Q^2}{m^2} F_2(Q^2)_{\text{QED}} \delta(1-x)
\]

(17)

From eqs. (11-13) and (17) the announced result of a vanishing first moment of \( g_2 \), according to the BC sum rule in eq.(1), follows identically at all orders in \( r = m^2/Q^2 \).

A linear version in \( m^2/Q^2 \) of eqs. (11-13) and (17) is more transparent and is useful to see the relation with the result of ref.[1]. We find

\[
\zeta_2\text{-real} = \frac{C_F a_s}{8\pi} \frac{\xi}{(1-\xi) x} \left[ \frac{1}{(1-x)(x + m^2/Q^2)} - 4 \log \frac{m^2 x(1-x)}{Q^2} \right]
\]

(18)

with the well known definition [4] of the "plus" distribution. Note that potentially important terms proportional to \( 1/(\xi + 1)^2 \) which could in principle contribute to the sum rule are in fact suppressed by a factor of \( 1-x \) and therefore have been dropped. The difference between eqs.(18-19) and the results of ref.[1] is simply in the 1 that multiplies \( \delta(1-x) \) in eq.(18).

The \( n \)-th moment of \( g_2 \) that is obtained from the sum of eqs. (18,19) is given by:

\[
M_n = \frac{1}{0} \int_0^1 x^n g_2(x,Q^2) = \frac{C_F a_s}{8\pi} \frac{Q^2}{m^2} \left[ A_n \log m^2/Q^2 + B_n \right]
\]

(20)

with

\[
A_n = \frac{2n}{n+1} ; \quad B_n = \frac{n+5}{n+1} \sum_{k=1}^n \frac{1}{(n+1)^2} + 4 \frac{n^2}{(n+1)^2} + 1 - \frac{13}{n+1}
\]

(21)

Of course, when \( g_2 \) is given, all its moments can be computed and not only the ones for \( n \) odd which appear in the light cone expansion eq.(6). Once again, note the vanishing of the first moment, \( n=1 \), which corresponds to the BC sum rule. The result for the logarithmic terms is established by the anomalous dimension of the twist-2 operator in eq.(4) (which determines \( V^\mu_{\text{WW}} \) in eq.(7)) and only the mass operator among the twist-3 operators in eq.(5). In fact these are the only operators whose coefficients start at order \( 0 \) in \( a_s \). However the mass operator has a non diagonal anomalous dimension matrix with mixings to additional operators involving gluons [7], so that the non diagonal anomalous dimension matrix element also enter.

In conclusion there is no sign of a violation of the BC sum rule in first order perturbative QCD on a target quark. It is well known [3] that the BC sum rule corresponds to a superconvergence sum rule for an amplitude whose imaginary part is \( g_2 \). A violation of
the sum rule corresponds to the presence of a fixed pole or a cut with non-vanishing residue or discontinuity at large $Q^2$ and clearly there are no such features in low order perturbative QCD. Of course the problem remains of the relation of the present results on $g_2$ on a target quark with those relevant for the real case of nucleon scattering. In fact we expect light quark mass contributions to the nucleon structure function to be suppressed by a factor $m/M$. We believe however the quark case is anyway interesting at the conceptual level and a complete clarification of this model is a prerequisite for an understanding of the proton case.

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References


Figure Caption

[1] Relevant real (c,d) and virtual (a,b) diagrams for $g_2$ at order $\alpha_s$. 